Econometrica Supplementary Material

SUPPLEMENT TO "CREDIBLE AUCTIONS: A TRILEMMA"

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S.0. EXTENSIONS AND OTHER APPLICATIONS

HERE, WE APPLY the notion of credibility to an auction setting with affiliated values, multi-item auctions with matroid constraints, as well as a simple public good setting.

S.1. AFFILIATED VALUES

Here, we use a discrete model of single-object auctions, as in Section 3.2. As is well-known, relaxing the independence assumption even slightly results in auctions that extract all bidder surplus (Cremer and McLean (1988)). The standard (static) mechanisms for full surplus extraction make each bidder's payment depend on the other bidders' types. The auctioneer can increase revenue by misrepresenting the other bidders' types, so these mechanisms are not credible. Even using extensive forms does not generally permit credible full surplus extraction.

DEFINITION S.1: (G, S_N) **extracts full surplus** if it is BIC, has voluntary participation, and $\pi(G, S_N) = \mathbf{E}_{\theta_N}[\max\{0, \max_{i \in N} \theta_i\}]$.

PROPOSITION S.1: *The Cremer and McLean (1988) conditions are not sufficient for the existence of a credible protocol that extracts full surplus.*

PROOF: There are two bidders i and j, each with two possible values $0 < \theta_i < \theta'_i < \theta_j < \theta'_j$. The joint distribution of types is $f_N(\theta_i, \theta_j) = f_N(\theta'_i, \theta'_j) = 1/3$, $f_N(\theta_i, \theta'_j) = f_N(\theta'_i, \theta_j) = 1/6$, which satisfies the full rank condition of Cremer and McLean (1988, Theorem 2).

For a given protocol (G, S_N), consider the induced allocation rule $\tilde{y}$ and transfer rule $\tilde{t}_N$. Suppose (G, S_N) is credible and extracts full surplus. By Propositions 1 and 2, it is without loss of generality to restrict (G, S_N) so that after j is called to play once, he is never called to play again.

Take any information set I_j at which j is called to play. Since (G, S_N) is credible, for each action that j takes at I_j, there is a unique transfer from j if j wins (Proposition 5). Since (G, S_N) extracts full surplus, j wins no matter whether he plays $S_j(I_j, \theta_j)$ or $S_j(I_j, \theta'_j)$. Since (G, S_N) is BIC, j's transfer after playing $S_j(I_j, \theta_j)$ is the same as j's transfer after playing $S_j(I_j, \theta'_j)$.

This argument applies to every information set at which j is called to play, so j's transfer does not depend on his own type; $\tilde{t}_j(\theta_i, \theta_j) = \tilde{t}_j(\theta_i, \theta'_j)$ and $\tilde{t}_j(\theta'_i, \theta_j) = \tilde{t}_j(\theta'_i, \theta'_j)$.

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Since j always wins the object, the auctioneer can safely deviate to communicate with j as though i's type is θ_i or as though i's type is θ'_i. Since (G, S_N) is credible, j's transfer does not depend on i's type; $\tilde{t}_j(\theta_i, \theta_j) = \tilde{t}_j(\theta'_i, \theta_j)$. Thus, j's transfer is some constant $\overline{t}_j$ across all type profiles. $\theta_j - \overline{t}_j = 0$, so $\theta'_j - \overline{t}_j > 0$, and (G, S_N) does not extract full surplus, a contradiction. *Q.E.D.*

Optimal auctions with correlation are better-behaved if we additionally require ex post incentive compatibility and ex post individual rationality.[1] The virtual values machinery generalizes, and a modified ascending auction is optimal under some standard assumptions (Roughgarden and Talgam-Cohen (2013)). That modified ascending auction is credible. We now make the claim precisely.

Consider some probability mass function $f_N : \Theta_N \to [0, 1]$. We assume symmetric type spaces, $K_i = K_j = K$ and $\theta_i^k = \theta_j^k$ for all i, j, k, as well as affiliated types (Milgrom and Weber (1982)).

DEFINITION S.2: f_N is **symmetric** if its value is equal under any permutation of its arguments. f_N is **affiliated** if, for all θ_N, θ'_N,

$$f_N(\theta_N \vee \theta'_N) f_N(\theta_N \wedge \theta'_N) \geq f_N(\theta_N) f_N(\theta'_N), \tag{S.1}$$

where $\vee$ is the component-wise maximum and $\wedge$ the component-wise minimum.

For a protocol (G, S_N), let $\tilde{y}_i^{G,S_N}(\theta_N)$ be an indicator variable equal to 1 if i wins the object at θ_N and 0 otherwise. (We suppress the independence on (G, S_N) to ease notation.)

DEFINITION S.3: (G, S_N) is **optimal among ex post auctions** if it maximizes expected revenue subject to the constraints:

1. Ex post incentive compatibility. For all i, θ_i, θ'_i, θ_{-i}:

$$\theta_i \tilde{y}_i(\theta_i, \theta_{-i}) - \tilde{t}_i(\theta_i, \theta_{-i}) \geq \theta_i \tilde{y}_i(\theta'_i, \theta_{-i}) - \tilde{t}_i(\theta'_i, \theta_{-i}). \tag{S.2}$$

2. Ex post individual rationality. For all i, θ_i, θ_{-i}:

$$\theta_i \tilde{y}_i(\theta_i, \theta_{-i}) - \tilde{t}_i(\theta_i, \theta_{-i}) \geq 0. \tag{S.3}$$

DEFINITION S.4: The **conditional virtual value** of θ_i^k given θ_{-i} is

$$\eta_i(\theta_i^k \mid \theta_{-i}) \equiv \theta_i^k - \frac{1 - F_i(\theta_i^k \mid \theta_{-i})}{f_i(\theta_i^k \mid \theta_{-i})} (\theta_i^{k+1} - \theta_i^k), \tag{S.4}$$

where $f_i(\cdot \mid \theta_{-i})$ is the conditional distribution of θ_i given θ_{-i} and $F_i(\cdot \mid \theta_{-i})$ is the conditional cumulative distribution. f_N is **regular** if, for all i and θ_{-i}, $\eta_i(\theta_i \mid \theta_{-i})$ is strictly increasing in θ_i.

[1]Ex post incentive compatibility and ex post individual rationality are implied by strategy-proofness and voluntary participation (Definition 5). For extensive forms, ex post incentive compatibility and strategy-proofness are not equivalent. An opponent strategy profile S_{-i} consists of complete contingent plans of action. Ex post incentive compatibility in effect considers only plans 'consistent with' some opponent type profile θ_{-i}.

We now define a modified ascending auction. When there is only one bidder left, the auctioneer sets a reserve so that she only sells to types with a positive conditional virtual value.[2] That reserve depends on the final bids from the bidders who quit.

DEFINITION S.5: (G, S_N) is a **quirky ascending auction** if:

1. All bidders start as active, with initial bids $(b_i)_{i\in N} := (\theta_i^1)_{i\in N}$.
2. Whenever there is more than one active bidder, some active bidder i is called to play, where $b_i \leq \max_{j\neq i} b_j$.
   (a) i chooses between two actions; he can either raise b_i by one increment[3] or quit.
   (b) If i quits, then he is no longer active.
3. When there is exactly one active bidder i, if $\eta_i(b_i \mid b_{-i}) \leq 0$, i chooses to either raise his bid to $\min b_i' \mid \eta_i(b_i' \mid b_{-i}) > 0$ or quit. Otherwise i wins and pays b_i.
4. Inactive bidders do not win the object, and have zero transfers.
5. S_i specifies that i bids b_i if and only if $\theta_i \geq b_i$.

PROPOSITION S.2: *Assume f_N is symmetric, affiliated, and regular. If (G, S_N) is a quirky ascending auction, then it is optimal among ex post auctions and is credible.*

PROOF: Define $\nu(\theta_i, \theta_{-i}) = \theta_i\tilde{y}_i(\theta_i, \theta_{-i}) - \tilde{t}_i(\theta_i, \theta_{-i})$.

We can use the same method as in Elkind (2007) to derive an upper bound on $\nu(\theta_i, \theta_{-i})$ under ex post incentive compatibility and ex post individual rationality, namely,

$$\nu\big(\theta_i^k, \theta_{-i}\big) \geq \sum_{l=2}^{k} \tilde{y}_i\big(\theta_i^{l-1}, \theta_{-i}\big)\big(\theta_i^l - \theta_i^{l-1}\big). \tag{S.5}$$

This implies a bound on i's expected utility conditional on θ_{-i}, namely,

$$\begin{aligned}\mathbf{E}_{\theta_i}\big[\nu\big(\theta_i^k, \theta_{-i}\big) \mid \theta_{-i}\big] &\geq \sum_{k=2}^{K} f_i\big(\theta_i^k\big) \sum_{l=1}^{k} \tilde{y}_i\big(\theta_i^{l-1}, \theta_{-i}\big)\big(\theta_i^l - \theta_i^{l-1}\big) \\ &= \sum_{k=1}^{K} f_i\big(\theta_i^k \mid \theta_{-i}\big) \frac{1 - F_i\big(\theta_i^k \mid \theta_{-i}\big)}{f_i\big(\theta_i^k \mid \theta_{-i}\big)} \big(\theta_i^{k+1} - \theta_i^k\big)\tilde{y}_i\big(\theta_i^k, \theta_{-i}\big),\end{aligned} \tag{S.6}$$

which gives an upper bound on expected revenue

$$\begin{aligned}\pi(G, S_N) &= \sum_{i\in N} \mathbf{E}_{\theta_N}\big[\theta_i\tilde{y}_i(\theta_N) - \nu(\theta_i, \theta_{-i})\big] \\ &= \sum_{i\in N} \mathbf{E}_{\theta_{-i}}\big[\mathbf{E}_{\theta_i}\big[\theta_i\tilde{y}_i(\theta_N) - \nu(\theta_i, \theta_{-i}) \mid \theta_{-i}\big]\big] \\ &\leq \sum_{i\in N} \mathbf{E}_{\theta_{-i}}\big[\mathbf{E}_{\theta_i}\big[\eta_i(\theta_i \mid \theta_{-i})\tilde{y}_i(\theta_N) \mid \theta_{-i}\big]\big] = \mathbf{E}_{\theta_N}\Big[\sum_{i\in N} \eta_i(\theta_i \mid \theta_{-i})\tilde{y}_i(\theta_N)\Big].\end{aligned} \tag{S.7}$$

[2] This definition is due to Roughgarden and Talgam-Cohen (2013), and differs only in that our construction is for finite type spaces to allow the use of extensive game forms.

[3] That is, from θ_i^k to θ_i^{k+1}, where we set $\theta_i^{K+1} > \theta_i^K$.

Moreover, the above equation holds with equality if the local downward incentive constraints bind and the participation constraints bind for the lowest type, where these constraints are conditional on θ_{-i}.

We now apply the argument in Roughgarden and Talgam-Cohen (2013), which is written for continuous densities but works also for the discrete case. For the reader's convenience, we repeat it here.

LEMMA S.1: *If f_N is affiliated and $\theta_j < \theta'_j$, then $\eta_i(\theta_i \mid \theta_j, \theta_{N\setminus\{i,j\}}) \geq \eta_i(\theta_i \mid \theta'_j, \theta_{N\setminus\{i,j\}})$.*

By affiliation, $F_i(\theta_i \mid \theta'_j, \theta_{N\setminus\{i,j\}})$ dominates $F_i(\theta_i \mid \theta_j, \theta_{N\setminus\{i,j\}})$ in terms of hazard rate (Krishna (2010, Appendix D)), that is,

$$\frac{1 - F_i(\theta_i \mid \theta_j, \theta_{N\setminus\{i,j\}})}{f_i(\theta_i \mid \theta_j, \theta_{N\setminus\{i,j\}})} \leq \frac{1 - F_i(\theta_i \mid \theta'_j, \theta_{N\setminus\{i,j\}})}{f_i(\theta_i \mid \theta'_j, \theta_{N\setminus\{i,j\}})}, \tag{S.8}$$

which implies $\eta_i(\theta_i \mid \theta_j, \theta_{N\setminus\{i,j\}}) \geq \eta_i(\theta_i \mid \theta'_j, \theta_{N\setminus\{i,j\}})$. This proves Lemma S.1.

LEMMA S.2: *Assume f_N is symmetric, regular, and affiliated. For all $\theta_{N\setminus\{i,j\}}$, if $k \geq k'$, then $\eta_i(\theta_i^k \mid \theta_{N\setminus\{i,j\}}, \theta_j^{k'}) \geq \eta_j(\theta_j^{k'} \mid \theta_{N\setminus\{i,j\}}, \theta_i^k)$.*

PROOF:

$$\begin{aligned}
\eta_i(\theta_i^k \mid \theta_{N\setminus\{i,j\}}, \theta_j^{k'}) &\geq \theta_i^{k'} - \frac{1 - F_i(\theta_i^{k'} \mid \theta_{N\setminus\{i,j\}}, \theta_j^{k'})}{f_i(\theta_i^{k'} \mid \theta_{N\setminus\{i,j\}}, \theta_j^{k'})}(\theta_i^{k'+1} - \theta_i^{k'}) \\
&\geq \theta_i^{k'} - \frac{1 - F_i(\theta_i^{k'} \mid \theta_{N\setminus\{i,j\}}, \theta_j^{k})}{f_i(\theta_i^{k'} \mid \theta_{N\setminus\{i,j\}}, \theta_j^{k})}(\theta_i^{k'+1} - \theta_i^{k'}) \\
&= \eta_j(\theta_j^{k'} \mid \theta_{N\setminus\{i,j\}}, \theta_i^k),
\end{aligned} \tag{S.9}$$

where the first inequality follows from regularity, the second inequality follows from Lemma S.1, and the equality follows from symmetry. This proves Lemma S.2. *Q.E.D.*

By Lemma S.2, the right-hand side of Equation (S.7) is maximized by, at each θ_N, selling to some bidder in $\arg\max_i \theta_i$ if $\max_i \eta_i(\theta_i \mid \theta_{-i}) > 0$, and keeping the object otherwise. The quirky ascending auction does this, and additionally the local incentive constraints bind downward and the participation constraint of the lowest type binds, so the left-hand side of Equation (S.7) is equal to the right-hand side. Thus, any quirky ascending auction is optimal among ex post mechanisms.

It remains to prove that the quirky ascending auction is credible. Once more, note that S_i is a best response to any safe deviation by the auctioneer. Under any safe deviation, if $b_i \leq \theta_i$, then bidder i's utility is non-negative if he continues bidding according to S_i, and zero if he quits now. If $b_i > \theta_i$, then bidder i's utility is non-positive if he continues bidding, and zero if he quits now. Thus, S_i is a best response to any safe deviation by the auctioneer, regardless of θ_{-i}. For any safe deviation S'_0, the corresponding protocol (G', S_N) is ex post incentive-compatible and ex post individually rational. Suppose that S'_0 is profitable, so (G', S_N) yields strictly more expected revenue than (G, S_N). Since (G, S_N) is optimal among ex post mechanisms, we have the desired contradiction. *Q.E.D.*

S.2. AUCTIONS WITH MATROID CONSTRAINTS

So far we have assumed that, in each feasible allocation, there is at most one winner. Suppose instead that multiple bidders can be satisfied at once; that is, the feasible sets of winners are a family $\mathcal{F} \subseteq 2^N$. Each bidder's type is independently distributed according to $f_i : \Theta_i \to (0, 1]$, where i's utility at allocation $Y \in \mathcal{F}$ is $\theta_i 1_{i \in Y} - t_i$. Each bidder observes whether or not he is in the allocation, and his own transfer.

DEFINITION S.6: $\mathcal{F}$ is a **matroid** if:
1. $\emptyset \in \mathcal{F}$.
2. If $Y' \subset Y$ and $Y \in \mathcal{F}$, then $Y' \in \mathcal{F}$.
3. For any $Y, Y' \in \mathcal{F}$, if $|Y| > |Y'|$, then there exists $i \in Y \setminus Y'$ such that $Y' \cup \{i\} \in \mathcal{F}$.

Here are some examples of matroids:
1. The auctioneer can sell at most k items; that is, $Y \in \mathcal{F}$ if and only if $|Y| \leq k$.
2. There are incumbent bidders and new entrants. The auctioneer sells k licenses, and at most l licenses can be sold to incumbents.
3. The auctioneer is selling the edges of a graph. Each edge is demanded by exactly one bidder, and the auctioneer can sell any set of edges that is acyclic.
4. There are bands of spectrum $\{1, \dots, K\}$, and each band k is acceptable to a subset of bidders N_k. Each bidder is indifferent between bands that he finds acceptable. At most one bidder can be assigned to each band.

PROPOSITION S.3: *If $\mathcal{F}$ is a matroid, then there exists a credible strategy-proof optimal protocol.*

We describe this protocol informally, since the fine details parallel Definition 17, and our construction draws heavily on Bikhchandani, De Vries, Schummer, and Vohra (2011) and Milgrom and Segal (2017). Each bidder's starting bid is equal to his lowest possible type. We score bids according to their ironed virtual values, and keep track of a set of active bidders $\hat{N}$.

Bidder i is **essential** at $\hat{N}$ if, for all $Y \subseteq \hat{N}$, if $Y \in \mathcal{F}$, then $Y \cup \{i\} \in \mathcal{F}$. At each step, we choose an active bidder i whose score is minimal in $\hat{N}$. If i's score is positive and i is essential at $\hat{N}$, then we guarantee that i is in the allocation and charge him his current bid, removing him from $\hat{N}$. Otherwise, i chooses to either raise his bid until his score is positive and no longer minimal, or quit (in which case he is also removed from $\hat{N}$). The auction ends when $\hat{N} = \emptyset$.

The above protocol outputs the same allocation as a greedy algorithm that starts with the empty set and at each step adds a bidder with the highest ironed virtual value among those that can be feasibly added, until no bidders with positive ironed virtual values can be added (we prove this in the Appendix). By a standard result in combinatorial optimization (Hartline (2016, p. 134)), this greedy algorithm maximizes the ironed virtual value when $\mathcal{F}$ is a matroid. Given that the relevant participation constraints and incentive constraints bind, maximizing ironed virtual values implies that the protocol is optimal (Elkind (2007)).

The auction we described is credible, for the same reasons as before: Since truthful bidding is best response to any safe deviation, if the auctioneer could improve revenue by a safe deviation, she could have committed from the beginning to an alternative mechanism and increased revenue. Since the original protocol was optimal, we have a contradiction. The formal proof of Proposition S.3 follows.

PROOF: Suppose we construct ironed virtual values for discrete type spaces as in Elkind (2007). Let the protocol break ties according to some fixed order on N, when two bids have the same ironed virtual value.

Fix some type profile θ_N. Let us label bidders in decreasing order of ironed virtual values, $\{1, 2, \dots, n\}$, breaking ties according to the fixed order. Let $\{i^1, i^2, \dots, i^J\}$ be the set picked by the greedy algorithm, in order of selection (where the algorithm breaks ties using the same fixed order). We must show that the protocol described in Section S.2 results in the same allocation.

Take the greedy algorithm's jth pick, $i^j = k$. We will show that k is essential with respect to the set of active bidders $\hat{N}$ before k is asked to place a bid strictly above his type. Consider any step of the algorithm at which k, if not essential, would be asked to place a bid strictly above his type. At this step, $\hat{N} \subseteq \{1, 2, \dots, k\}$, since bidders with lower ironed virtual values have either been put in the allocation or quit (and similarly bidders with equal ironed virtual values but who lose ties to k).

Take any $Y \subseteq \{1, 2, \dots, k\}$ such that $Y \in \mathcal{F}$. We assert that $Y \cup \{k\} \in \mathcal{F}$. There are two cases: either $|Y| \geq j$ or $|Y| < j$.

Suppose $|Y| \geq j > |\{i^1, \dots, i^{j-1}\}|$. Since $\mathcal{F}$ is a matroid, there exists $l \in Y \setminus \{i^1, \dots, i^{j-1}\}$ such that $\{i^1, \dots, i^{j-1}\} \cup \{l\} \in \mathcal{F}$. If $Y \cup \{k\} \notin \mathcal{F}$, then $k \notin Y$, so $k \neq l$. Thus, $i^j = k$ is not the greedy algorithm's jth pick, a contradiction.

If $|Y| < j = |\{i^1, \dots, i^j\}|$, then since $\mathcal{F}$ is a matroid, there exists $l \in \{i^1, \dots, i^j\} \setminus Y$ such that $Y \cup \{l\} \in \mathcal{F}$ and $Y \cup \{l\} \subseteq \{1, \dots, k\}$. Thus, we can find $Y' \supset Y$ such that $|Y'| = j$, $Y' \subseteq \{1, \dots, k\}$, and $Y' \in \mathcal{F}$. From the argument in the previous paragraph, $Y' \cup \{k\} \in \mathcal{F}$, and, since $\mathcal{F}$ is a matroid, $Y \cup \{k\} \in \mathcal{F}$.

We have now established that, since $\hat{N} \subseteq \{1, 2, \dots, k\}$, k is essential with respect to $\hat{N}$. Thus, the jth pick of the greedy algorithm is in the allocation produced by the protocol. This argument holds for all j, so the protocol's allocation is a superset of $\{i^1, \dots, i^J\}$. But the protocol only sells to bidders with positive ironed virtual values, so its allocation is exactly $\{i^1, \dots, i^J\}$, and the protocol is optimal.

Finally, note that for any safe deviation, each bidder's 'truth-telling' strategy is a best response. That is, each bidder should keep bidding so long as the price he faces is weakly below his value, and quit otherwise. Thus, if the auctioneer has a profitable safe deviation, then the original protocol is not optimal, a contradiction. *Q.E.D.*

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