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SUPPLEMENT TO "IDENTIFICATION OF NONPARAMETRIC SIMULTANEOUS EQUATIONS MODELS WITH A RESIDUAL INDEX STRUCTURE"

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Berry and Haile (2018) considered identification in a class of nonparametric simultaneous equations models, providing several combinations of sufficient conditions on the joint density of structural errors and the support of instruments. We show here that, even when the instruments vary only over a small open ball, the requirements on the joint density may be viewed as mild in at least one formal sense.

S.1. INTRODUCTION

THE NOTATION AND MAINTAINED HYPOTHESES (Assumption 1) of the model are as given in Berry and Haile (2018). Here we demonstrate that, if $\mathbb{Y}'$ is the pre-image under r of any bounded open connected subset of $\mathbb{R}^J$, log densities satisfying the requirements of Corollary 2 in Berry and Haile (2018) form a dense open subset of all log densities on $\mathbb{R}^J$ that are twice continuously differentiable and possess a local maximum. This is true even when instruments vary only over an arbitrarily small open ball. This implies a form of generic identification of g on $\mathbb{X}$ and of r on $\mathbb{Y}'$.[1]

In the following section, we state and prove the genericity result (Theorem 4).[2] Section S.3 provides additional discussion of a perturbation function used in our proof.

S.2. GENERIC IDENTIFICATION

Let $\mathcal{G} = \times_j [g_j(\underline{x}_j), g_j(\overline{x}_j)] \subset g(\mathbb{X})$ be a compact "square" in $\mathbb{R}^J$ with width $w_x = g_j(\overline{x}_j) - g_j(\underline{x}_j) > 0$ for all j.[3] We form a tessellation of $\mathbb{R}^J$ using squares of width $w_x/2$.[4] Let $\sigma = (\sigma_1, \dots, \sigma_J)$ denote a J-vector of integers. For each $\sigma \in \mathbb{Z}^J$, define the square

$$sq_\sigma = \times_j \left[\frac{2\sigma_j - 1}{4} w_x, \frac{2\sigma_j + 1}{4} w_x\right].$$

Then $\{sq_\sigma\}_{\sigma \in \mathbb{Z}^J}$ forms a regular tessellation of $\mathbb{R}^J$ such that, for every $y \in \mathbb{Y}$, the set $\{r(y) - \mathcal{G}\}$ (and therefore $\{r(y) - g(\mathbb{X})\}$) covers some square sq_σ.

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[1]Because $r(\mathbb{Y}')$ may be arbitrarily large, the gap between generic identification on $\mathbb{Y}'$ and generic identification on the pre-image of $\mathbb{R}^J$ (i.e., on $\mathbb{Y}$) may be of little importance.

[2]For clarity, we number results by continuing the sequences begun in Berry and Haile (2018).

[3]Such a square must exist. Because $\mathbb{X}$ is open, it contains a rectangle $\hat{\mathcal{X}} = \times_j (x_j^L, x_j^H)$ with $x_j^H > x_j^L$ $\forall j$. By continuity and strict monotonicity of each g_j, $g(\hat{\mathcal{X}}) = \times_j (g_j(x_j^L), g_j(x_j^H))$, which contains a compact square.

[4]In three or more dimensions, a tessellation is also known as a honeycomb, and what we call a square is a cube or hypercube. For simplicity, we use the language of the two-dimensional case.

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Given any open set $S \subset \mathbb{R}^J$ that is bounded and connected, let $\mathbb{Y}_S$ denote the pre-image of S under r. Because r is continuous, $\mathbb{Y}_S$ is open; and because r has continuous inverse (see the proof of Lemma 1 in Berry and Haile (2018)), $\mathbb{Y}_S$ is connected. Boundedness of S and $\mathcal{G}$ implies that there is a finite set $\mathbb{Z}_S \subset \mathbb{Z}^J$ such that

$$\bigcup_{\sigma \in \mathbb{Z}_S} sq_q \supset \bigcup_{y \in \mathbb{Y}_S} \{r(y) - \mathcal{G}\}. \tag{S.1}$$

Then, by construction, for every $y \in \mathbb{Y}_S$ there exists $\sigma \in \mathbb{Z}_S$ such that $\{r(y) - g(\mathbb{X})\}$ covers sq_σ. So if Berry and Haile's (2017) Assumption 1 and Condition M hold, the following is sufficient for their Corollary 2 to apply, yielding identification of g on $\mathbb{X}$ and of r on $\mathbb{Y}_S$.

CONDITION H: For every $\sigma \in \mathbb{Z}_S$, $\partial^2 \ln f(u)/\partial u\, \partial u^\intercal$ is nonsingular at some $u \in sq_\sigma$.

We show below that simultaneous satisfaction of Conditions M and H is generic in the space of log densities on $\mathbb{R}^J$ that are twice continuously differentiable and possess a local maximum. To define this space, first let $C^2(\mathbb{R}^J)$ denote the space of twice continuously differentiable real-valued functions on $\mathbb{R}^J$. We define a topology on $C^2(\mathbb{R}^J)$ using the C^2 extended norm $\|\cdot\|_{C^2}$, where

$$\|h\|_{C^2} = \sup_{u \in \mathbb{R}^J} \big|h(u)\big| + \max_{j \in \{1,\dots,J\}} \sup_{u \in \mathbb{R}^J} \left|\frac{\partial h(u)}{\partial u_j}\right| + \max_{j,k \in \{1,\dots,J\}} \sup_{u \in \mathbb{R}^J} \left|\frac{\partial^2 h(u)}{\partial u_j\, \partial u_k}\right|$$

for any $h \in C^2(\mathbb{R}^J)$. Under the induced extended metric, two functions h and $\hat{h}$ in $C^2(\mathbb{R}^J)$ are deemed to be "close" if these functions and their partial derivatives up to order 2 are uniformly close.[5] Let $\mathcal{F} \subset C^2(\mathbb{R}^J)$ denote the subspace (with subspace topology) of twice continuously differentiable log densities on $\mathbb{R}^J$ that possess a local maximum. We say that functions in a set $\mathcal{H} \subset \mathcal{F}$ are *generic* in $\mathcal{F}$ if $\mathcal{H}$ is a dense open subset of $\mathcal{F}$ (see, e.g., Mas-Colell (1985)).[6]

THEOREM 4: *Let Assumption* 1 *hold. Let* $S \subset \mathbb{R}^J$ *be bounded, open, and connected, with pre-image* $\mathbb{Y}_S$ *under* r. *Let the finite set* $\mathbb{Z}_S \subset \mathbb{Z}^J$ *satisfy* (S.1). *Then the set* $\mathcal{F}_S^* = \{\ln f \in \mathcal{F} :$ *Conditions* M *and* H *hold*$\}$ *is dense and open in* $\mathcal{F}$.

To prove this result, define the following subsets of $\mathcal{F}$:

$$\mathcal{F}_\sigma^H = \big\{\ln f \in \mathcal{F} : \partial^2 \ln f(u)/\partial u\, \partial u^\intercal \text{ is nonsingular at some } u \in sq_\sigma\big\},$$

$$\mathcal{F}^H = \{\ln f \in \mathcal{F} : \text{Condition H holds}\},$$

$$\mathcal{F}^M = \{\ln f \in \mathcal{F} : \text{Condition M holds}\}.$$

[5] A metric inducing the same topology is $\tilde{d}(h', h) = \frac{\|h'-h\|_{C^2}}{1+\|h'-h\|_{C^2}}$. We work with the C^2 extended metric to simplify exposition. Genericity can also be shown under the (coarser) topology of compact convergence (of sequences of functions in $C^2(\mathbb{R}^J)$ and their partial derivatives up to order 2).

[6] A weaker notion of genericity is that of a *residual* set (countable intersection of dense open subsets). With minor modification, our arguments below demonstrate that even the set $\mathcal{F}^* \subset \mathcal{F}_S^*$ (defined in (S.4)) is residual in $\mathcal{F}$. As discussed by Mas-Colell (1985), these topological definitions of genericity are standard in infinite-dimensional spaces but fall short of fully satisfactory notions of "typical" (see also Hunt, Sauer, and Yorke (1992), Anderson and Zame (2001), or Stinchcombe (2002)). Classification of our density conditions according to alternative notions of genericity is a potentially interesting direction for future work.

Then we have

$$\mathcal{F}^H = \bigcap_{\sigma \in \mathbb{Z}_S} \mathcal{F}^H_\sigma \tag{S.2}$$

and

$$\mathcal{F}^*_S = \mathcal{F}^M \cap \mathcal{F}^H. \tag{S.3}$$

Thus $\mathcal{F}^*_S$ is a finite intersection of subsets of $\mathcal{F}$. In Corollary 3 below (Section S.2.1), we show that $\mathcal{F}^*_S$ is dense in $\mathcal{F}$. Lemmas 12 and 13 below (Section S.2.2) show that $\mathcal{F}^M$ and each $\mathcal{F}^H_\sigma$ (for $\sigma \in \mathbb{Z}^J$) is open in $\mathcal{F}$. Theorem 4 then follows from (S.2) and (S.3).[7]

S.2.1. *Dense*

Let

$$\mathcal{F}^* = \mathcal{F}^M \cap \left\{ \bigcap_{\sigma \in \mathbb{Z}^J} \mathcal{F}^H_\sigma \right\}. \tag{S.4}$$

In this subsection, we prove the following result, whose corollary is immediate from the fact that $\mathcal{F}^* \subset \mathcal{F}^*_S$.

LEMMA 10: *$\mathcal{F}^*$ is dense in $\mathcal{F}$.*

COROLLARY 3: *$\mathcal{F}^*_S$ is dense in $\mathcal{F}$.*

Fix any $\ln f \in \mathcal{F}$ and let u^* be a point at which $\ln f$ has a local max. To prove Lemma 10, it is sufficient to show that for any $\varepsilon > 0$, there exists $\ln f^* \in \mathcal{F}^*$ such that $\| \ln f^* - \ln f \|_{C^2} < \varepsilon$. Let $w_f > 0$ be such that $\ln f(u^*) \geq \ln f(u)$ for all u in the square $\times_j [u^*_j - \frac{w_f}{2}, u^*_j + \frac{w_f}{2}]$.[8] Let s^* be a closed square with center u^* and width

$$w = \min\left\{ \frac{w_x}{4}, w_f \right\}.$$

For all j, let $\underline{u}^*_j$ and $\overline{u}^*_j$ be defined such that $s^* = \times_j [\underline{u}^*_j, \overline{u}^*_j]$.

Starting from s^*, form another tessellation of $\mathbb{R}^J$ using squares of width w. Let $\tau = (\tau_1, \dots, \tau_J)$ denote a J-vector of integers. For each $\tau \in \mathbb{Z}^J$, define the square $s_\tau = \times_j [\underline{u}^*_j + \tau_j w, \overline{u}^*_j + \tau_j w]$. Then $\{s_\tau\}_{\tau \in \mathbb{Z}^J}$ forms a regular tessellation of $\mathbb{R}^J$.[9] Let $\dot{u}_\tau = (\dot{u}_{\tau 1}, \dots, \dot{u}_{\tau J})$ denote the center of square s_τ. For $\tau = (0, \dots, 0)$, we then have $s_\tau = s^*$ and $\dot{u}_\tau = u^*$. For all $u \in \mathbb{R}^J$, let $\tau(u) \in \mathbb{Z}^J$ denote the index of a square such that $u \in s_{\tau(u)}$.[10] Observe that every cell of the tessellation $\{sq_\sigma\}_{\sigma \in \mathbb{Z}^J}$ covers at least one cell of the tessellation $\{s_\tau\}_{\tau \in \mathbb{Z}^J}$. We prove Lemma 10 by constructing an arbitrarily small perturbation of

[7] Although we focus on genericity of the conditions required by Berry and Haile's (2018, Corollary 2), Lemmas 10 and 12 below imply that $\mathcal{F}^M$ is a dense open subset of $\mathcal{F}$ (see Corollary 1 in Berry and Haile (2018)).

[8] Such w_f must exist since around any local max is an open ball on which $\ln f(u^*)$ is (at least weakly) maximal.

[9] Unlike the tessellation $\{sq_\sigma\}_{\sigma \in \mathbb{Z}^J}$, this tessellation may vary with the choice of $\ln f$ through the choice of the point u^* and width w.

[10] There will be only one such square for almost all u. However, any u on the boundary of one square will also be on the boundary of at least one other square. How the function $\tau(u)$ resolves this ambiguity does not matter for what follows.

$\ln f$ that (i) lies in $\mathcal{F}$, (ii) has a nondegenerate local max at u^*, and (iii) has a nonsingular Hessian matrix at the center of every square s_τ.

Let f denote the probability density function associated with the fixed log density $\ln f$ (i.e., $f = \exp(\ln f)$). For $v \in \mathbb{R}$, define[11]

$$p(v) = 1\{|v| \leq 1\}(1 - v^2)^3.$$

Given any $\lambda > 0$ and $\{\lambda_\tau \in (0, \lambda]\}_{\tau \in \mathbb{Z}^J}$, for all $u \in \mathbb{R}^J$ let

$$f_\lambda(u) = \kappa f(u) \exp\left(\lambda_{\tau(u)} \prod_{j=1}^{J} p\left(\frac{u_j - \dot{u}_{\tau(u)j}}{w/2}\right)\right), \tag{S.5}$$

with particular values of each λ_τ to be determined below. The scalar κ is chosen to ensure that $f_\lambda(u)$ integrates to 1 on $\mathbb{R}^J$, that is,

$$\kappa = \left[\int_{\mathbb{R}^J} f(u) \exp\left(\lambda_{\tau(u)} \prod_{j=1}^{J} p\left(\frac{u_j - \dot{u}_{\tau(u)j}}{w/2}\right)\right) du\right]^{-1}. \tag{S.6}$$

Because the term $\lambda_{\tau(u)} \prod_{j=1}^{J} p(\frac{u_j - \dot{u}_{\tau(u)j}}{w/2})$ takes only values between 0 and $\lambda_{\tau(u)}$ (see Section S.3), κ must lie in the interval $[\exp(-\lambda), 1]$. Thus, by construction the perturbed function f_λ is positive on $\mathbb{R}^J$, integrates to 1, and (see Section S.3) is twice continuously differentiable on $\mathbb{R}^J$.

We first show that $\ln f_\lambda$ is made arbitrarily close to $\ln f$ by setting λ sufficiently small.

CLAIM 1: *For any $\varepsilon > 0$, there exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$ and any $\{\lambda_\tau \in (0, \lambda]\}_{\tau \in \mathbb{Z}^J}$, $\|\ln f_\lambda - \ln f\|_{C^2} < \varepsilon$.*

PROOF: Fix any $\varepsilon > 0$. From (S.5),

$$\ln f_\lambda(u) - \ln f(u) = \lambda_{\tau(u)} \prod_{j} p\left(\frac{u_j - \dot{u}_{\tau(u)j}}{w/2}\right) + \ln \kappa. \tag{S.7}$$

So because $\ln \kappa \in [-\lambda, 0]$ and $\lambda_{\tau(u)} \prod_j p(\frac{u_j - \dot{u}_{\tau(u)j}}{w/2}) \in [0, \lambda]$,

$$\sup_{u \in \mathbb{R}^J} \left|\ln f_\lambda(u) - \ln f(u)\right| \leq \lambda. \tag{S.8}$$

Further, differentiating (S.7) (see Section S.3), we have

$$\frac{\partial \ln f_\lambda(u)}{\partial u_j} - \frac{\partial \ln f(u)}{\partial u_j} = \lambda_{\tau(u)} \left[\prod_{\ell \neq j} p\left(\frac{u_\ell - \dot{u}_{\tau(u)\ell}}{w/2}\right)\right] p'\left(\frac{u_j - \dot{u}_{\tau(u)j}}{w/2}\right)\left(\frac{w}{2}\right)^{-1}, \tag{S.9}$$

$$\frac{\partial^2 \ln f_\lambda(u)}{\partial u_j^2} - \frac{\partial^2 \ln f(u)}{\partial u_j^2} = \lambda_{\tau(u)} \left[\prod_{\ell \neq j} p\left(\frac{u_\ell - \dot{u}_{\tau(u)\ell}}{w/2}\right)\right] p''\left(\frac{u_j - \dot{u}_{\tau(u)j}}{w/2}\right)\left(\frac{w}{2}\right)^{-2}, \tag{S.10}$$

[11] The function p is proportional to a triweight kernel (see, e.g., Silverman (1986)). In Section S.3, we discuss some relevant properties of p and of a log density perturbed on a square using this function as in (S.5).

while for $j \neq k$,

$$\frac{\partial^2 \ln f_\lambda(u)}{\partial u_k\,\partial u_j} - \frac{\partial^2 \ln f(u)}{\partial u_k\,\partial u_j} = \lambda_{\tau(u)}\left[\prod_{\ell\neq j,k} p\left(\frac{u_\ell - \dot{u}_{\tau(u)\ell}}{w/2}\right)\right] p'\left(\frac{u_k - \dot{u}_{\tau(u)k}}{w/2}\right) p'\left(\frac{u_j - \dot{u}_{\tau(u)j}}{w/2}\right)\left(\frac{w}{2}\right)^{-2}. \tag{S.11}$$

The function p is bounded, as are its first and second derivatives (see Section S.3). So because $\lambda_{\tau(u)} \in (0, \lambda]$, (S.8)–(S.11) demonstrate that

$$\sup_{u\in\mathbb{R}^J}\left|\ln f_\lambda(u) - \ln f(u)\right| + \max_{j\in\{1,\dots,J\}} \sup_{u\in\mathbb{R}^J}\left|\frac{\partial \ln f_\lambda(u)}{\partial u_j} - \frac{\partial \ln f(u)}{\partial u_j}\right| + \max_{j,k\in\{1,\dots,J\}} \sup_{u\in\mathbb{R}^J}\left|\frac{\partial^2 \ln f_\lambda(u)}{\partial u_j\,\partial u_k} - \frac{\partial^2 \ln f(u)}{\partial u_j\,\partial u_k}\right| < \varepsilon$$

for all sufficiently small $\lambda > 0$. *Q.E.D.*

To complete the proof of Lemma 10, we show that, given any $\lambda > 0$, the scaling parameters $\{\lambda_\tau\}_{\tau\in\mathbb{Z}^J}$ can be chosen to ensure that $\ln f_\lambda \in \mathcal{F}^*$.

CLAIM 2: *For any* $\lambda > 0$, *there exist* $\{\lambda_\tau \in (0, \lambda]\}_{\tau\in\mathbb{Z}^J}$ *such that* $\ln f_\lambda \in \mathcal{F}^*$.

PROOF: Fix any $\lambda > 0$. We first show that there exist $\{\lambda_\tau \in (0, \lambda]\}_{\tau\in\mathbb{Z}^J}$ such that, for all $\sigma \in \mathbb{Z}^J$, $\partial^2 \ln f_\lambda(u)/\partial u\,\partial u^\top$ is nonsingular at some $u \in sq_\sigma$. Because every square sq_σ covers the square s_τ for some $\tau \in \mathbb{Z}^J$, it is sufficient that, for each τ and some $\lambda_\tau \in (0, \lambda]$, $\partial^2 \ln f_\lambda(u)/\partial u\,\partial u^\top$ is nonsingular at $u = \dot{u}_\tau$. Take any $\tau \in \mathbb{Z}^J$. Equations (S.9)–(S.11) imply (see the values of $p(0)$, $p'(0)$ and $p''(0)$ given in Section S.3)

$$\frac{\partial^2 \ln f_\lambda(\dot{u}_\tau)}{\partial u\,\partial u^\top} = \frac{\partial^2 \ln f(\dot{u}_\tau)}{\partial u\,\partial u^\top} - \left(\frac{24\lambda_\tau}{w^2}\right) I_J,$$

where I_J is the identity matrix. The eigenvalues of $\partial^2 \ln f_\lambda(\dot{u}_\tau)/\partial u\,\partial u^\top$ are therefore equal to those of $\partial^2 \ln f(\dot{u}_\tau)/\partial u\,\partial u^\top$ minus $\frac{24\lambda_\tau}{w^2}$. So for almost all values of $\lambda_\tau \in (0, \lambda]$, all eigenvalues of $\partial^2 \ln f_\lambda(\dot{u}_\tau)/\partial u\,\partial u^\top$ are nonzero, ensuring that $\ln f_\lambda \in \mathcal{F}^H$.[12] Fixing any such $\{\lambda_\tau \in (0, \lambda]\}_{\tau\in\mathbb{Z}^J}$, we then complete the proof by showing that $\ln f_\lambda \in \mathcal{F}^M$. By our choice of the point u^* and square s^*, $u^* \in \arg\max_{u\in s^*} \ln f(u)$. And because $u^* = \dot{u}_\tau$ for $\tau = (0, \dots, 0)$, $u^* = \arg\max_{u\in s^*} [\ln f_\lambda(u) - \ln f(u)]$. Thus, u^* is a local maximum of $\ln f_\lambda$ and, by the choice of λ_τ above for $\tau = (0, \dots, 0)$, $\partial^2 \ln f_\lambda(u^*)/\partial u\,\partial u^\top$ is nonsingular. *Q.E.D.*

S.2.2. *Open*

To prove the required openness results, we begin with a result from the literature on Morse functions.[13] Given any compact $K \subset \mathbb{R}^J$, let $C^2(K)$ denote the space of twice con-

[12] Since we needed only *one* such λ_τ, we have shown more than necessary. Thus the "abundance" of perturbations lying in $\mathcal{F}^*$ is even greater than required by the notion of C^2-denseness.

[13] See, for example, Lemma 5.32 in Banyaga and Hurtubise (2004). We provide a proof here for completeness.

tinuously differentiable real-valued functions on K. For $h \in C^2(K)$, let

$$\|h\|_{C_K^2} = \sup_{u\in K}\big|h(u)\big| + \max_{j\in\{1,\dots,J\}} \sup_{u\in K}\left|\frac{\partial h(u)}{\partial u_j}\right| + \max_{j,k\in\{1,\dots,J\}} \sup_{u\in K}\left|\frac{\partial^2 h(u)}{\partial u_j\,\partial u_k}\right|.$$

LEMMA 11: *Suppose that $\mathfrak{f} \in C^2(K)$ has no degenerate critical point. Then there exists $\varepsilon > 0$ such that any $\mathfrak{g} \in C^2(K)$ satisfying $\|\mathfrak{f} - \mathfrak{g}\|_{C_K^2} < \varepsilon$ has no degenerate critical point.*

PROOF: For any $\mathfrak{h} \in C^2(K)$ and $u \in K$, define $\rho_{\mathfrak{h}}(u) = \sum_j |\frac{\partial \mathfrak{h}(u)}{\partial u_j}| + |\det(\frac{\partial^2 \mathfrak{h}(u)}{\partial u\,\partial u^\top})|$. A function $\mathfrak{h} \in C^2(K)$ has no degenerate critical point on K if and only if $\rho_{\mathfrak{h}}(u) > 0$ for all $u \in K$. So by the hypothesis of the lemma, $\rho_{\mathfrak{f}}(u) > 0$ for all $u \in K$. Because K is compact and $\rho_{\mathfrak{f}}$ is continuous, there must exist $\delta > 0$ such that $\rho_{\mathfrak{f}}(u) > \delta$ for all $u \in K$. If $\|\mathfrak{f} - \mathfrak{g}\|_{C_K^2} < \varepsilon$, then for all $u \in K$,

$$\left|\frac{\partial \mathfrak{f}(u)}{\partial u_j} - \frac{\partial \mathfrak{g}(u)}{\partial u_j}\right| < \varepsilon \quad \forall j,$$
$$\left|\frac{\partial^2 \mathfrak{f}(u)}{\partial u_j\,\partial u_k} - \frac{\partial^2 \mathfrak{g}(u)}{\partial u_j\,\partial u_k}\right| < \varepsilon \quad \forall j, k.$$

For sufficiently small $\varepsilon > 0$, these inequalities imply

$$\sum_j \left|\left|\frac{\partial \mathfrak{f}(u)}{\partial u_j}\right| - \left|\frac{\partial \mathfrak{g}(u)}{\partial u_j}\right|\right| < \frac{\delta}{2},$$
$$\left|\left|\det\left(\frac{\partial^2 \mathfrak{f}(x)}{\partial x\,\partial x^\top}\right)\right| - \left|\det\left(\frac{\partial^2 \mathfrak{g}(u)}{\partial u\,\partial u^\top}\right)\right|\right| < \frac{\delta}{2}$$

for all $u \in K$, which require

$$\sum_j \left|\frac{\partial \mathfrak{g}(u)}{\partial u_j}\right| > \sum_j \left|\frac{\partial \mathfrak{f}(u)}{\partial u_j}\right| - \frac{\delta}{2}, \tag{S.12}$$

$$\left|\det\left(\frac{\partial^2 \mathfrak{g}(u)}{\partial u\,\partial u^\top}\right)\right| > \left|\det\left(\frac{\partial^2 \mathfrak{f}(x)}{\partial x\,\partial x^\top}\right)\right| - \frac{\delta}{2}. \tag{S.13}$$

Summing (S.12) and (S.13), for all $u \in K$ we have $\rho_{\mathfrak{g}}(u) > \rho_{\mathfrak{f}}(u) - \delta > 0$. *Q.E.D.*

Next we show that $\mathcal{F}^M$ is an open subset of $\mathcal{F}$.

LEMMA 12: *For every $\ln f \in \mathcal{F}^M$, there exists $\varepsilon > 0$ such that if $\ln \hat{f} \in \mathcal{F}$ and $\|\ln \hat{f} - \ln f\|_{C^2} < \varepsilon$, then $\ln \hat{f} \in \mathcal{F}^M$.*

PROOF: Take any $\ln f \in \mathcal{F}^M$ and let u^* denote a point at which it has a nondegenerate local maximum. The proof of Lemma 5 in Berry and Haile (2018) showed that, for some compact set Σ with nonempty interior, there exists $\underline{c}$ such that (i) the upper contour set $A(\underline{c}, \Sigma) = \{u \in \Sigma : \ln f(u) \geq \underline{c}\}$ lies in the interior of Σ and (ii) the restriction of $\ln f$ to $A(\underline{c}, \Sigma)$ attains a maximum $\bar{c} = \ln f(u^*) > \underline{c}$ at its unique critical point. Let

$K = A(\underline{c}, \Sigma)$. Because $\ln f$ is continuous, $A(\underline{c}, \Sigma)$ is closed in $\mathbb{R}^J$. And because $A(\underline{c}, \Sigma)$ lies on the interior of the compact set Σ, $A(\underline{c}, \Sigma)$ is bounded. Thus K is compact, and $\ln f$ has no degenerate critical point on K. So by Lemma 11, for all sufficiently small $\varepsilon > 0$, $\| \ln \hat{f} - \ln f \|_{C^2} < \varepsilon$ (which implies $\| \ln \hat{f} - \ln f \|_{C^2_K} < \varepsilon$) ensures that $\ln \hat{f}$ also has no degenerate critical point on K. To complete the proof, it suffices to show that, for all sufficiently small $\varepsilon > 0$, $\| \ln \hat{f} - \ln f \|_{C^2} < \varepsilon$ ensures that the restriction of $\ln \hat{f}$ to K has a maximum on the interior of K. By continuity of $\ln f$, $\ln f(u) = \underline{c}$ for all u on the boundary of K. So when $\| \ln \hat{f} - \ln f \|_{C^2} < \varepsilon$, $\ln \hat{f}$ must take values no more than $\underline{c} + \varepsilon$ on the boundary of K and no less than $\bar{c} - \varepsilon$ at u^*. For sufficiently small $\varepsilon > 0$, we have $\underline{c} + \varepsilon < \bar{c} - \varepsilon$, requiring that the restriction of $\ln \hat{f}$ to K have an interior maximum. *Q.E.D.*

Finally, we show that for every $\sigma \in \mathbb{Z}^J$, $\mathcal{F}_\sigma^H$ is an open subset of $\mathcal{F}$.

LEMMA 13: *For any* $\sigma \in \mathbb{Z}^J$ *and* $\ln f \in \mathcal{F}_\sigma^H$, *there exists* $\varepsilon > 0$ *such that if* $\ln \hat{f} \in \mathcal{F}$ *and* $\| \ln \hat{f} - \ln f \|_{C^2} < \varepsilon$, $\ln \hat{f} \in \mathcal{F}_\sigma^H$.

PROOF: Fix $\sigma \in \mathbb{Z}^J$ and $\ln f \in \mathcal{F}_\sigma^H$, the latter implying that, for some $\delta > 0$ and some $\hat{u} \in sq_\sigma$, $|\det(\partial^2 \ln f(\hat{u})/\partial u \, \partial u^\top)| > \delta$. Recall that $\|h\|_{C^2} < \varepsilon$ requires $\max_{j,k \in \{1,\ldots,J\}} \sup_{u \in sq_\sigma} |\frac{\partial^2 h(u)}{\partial u_j \partial u_k}| < \varepsilon$. So for sufficiently small $\varepsilon > 0$, $\| \ln f - \ln \hat{f} \|_{C^2} < \varepsilon$ implies $|\det(\frac{\partial^2 \ln f(\hat{u})}{\partial u \, \partial u^\top}) - \det(\frac{\partial^2 \ln \hat{f}(\hat{u})}{\partial u \, \partial u^\top})| < \delta$, ensuring that $|\det(\frac{\partial^2 \ln \hat{f}(\hat{u})}{\partial u \, \partial u^\top})| > 0$. *Q.E.D.*

S.3. TRIWEIGHT PERTURBATION ON A SQUARE

The proof of Lemma 10 uses a particular perturbation of a log density on squares in $\mathbb{R}^J$. Here we provide some additional discussion of this perturbation and derive some elementary properties referenced in the proof.

Recall that for $v \in \mathbb{R}$ we defined

$$p(v) = 1\{|v| \leq 1\}\big(1 - v^2\big)^3.$$

The function $p(v)$ is equal to zero at -1 and 1, strictly increasing for $v \in (-1, 0)$, and strictly decreasing for $v \in (0, 1)$. It attains a maximum (of 1) at $v = 0$. For $v \in [-1, 1]$, its first and second derivatives are given by

$$p'(v) = -6v\big(1 - v^2\big)^2,$$
$$p''(v) = 24v^2\big(1 - v^2\big) - 6\big(1 - v^2\big)^2,$$

which are continuous and bounded. The first and second derivatives of p at -1, 0, and 1 are given in Table S.I.

Let s denote a square $\times_j[\underline{u}_j, \overline{u}_j]$ in $\mathbb{R}^J$, with $\overline{u}_j - \underline{u}_j = \bar{w} > 0$ for all $j = 1, \ldots, J$. Let $\dot{u}_s = (\frac{\overline{u}_1 + \underline{u}_1}{2}, \ldots, \frac{\overline{u}_J + \underline{u}_J}{2})$ denote the center of this square. Let $\ln f$ be a twice continuously differentiable log density defined on $\mathbb{R}^J$, with $f = \exp(\ln f)$ its associated probability density function. Given any finite scalar $\lambda_s > 0$ and $\kappa > 0$, let

$$f_{\lambda_s}(u) = \kappa f(u) \exp\left[\lambda_s \prod_{j=1}^J p\left(\frac{u_j - \dot{u}_{sj}}{\bar{w}/2}\right)\right], \quad u \in s.$$

TABLE S.I

SOME VALUES OF $p(v)$ AND ITS DERIVATIVES

v	$p(v)$	$p'(v)$	$p''(v)$
-1	0	0	0
0	1	0	-6
1	0	0	0

Then, on the square s, $\ln f_{\lambda_s}$ is equal to the sum of $\ln f$, the rescaled multivariate triweight function $\lambda_s \prod_{j=1}^{J} p(\frac{u_j - \dot{u}_{sj}}{\bar{w}/2})$, and the constant $\ln(\kappa)$. Observe that $\lambda_s \prod_{j=1}^{J} p(\frac{u_j - \dot{u}_{sj}}{\bar{w}/2})$ takes values in the interval $[0, \lambda_s]$, attaining λ_s only at the center of the square. Figure S.1 illustrates the scaled multivariate triweight function for the case $J = 2$ with $\lambda_s = 1$.

Recalling Table S.I, observe that, regardless of λ_s, for any u on the boundary of the square s, we have

$$\ln f_{\lambda_s}(u) = \ln f(u) + \ln(\kappa),$$
$$\frac{\partial}{\partial u_j} \ln f_{\lambda_s}(u) = \frac{\partial}{\partial u_j} \ln f(u) \quad \forall j,$$
$$\frac{\partial^2}{\partial u_j \, \partial u_k} \ln f_{\lambda_s}(u) = \frac{\partial^2}{\partial u_j \, \partial u_k} \ln f(u) \quad \forall j, k.$$

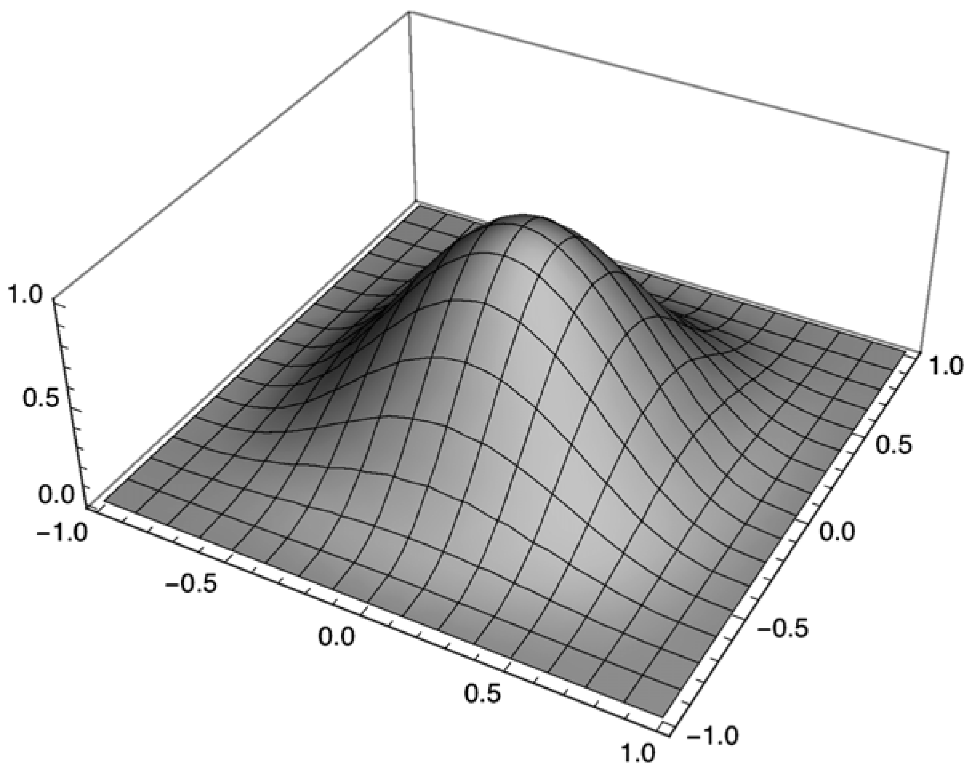

FIGURE S.1.—Plot of a scaled bivariate triweight function.

These properties ensure that when we perturb $\ln f$ on adjacent squares—potentially with different scaling factors λ_s for each square (but the same κ for all squares)—the perturbed log density function will remain twice continuously differentiable, even on the boundaries of the squares.

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