Matching Theory and Applications

by Fuhito Kojima (Stanford)

Econometric Society World Congress, Montréal
Introduction
Introduction

- Matching and Market Design
- Theory
- Applications
Introduction

- Matching and Market Design
  - Theory
  - Applications
- Applications stimulate new theory
  1. Large markets and “approximate market design”
  2. Matching with constraints
Part 1: Standard Model

based on (mostly)


Roth, “The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory” JPE, 1984
Standard model: setting

- Doctors (1, 2, ..., i, j, ...) and hospitals (A, B, ...)
  (students/workers) (schools/firms)
- Many-to-one matching
- Preferences over each other (& outside option \( \emptyset \))
- Matching; specify who matches with whom
Standard Model: Stability
Standard Model: Stability

A matching is **stable** if

- Individually rational
- No blocking pairs
Standard Model: Stability

A matching is **stable** if

- Individually rational
- No blocking pairs

**Interpretations**

- Core (so Pareto efficient)
- Fairness
Deferred Acceptance Algorithm ("DA")

Step 1:

- Doctors apply to their 1st choice hospitals
- Each hospital (tentatively) holds its most preferred acceptable applicants up to capacity and rejects the rest

Step $t \geq 2$:

- Rejected doctors apply to their next choice hospitals
- Each hospital combines both existing and new applicants, holds its most preferred acceptable ones up to capacity, and rejects the rest

Repeat until no more applications are made and finalize the match.
Results

- The outcome of DA is stable (Gale and Shapley 1962 AMM)
- DA is strategy-proof for the doctors (Roth 1982 MOR; Dubins and Freedman 1981 AMM)
- DA is used in NRMP (Roth 1984 JPE)
- Other markets: UK medical match (Roth 1991 AER), Turkish college admission (Balinski and Sonmez 1999 JET), ...
- Intentional market design adopts DA; UK medical match, JRMP, APPIC, NYC, Boston,...
DA is not strategy-proof

Preference table
(each hospital has 1 seat)
DA is not strategy-proof

Under truth-telling:

\[
\begin{array}{c|c|c}
A & i & A \\
B & j & B \\
\end{array}
\]

Preference table
(each hospital has 1 seat)
DA is not strategy-proof

Under truth-telling:

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Preference table (each hospital has 1 seat)
DA is not strategy-proof

Under truth-telling:

\[ \begin{array}{cc}
A & i \heartsuit \\
B & j \heartsuit \\
\end{array} \]

Preference table (each hospital has 1 seat)
DA is not strategy-proof

Under truth-telling:

A  i  ❤️
B  j  ❤️

When A misreports:

A  i
B  j

Preference table
(each hospital has 1 seat)
DA is not strategy-proof

### Under truth-telling:

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### When A misreports:

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DA is not strategy-proof

Under truth-telling:

A i ❤
B j ❤

When A misreports:

A i 💔
B j

Preference table
(each hospital has 1 seat)
DA is not strategy-proof

Under truth-telling:

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B  j ❤️

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Preference table (each hospital has 1 seat)
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Under truth-telling:
- A i ❤
- B j ❤

When A misreports:
- A
- B j i

Preference table (each hospital has 1 seat)
DA is not strategy-proof

- Under truth-telling:
  - A: i
  - B: j

- When A misreports:
  - A
  - B: j

Preference table
(each hospital has 1 seat)
DA is not strategy-proof

Under truth-telling:

A  i ❤
B  j ❤

When A misreports:

A
B  i  j
B  i ❤

Preference table (each hospital has 1 seat)
DA is not strategy-proof

- **Under truth-telling:**
  - A: i
  - B: j

- **When A misreports:**
  - A: j
  - B: i

Preference table
(each hospital has 1 seat)
DA is not strategy-proof

- Under truth-telling:
  - A: i
  - B: j

- When A misreports:
  - A: j
  - B: i

Preference table (each hospital has 1 seat)
DA is not strategy-proof

- Under truth-telling:
  - A prefers i over j
  - B prefers j over i

- When A misreports:
  - Profitable misreport!
  - A prefers j over i
  - B prefers i over j

Preference table (each hospital has 1 seat)
Impossibility Theorem

Theorem (Roth 1982 MOR): There exists no stable and strategy-proof mechanism.
Impossibility Theorem

Theorem (Roth 1982 MOR): There exists no stable and strategy-proof mechanism.

So, a “perfect mechanism” is impossible!
Part 2: Large Markets and “Approximate Market Design”

based on (mostly)

- Kojima and Pathak “Incentives and Stability in Large Two-Sided Matching Markets” AER 2009
- Kojima Pathak, and Roth “Matching with Couples: Stability and Incentives in Large Markets” QJE 2013
Why are DAs used?
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Puzzle; DA isn’t strategy-proof, but still used widely in practice
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- NRMP, JRMP, UK medical match, APPIC, NYC, Boston, ...
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- Roth and Peranson (1999 AER); DA is “approximately IC”: less than 25 hospitals have profitable manipulations out of 3000 hospitals in NRMP data (→Detail).
Why are DAs used?

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- Roth and Peranson (1999 AER): DA is “approximately IC”: less than 25 hospitals have profitable manipulations out of 3000 hospitals in NRMP data (Detail).

- Is this robust? What is the reason?
Simulation
Simulation

- \( n \) doctors, \( n \) hospitals
  (one seat in each hospital)
- \( k = \#\{\text{hospitals acceptable to each doctor}\} \)
- iid uniform random preferences
- \( p(n) = \text{proportion of hospitals that can profitably manipulate} \)
Simulation

- $n$ doctors, $n$ hospitals (one seat in each hospital)
- $k = \#\{\text{hospitals acceptable to each doctor}\}$
- iid uniform random preferences
- $p(n) =$ proportion of hospitals that can profitably manipulate

- Large market $\rightarrow$ approximate IC(?)
Theory: Setup
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Sequence of markets, indexed by # of hospitals, n.
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Theory: Setup

- Sequence of markets, indexed by # of hospitals, n.
- $k=#\{\text{hospitals acceptable to each doctor}\}$ constant; “short list” (→Detail)
- Game: each participant submits preference, and DA produces a matching
Theorem (Immorlica and Mahdian 2005 SODA, Kojima and Pathak 2009 AER);
For any $\varepsilon > 0$, there is $n$ such that truth-telling is an $\varepsilon$-Bayes-Nash equilibrium for any market with more than $n$ hospitals.
Theory: Result

Theorem (Immorlica and Mahdian 2005 SODA, Kojima and Pathak 2009 AER);
For any \( \varepsilon > 0 \), there is \( n \) such that truth-telling is an \( \varepsilon \)-Bayes-Nash equilibrium for any market with more than \( n \) hospitals.

So a stable mechanism may be approximately incentive compatible in large markets!
Intuition
Intuition

- Recall DA is strategy-proof for doctors.
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- Reason for profitable manipulations: rejection chain.
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A rejected doctor applies to the manipulating hospital
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- In large markets; Doctors along the rejection chain are likely to apply to a hospital with a vacant position and be accepted.
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→ rejection chain stops; no benefit of manipulation.
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- In large markets; Doctors along the rejection chain are likely to apply to a hospital with a vacant position and be accepted.
  \[\rightarrow\text{rejection chain stops; no benefit of manipulation.}\]

\[\rightarrow\text{Discussion (Lee (2014), Ashlagi et al. (2015))}\]
Existence of stable matchings
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- Existence of stable matching fails when standard conditions are violated
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- Couples who want two jobs
Existence of stable matchings

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Existence of stable matchings

- Existence of stable matching fails when standard conditions are violated
- Couples who want two jobs
  - Couples are
    - 5-10% in NRMP
    - 1-2% in APPIC

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Nonexistence with couples

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Each hospital has one seat
Nonexistence with couples

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Each hospital has one seat

- Can verify each matching is unstable
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- Each hospital has one seat

- Can verify each matching is unstable

- Failure of DA-like algorithm

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❤️💔
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Nonexistence with couples

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Nonexistence with couples

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- Can verify each matching is unstable
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Cycle!
Existence in practice
Existence in practice

- Stable matchings exist for most cases even with couples
Existence in practice

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- NRMP, APPIC...
Existence in practice

- Stable matchings exist for most cases even with couples
  - NRMP, APPIC...
- Why is this?
Existence in practice

- Stable matchings exist for most cases even with couples
  - NRMP, APPIC...

- Why is this?
  - Preference restriction? (Detail)
Theoretical Result

Consider a large market setting as in Kojima and Pathak (2009), and assume the proportion of couples is small compared to the market size.
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Theorem (Kojima et al. 2013); The probability that a stable matching exists converges to one as the market size goes to infinity.
Theoretical Result

Consider a large market setting as in Kojima and Pathak (2009), and assume the proportion of couples is small compared to the market size.

Theorem (Kojima et al. 2013); The probability that a stable matching exists converges to one as the market size goes to infinity.

So a stable matching existence is robust!
Intuition
Intuition

A DA-like algorithm finds a stable matching if no couples are displaced later.
Intuition

- A DA-like algorithm finds a stable matching if no couples are displaced later.
- Large markets: rejection chains are likely to stop at a vacant position.
**Intuition**

- A DA-like algorithm finds a stable matching if no couples are displaced later.
- Large markets: rejection chains are likely to stop at a vacant position → produces a stable matching.
Intuition

- A DA-like algorithm finds a stable matching if no couples are displaced later.

- Large markets: rejection chains are likely to stop at a vacant position

→ produces a stable matching.

→ Discussion (Ashlagi et al. 2014 OR)
Large market approach


Part 3: Matching with Constraints

based on (mostly)


Overview

- Many matching markets are subject to constraints
  - Medical specialties
  - Multiple school programs sharing one building
  - Affirmative action (diversity constraints)
  - ...

Case study: Japan
Case study: Japan

Japan residency matching program (JRMP) adopted DA in 2003
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- Criticism: DA places too many doctors in urban regions (cf. Roth 1986 ECMA).
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- Government introduced a regional cap as a constraint (numbers)
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- ... and modified DA → JRMP mechanism.
Case study: Japan

- Japan residency matching program (JRMP) adopted DA in 2003
- Criticism: DA places too many doctors in urban regions (cf. Roth 1986 ECMA).
- Government introduced a regional cap as a constraint (→numbers)
  - ... and modified DA → JRMP mechanism.
- Other examples (→Detail)
Model of constraints

- Standard two-sided matching except
- Each hospital belongs to a region
- Each region has exogenous regional cap (positive integer)
- A matching is feasible if, for each region,
  \[(\text{# of doctors in the region}) \leq (\text{regional cap})\]
Example: JRMP mechanism
Example: JRMP mechanism

- Government imposes a **target capacity** for each hospital
- Smaller than real (physical) hospital capacity
- Sums at most to the regional caps
Example: JRMP mechanism

- Government imposes a target capacity for each hospital
  - Smaller than real (physical) hospital capacity
  - Sums at most to the regional caps
- **JRMP mechanism** = DA using target capacity (instead of real capacity)
- Feasible
Example: JRMP mechanism

- Government imposes a target capacity for each hospital
  - Smaller than real (physical) hospital capacity
  - Sums at most to the regional caps

**JRMP mechanism** = DA using target capacity (instead of real capacity)

- Feasible

- Is JRMP stable? (constrained) efficient?
JRMP is inefficient and unstable!

Let there be one region, with cap 10.

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Each hospital has 10 seats

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JRMP is inefficient and unstable!

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JRMP (target capacity=5 each):

Each hospital has 10 seats

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JRMP is inefficient and unstable!

Let there be one region, with cap 10.

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Each hospital has 10 seats

A 1 2 3 4 5 6 7 8 9 10
JRMP is inefficient and unstable!

Let there be one region, with cap 10.

JRMP (target capacity=5 each):

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JRMP (target capacity=5 each):

A  1 2 3
B  4 5 6 7 8 9 10

Each hospital has 10 seats

A  1 1
B  2 2
... ...
10 10
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Each hospital has 10 seats

![Diagram showing inefficient and unstable placement]
FDA mechanism
(flexible DA)
FDA mechanism
(flexible DA)

Begin with an empty matching, and repeat Steps below:

Application: Each currently unmatched doctor applies to her favorite hospital that has not rejected her yet (if any).

Acceptance/Rejection: Each hospital (tentatively) accepts from both its tentatively matched doctors and new applicants (if any):

- Phase 1 ("regular" phase): each hospital (tentatively) accepts its favorite acceptable applicants up to its target capacity.
FDA mechanism  
(flexible DA)

Begin with an empty matching, and repeat Steps below:

**Application:** Each currently unmatched doctor applies to her favorite hospital that has not rejected her yet (if any).

**Acceptance/Rejection:** Each hospital (tentatively) accepts from both its tentatively matched doctors and new applicants (if any):

- **Phase 1 ("regular" phase):** each hospital (tentatively) accepts its favorite acceptable applicants up to its target capacity.
- **Phase 2 ("waitlist" phase):** hospitals take turns to (tentatively) accept favorite applicants from waitlist until (i) the regional cap becomes full or (ii) no doctor remains to be processed.
**FDA example**

Let there be one region, with cap 10.

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FDA example

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FDA (target capacity=5 each):

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Each hospital has 10 seats
FDA example

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FDA (target capacity=5 each):

A  B

1  2  3  4  5  6  7  8  9  10

Each hospital has 10 seats
FDA example

Let there be one region, with cap 10.

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A 1 2 3
B

4 5 6 7 8 9 10
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A 1 2 3
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FDA example

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Each hospital has 10 seats

FDA (target capacity=5 each):

regular phase

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B  4  5  6  7  8  9  10
Let there be one region, with cap 10.

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Each hospital has 10 seats

FDA (target capacity=5 each):

- **Regular phase**
- **Waitlist phase**
FDA example

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Each hospital has 10 seats

FDA (target capacity=5 each):

Each hospital

regular phase

waitlist phase

efficient & “stable”!
FDA results

- (constrained) Pareto efficient
- “stable under constraints”
FDA results

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- “stable under constraints”
- strategy-proof for doctors
FDA results

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- strategy-proof for doctors
- Every doctor weakly prefers FDA to JRMP
FDA results

- (constrained) Pareto efficient
- "stable under constraints"
- strategy-proof for doctors
- Every doctor weakly prefers FDA to JRMP

- So matched doctors increases (simulation; 1400 unmatched in JRMP, 1000 in FDA, \(\rightarrow\) detail)
Matching with Constraints


Conclusion
Conclusion

Matching theory has been applied to many markets in practice
Conclusion

- Matching theory has been applied to many markets in practice.
- Classical theory may not directly apply to markets in practice, suggesting new kinds of theory.
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Conclusion

- Matching theory has been applied to many markets in practice.
- Classical theory may not directly apply to markets in practice, suggesting new kinds of theory:
  - Large market and “approximate market design”
  - Matching with constraints
- Interaction with markets in practice continues to enrich theory.

Thank You! 😊
Additional Slides
NRMP data

Roth and Peranson (1999 AER); # of hospitals with profitable misreporting in NRMP
NRMP data

Roth and Peranson (1999 AER); # of hospitals with profitable misreporting in NRMP

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<tbody>
<tr>
<td># of hospitals</td>
<td>3170</td>
<td>3622</td>
<td>3662</td>
<td>3745</td>
<td>3758</td>
</tr>
<tr>
<td># manipulable</td>
<td>15</td>
<td>12</td>
<td>15</td>
<td>23</td>
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How robust is approximate IC?
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- Rely on various assumptions
How robust is approximate IC?

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- I focus on “short list”, k=constant
How robust is approximate IC?

- Rely on various assumptions
- I focus on “short list”, k=constant
- Motivated by real-life markets
How robust is approximate IC?

- Rely on various assumptions
- I focus on “short list”, k=constant
- Motivated by real-life markets
  - k<15 (NRMP), 12 (NYC), 8 (APPIC), 4 (JRMP), ...
"Necessity" of short lists

Simulation for $k=n$
“Necessity” of short lists

Knuth, Motwani, and Pittel (1990 RSA)

Simulation for k=n
“Necessity” of short lists

Knuth, Motwani, and Pittel (1990 RSA)

Intuition: Rejection chain doesn’t get absorbed by vacancy!

Simulation for $k=n$
“Necessity” of short lists

Knuth, Motwani, and Pittel (1990 RSA)

Intuition:
Rejection chain doesn’t get absorbed by vacancy!

Then, is approximate IC sensitive to the short list assumption?
Approximate IC with long lists
Approximate IC with long lists

- Lee (2014): Many agents have profitable manipulations, but utility gains are small
Approximate IC with long lists

Lee (2014): Many agents have profitable manipulations, but utility gains are small

Consistent with previous results
Approximate IC with long lists

Lee (2014): Many agents have profitable manipulations, but utility gains are small

Consistent with previous results

Ashlagi, Kanoria, and Leshno (2014): Assume the market is unbalanced, i.e. \(#\{doctors\} \neq \#\{hospitals\} \)
Approximate IC with long lists

- Lee (2014): Many agents have profitable manipulations, but utility gains are small.
  - Consistent with previous results.

- Ashlagi, Kanoria, and Leshno (2014): Assume the market is\textbf{ unbalanced}, i.e. \#\{doctors\} \#\# \#\{hospitals\}
  - With vacancy, rejection chain works (and much better than previously shown!)

Back to Setup, Intuition
Approximate IC with long lists

- Lee (2014): Many agents have profitable manipulations, but utility gains are small
  - Consistent with previous results
- Ashlagi, Kanoria, and Leshno (2014): Assume the market is unbalanced, i.e. $\#{\text{doctors}} \neq \#{\text{hospitals}}$
  - With vacancy, rejection chain works (and much better than previously shown!)
- Failure of approximate IC is a “knife edge” case

Back to Setup, Intuition
Preference restriction
Preference restriction

Klaus and Klijn (2005 JET) find a sufficient and “almost necessary” condition for couple preferences: responsiveness
Preference restriction

Klaus and Klijn (2005 JET) find a sufficient and “almost necessary” condition for couple preferences: responsiveness

APPIC data: 1/167 couples have responsive preferences (Kojima, Pathak, and Roth 2013 QJE)
Explaining data better
Explaining data better

Kojima et al. (2013) result obtained with assumption \( \#\{\text{couples}\} = o(\sqrt{n}) \), so couples quickly vanishes in proportion.
Explaining data better

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- Also nonexistence probability is bounded below if couples proportion is constant.
Japanese Data on Regional Caps

Total capacity

Regional Cap
More examples of constraints

- Chinese graduate school admission
- academic/professional programs
- College admission in Hungary & Ukraine
- state-financed/privately-financed seats
- Medical match in U.K. (regional cap)
- Teacher assignment in Scotland (regional cap)
Simulation: number of matched/unmatched doctors

Number of doctors

- Unconstrained
- JRMP
- FDA

Matched

Unmatched
Simulation: Rank distributions

Number of doctors vs. Ranking of matched hospitals for DA, FDA, and JRMP.
Simulation: Doctor distribution

\[ y = 0.0001x - 0.0211 \]
\[ R^2 = 0.00327 \]

# of doctors/100,000 persons