SUPPLEMENT TO “TASK TRADE BETWEEN SIMILAR COUNTRIES”  
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IN THIS SUPPLEMENTAL MATERIAL, we prove Lemmas 1, 2, and 3.

**Lemma 1**: If \( w > w^* \), either (i) \( \tilde{\pi}(i) > 0 \) or \( \tilde{\pi}(i) < 0 \) for all \( i \) or (ii) \( J > 0 \) and \( \tilde{\pi}(i) > 0 \) for \( i < J \) while \( \tilde{\pi}(i) < 0 \) for \( i > J \).

**Proof**: Without loss of generality, assume \( w > 1 \) (given \( w^* = 1 \)). The aggregate cost of performing task \( i \) in East minus the aggregate cost of performing it in West is proportional to

\[
\Lambda(i; nx, n^* x^*, w) \equiv \tilde{\pi}(i; nx, n^* x^*, w) A(nx + n^* x^* )
= (wnx - n^* x^*) - \beta t(i)(nx - wn^* x^*) .
\]

First assume that \( n^* x^* \geq nx \). Then \( nx - wn^* x^* < 0 \), which implies \( \min_i \Lambda(i; nx, n^* x^*, w) = \Lambda(0; nx, n^* x^*, w) \) since \( t'(i) > 0 \) for all \( i \). Then, since \( \beta t(0) > 1 \),

\[
\Lambda(0; nx, n^* x^*, w) > wnx - n^* x^* - nx + wn^* x^*
= (w - 1)(nx + n^* x^*) > 0 .
\]

So all tasks have higher aggregate cost in East; that is, \( \tilde{\pi}(i) > 0 \) for all \( i \) and \( J = 1 \).

Now suppose instead that \( nx > n^* x^* \). Then \( wnx - n^* x^* > nx - wn^* x^* \). Suppose first that \( \beta t(0) > 1 \) is close enough to 1 that \( \Lambda(0; nx, n^* x^*, w) > 0 \). Then tasks in the neighborhood of task 0 yield lower costs in West. Since \( t'(i) > 0 \) for all \( i \), either there exists \( J > 0 \) such that \( \Lambda(J; nx, n^* x^*, w) = 0 \), in which case tasks with \( i > J \) have lower cost in East (\( \tilde{\pi}(i) < 0 \)) and tasks with \( i < J \) have lower cost in West (\( \tilde{\pi}(i) > 0 \)), or \( wnx - n^* x^* > \beta t(1)(nx - wn^* x^*) \), in which case \( \Lambda(i; nx, n^* x^*, w) > 0 \) for all \( i \) and all tasks have lower cost in West (\( \tilde{\pi}(i) > 0 \) and \( J = 1 \)). If \( \beta t(0) \) is such that \( \Lambda(0; nx, n^* x^*, w) < 0 \), then since \( t'(i) > 0 \) for all \( i \), all tasks have lower costs in East, namely, \( \tilde{\pi}(i) < 0 \) and \( J = 0 \).

**Q.E.D.**

**Lemma 2**: If \( w > w^* \), then \( J < I \) implies \( I > I^* \).

**Proof**: The proof of Lemma 1 guarantees that if \( w > 1 \), then \( n^* x^* > nx \) implies \( J = 1 \). So we can limit our attention to circumstances with \( nx > n^* x^* \). To establish a contradiction, we suppose that \( J < I \) and \( I^* > I \). Then (1) and (3) imply that \( w^2 > A(nx)/A(n^* x^*) \).

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From the definition of $J$, we know that

\begin{equation}
\beta_t(J) - \beta_t(I) = \frac{wnx - n^*x^*}{nx - wn^*x^*} - \frac{A(nx + n^*x^*)}{wA(n^*x^*)}.
\end{equation}

Since the denominators are both positive for $J \in (0, 1)$, the left-hand side has the same sign as

\[ \Delta(n^*x^*, nx, w) \equiv w^2 A(n^*x^*)nx - wA(n^*x^*)n^*x^* - A(nx + n^*x^*)nx + wA(nx + n^*x^*)n^*x^*. \]

But then $w^2 > A(nx)/A(n^*x^*)$ implies that

\[ \Delta(n^*x^*, nx, w) = n^*x^* [A(nx + n^*x^*) - A(n^*x^*)] + nx [A(nx) - A(nx + n^*x^*)]. \]

Define the the right-hand side as $\Omega(n^*x^*, nx)$ and note that $\Omega(\cdot)$ is continuously differentiable in both arguments and $\Omega(nx, nx) = 0$. Calculate the partial derivative of $\Omega(n^*x^*, nx)$ with respect to the second argument. Then $\Omega_2(0, nx) = 0$ and $\Omega_2(nx, nx) = A(nx) + nx A'(nx) - A(2nx) \geq 0$, where the inequality follows from the concavity of $A(\cdot)$. Note also that $\Omega_2(n^*x^*, nx) = -(nx - n^*x^*)A''(n^*x^* + nx) \geq 0$ by the concavity of $A(\cdot)$. Then, since $\Omega_2(\cdot)$ is continuous, $\Omega_2(n^*x^*, nx) \geq 0$ for all $n^*x^* \geq 0$ and $nx \geq n^*x^*$. Since $\Omega(nx, nx) = 0$ and $\Omega_2(n^*x^*, nx) \geq 0$ for all $nx \geq n^*x^*$, it follows by continuity that $\Omega(n^*x^*, nx) \geq 0$ for all $nx \geq n^*x^*$. Hence, if $w > 1$, $I^* > I$, and $nx > n^*x^*$, we obtain that $\Delta(n^*x^*, nx, w) > 0$, which implies by (15) that $J > I$. This establishes our contradiction.

**Q.E.D.**

**LEMMA 3:** $w > 1$ if and only if $nx > n^*x^*$.

**PROOF:** We consider three mutually exhaustive cases: (i) $I \geq I^*$, (ii) $I < I^*$ and $L > L^*$, and (iii) $I < I^*$ and $L \leq L^*$.

(i) From the definitions of $I$ and $I^*$ in (1) and (3), $I \geq I^*$ implies

\[ \frac{A(nx + n^*x^*)}{wA(n^*x^*)} \geq \beta_t(I) \geq \beta_t(I^*) \geq \frac{wA(nx + n^*x^*)}{A(nx)}, \]

which implies that $A(nx)/A(n^*x^*) \geq w^2 > 1$. So $nx > n^*x^*$.

(ii) To establish a contradiction, suppose that $nx \leq n^*x^*$. From Figure 3(d) and (e), $I < I^*$ implies $E = \emptyset$. Then

\[ L = \frac{M(D)nx}{A(nx)} > L^* > \frac{M(D)n^*x^*}{A(n^*x^*)}, \]
which implies \( A(nx)/(nx) < A(n^*x^*)/(n^*x^*) \). But \( A(\cdot) \) concave, \( A(0) \geq 0 \),
and \( nx \leq n^*x^* \) imply that \( A(nx)/(nx) \geq A(n^*x^*)/(n^*x^*) \). This contradicts the supposition that \( nx < n^*x^* \).

(iii) To establish a contradiction, suppose that \( nx \leq n^*x^* \). Labor-market clearing implies \( L = (1 - I^*)(nx)/A(nx) \) and

\[
L^* > (1 - I^*) \frac{n^*x^*}{A(n^*x^*)} + I^* \frac{nx + n^*x^*}{A(nx + n^*x^*)},
\]

since \( T(I^*) > I^* \) for all \( I^* \). From manager-market clearing, and \( H = L \) and \( H^* = L^* \), this implies that

\[
\frac{x}{x^*} > \frac{1 - I^*}{A(n^*x^*)} + \frac{I^*}{A(nx + n^*x^*)} \frac{1}{1 - I^* A(nx)}.
\]

Note that \( nx \leq n^*x^* \) and \( w > 1 \) imply that

\[
\frac{c}{c^*} = \frac{w(1 - I^*) + \beta T(I^*)}{A(nx) + A(nx + n^*x^*)} \geq 1.
\]

Equation (7) implies, since \( \sigma > 1 \), that \( x^*/x \geq c/c^* \). Given that \( T(I^*) > I^* \) and \( w > 1 \), then

\[
\frac{x}{x^*} < \frac{1 - I^*}{A(n^*x^*)} + \frac{I^*}{A(nx + n^*x^*)} \frac{1}{1 - I^* A(nx)}.
\]

Therefore, for an equilibrium to exhibit \( nx < n^*x^* \), it has to be the case that

\[
\frac{1 - I^*}{A(n^*x^*)} + \frac{I^*}{A(nx + n^*x^*)} \frac{1}{1 - I^* A(nx)} > \frac{x}{x^*} > \frac{1 - I^*}{A(n^*x^*)} + \frac{I^*}{A(nx + n^*x^*)} \frac{1}{1 - I^* A(nx)}.
\]
But note that $I^*/A(nx + n^*x^*) > 0$ and $(nx + n^*x^*)/n^*x^* > 1$, so

$$
\frac{1 - I^*}{A(n^*x^*)} + \frac{I^*}{A(nx + n^*x^*)} > 0 \quad \text{and} \quad \frac{1 - I^*}{A(nx)} + \frac{I^*}{A(nx + n^*x^*)} > \frac{1}{A(nx)}
$$

which contradicts the previous string of inequalities. \textit{Q.E.D.}

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