Online Appendix to "Uneven Growth: Automation's Impact on Income and Wealth Inequality"

Benjamin Moll, Lukasz Rachel and Pascual Restrepo

June 16, 2022

A Derivations and Proofs for the Baseline Model

A.1 Task Model Micro-foundation and Details

This appendix provides a micro-foundation for the aggregate production function in equation (2). The appendix also provides a derivation of equation (9) and primitive conditions to ensure Assumption 1 holds.

Each skill type z works in a different sector that produces output Y_z. The economy produces a final good Y using these sectoral outputs according to a Cobb-Douglas aggregator

$$Y = A \prod_z Y_z^{\eta_z} \quad \text{with} \quad \sum_z \eta_z = 1.$$

Here, η_z denotes the importance of the sectoral output produced by skill type z in production. The productivity shifter A captures the role of factor-neutral technological improvements.

The production of sectoral output Y_z involves the completion of a unit continuum of tasks u, which are then combined via a Cobb-Douglas aggregator:

$$\ln Y_z = \int_0^1 \ln \mathcal{Y}_z(u) du.$$

These tasks can be produced using capital and skill-z labor as follows:

$$\mathcal{Y}_z(u) = \begin{cases} \psi_z \ell_z(u) + k_z(u) & \text{if } u \in [0, \alpha_z] \\ \psi_z \ell_z(u) & \text{if } u \in (\alpha_z, 1]. \end{cases}$$

The threshold α_z summarizes the possibilities for the automation of tasks performed by workers of skill z. Tasks $u \in [0, \alpha_z]$ are technologically automated and can be produced by capital $k_z(u)$ or labor $\ell_z(u)$. The remaining tasks are not technologically automated and must be produced by labor.

The unit cost of producing a task with capital is R and that of producing it with labor is w_z/ψ_z. Denote by $p_z(u)$ the price of task $\mathcal{Y}_z(u)$, and by p_z the price of sector z output Y_z. Cost minimization in the production of sector z output implies that the quantity of task u used is given by

$$\mathcal{Y}_z(u) = \frac{p_z Y_z}{p_z(u)}$$

Assumption 1 implies that all tasks $u \in [0, \alpha_z]$ are produced with capital. It follows that for those tasks, $p_z(u) = R$ and the quantity of capital required to produce $\mathcal{Y}_z(u)$ is $p_z Y_z / R$. It follows that the total amount of capital used in sector z is:

$$K_z = \frac{\alpha_z p_z Y_z}{R} \tag{A1}$$

Assumption 1 implies that all tasks $u \in (\alpha_z, 1]$ are produced with labor. It follows that for those tasks, $p_z(u) = \frac{w_z}{\psi_z}$ and the quantity of labor required to produce $\mathcal{Y}_z(u)$ is $p_z Y_z / w_z$. It follows that the total amount of labor of skill z used in sector z is:

$$\ell_z = \frac{(1 - \alpha_z) p_z Y_z}{w_z} \tag{A2}$$

With perfect competition, the price of sector z output equals the marginal cost of production. Because tasks are combined via a Cobb-Douglas aggregator, the price is given by the dual $\ln(p_z) = \int_0^1 \ln(p_z(u)) du$. It follows that

$$p_z = R^{\alpha_z} \left(\frac{w_z}{\psi_z} \right)^{1-\alpha_z}. \tag{A3}$$

Combining the formula for p_z in (A3) with capital and labor demand conditions (A1) and (A2) gives the production of sector z as a function of the total capital and labor used in this sector, K_z and ℓ_z:

$$Y_z = \left(\frac{K_z}{\alpha_z} \right)^{\alpha_z} \left(\frac{\psi_z \ell_z}{1 - \alpha_z} \right)^{1-\alpha_z} \tag{A4}$$

We now turn to aggregate output. We normalize the price of the final good to 1, so that the demand for sector z output satisfies $p_z Y_z = \eta_z Y$.

Using equations (A1), we can compute the demand for capital in sector z as

$$K_z = \frac{\alpha_z p_z Y_z}{R} = \frac{\alpha_z \eta_z Y}{R}.$$

Adding this formula across sectors, it follows that the total amount of capital used in the economy is

$$K = \alpha \frac{Y}{R}, \tag{A5}$$

where recall that $\alpha := \sum_z \alpha_z \eta_z$. The share of capital allocated to sector z is therefore equal to

$$K_z = K \frac{\alpha_z \eta_z}{\alpha}. \tag{A6}$$

Substituting this formula into (A4) we get:

$$Y_z = \left(K \frac{\eta_z}{\alpha} \right)^{\alpha_z} \left(\frac{\psi_z L_z}{1 - \alpha_z} \right)^{1-\alpha_z}. \tag{A7}$$

Substituting sectoral outputs into the aggregate production function we obtain the formula in equation (2), with

$$\mathcal{A} := A\alpha^{-\alpha} \prod_z (\eta_z(1-\alpha_z))^{-\eta_z(1-\alpha_z)} \prod_z \eta_z^{\eta_z} \tag{A8}$$

Proof of the expression for productivity gains from automation. The formula in equation (2) can be written as

$$Y = A \prod_z \eta_z^{\eta_z} \left(\frac{K}{\alpha}\right)^{\alpha} \prod_z \left(\frac{\psi_z \ell_z}{\eta_z(1-\alpha_z)}\right)^{\eta_z(1-\alpha_z)}.$$

It follows that

$$\begin{aligned} d\ln Y =& \eta_z \ln\left(\frac{K}{\alpha}\right) - \ln\left(\frac{\psi_z \ell_z}{\eta_z(1-\alpha_z)}\right) + \alpha d\ln K \\ =& \eta_z \ln\left(\frac{Y}{R}\right) - \ln\left(\psi_z \frac{Y}{w_z}\right) + \alpha d\ln K \\ =& \eta_z \ln\left(\frac{w_z}{\psi_z R}\right) + \alpha d\ln K > 0. \end{aligned}$$

The third row substitutes factor prices for their marginal products. Subtracting $\alpha d\ln Y$ from both sides of this equation and dividing through by $1-\alpha$ yields the formula in (9). ■

Lemma A1 (Lemma ensuring adoption of automation technologies) *Suppose that for all z, the following inequality holds:*

$$(\rho + p\sigma + \delta)^{-\frac{1}{1-\alpha}} \mathcal{A}^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} > \frac{1}{(1-\alpha_z)\eta_z} \frac{\ell_z \psi_z}{\prod_v (\ell_v \psi_v)^{\frac{\eta_v(1-\alpha_v)}{1-\alpha}}}, \tag{A9}$$

where $\mathcal{A}$ is defined in (A8). The equilibrium will involve the adoption of all available automation technologies. The above inequality holds for values of A above a threshold $\bar{A}$.

Proof. We assume that all automation technologies are adopted and verify that in equilibrium, the condition above ensures that $w_z^* > \psi_z R^*$.

In steady state, we have that $\rho + p\sigma > r^*$, which can be seen from the fact that $r^* = \rho + p\sigma\alpha^*_{net}$. Using the fact that $r^* + \delta = \alpha\frac{Y}{K}$, we can rewrite $\rho + p\sigma > r^*$ as

$$(K/Y)^* > \frac{\alpha}{\rho + p\sigma + \delta}. \tag{A10}$$

Turning to wages, we have that

$$
\begin{aligned}
w_z^* =& (1-\alpha_z)\frac{\eta_z}{\ell_z}Y^* \\
=& (1-\alpha_z)\frac{\eta_z}{\ell_z}\mathcal{A}^{\frac{1}{1-\alpha}}(K/Y)^{*\frac{\alpha}{1-\alpha}}\prod_v(\psi_v\ell_v)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}} \\
>& \frac{(1-\alpha_z)\eta_z}{\ell_z}\mathcal{A}^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}\prod_z(\psi_z\ell_z)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}}(\rho+p\sigma+\delta)^{-\frac{\alpha}{1-\alpha}},
\end{aligned}
$$

where the last line uses inequality (A10).

Finally, because $\psi_z(\rho+p\sigma+\delta) > \psi_z R^*$, a sufficient condition to ensure $w_z^* > \psi_z R^*$ is

$$
\frac{(1-\alpha_z)\eta_z}{\ell_z}\mathcal{A}^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}\prod_z(\psi_z\ell_z)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}}(\rho+p\sigma+\delta)^{-\frac{\alpha}{1-\alpha}} > \psi_z(\rho+p\sigma+\delta).
$$

This inequality is equivalent to (A9). Finally, the definition of $\mathcal{A}$ in (A8) implies that (A9) holds for large values of A, concluding the proof of the lemma. ■

A.2 Propositions for Baseline Model A

Proof of Lemma 1. Let $x_{z,t} := w_z/r + a_{z,t}$ denote effective wealth. Equation (1) can be rewritten as:

$$
\begin{aligned}
&\max_{\{c_{z,t},x_{z,t}\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\frac{c_{z,t}^{1-\sigma}}{1-\sigma}dt \\
&\text{s.t. } \dot{x}_{z,t} = rx_{z,t} - c_{z,t}, \text{ and } x_{z,t} \geq 0
\end{aligned}
\tag{A11}
$$

The Hamiltonian associated with this maximization problem is

$$
H(c_z, x_z, \lambda_z) := \frac{c_z^{1-\sigma}}{1-\sigma} + \lambda(rx_z - c_z), \tag{A12}
$$

where λ_z is the co-state for effective wealth.

We can write the candidate solution given in Lemma 1 as (time arguments are ignored to save on notation)

$$
\dot{x}_z = \frac{r-\rho}{\sigma}x_z \qquad\qquad c_z = \left(r - \frac{r-\rho}{\sigma}\right)x_z. \tag{A13}
$$

We will show that the unique solution to this system of differential equations starting from $x_{z,0} = w_z/r$ solves the maximization problem in (A11).

Theorem 7.14 in Acemoglu (2009) implies that this candidate path reaches an optimum if there exists a co-state variable λ_z such that:

1. the path satisfies the restrictions $\dot{x}_z = rx_z - c_z$, and $x_z \geq 0$;
2. the following necessary conditions hold:

$$
\begin{aligned}
c_z^{-\sigma} =& \lambda_z, \\
\rho\lambda_z - \dot{\lambda}_z =& r\lambda_z;
\end{aligned}
$$

3. the maximized Hamiltonian $M(x_z, \lambda_z) = \max_c H(c, x_z, \lambda_z)$ is concave in x_z along the candidate path;
4. the transversality condition holds. That is, for the candidate path, we have

$$
\lim_{s\to\infty} e^{-\rho s} x_z \lambda_z = 0.
$$

and for all other feasible paths, $\hat{x}_z$, we have

$$
\lim_{s\to\infty} e^{-\rho s} \hat{x}_z \lambda_z \geq 0.
$$

To prove condition 1, note that starting from any $x_{z,0} \geq 0$, we will have $x_z \geq 0$. Moreover, for any path satisfying equations (A13) the flow budget constraint holds:

$$
\begin{aligned}
rx_z - c_z &= rx_z - \left(r - \frac{r-\rho}{\sigma}\right) x_z \\
&= \frac{r-\rho}{\sigma} x_z \\
&= \dot{x}_z.
\end{aligned}
$$

To prove condition 2, define $\lambda_z := (r - (r-\rho)/\sigma)^{-\sigma} x_z^{-\sigma} > 0$ (here we used the condition $r > (r-\rho)/\sigma$). By construction, $c_z^{-\sigma} = \lambda_z$. Moreover:

$$
\begin{aligned}
\rho\lambda_z - \dot{\lambda}_z =& \rho\left(r - \frac{r-\rho}{\sigma}\right)^{-\sigma} x_z^{-\sigma} + \left(r - \frac{r-\rho}{\sigma}\right)^{-\sigma} \sigma x_{z,t}^{-\sigma-1} \dot{x}_z \\
=& \left(\rho + \sigma\frac{\dot{x}_z}{x_z}\right)\left(r - \frac{r-\rho}{\sigma}\right)^{-\sigma} x_z^{-\sigma} \\
=& \left(\rho + \sigma\frac{\dot{x}_z}{x_z}\right)\lambda_z \\
=& \left(\rho + \sigma\frac{r-\rho}{\sigma}\right)\lambda_z \\
=& r\lambda_z.
\end{aligned}
$$

To prove condition 3, note that

$$\max_c H(c, x_z, \lambda_z) = \frac{\lambda_z^{\frac{\sigma-1}{\sigma}}}{1-\sigma} + \lambda_z(r x_z - \lambda_z^{-\frac{1}{\sigma}}),$$

which is concave (linear) in x_z.

To prove the first part of condition 4, note that along the candidate path, x_z grows at a rate $\frac{r-\rho}{\sigma}$, and λ_z at a rate $\rho - r$. It follows that the first part of the transversality condition holds if

$$-\rho + \frac{r-\rho}{\sigma} + \rho - r < 0,$$

which is equivalent to the condition $r > (r-\rho)/\sigma$.

The second part of the transversality condition follows from the fact that, along any feasible path, we have $\hat{x}_z \geq 0$.

It follows that the candidate paths given in Lemma 1 provide optimal paths for consumption and asset accumulation in a steady state. ■

Proof of Proposition 1.

The main text presents the derivations of the supply curve (equation 6) and the demand curve (equation 7).

The supply curve $(K/\bar{w})^s$ increases from zero to infinity as r increases from ρ to $\rho + p\sigma$. For $r < \rho$ households supply no capital. For $r > \rho + p\sigma$, households amass a divergent amount of capital.

The demand curve $(K/\bar{w})^s$ decreases from $(\alpha/(1-\alpha))/(\rho+\delta) > 0$ to $(\alpha/(1-\alpha))/(\rho + p\sigma + \delta) > 0$ as r increases from ρ to $\rho + p\sigma$.

These observations imply that equation (4) has a unique solution r^* and that this unique solution lies in $(\rho, \rho + p\sigma)$. In fact, r^* can be computed analytically as

$$r^* = \frac{-((1-\alpha)\delta - \rho - \alpha p\sigma) + \sqrt{((1-\alpha)\delta - \rho - \alpha p\sigma)^2 + 4(1-\alpha)\rho\delta}}{2} \tag{A14}$$

The equilibrium return r^* determines the remaining aggregates as follows. First, the capital-output ratio is given by

$$(K/Y)^* = \frac{\alpha}{r^* + \delta}.$$

The output level is given by

$$Y^* = \mathcal{A}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{r^*+\delta}\right)^{\frac{\alpha}{1-\alpha}} \prod_z (\ell_z \psi_z)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}}.$$

These two equations combined imply

$$K^* = \mathcal{A}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{r^* + \delta} \right)^{\frac{1}{1-\alpha}} \prod_z (\ell_z \psi_z)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}}. \tag{A15}$$

Turning to wages, we have that $w_z^* = (1-\alpha_z)\frac{\eta_z}{\ell_z} Y^*$, which implies

$$w_z^* = (1-\alpha_z)\frac{\eta_z}{\ell_z} \mathcal{A}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{r^* + \delta} \right)^{\frac{\alpha}{1-\alpha}} \prod_z (\ell_z \psi_z)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}}.$$

Finally, equation (5) can be derived from the household side. As explained in the main text, in steady state we must have

$$0 = \frac{r^* - \rho}{\sigma} \left(K^* + \frac{\bar{w}^*}{r^*} \right) - pK^*.$$

This expression can be rearranged as

$$r^* = \rho + p\sigma \frac{r^* K^*}{r^* K^* + \bar{w}^*} = \rho + p\sigma \alpha_{net}^*.$$

Note that the condition $r^* > (r^* - \rho)/\sigma$, which is needed to ensure the households' policy functions are an optimum is equivalent to $\rho + p\alpha_{net}^*(\sigma - 1) > 0$, which we assume holds throughout.

Turning to the comparative statics exercises, we can rearrange (4) as:

$$\frac{\left(1 - \frac{\rho}{r^*}\right)(r^* + \delta)}{p\sigma + \rho - r^*} = \frac{\alpha}{1-\alpha}.$$

The right hand side of this equation is increasing in r^*, and the left is increasing in α. It follows that r^* is increasing in α.

The household accumulation rate is given by $(r^* - \rho)/\sigma$, and so it also increases with α.

The net capital share satisfies the identity in equation (5), and so it increases with α.

Denote the capital-output ratio by $(K/Y)^*$. We have

$$\alpha_{net}^* = \frac{r^*(K/Y)^*}{1 - \delta(K/Y)^*}.$$

Rearranging this equation and using the fact that $r^* = \rho + p\sigma\alpha_{net}^*$, we obtain

$$(K/Y)^* = \frac{\alpha_{net}^*}{\rho + (p\sigma + \delta)\alpha_{net}^*}, \tag{A16}$$

which is an increasing function of α_{net}^*, and hence α.

Finally, turning to output, from equation (2) we have

$$d \ln Y^* = \frac{1}{1-\alpha} d \ln \mathrm{TFP}_\alpha + \frac{\alpha}{1-\alpha} d \ln(K/Y)^*.$$

Because $d \ln \mathrm{TFP}_\alpha > 0$ and $d \ln(K/Y)^* > 0$ following any increase in the α_z's, we have that automation always increase output. ■

Proof of Proposition 2. Equilibrium factor prices imply that relative wages satisfy

$$\frac{w_z}{w_v} = \frac{(1-\alpha_z)}{(1-\alpha_v)} \frac{\eta_z}{\eta_v} \frac{\ell_v}{\ell_z}.$$

It follows that an increase in α_z reduces w_z/w_v for $v \neq z$.

Turning to the wage bill, we can add up individual wages for all z to obtain:

$$\bar{w} = (1-\alpha)Y.$$

It follows that

$$d \ln \bar{w} = -\frac{1}{1-\alpha} \sum \eta_z d\alpha_z + \frac{1}{1-\alpha} d \ln \mathrm{TFP}_\alpha + \frac{\alpha}{1-\alpha} d \ln(K/Y)^*.$$

We now show that the terms $d \ln \mathrm{TFP}_\alpha$ and $d \ln(K/Y)^*$ are both decreasing in p and converge to zero as p increases. Because the term $-(1/(1-\alpha_z)) \sum \eta_z d\alpha_z$ is negative, this establishes the existence of the threshold $\bar{p}$.

We first analyze the term $d \ln \mathrm{TFP}_\alpha$. This is given by

$$d \ln \mathrm{TFP}_\alpha = \sum_z \eta_z \ln\left(\frac{w_z^*}{\psi_z R^*}\right) = \sum_z \eta_z \ln\left(\frac{K^*}{\alpha}\right) - \eta_z \ln\left(\frac{\psi_z \ell_z}{\eta_z(1-\alpha_z)}\right),$$

where we used the formulas for equilibrium factor prices. It is enough to show that K^* is decreasing in p and that K^* converges to zero as p increases. Because K^* is given by (A15), it is enough to show that r^* is increasing in p and that r^* converges to infinity as p increases.

The fact that r^* increases in p follows from equation (4). An increase in p contracts the supply of capital, which results in a higher r^*. Moreover, equation (A14) shows that $r^* \to \infty$ as $p \to \infty$. Note that the formal limit of $d \ln \mathrm{TFP}_\alpha$ as $p \to \infty$ is zero, since as K^* declines, we eventually reach a point where Assumption 1 starts failing and increases in α_z do not affect productivity.

We now turn to the term $d \ln(K/Y)^*$. We have that

$$\alpha = (K/Y)^*(\rho + p\sigma\alpha_{net}^* + \delta).$$

Differentiating this expression we obtain

$$1=\frac{p\sigma\alpha^*_{net}}{\rho+p\sigma\alpha^*_{net}+\delta}\frac{\partial\ln\alpha^*_{net}}{\partial\ln\alpha}+\frac{\partial\ln(K/Y)^*}{\partial\ln\alpha}. \tag{A17}$$

Moreover, equation (A16) implies that

$$\frac{\partial\ln(K/Y)^*}{\partial\ln\alpha}=\frac{\rho}{\rho+(p\sigma+\delta)\alpha^*_{net}}\frac{\partial\ln\alpha^*_{net}}{\partial\ln\alpha}. \tag{A18}$$

Solving equations (A17) and (A18), we obtain:

$$\frac{\partial\ln(K/Y)^*}{\partial\ln\alpha}=\frac{1}{1+\dfrac{p\sigma\alpha^*_{net}}{\rho+p\sigma\alpha^*_{net}+\delta}\dfrac{\rho+(p\sigma+\delta)\alpha^*_{net}}{\rho}}.$$

We now show that the elasticity $\frac{\partial\ln(K/Y)^*}{\partial\ln\alpha^*_{net}}$ converges to zero as p rises. A sufficient condition for this to be the case is that α^*_{net} is nondecreasing in p, which holds when $\delta=0$ and $\alpha^*_{net}=\alpha$. To show this is the case more generally, start from the fact that $\alpha=R(K/Y)$. Rewriting the right hand side in terms of α^*_{net}, we obtain

$$\alpha=(\rho+p\sigma\alpha^*_{net}+\delta)\frac{\alpha^*_{net}}{\rho+(p\sigma+\delta)\alpha^*_{net}}.$$

This equation can be rearranged as

$$\alpha\left(1-\frac{\delta(1-\alpha^*_{net})}{\rho+p\sigma\alpha^*_{net}+\delta}\right)=\alpha^*_{net}.$$

This equation defines α^*_{net} implicitly as a function of α and p. The left hand side is increasing in α^*_{net} and intercepts the right-hand side (the 45-degree line) from above at a single equilibrium point. An increase in p shifts the left-hand side upwards, which results in a higher equilibrium value for α^*_{net} as claimed.

The above argument shows that there exists some $\bar{p}$ such that, for $p>\bar{p}$, $d\ln\bar{w}<0$. To conclude the proof, we show that $\bar{p}>0$. This follows from the fact that, for $p=0$, $d\ln\bar{w}>0$.

To show this, note that for $p=0$ we get

$$d\ln(K/Y)^*=\frac{d\alpha}{\alpha}=\frac{1}{\alpha}\sum_z\eta_z d\alpha_z,$$

and therefore

$$d\ln\bar{w}=-\frac{1}{1-\alpha}\sum\eta_z d\alpha_z+\frac{1}{1-\alpha}d\ln\text{TFP}_\alpha+\frac{\alpha}{1-\alpha}\frac{1}{\alpha}\sum_z\eta_z d\alpha_z=\frac{1}{1-\alpha}d\ln\text{TFP}_\alpha>0.$$

■

Proof of Proposition 3. Below we derive the effective wealth distribution, the wealth distribution, and the income distribution. To save on notation, we do not include asterisks when denoting steady state objects.

Effective wealth distribution: Denote the stationary density of effective wealth conditional on a given skill type z by $f_z(x)$. f_z satisfies the Kolmogorov Forward Equation (KFE)

$$0 = -\partial_x \left(\frac{r-\rho}{\sigma} x f_z(x) \right) - p f_z(x)$$

on $(w_z/r, \infty)$. We guess and verify that f is Pareto, i.e. $f_z(x) = c\zeta x^{-\zeta-1}$ for some constants c and ζ. Substituting in the guess

$$\begin{aligned} 0 &= \zeta \frac{r-\rho}{\sigma} c\zeta x^{-\zeta-1} - pc\zeta x^{-\zeta-1} \\ 0 &= \zeta \frac{r-\rho}{\sigma} - p \\ \frac{1}{\zeta} &= \frac{r-\rho}{p\sigma} = \alpha_{net} \end{aligned}$$

Since $f_z(x) = c\zeta x^{-\zeta-1}$ must integrate to 1 on $(w_z/r, \infty)$, we must have $c = (w_z/r)^{-\zeta}$. Hence this is a Pareto distribution with tail parameter $\zeta = \frac{1}{\alpha_{net}}$ and scale parameter $x_z(0) = w_z/r$.

Because the distribution of effective wealth is Pareto, the conditional counter-CDF for effective wealth of each skill type z is of the form:

$$\Pr(\text{effective wealth} \geq x|z) = \left(\frac{x}{w_z/r} \right)^{-\frac{1}{\alpha_{net}}}, \quad x \geq w_z/r. \tag{A19}$$

Wealth distribution: We now derive the counter-CDF for wealth. Recall that effective wealth x is $x := a + w_z/r$. Therefore

$$\Pr(\text{wealth} \geq a|z) = \Pr(\text{effective wealth} \geq a + w_z/r|z) = \left(\frac{a + w_z/r}{w_z/r} \right)^{-\frac{1}{\alpha_{net}}}, \quad a \geq 0$$

To find the unconditional distribution, we add across the different skill-types, which yields

$$\Pr(\text{assets} \geq a) = \sum_z \ell_z \left(\frac{a + w_z/r}{w_z/r} \right)^{-\frac{1}{\alpha_{net}}}.$$

Income Distribution: We now derive the counter-CDF for income. The income of a person with effective wealth x is rx. Therefore

$$\Pr(\text{income} \geq y|z) = \Pr(\text{effective wealth} \geq y/r|z) = \left(\frac{y/r}{w_z/r} \right)^{-\frac{1}{\alpha_{net}}}, \quad y \geq w_z.$$

To find the unconditional distribution, we add across the different skill-types, which yields

$$\Pr(\text{income} \geq y) = \sum_z \left(\frac{\max\{y, w_z\}}{w_z} \right)^{-\frac{1}{\alpha_{net}}}.$$

Finally, when $\delta = 0$, we have $\alpha_{net} = \alpha$ and $\frac{1}{\zeta} = \alpha$. When $\delta > 0$, Proposition 1, implies that $\frac{1}{\zeta}$ is increasing in α. ■

B Derivations and Proofs for the Extended Model

B.1 Derivations and Lemmas

Before presenting the proofs, we generalize the model in the text so that investors could also face a borrowing constraint of the form

$$-b_{z,t} \leq \theta a_{z,t} + \frac{w_z + T}{r_B - g},$$

where $\theta \in (0, 1]$ parameterizes the extent to which investors can pledge their capital.[44] The results in the main text follow in the special case with $\theta = 1$. We also provide a lemma characterizing investors policy functions.

Lemma A2 (Achdou et al. (2022)) *Let $r_I = \kappa r_K + (1-\kappa) r_B$. Investors' policy functions are given by*

$$c_{z,t} = \frac{\rho + (\sigma - 1) r_I - \frac{1}{2}(\sigma - 1)\gamma \nu^2 \kappa^2}{\sigma} x_{z,t},$$
$$\kappa = \min\left\{ \frac{1}{1-\theta}, \frac{r_K - r_B}{\gamma \nu^2} \right\}.$$

which imply that effective wealth follows a random growth process:

$$dx_{z,t} = \frac{r_I - \rho + \frac{1}{2}(\sigma - 1)\gamma \nu^2 \kappa^2}{\sigma} x_{z,t} dt + \kappa \nu x_{z,t} dW_t.$$

Proof. Using the definition of effective wealth and the fact that in a balanced growth equilibrium wages and tax revenue grow at a constant rate g, we have

$$dx_{z,t} = da_{z,t} + db_{z,t} + g \frac{w_z + T}{r_B - g} dt.$$

[44] The usual formulation used in the literature is $-b_{z,t} \leq \theta a_{z,t}$. Relative to this, our formulation assumes that human wealth is pledgeable, which makes the model more tractable.

Substituting the investors' budget constraint in place of $da_{z,t} + db_{z,t}$ and rearranging terms we obtain

$$
\begin{aligned}
dx_{z,t} =& r_B \left(a_{z,t} + b_{z,t} + \frac{w_z + T}{r_B - g} \right) dt + (r_K - r_B) a_{z,t} dt + \nu a_{z,t} dW_t - c_{z,t} dt \\
=& r_B x_{z,t} dt + (r_K - r_B) a_{z,t} dt + \nu a_{z,t} dW_t - c_{z,t} dt.
\end{aligned}
$$

Let κ denote the share of effective wealth held in equity, so that $a_{z,t} = \kappa x_{z,t}$. We can rewrite the budget constraint as

$$
dx_{z,t} = (r_I x_{z,t} - c_{z,t}) dt + \nu \kappa x_{z,t} dW_t. \tag{A20}
$$

In what follows, we drop the subscript z and time t, and examine the savings problem in terms of the state variable x—effective wealth. The HJB equation for this problem is

$$
0 = \max_{c>0, \kappa \in [0, 1/(1-\theta)]} f(c, v(x)) + (r_I x - c) v'(x) + \frac{1}{2} \nu^2 \kappa^2 x^2 v''(x).
$$

Using the Duffie–Lions aggregator and guessing $v(x) = \Lambda x^{1-\gamma}/(1-\gamma)$, the HJB equation becomes

$$
\frac{\rho \Lambda x^{1-\gamma}}{1-\sigma} = \max_{c>0, \kappa \in [0, 1/(1-\theta)]} \frac{\rho \Lambda x^{1-\gamma}}{1-\sigma} \left(\frac{c}{\Lambda^{1/(1-\gamma)} x} \right)^{1-\sigma} + (r_I x - c) \Lambda x^{-\gamma} - \frac{\gamma}{2} \nu^2 \kappa^2 \Lambda x^{1-\gamma}.
$$

The optimal consumption and portfolio choice are given by

$$
c = \rho^{\frac{1}{\sigma}} \Lambda^{-\frac{1}{\sigma} \frac{1-\sigma}{1-\gamma}} x \qquad \kappa = \min \left\{ \frac{1}{1-\theta}, \frac{r_K - r_B}{\gamma \nu^2} \right\}.
$$

Plugging into the HJB equation and canceling terms, we get

$$
\frac{\rho}{1-\sigma} = \frac{\rho}{1-\sigma} \left(\frac{\rho^{\frac{1}{\sigma}} \Lambda^{-\frac{1}{\sigma} \frac{1-\sigma}{1-\gamma}}}{\Lambda^{1/(1-\gamma)}} \right)^{1-\sigma} + r_I - \rho^{\frac{1}{\sigma}} \Lambda^{-\frac{1}{\sigma} \frac{1-\sigma}{1-\gamma}} - \frac{1}{2} \gamma \nu^2 \kappa^2.
$$

This is an equation in $c/x = \rho^{\frac{1}{\sigma}} \Lambda^{-\frac{1}{\sigma} \frac{1-\sigma}{1-\gamma}}$, which yields

$$
c/x = \frac{\rho + (\sigma - 1) r_I - \frac{1}{2} (\sigma - 1) \gamma \nu^2 \kappa^2}{\sigma}.
$$

The policy function for $c_{z,t}$ follows from this expression, and the behavior of $x_{z,t}$ follows after plugging this policy function in the the budget constraint in equation (A20). ■

B.2 Propositions for Extended Model

Proof of Proposition 4. Lemma A2 implies that investors accumulate wealth at a rate

$$\mu_I = \frac{r_W - \rho}{\sigma}. \tag{A21}$$

Lemma 1 applies to the remaining households, whose wealth then grows at a rate

$$\mu_H = \frac{r_B - \rho}{\sigma}.$$

In what follows, we will describe the BGE in terms of r_W and r_B. In particular, we define $h(r_W - r_B) = r_K - r_B$ and $m(r_W - r_B) = \kappa$ implicitly as the solution to

$$r_W - r_B = mh + \frac{1}{2}(\sigma - 1)\gamma\nu^2 m^2 \qquad\qquad m = \min\left\{\frac{1}{1-\theta}, \frac{h}{\gamma\nu^2}\right\}.$$

We can then write the return to capital and portfolio choice implicitly as:

$$r_K = h(r_W - r_B) + r_B \qquad\qquad \kappa = m(r_W - r_B),$$

where h is a continuous and increasing function and m is continuous and nondecreasing.

Denote by X_I the aggregate effective wealth of investors and by X_H the aggregate effective wealth of the remaining households, and bond holdings by B_I and B_H, respectively. Optimal household saving behavior combined with the dissipation shocks implies that

$$\dot{X}_I = \mu_I X_I - p(K + B_I), \qquad\qquad \dot{X}_H = \mu_H X_H - pB_H.$$

Moreover, because the value of capital installed in firms must be equal to the total capital owned by investors, aggregate effective wealths are given by

$$X_I = K + B_I + \chi\frac{\bar{w} + T}{r_B - g} \qquad\qquad X_H = B_H + (1-\chi)\frac{\bar{w} + T}{r_B - g}.$$

Human wealth now depends on the sum of wages and transfers, whose present discounted value is obtained by dividing them by $r_B - g$ to account for their growth over time. It is convenient to analyze the BGE in terms of capital and bonds normalized by the value of human wealth, which are constant along a BGE:

$$k_n = \frac{K}{(\bar{w} + T)/(r_B - g)} \qquad b_I = \frac{B_I}{(\bar{w} + T)/(r_B - g)} \qquad b_H = \frac{B_H}{(\bar{w} + T)/(r_B - g)}.$$

Along a BGE, the effective wealth of investors grows at a rate g, so that $\dot{X}_I = gX_I$ and $\dot{X}_H = gX_H$. The effective wealth of investors grows at a rate g if and only if $r_W > \rho + \sigma g$

and the BGE values of k_n and b_I satisfy

$$\left(\frac{r_W-\rho}{\sigma}-g\right)(k_n+b_I+\chi)=p(k_n+b_I). \tag{A22}$$

In addition, because a fraction $\kappa=m(r_W-r_B)$ of investors wealth is held in equity,

$$m(r_W-r_B)\,(k_n+b_I+\chi)=k_n. \tag{A23}$$

Likewise, the effective wealth of households grows at a rate g if and only if $r_B>\rho+\sigma g$ and

$$\left(\frac{r_B-\rho}{\sigma}-g\right)(b_H+1-\chi)=pb_H, \tag{A24}$$

or $r_B\le\rho+\sigma g$ and $b_H=0$.

We now characterize the production side of the economy. We focus on a balanced-growth equilibrium in which Assumption 1 holds, so that output is given by 2. Because of markups, total wage payments are now given by

$$\bar{w}=\frac{1-\alpha}{\varphi}Y,$$

which implies that firms pay a share $(1-\alpha)/\varphi$ of their revenue to labor; while the remaining share of revenue $1-(1-\alpha)/\varphi$ constitutes gross capital income which is taxed at a rate $1-\tau$ (recall that we assumed a gross tax on capital income) and must cover for the depreciation of capital. Thus, we can compute the after-tax gross income from capital as

$$(r_K+\delta)K=(1-\tau)\left(1-\frac{1-\alpha}{\varphi}\right)Y.$$

Tax revenue from capital taxation, and thus the lump-sum transfers, are equal to

$$T=\tau\left(1-\frac{1-\alpha}{\varphi}\right)Y.$$

We can combine the equations for after-tax gross capital income and labor income and transfers to obtain an expression of the demand for financing by firms:

$$k_n=\frac{\widetilde{\alpha}}{1-\widetilde{\alpha}}\frac{r_B-g}{h(r_W-r_B)+r_B+\delta}. \tag{A25}$$

A balanced-growth equilibrium is characterized by constant values for r_W,r_B,k_n,b_I,b_H that solve equations (A22), (A23), (A24) and (A25) and where r_B is endogenous and ensures market clearing in the bonds market

$$b_I+b_H=0.$$

In this case, we will assume that $\rho + (\sigma - 1)g > 0$, which is a sufficient condition to ensure that the equilibrium exists and features finite wealth.

We start with the case in which investors are risk neutral ($\gamma = 0$ in the Duffie–Lions aggregator) and their borrowing constraint does not bind. We will provide necessary and sufficient conditions for this to be the case below.

Because we assumed that the borrowing constraint does not bind, it must be the case that $r_W = r_K = r_B = r^*$.

Adding equations (A22) and (A24), we obtain

$$\left(\frac{r^* - \rho}{\sigma} - g\right)(k_n + 1) = pk_n. \tag{A26}$$

Solving for k_n and substituting into equation (A25) shows that an equilibrium is fully determined by a level of returns that satisfies:

$$\frac{1 - (\rho + (\sigma - 1)g)/(r^* - g)}{\sigma(p + g) + \rho - r^*} = \frac{\widetilde{\alpha}}{1 - \widetilde{\alpha}} \frac{1}{r^* + \delta}. \tag{A27}$$

The left hand side of this equation is increasing in r^*; while the right-hand side is decreasing in r^*. Moreover, at $r^* = \rho + \sigma g$, the left-hand side is lower than the right-hand side; and at $r^* = \rho + \sigma(p + g)$, the the left-hand side converges to infinity and exceeds the right-hand side. This implies a unique solution exists and satisfies $r^* \in (\rho + \sigma g, \rho + \sigma(p + g))$.

We now derive conditions for θ that ensure the borrowing constraint does not bind. Denote by k_n^* the value of k_n in the balanced growth equilibrium above, which is given by

$$k_n^* = \frac{\widetilde{\alpha}}{1 - \widetilde{\alpha}} \frac{r^* - g}{r^* + \delta},$$

and is independent of θ by construction. Equations (A22) and (A24) imply that investors must borrow and amount

$$b^* = (1 - \chi)k_n^*$$

from households. and so we must have

$$\kappa^* = \frac{k_n^*}{k_n^* - b^* + \chi} = \frac{1}{\chi} \frac{k_n^*}{k_n^* + 1}.$$

It follows that the borrowing constraint will be slack if and only if

$$\frac{1}{1 - \theta} \geq \frac{1}{\chi} \frac{k_n^*}{k_n^* + 1} \Leftrightarrow \theta \geq 1 - \chi \frac{k_n^* + 1}{k_n^*} := \bar{\theta}.$$

Note that $\bar{\theta} \leq 1$, as claimed in the proposition.

We now turn to the case in which investors are risk averse and/or $\theta < \bar{\theta}$ and we have a closed economy. In what follows, we will assume that $r_W > \rho + \sigma g$, which must hold in any equilibrium. To see this, notice that for $r_W \leq \rho + \sigma g$, investors do not accumulate wealth and the supply of capital is zero, which cannot be the case in a BGE.

Combining equations (A22) and (A24), we obtain a supply of (normalized) capital

$$k_n = \frac{p\sigma m(r_W - r_B)}{\sigma(p+g) + \rho - r_W}\chi. \tag{A28}$$

Combining this with the demand for firm financing in equation (A25) yields the market clearing condition in the capital market:

$$D_K(r_W, r_B) = \frac{\widetilde{\alpha}}{1-\widetilde{\alpha}} \frac{r_B - g}{h(r_W - r_B) + r_B + \delta} - \frac{p\sigma m(r_W - r_B)}{\sigma(p+g) + \rho - r_W}\chi = 0 \tag{A29}$$

where $D_K(r_W, r_B)$ is the excess demand for capital, and the curve $D_K(r_W, r_B) = 0$ defines the locus of points for which the capital market clears. Likewise, the market clearing condition in the bond market is given by

$$D_B(r_W, r_B) = \frac{p\sigma(m(r_W - r_B) - 1)}{\sigma(p+g) + \rho - r_W}\chi + \chi - \frac{r_B - \rho - \sigma g}{\sigma(p+g) + \rho - r_B}(1-\chi) = 0, \tag{A30}$$

where $D_B(r_W, r_B)$ is the excess demand for bonds, and the curve $D_B(r_W, r_B) = 0$ defines the locus of points for which the bond market clears.

The following Lemmas characterize the behavior of these loci.

Lemma A3 *The curve $D_K(r_W, r_B) = 0$ gives a continuous and upward sloping locus in the (r_W, r_B) space defined for $r_W \in (g, \rho + \sigma(p+g))$ and $r_B \in (g, r_W)$. Moreover:*

1. *as $r_W \downarrow g$, the locus converges to the point (g, g)*
2. *as $r_W \uparrow \rho + \sigma(p+g)$, the locus converges to the point $(\rho + \sigma(p+g), \rho + \sigma(p+g))$.*

Lemma A4 *The curve $D_B(r_W, r_B) = 0$ gives a continuous and initially decreasing locus in the (r_W, r_B) space defined for $r_W \in (\rho + \sigma g, \rho + \sigma(p+g))$ and $r_B < r_W$. Moreover:*

1. *as $r_W \downarrow \rho + \sigma g$, the locus converges to the point $(\rho + \sigma g, \rho + \sigma g)$;*
2. *as $r_W \uparrow \rho + \sigma(p+g)$, the locus converges to the point $(\rho + \sigma(p+g), \tilde{r}_B)$, where*

$$\tilde{r}_B := \rho + \sigma(p+g) - \frac{1}{2}(\sigma+1)\gamma\nu^2.$$

3. *let $\bar{\gamma} := 2p\frac{\sigma}{1+\sigma}(1-\chi)$.*
    - *if $\gamma\nu^2 > \bar{\gamma}$, then $r_W - r_B$ increases from zero to a maximum of $\frac{1}{2}(\sigma+1)\gamma\nu^2$ along the locus $D_B(r_W, r_B) = 0$;*

- *if $\gamma\nu^2 < \bar{\gamma}$, then $r_W - r_B$ increases from zero up to a maximum and then decreases and reaches $\frac{1}{2}(\sigma+1)\gamma\nu^2$ along the locus $D_B(r_W, r_B) = 0$.*

The proof of these lemmas is technical and relegated to Appendix F.

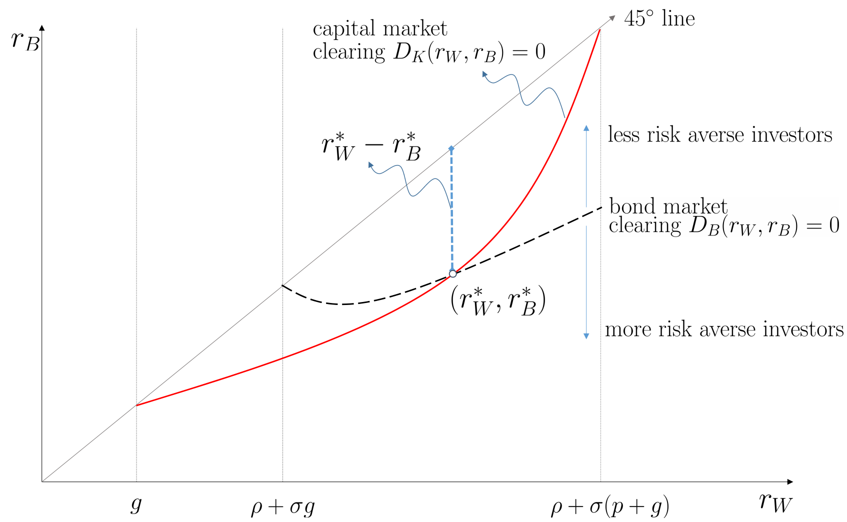

Figure A1: Typical configurations of the locus for $D_K(r_W, r_B) = 0$ and $D_B(r_W, r_B) = 0$.

The two lemmas combined imply that the loci $D_K(r_W, r_B) = 0$ and $D_B(r_W, r_B) = 0$ are as depicted in Figure A1. The lemmas imply that at $r_W = \rho + \sigma g$, the locus for $D_B(r_W, r_B) = 0$ is above that for $D_K(r_W, r_B) = 0$ (recall that, by assumption, $\rho + \sigma g > g$). On the other hand, at $r_W = \rho + \sigma(p+g)$, the locus for $D_K(r_W, r_B) = 0$ is above that for $D_B(r_W, r_B) = 0$. The intermediate value theorem then implies that these loci intercept at a point r_W^*, r_B^* and an equilibrium exists. Moreover, as the figure shows, $r_W^* > r_B^* > g$ and $r_W^* \in (\rho + \sigma g, \rho + \sigma(p+g))$.

Finally, the characterization of the tail properties of the income and wealth distribution follows as a corollary of Proposition A1. ■

Proposition A1 *Let x denote effective wealth and define* normalized wealth *by*

$$\widetilde{x} = x \Big/ \frac{w_z}{r_B}.$$

Let $f_H(\widetilde{x})$ and $f_I(\widetilde{x})$ denote the PDFs of the distributions of normalized wealth for households and investors. The distribution of normalized wealth for households is given by

$$f_H(\widetilde{x}) = \zeta_H \widetilde{x}^{-\zeta_H - 1} \text{ for } \widetilde{x} \geq 1, \tag{A31}$$

where

$$\frac{1}{\zeta_H} := \frac{r_B - \rho - \sigma g}{p\sigma},$$

and the distribution of normalized wealth for investors is given

$$f_I(\widetilde{x}) = \begin{cases} \dfrac{1}{1/\zeta_P - 1/\zeta_N}\widetilde{x}^{-\zeta_P-1} & \textit{for } \widetilde{x} \geq 1 \\ \dfrac{1}{1/\zeta_P - 1/\zeta_N}\widetilde{x}^{-\zeta_N-1} & \textit{for } \widetilde{x} \in [0,1), \end{cases} \tag{A32}$$

where

$$\frac{1}{\zeta_P} := \frac{r_W - \rho - \sigma g - \frac{\sigma\kappa^2\nu^2}{2} + \sqrt{\left(r_W - \rho - \sigma g - \frac{\sigma\kappa^2\nu^2}{2}\right)^2 + 2\sigma^2\kappa^2\nu^2 p}}{2p\sigma} > 0$$

and

$$\frac{1}{\zeta_N} := \frac{r_W - \rho - \sigma g - \frac{\sigma\kappa^2\nu^2}{2} - \sqrt{\left(r_W - \rho - \sigma g - \frac{\sigma\kappa^2\nu^2}{2}\right)^2 + 2\sigma^2\kappa^2\nu^2 p}}{2p\sigma} < 0.$$

Moreover, the distribution for investors' income flows has a Pareto tail with index $1/\zeta_P$.

Proof. The evolution of (normalized) effective wealth for households is given by

$$\dot{\widetilde{x}}_t = (\mu_H - g)\widetilde{x}_t,$$

where

$$\mu_H := \frac{r_B - \rho}{\sigma}$$

and $\widetilde{x}_t$ resets to 1 with probability p.

The Kolmogorov-forward equation characterizing f_H in steady state is then given by

$$0 = -\partial_x\left((\mu_H - g)x f_H(x)\right) - p f_H(x) + p\wp(x-1), \tag{A33}$$

where $\wp(.)$ is the Dirac's delta function (a mass of 1 at 0). To solve this differential equation, we guess and verify that

$$f_H(x) = C_H x^{-\zeta_H - 1}, \text{ for } x \geq 1.$$

Plugging this guess in the KFE equation (A33), we obtain

$$0 = \zeta_H(\mu_H - g)C_H x^{-\zeta_H-1} - pC_H x^{-\zeta_H-1}, \text{ for } x \geq 1.$$

This implies

$$\frac{1}{\zeta_H} = \frac{\mu_H - g}{p} = \frac{r_B - \rho - \sigma g}{p\sigma},$$

as claimed in the proposition. Moreover, because the density $f_H(x)$ must integrate to 1, we obtain

$$1 = \int_1^{\infty} C_H x^{-\zeta_H - 1} dx \Rightarrow C_H = \zeta_H.$$

The evolution of (normalized) effective wealth for investors is given by

$$d\widetilde{x}_t = (\mu_I - g)\widetilde{x}_t dt + \kappa\nu\widetilde{x}_t dW_t,$$

where

$$\mu_I := \frac{r_W - \rho}{\sigma}$$

and $\widetilde{x}_t$ resets to 1 with probability p.

The Kolmogorov-forward equation characterizing f_I in steady state is then given by

$$0 = -\partial_x \left((\mu_I - g)x f_I(x)\right) + \frac{1}{2}\partial_{xx}\left(\nu^2\kappa^2 x^2 f_I(x)\right) - p f_I(x) + p\wp(x-1), \tag{A34}$$

where $\wp(.)$ is the Dirac's delta function (a mass of 1 at 0). To solve this differential equation, we guess and verify a piece-wise solution of the form

$$f_I(x) = C_P x^{-\zeta_P - 1}, \text{ for } x \geq 1,$$

and

$$f_I(x) = C_N x^{-\zeta_N - 1}, \text{ for } x \in (0,1),$$

which allows for the possibility that the distribution might be different to the left and to the right of the reinjection point (note also that the process for effective wealth implies that $x > 0$).

Plugging this guess in the KFE equation (A33), we obtain

$$0 = \zeta_P(\mu_I - g)C_P x^{-\zeta_P - 1} + (\zeta_P - 1)\zeta_P \frac{1}{2}\nu^2\kappa^2 C_P x^{-\zeta_P - 1} - p C_P x^{-\zeta_P - 1}, \text{ for } x \geq 1.$$

This implies a quadratic equation for ζ_P given by

$$0 = \zeta_P(\mu_I - g) + (\zeta_P - 1)\zeta_P \frac{1}{2}\nu^2\kappa^2 - p.$$

Because the integral of $f_I(x)$ must converge on $(1, \infty)$, ζ_P must be equal to the unique positive root of the above quadratic equation, which is given by

$$\zeta_P = \frac{\left(\frac{\kappa^2\nu^2}{2} - \mu_I + g\right) + \sqrt{\left(\frac{\kappa^2\nu^2}{2} - \mu_I + g\right)^2 + 2\kappa^2\nu^2 p}}{\kappa^2\nu^2}.$$

Multiplying the numerator and denominator by

$$\left(\frac{\kappa^2\nu^2}{2} - \mu_I + g\right) - \sqrt{\left(\frac{\kappa^2\nu^2}{2} - \mu_I + g\right)^2 + 2\kappa^2\nu^2 p}$$

and rearranging, this formula yields

$$\frac{1}{\zeta_P} = \frac{\mu_I - g - \frac{\kappa^2\nu^2}{2} + \sqrt{\left(\mu_I - g - \frac{\kappa^2\nu^2}{2}\right)^2 + 2\kappa^2\nu^2 p}}{2p},$$

which is the same as the formula provided in the lemma.

Likewise, plugging our guess for $x \in (0,1)$, we obtain the same quadratic equation for ζ_N given by

$$0 = \zeta_N(\mu_I - g) + (\zeta_N - 1)\zeta_N \frac{1}{2}\nu^2\kappa^2 - p.$$

Because the integral of $f_I(x)$ must converge on $(0,1)$, ζ_N must be equal to the unique negative root of the above quadratic equation, which is given by

$$\frac{1}{\zeta_N} = \frac{\mu_I - g - \frac{\kappa^2\nu^2}{2} - \sqrt{\left(\mu_I - g - \frac{\kappa^2\nu^2}{2}\right)^2 + 2\kappa^2\nu^2 p}}{2p},$$

which is the same as the formula provided in the lemma.

Finally, we turn to the constants C_P and C_N. First, because $f_I(x)$ must be continuous at $x = 1$, we obtain $C_P = C_N$. Second, because the density $f_I(x)$ must integrate to 1, we obtain

$$1 = \int_0^1 C_N x^{-\zeta_N - 1} dx + \int_1^\infty C_P x^{-\zeta_P - 1} dx \Rightarrow C_P = C_N = \frac{1}{1/\zeta_P - 1/\zeta_N}.$$

Turning to the income distribution, Lemma S4 shows that, over a short period of time t, the income received by an investor with effective wealth x_z can be approximated as

$$y_{z,t} = x_{z,0}(r_I t + \kappa\nu\sqrt{t}u),$$

where $u \sim N(0,1)$. For large y, we have that

$$\Pr(y_{z,t} \geq y) \propto \int_{-\frac{r_I\sqrt{t}}{\kappa\nu}}^{\infty} (r_I t + \kappa\nu\sqrt{t}u)^{\zeta_P} y^{-\zeta_P} \phi(u) du \propto y^{-\zeta_P},$$

where $\phi(u)$ is the pdf of a standard normal, and the second equality follows from the fact that $\int_0^\infty (r_I t + \kappa\nu\sqrt{t}u)^{\zeta_P}\phi(u)du$ is a finite constant for any value of $\zeta_P \geq 0$ (which in turn follows from the fact that the normal distribution has finite moments). Thus, the income distribution also has a Pareto tail with tail index $1/\zeta_P$. ■

Proof of Proposition 5. The equilibrium equations for $r_K, r_B, \kappa, k_n, b_I$ and b_H are

$$\begin{aligned}
\left(\frac{r_B + \kappa\cdot(r_K - r_B) + \frac{1}{2}(\sigma-1)\gamma\nu^2\kappa^2 - \rho}{\sigma} - g\right)\cdot(k_n + b_I + \chi) =& p\cdot(k_n + b_I)\\
\left(\frac{r_B - \rho}{\sigma} - g\right)\cdot(b_H + 1 - \chi) =& p\cdot b_H\\
\kappa\cdot(k_n + b_I + \chi) =& k_n\\
\frac{1}{\gamma\nu^2}(r_K - r_B) =& \kappa\\
b_I + b_H =& 0\\
\frac{\alpha}{1-\alpha}\cdot\frac{r_B - g}{r_K + \delta} =& k_n.
\end{aligned}$$

For $\alpha = 0$, we get $r_K = r_B = \rho + \sigma g$, $\kappa = k_n = b_H = b_I = 0$.

Linearizing the system of equations around this equilibrium, we get

$$\begin{aligned}
\frac{dr_B}{\sigma}\cdot\chi =& p\cdot(dk_n + db_I) & \frac{dr_B}{\sigma}\cdot(1-\chi) =& p\cdot db_H\\
d\kappa\cdot\chi =& dk_n & dr_K - dr_B =& \gamma\nu^2 d\kappa\\
db_I + db_H =& 0 & dk_n =& \frac{\rho + (\sigma-1)g}{\rho + \sigma g + \delta}\alpha.
\end{aligned}$$

Which yields the solution

$$\begin{aligned}
dr_K =& \left[p\sigma + \frac{\gamma\nu^2}{\chi}\right]\frac{\rho + (\sigma-1)g}{\rho + \sigma g + \delta}\alpha\\
dr_B =& p\sigma\cdot\frac{\rho + (\sigma-1)g}{\rho + \sigma g + \delta}\alpha\\
d\kappa =& \frac{1}{\chi}\cdot\frac{\rho + (\sigma-1)g}{\rho + \sigma g + \delta}\alpha.
\end{aligned}$$

It follows that for small values of α we can approximate all equilibrium objects as

$$\begin{aligned}
r_K =& \rho + \sigma g + \left[p\sigma + \frac{\gamma\nu^2}{\chi}\right]\frac{\rho + (\sigma-1)g}{\rho + \sigma g + \delta}\alpha\\
r_B =& \rho + \sigma g + p\sigma\cdot\frac{\rho + (\sigma-1)g}{\rho + \sigma g + \delta}\alpha\\
\kappa =& \frac{1}{\chi}\cdot\frac{\rho + (\sigma-1)g}{\rho + \sigma g + \delta}\alpha\\
k_n =& \frac{\rho + (\sigma-1)g}{\rho + \sigma g + \delta}\alpha.
\end{aligned}$$

Let α^*_{net} denote the capital share net of growth and depreciation in equation (16). By definition, this is equal to $k_n/(k_n+1)$, and so for small values of α we get the approximation $\alpha^*_{net} = k_n$ which implies $\alpha^*_{net} = \frac{\rho+(\sigma-1)g}{\rho+\sigma g+\delta}\alpha$.

We conclude that

$$\begin{aligned} r_K =& \rho + \sigma g + \left[p\sigma + \frac{\gamma\nu^2}{\chi} \right] \alpha^*_{net} \\ r_B =& \rho + \sigma g + p\sigma \cdot \alpha^*_{net} \\ \kappa =& \frac{1}{\chi}\alpha^*_{net}, \end{aligned}$$

as claimed in the proposition.

Finally, the formula for the return gap in equation 13 implies

$$r_W - \rho - \sigma g = p\sigma\alpha^*_{net} + \frac{1}{\chi^2}{\alpha^*_{net}}^2\gamma\nu^2.$$

Using this expression for the return gap, we can compute the tail index for inequality as

$$\frac{1}{\zeta} = \frac{p\sigma\alpha^*_{net} + \frac{1}{\chi^2}{\alpha^*_{net}}^2\gamma\nu^2 - \frac{1}{\chi^2}{\alpha^*_{net}}^2\frac{\sigma}{2}\nu^2 + \sqrt{(p\sigma\alpha^*_{net} + \frac{1}{\chi^2}{\alpha^*_{net}}^2\gamma\nu^2 - \frac{1}{\chi^2}{\alpha^*_{net}}^2\frac{\sigma}{2}\nu^2)^2 + \frac{2}{\chi^2}{\alpha^*_{net}}^2\sigma^2\nu^2 p}}{2p\sigma}.$$

For small values of α this can be linearized as

$$\frac{1}{\zeta} = \alpha^*_{net} + \alpha^*_{net} \cdot \frac{\frac{\nu^2}{p\chi^2}}{1 + \sqrt{1 + 2\frac{\nu^2}{p\chi^2}}}$$

■

Proposition 5 provides explicit formulas that are valid for small values of $\tilde{\alpha}$. We now provide an additional proposition characterizing the comparative statics of our extended model away from $\tilde{\alpha} = 0$.

Proposition A2 *Suppose that investors are risk averse and/or $\theta < \bar{\theta}$. There exists a threshold $\bar{\alpha} \in (0, 1]$ such that, for $\tilde{\alpha} < \bar{\alpha}$, the balanced-growth equilibrium in the closed economy is unique, and following an increase in $\tilde{\alpha}$, we have that:*

- *The return to wealth r^*_W, the return gap $r^*_W - \rho - \sigma g$, and the return spread $r^*_K - r^*_B$ all strictly increase, and the portfolio share of capital κ^* weakly increases;*
- *Top tail inequality $1/\zeta^*$ in (15) strictly increases.*

Proof of Proposition A2. Let $\bar{r}_W$ denote the point at which $r_W - r_B$ is maximized along the locus $D_B(r_W, r_B) = 0$. This definition implies that $\bar{r}_W = \rho + \sigma(p + g)$ if $\gamma\nu^2 < \bar{\gamma}$, and $\bar{r}_W < \rho + \sigma(p + g)$ if $\gamma\nu^2 > \bar{\gamma}$.

We first show that there exists a threshold $\bar{\alpha}$ such that, for $\widetilde{\alpha} < \bar{\alpha}$, there is a unique equilibrium, and this equilibrium satisfies $r_W^* < \bar{r}_W$.

Suppose there are multiple equilibria, and let $r_W^M(\widetilde{\alpha})$ denote the value of r_W^* in the equilibrium with the highest return to wealth. In this equilibrium, the locus for $D_K(r_W, r_B)$ cuts the locus for $D_B(r_W, r_B)$ from below, and so an increase in $\widetilde{\alpha}$ shifting the locus for $D_K(r_W, r_B)$ outwards results in a higher $r_W^M(\widetilde{\alpha})$. Moreover, as $\widetilde{\alpha} \to 0$, $r_W^M(\widetilde{\alpha}) \to \rho + \sigma g$, which implies that for small values of all $\widetilde{\alpha}$, we have $r_W^* < \bar{r}_W$ in any equilibria. Finally, note that as $\widetilde{\alpha} \to 0$, the locus for $D_K(r_W, r_B)$ converges to the 45 degree line and so we will have a unique equilibrium. This follows from the fact that, as shown in Lemma A4, the locus for $D_B(r_W, r_B) = 0$ moves away from the 45 degree line for $r_W \in (\rho + \sigma g, \check{r}_W)$, and retains a gap of at least $\frac{1}{2}(\sigma + 1)\gamma\nu^2$ from there on. It follows that we can pick a $\bar{\alpha}$ such that, for $\widetilde{\alpha} < \bar{\alpha}$, the equilibrium is unique and satisfies $r_W^* < \bar{r}_W$.

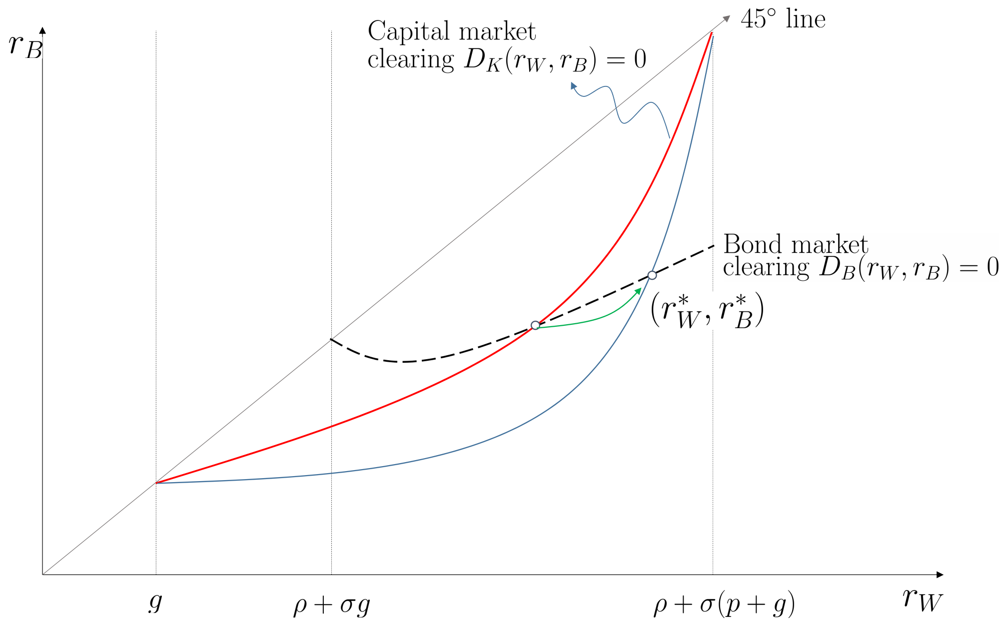

Figure A2: Effects of changes in the demand for capital on r_W^* and r_B^*.

For $\widetilde{\alpha} < \bar{\alpha}$, the equilibrium will look as in Figure A2, and increases in $\widetilde{\alpha}$ will result in a higher r_W^*. Moreover, because $r_W^* < \bar{r}_W$, we have that the gap $r_K^* - r_B^*$ rises following an increase in automation (so long as $\widetilde{\alpha}$ remains below $\bar{\alpha}$). ■