B Online Appendix

B.1 Schur-Convex Functions and Functionals

Consider $X_F$ and $X_G$ to be uniform, discrete random variables, each taking $n$ values $x_F = (x_{F1}, ..., x_{Fn})$ and $x_G = (x_{G1}, ..., x_{Gn})$, respectively. Then

$$x_F \prec_{dm} x_G \iff F^{-1} \prec G^{-1} \iff G \prec F$$

where $\prec_{dm}$ denotes the classical discrete majorization relation due to Hardy, Littlewood and Polya. Thus, discrete majorization is equivalent to the present majorization relation applied to quantile functions. A function $V : \mathbb{R}^n \to \mathbb{R}$ is Schur-convex (concave) if $V(x) \geq V(y)$ ($V(x) \leq V(y)$) whenever $x \succ_{dm} y$. If $V$ is a symmetric function, and if all its partial derivatives exist, then the Schur-Ostrovski criterion says that $V$ is Schur-convex (concave) if and only if

$$(x_i - x_j) \left( \frac{\partial V}{\partial x_i} - \frac{\partial V}{\partial x_j} \right) \geq (\leq) 0 \text{ for all } x.$$ 

It is useful to have a similar characterization for continuous majorization. Chan et al. (1987) showed that a law-invariant, Gâteaux-differentiable functional $V : L^1(0,1) \to \mathbb{R}$ respects the majorization relation on $L^1(0,1)$, if and only if its Gâteaux-derivatives in specially defined directions are non-positive. The considered directions are of the form

$$h = \lambda_1 \mathbf{1}_{(a,b)} + \lambda_2 \mathbf{1}_{(c,d)}$$

with $0 \leq a < b < c < d \leq 1$ and $\lambda_1 \geq 0 \geq \lambda_2$ such that $\lambda_1(b-a) + \lambda_2(d-c) = 0$. Note that the function $h$ takes at most two values that are different from zero, and is decreasing on $[a, b] \cup [c, d]$. Moreover, $\int_0^1 h(t) \, dt = 0$.

This result also yields a simple intuition for the Fan Lorentz Theorem in the case where $K$ is differentiable. Consider a monotonic $f$ and note that, for any direction $h$, the Gâteaux-derivative of the functional $V(f) = \int_0^1 K(f(t), t) \, dt$ is given by

$$\delta V(f, h) = \frac{d}{d\varepsilon} \int_0^1 K(f(t) + \varepsilon h(t), t) \, dt \big|_{\varepsilon=0} = \int_0^1 K_f(f(t), t) h(t) \, dt,$$

where the last equality follows by interchanging the order of differentiation and integration.\footnote{This means that the functional is constant over the equivalence class of functions with the same non-decreasing re-arrangement. This replaces the symmetry in the discrete formulation.}

\footnote{This is allowed since $K$ is convex in $f$.}
The Fan-Lorentz conditions imply together that

\[ \frac{dK_f}{dt} = f_t \cdot K_{ff} + K_{ft} \geq 0. \]

For a direction \( h \) such that \( \int_0^1 h(t) \, dt = 0 \), and such that \( h \) is a decreasing two-step function as defined above, we obtain that

\[ \delta V(f, h) = \int_0^1 K_f(f(t), t) h(t) \, dt \leq 0. \]

Hence the Fan-Lorentz functional \( V(f) = \int_0^1 K(f(t), t) \, dt \) is Schur-concave by the result of Chan et al. (1987)

### B.2 Decision-Making Under Uncertainty

We briefly illustrate here how our insights can be applied in order to understand how agents with non-expected utility preferences choose among risky prospects.

#### B.2.1 Rank-Dependent Utility and Choquet Capacities

Quiggin (1982) and Yaari (1987) axiomatically derived utility functionals with rank-dependent assessments of probabilities of the form\(^3\)

\[ U(F) = \int_0^1 v(t) \, d(g \circ F)(t) \]

where \( F \) is the distribution of a random variable on the interval \([0, 1]\), \( v : [0, 1] \to R \) is continuous, strictly increasing and bounded, and where \( g : [0, 1] \to [0, 1] \) is strictly increasing, continuous and onto. The function \( v \) represents a transformation of monetary payoffs, while the function \( g \) represents a transformation of probabilities\(^4\).

The case \( g(x) = x \) yields the classical von-Neumann and Morgenstern expected utility model where risk-aversion is equivalent to \( v \) being concave. The case \( v(x) = x \) yields Yaari’s (1987) dual utility theory, where risk aversion is equivalent to \( g \) being concave. Because of the possible interactions between \( v \) and \( g \), it is not clear what properties yield risk aversion

\(^3\)Their theory is a bit more general (for example it allows a more general domain for the functions \( v \) and \( F \)). We keep here a framework that is compatible with the rest of the paper.

\(^4\)For the sake of brevity we assume below that both \( g \) an \( v \) are twice differentiable. Since the Fan-Lorentz result does not require differentiability, the observations below generalize.
in the general rank-dependent model. Using integration by parts, we can also write:

\[ U(F) = \int_0^1 v(t) d(g \circ F)(t) = v(1) - \int_0^1 v'(t)(g \circ F)(t) \, dt \]

\[ = v(1) + \int_0^1 K(F(t), t) \, dt \]

where

\[ K(F, t) = -v'(t)(g \circ F) \]

and where we used \( g(0) = 0 \) and \( g(1) = 1 \). Then

\[ \frac{\partial^2 K(F, t)}{\partial F \partial t} = -g'(F(t))v''(t) \geq 0 \]

for all \( t \) if and only if \( v \) is concave. Similarly

\[ \frac{\partial^2 K(F, t)}{\partial^2 F} = -g''(F(t))v'(t) \geq 0 \]

for all \( t \) if and only if \( g \) is concave.

Hence, the Fan-Lorentz conditions are satisfied if and only if \( v'' \leq 0 \) and \( g'' \leq 0 \). As a consequence, the utility functional \( U = \int_0^1 v(t) d(g \circ F)(t) \) is Schur-concave, and the agent whose preferences are represented by \( U \) is risk averse, exactly as under standard expected utility\[^5\].

Another important strand of the literature on non-expected utility considers ambiguity aversion. The main tool is the Choquet integral with respect to a (convex) capacity (this is unrelated to the Choquet representation used above!) Analogously to the derivations above, it can be shown that the Choquet integral yields a Schur-concave functional if and only if it is computed with respect to a convex capacity.

### B.2.2 A Portfolio Choice Problem

Dybvig (1988) studies a simplified version of the following problem:

\[ \min_X \mathbb{E}[ XY ] \]

\[ \text{s.t. } X \geq_{cv} Z \]

\[^5\text{The equivalence between the concavity of the functions } v \text{ and } g, \text{ and risk-aversion has been pointed out by Hong et al (1987), who build on Machina (1982).} \]
where $Y$ and $Z$ are given random variables. $Y$ represents here the distribution of a pricing function over the states of the world, and the goal is to choose, given $Y$, the cheapest contingent claim $X$ that is less risky than a given claim $Z$. To make the problem well-defined, $Y$ needs to be essentially bounded and $X, Z$ must be integrable. Recalling that

$$ X \gtrless_{cv} Z \iff F_X > F_Z \iff F_X^{-1} < F_Z^{-1} $$

we obtain that:

$$ \mathbb{E}[XY] \geq \int_0^1 F_Y^{-1}(1-t)F_X^{-1}(t)\,dt \geq \int_0^1 F_Y^{-1}(1-t)F_Z^{-1}(t)\,dt $$

where the first inequality follows by the rearrangement inequality of Hardy, Littlewood and Polya (1929) (the anti-assortative part!), and where the second inequality follows by the Fan-Lorentz Theorem.

By choosing a random variable $X$ that has the same distribution as $Z$ and that is anti-comonotonic with $Y$,\(^6\) the lower bound $\int_0^1 F_Y^{-1}(1-t)F_Z^{-1}(t)\,dt$ is attained, and hence such a choice solves the portfolio choice problem.\(^7\)

If $Y' \leq_{cv} Y$, we obtain by the Fan-Lorentz inequality (now applied to the functional with argument $F_Y^{-1}$) that

$$ \sup_{X \gtrless_{cv} Z} \mathbb{E}[XY] = \int_0^1 F_Y^{-1}(1-t)F_Z^{-1}(t)\,dt \geq \int_0^1 F_{Y'}^{-1}(1-t)F_Z^{-1}(t)\,dt = \sup_{X \gtrless_{cv} Z} \mathbb{E}[XY'] $$

In other words, a decision maker that becomes more informed (in the Blackwell sense) will bear a lower cost.

**References**


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\(^6\)This can always be done if the underlying probability space is non-atomic. A random vector $(X, Y)$ is anti-comonotonic if there exists a random variable $W$ and non-decreasing functions $h_1, h_2$ such that $(X, Y) \overset{\text{dist}}{=} (h_1(W), -h_2(W))$.

\(^7\)For more details on this problem see Dana (2005) and the literature cited there. It does not use the Fan-Lorentz inequality.


