The Reputation Trap: Online Appendix[1]

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Online Appendix

Problem of the Long Run Player

We examine the problem of the normal type of long-run player. Recall the Bellman equation

$$V(\alpha_2) = \max_{a_1}(1-\delta)\left[\alpha_2 - ca_1\right] + \delta \sum_{z'} P(z'|z,a_1)V(\alpha_2(z')).$$

We may write this out as

$$V(\alpha_2) =$$

$$\max_{a_1}(1-\delta)\left[\alpha_2 - ca_1\right]+\delta\left[\left(\alpha_2+(1-\alpha_2)\pi\right)V(\alpha_2(a_1))+(1-\alpha_2)(1-\pi)V(\alpha_2(N))\right].$$

Lemma 1. *The optimum for the normal type of long run-player depends on the state only through α_2 and one of three cases applies:*

(i) $V(\alpha_2(1)) - V(\alpha_2(0)) < c(1-\delta)/\delta$: it is strictly optimal provide no effort in every state. In particular if $\alpha_2(1) = \alpha_2(0)$ this is the case.

(ii) $V(\alpha_2(1)) - V(\alpha_2(0)) > c(1-\delta)/(\delta\pi)$: it is strictly optimal to provide effort in every state

Defining

$$\tilde{\alpha}_2 = \frac{1-\delta}{\delta(1-\pi)\left(V(\alpha_2(1)) - V(\alpha_2(0))\right)}c - \frac{\pi}{1-\pi}$$

(iii) it is strictly optimal to provide effort if $\alpha_2(z) > \tilde{\alpha}_2$ and conversely. In particular the strategy $\alpha_1(0) > \alpha_1(1)$ is never optimal.

In addition

(iv) if $\alpha_2(0) = 1$ then it is strictly optimal provide no effort in every state.

Finally, if the short-run player uses a pure strategy then the optimum of the long-run player is strict and pure.

Proof. The argmax is derived from:

$$\max_{a_1} -(1-\delta)ca_1 + \delta\left(\alpha_2 + (1-\alpha_2)\pi\right)V(\alpha_2(a_1)).$$

The gain to no effort is

$$G(\alpha_2) = (1-\delta)c - \delta\left(\alpha_2 + (1-\alpha_2\pi)\left[V(\alpha_2(1)) - V(\alpha_2(0))\right]\right.$$

We then solve this equation form α_2 to see when effort is and is not optimal.

Turning to the details, it follows that no effort is strictly optimal if

$$V(\alpha_2(1)) - V(\alpha_2(0)) < \frac{1-\delta}{\delta\left(\alpha_2 + (1-\alpha_2)\pi\right)}c$$

and conversely.

The RHS is strictly decreasing function of α_2. The value $\tilde{a}_2$ is the unique value for which the two sides are equal, so the results (i) to (iii) follow directly.

For part (iv) suppose that $\alpha_2(0) = 1$. Then in state 0 choosing $\alpha_1 = 0$ gives 1 in very period so $V(0) = 1$. Since that is the greatest possible one period payoff $V(\alpha_2(1)) \leq 1 = V(\alpha_2(0))$ so the result follows from (i).

Finally, we analyze best response of the long-run player when the short-run player uses a pure strategy. From (i) and (iv) if $\alpha_2(0) \geq \alpha_2(1)$ it is strictly best not to provide effort. That leaves only the case $\alpha_2(a_1) = a_1$, or rather two cases, depending on $\alpha_2(N)$. This is a matter of solving the Bellman equations for each case to determine the value of c (if any) there can be a tie. This are the "non-generic" values listed in the text.

Turning to the details, If the response is not strict the condition for the gain to no effort must be zero

$$V(\alpha_2(1)) - V(\alpha_2(0)) = \frac{1-\delta}{\delta\left(a_2 + (1-a_2)\pi\right)}c.$$

Observe this cannot be the case at both states a_2.

(a) The tie is for $a_2 = 1$

In this case we have

$$V(\alpha_2(1)) = V(\alpha_2(0)) + \frac{1-\delta}{\delta}c$$

Moreover since $a_1 = 0$ must solve the Bellman equation for $a_2 = 1$ we have $V(\alpha_2(1)) = (1-\delta) + \delta V(\alpha_2(0))$. Solving we find $V(\alpha_2(0)) = 1 - c/\delta$.

Since $a_1 = 0$ is optimal at $a_2 = 1$ it must be that $a_1 = 0$ is strictly optimal at $a_1 = 0$. Hence

$$V(\alpha_2(0)) = \delta\left[\pi V(\alpha_2(0)) + (1-\pi)V(\alpha_2(N))\right].$$

There are two sub-cases depending on whether $\alpha_2(N) = 0, 1$.

If $\alpha_2(N) = 0$ then $V(\alpha_2(0)) = \delta V(\alpha_2(0))$ implies $V(\alpha_2(0)) = 0$. Since we previously found $V(\alpha_2(0)) = 1 - c/\delta$ this implies that $c = 1/\delta$ which is ruled out by generic cost.

If $\alpha_2(N) = 1$ then we have

$$V(\alpha_2(0)) = \delta\left[\pi V(\alpha_2(0)) + (1-\pi)V(\alpha_2(0)) + (1-\pi)\frac{1-\delta}{\delta}c\right]$$

which we solve to find

$$V(\alpha_2(0)) = (1-\pi)c/\delta.$$

Again this must also be equal to $1 - c/\delta$ so we have $(1-\pi)c/\delta = 1 - c/\delta$ or $c = \delta/(2-\pi)$ also ruled out by generic cost.

(b) The tie is for $a_2 = 0$

In this case we have

$$V(\alpha_2(1)) = V(\alpha_2(0)) + \frac{1-\delta}{\delta\pi}c.$$

Moreover since $a_1 = 1$ is optimal for $a_2 = 0$ it must also solve the Bellman equation for $a_2 = 1$, that is,

$$V(\alpha_2(1)) = (1-\delta)(1-c) + \delta V(\alpha_2(1))$$

so that $V(1) = 1 - c$. Hence

$$V(\alpha_2(0)) + \frac{1-\delta}{\delta\pi}c = 1 - c,$$

or

$$V(\alpha_2(0)) = 1 - c - \frac{1-\delta}{\delta\pi}c.$$

Again, there are two sub-cases depending on whether $\alpha_2(N) = 0, 1$.

If $\alpha_2(N) = 0$ then again $V(\alpha_2(0)) = \delta V(\alpha_2(0))$ implies $V(\alpha_2(0)) = 0$, giving

$$c\left[\frac{1-\delta+\delta\pi}{\delta\pi}\right] = 1$$

which is ruled out by generic cost.

If $\alpha_2(N) = 1$ since $a_1 = 0$ is optimal at $a_2 = 0$ wand $V(\alpha_2(1)) = 1 - c$

$$V(\alpha_2(0)) = \delta\left[\pi V(0) + (1-\pi)(1-c)\right]$$

or

$$V(\alpha_2(0)) = \frac{1-\pi}{1-\delta\pi}(1-c).$$

This must be equal to

$$1 - c - \frac{1-\delta}{\delta\pi}c$$

and equating the two we find

$$1 - c - \frac{1-\delta}{\delta\pi}c = \frac{1-\pi}{1-\delta\pi}(1-c)$$

$$c + \frac{1-\delta}{\delta\pi}c - \frac{1-\pi}{1-\delta\pi}c = 1 - \frac{1-\pi}{1-\delta\pi}$$

$$\frac{1-\delta}{\delta\pi}c + \frac{\pi-\delta\pi}{1-\delta\pi}c = \frac{\pi-\delta\pi}{1-\delta\pi}$$

$$(1-\delta\pi)(1-\delta)c + \delta\pi(\pi-\delta\pi)c = \delta\pi(\pi-\delta\pi)$$

$$c = \frac{\delta\pi(\pi-\delta\pi)}{(1-\delta\pi)(1-\delta)+\delta\pi(\pi-\delta\pi)}$$

ruled out by the generic cost assumption. □

Ergodic Beliefs of the Short-Run Player

Next we examine the beliefs of the short-run player. For given pure strategies of both players the signal type pairs (z, τ) are a Markov chain with transition probabilities independent of δ and depending only on ϵ, π and the strategies of the two players. Excluding the state N in case the short-run player always enters the chain is irreducible and aperiodic so it has a unique ergodic distribution $\mu_{z\tau}$. We first analyze the marginals μ_τ and μ_z.

Lemma 2. *The marginals μ_τ are independent of ϵ. Let $\underline{\mu} = \min_{\tau \neq n} \mu_\tau$. Then $\underline{\mu} > 0$, $\mu_0, \mu_1 \geq \pi\underline{\mu}$, if $\alpha_2(0) = \alpha_2(1) = 1$ then $\mu_N = 0$, otherwise if the short-run player plays a pure strategy then $\mu_N \geq (1-\pi)\underline{\mu}$.*

Proof. The type transitions are independent of the signals, so we analyze those first. For $\epsilon > 0$ we have $\mu_\tau > 0$ since every type transition has positive probability. This ergodic distribution is the unique fixed point of the 3×3 transition matrix A, which is to say given by the intersection of the null space of $I - A$ with the unit simplex. Since $A = I + Q\epsilon$ it follows that it is given by the intersection of the null space of $Q\epsilon$ with the unit simplex. As the null space of $Q\epsilon$ is independent of ϵ the marginals μ_τ are independent of ϵ as well.

For the signals we have $\mu_1 \geq \pi\mu_g$ and $\mu_0 \geq \pi\mu_b$. If if $a_2(0) = a_2(1) = 1$ then the state N is transient. If $\alpha_2(1) = 0$ then $\mu_N \geq (1-\pi)\mu_g$ while if $\alpha_2(0) = 0$ then $\mu_N \geq (1-\pi)\mu_b$. □

It will be convenient to normalize so that $\max(\mu_\sigma/\mu_\tau)Q_{\tau\sigma} = 1$. Next we show how the conditional probabilities $\mu_{z|\tau}$ can be computed approximately by using the ergodic conditions for $\epsilon = 0$.

Lemma 3. *When $z = N$*

$$\mu_{N|\tau} = (1-\pi)\left(\sum_y (1-\alpha_2(y))\mu_{y|\tau} + \epsilon H_{N\tau}\right)$$

when $z \neq N$

$$\mu_{z|\tau} = \sum_y 1\left((z=1)\alpha_1(\tau,y) + 1(z=0)(1-\alpha_1(\tau,y))\right)\left[\alpha_2(y) + \pi(1-\alpha_2(y))\right]\mu_{y|\tau} + \epsilon H_{z\tau}.$$

where $|H_{z\tau}| \leq 2$ for all z.

Proof. The idea is that the process for types is exogenous, so the stationary probabilities can be computed directly. This enables us to find a linear recursive relationship for the conditionals where the coefficients depend upon the strategies and the (already known) marginals over types. We then show that when ϵ is small to a good approximation we can do the computation for $\epsilon = 0$, that is, ignoring the type transitions, with the result above showing how good the approximation is for given ϵ.

For given strategies of the players define $P(z, \sigma|y, \tau)$ to be the conditional probability that $z_{t+1} = z, \sigma_{t+1} = \sigma$ conditional on $z_t = y, \tau_t = \tau$. We have

$$\mu_{z\tau} = \mu_{z|\tau}\mu_\tau = \sum_\sigma \sum_y P(z|y,\sigma)P(\tau|\sigma)\mu_{y\sigma}$$

$$= \sum_{\sigma}\sum_{y} P(z|y,\sigma)\Pr(\tau|\sigma)\mu_{y|\sigma}\mu_{\sigma} = \sum_{\sigma} P(\tau|\sigma)\mu_{\sigma}\sum_{y} P(z|y,\sigma)\mu_{y|\sigma}.$$

Since we know that $\mu_\tau > 0$ we may divide to find

$$\mu_{z|\tau} = \sum_{\sigma} P(\tau|\sigma)\frac{\mu_\sigma}{\mu_\tau}\sum_{y} P(z|y,\sigma)\mu_{y|\sigma}$$

$$= \sum_{\sigma} P(\tau|\sigma)\frac{\mu_\sigma}{\mu_\tau}\sum_{y} P(z|\alpha_2(y),\alpha_1(\sigma,y))\mu_{y|\sigma}.$$

Define $\overline{h}(\tau|\tau) = -\sum_{\sigma\neq\tau} Q_{\tau\sigma} = (P(\tau|\tau)-1)/\epsilon$ and for $\tau \neq \sigma$ define $\overline{h}(\tau|\sigma)\epsilon = (\mu_\sigma/\mu_\tau)Q_{\sigma\tau} = P(\tau|\sigma)/\epsilon$. Observe that $\overline{h}$ depends only on Q and that

$$|\overline{h}(\tau|\sigma)| \leq \max\{2(\mu_\sigma/\mu_\tau)Q_{\tau\sigma}|\tau\neq\sigma\} = 2.$$

Then

$$\mu_{z|\tau} = \sum_{y} P(z|\alpha_2(y),\alpha_1(\sigma,\tau))\mu_{y|\tau} + \epsilon\sum_{\sigma}\overline{h}(\tau|\sigma)\sum_{y} P(z|\alpha_2(y),\alpha_1(\sigma,y))\mu_{y|\sigma}.$$

For $z = N$ this is

$$\mu_{N|\tau} = \sum_{y}(1-\pi)(1-\alpha_2(y))\mu_{y|\tau} + \epsilon\sum_{\sigma}\overline{h}(\tau|\sigma)\sum_{y}(1-\pi)(1-\alpha_2(y))\mu_{y|\sigma}$$

$$= (1-\pi)\left(\sum_{y}(1-\alpha_2(y))\mu_{y|\tau} + \epsilon H_{N\tau}\right).$$

For $z \neq N$ this is

$$\mu_{z|\tau} = \sum_{y} P(z|\alpha_2(y),\alpha_1(\sigma,\tau))\mu_{y|\tau} + \epsilon\sum_{\sigma}\overline{h}(\tau|\sigma)\sum_{y} P(z|\alpha_2(y),\alpha_1(\sigma,y))\mu_{y|\sigma}$$

$$\sum_{y}\left(1(z=1)\alpha_1(\tau,y) + 1(z=0)(1-\alpha_1(\tau,y))\right)\left[\alpha_2(y)+\pi(1-\alpha_2(y))\right]\mu_{y|\tau} + \epsilon H_{z\tau}.$$

In both cases $|H_{z\tau}| \leq 2$. □

To apply Bayes Law we will need to bound marginal probabilities of signals from below. The hard case is that of no signal where we must solve the equations for the conditionals simultaneously. Here we analyze the short-run pure strategy case. If the short-run player enters for both $z = 0, 1$ then no signals are unlikely as they are generated only from type transitions, so we rule that out.

Lemma 4. *Suppose $\alpha_2(a_1) = 0$ for some $a_1 \in \{0,1\}$. Then*

$$\mu_N \geq \frac{1-\pi}{2}\left(1-\frac{4\epsilon}{\pi}\right)\underline{\mu}.$$

Proof. Let τ be the type that plays a_1. We have

$$\mu_{a_1|\tau} = \sum_y \left[\alpha_2(y) + \pi(1-\alpha_2(y))\right] \mu_{y|\tau} + \epsilon H_{a_1\tau}$$

$$\mu_{N|\tau} = (1-\pi)\left(\sum_y (1-\alpha_2(y))\mu_{y|\tau} + \epsilon H_{N\tau}\right)$$

These imply the inequalities

$$\mu_{a_1|\tau} \geq \pi(1-\mu_{N|\tau}) + \left[\alpha_2(N) + \pi(1-\alpha_2(N))\right] \mu_{N|\tau} + \epsilon H_{a_1\tau}$$

$$\mu_{N|\tau} \geq (1-\pi)\left((1-\alpha_2(N))\mu_{N|\tau} + \mu_{a_1|\tau} + \epsilon H_{N\tau}\right).$$

Hence

$$\begin{aligned}
\mu_{N|\tau} &\geq (1-\pi)\left((1-\alpha_2(N))\mu_{N|\tau} + \pi(1-\mu_{N|\tau}) + \left[\alpha_2(N) + \pi(1-\alpha_2(N))\right] \mu_{N|\tau} + \epsilon H_{N\tau} + \epsilon H_{a_1\tau}\right) \\
&= (1-\pi)\left(\pi + \left[\alpha_2(N) + (1+\pi)(1-\alpha_2(N)) - \pi\right] \mu_{N|\tau} + \epsilon H_{N\tau} + \epsilon H_{a_1\tau}\right) \\
&\geq (1-\pi)\left(\pi + (1-\pi)\mu_{N|\tau} + \epsilon H_{N\tau} + \epsilon H_{a_1\tau}\right).
\end{aligned}$$

It follows that

$$\mu_{N|\tau} \geq \frac{1-\pi}{1-(1-\pi)^2}(\pi + 4\epsilon) = \frac{1-\pi}{(2-\pi)\pi}(\pi+4\epsilon)$$

$$\geq \frac{1-\pi}{2-\pi}\left(1 - \frac{4\epsilon}{\pi}\right) \geq \frac{1-\pi}{2}\left(1-\frac{4\epsilon}{\pi}\right).$$

The result now follows from $\mu_N \geq \mu_{N|\tau}\mu_\tau \geq \mu_{N|\tau}\underline{\mu}$. □

Finally we compute bounds on beliefs about types that play the same action independent of the signal. Here we combine bounds from the equations for the conditionals with Bayes Law.

Lemma 5. *A long-run type τ that plays the pure action a_1 regardless of the signal has*

$$\mu_{\tau|-a_1} \leq \frac{2}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right)$$

and if $\alpha_2(1) = 1$ and $\alpha_2(0) = 0$ then a type τ that plays the action 1 regardless of signal has

$$\mu_{\tau|N} \leq \frac{8}{\left(1 - 4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}}\left(\frac{\epsilon}{\pi}\right).$$

Proof. If long-run type τ plays the pure action a_1 from Lemma 3

$$\mu_{-a_1|\tau} = \left(1(a_1=0)1(a_1=1) + 1(a_1=1)1(a_1=0)\right)\sum_y\right) \left[\alpha_2(y) + \pi(1-\alpha_2(y))\right]\mu_{y|\tau} + \epsilon H_{z\tau}.$$

$$= \epsilon H_{-a_1 \tau} \leq 2\epsilon.$$

From Lemma 2 $\mu_{-a_1} \geq \pi \underline{\mu}$ and Bayes law then implies

$$\mu_{\tau|-a_1} \leq \frac{\epsilon 2}{\pi \underline{\mu}}.$$

For the second part we have from Lemma 3

$$\mu_{N|\tau} = (1-\pi)\left(\sum_y (1-\alpha_2(y))\mu_{y|\tau} + \epsilon H_{N\tau}\right)$$

$$\mu_{0|\tau} = \epsilon H_{0\tau}.$$

Hence

$$\mu_{N|\tau} = (1-\pi)\left(\mu_{0|\tau} + [1-\alpha_2(N)]\,\mu_{N|\tau}\right) + (1-\pi)\epsilon H_{N\tau}.$$

Plugging in

$$\mu_{N|\tau} = (1-\pi)\,[1-\alpha_2(N)]\,\mu_{N|\tau} + (1-\pi)\epsilon H_{0\tau} + (1-\pi)\epsilon H_{N\tau}$$

$$\mu_{N|\tau} \leq (1-\pi)\mu_{N|\tau} + (1-\pi)\epsilon H_{0\tau} + (1-\pi)\epsilon H_{N\tau}$$

so

$$\mu_{N|\tau} \leq \frac{(1-\pi)4\epsilon}{\pi}.$$

From Lemma 4

$$\mu_N \geq \frac{1-\pi}{2}\left(1-\frac{4\epsilon}{\pi}\right)\underline{\mu}.$$

Hence Bayes law implies

$$\mu_{\tau|N} \leq \frac{8\epsilon}{\pi\left(1-\frac{4\epsilon}{\pi}\right)\underline{\mu}}.$$

□

Short-Run Player Optimality

Recall that $\mu^1(z)$ is the probability of $a_1 = 1$ in state z and that $\overline{B} = 1/(V+1)$ is the critical value of $\mu^1(z)$ such that

Lemma 6. *If $\mu^1(z) > \overline{B}$ the short-run player strictly prefers to enter; if $\mu^1(z) < \overline{B}$ the short-run player strictly prefers to stay out, and if $\mu^1(z) = \overline{B}$ the short-run player is indifferent.*

We next show that it cannot be optimal for the short-run player always to enter. Set $B \equiv \underline{\mu} \min\{\pi, 1-\pi\} \min\{\overline{B}, 1-\overline{B}\}$.

Lemma 7. *For $\epsilon < (1/2)B$ always enter $a_2(z) = 1$ for all z is not an equilibrium.*

Proof. By Lemma 1 always enter implies no effort by the normal long-run player. As there are few good types at $z = 0$ we show that this forces the short-run player to stay out there so the short-run player should not in fact enter.

Turning to the details, Lemma 5 gives

$$\mu_{g|0} \leq \frac{2}{\underline{\mu}} \left(\frac{\epsilon}{\pi}\right).$$

Hence

$$\mu^1(0) \leq \frac{2}{\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

also. From Lemma 6 it follows that $\epsilon/\pi < \underline{\mu}\overline{B}/2$ implies $a_2(0) = 0$ a contradiction. □

Lemma 8. *For $\epsilon < (1/16)B$ the strict equilibrium response to never provide effort is to enter only on $z = 1$ and do so with probability 1.*

Proof. As the normal and bad types never provide effort the signal $z = 1$ implies a good type with high probability so the short-run player should enter there. This means that the long-run player can have the signal $z = 1, N$ only through a type transition. In particular the bad signal is dominated by normal and bad types so the short run player should stay out. This in turn means that most of the N signals are generated by normal and bad types, so the short-run player should stay out there too.

Turning to the details, from Lemma 5 no effort implies

$$\mu_{n|1} \leq \frac{2}{\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

and the same inequality holds for $\mu_{b|1}$. Hence

$$\mu_{g|1} \geq 1 - \frac{4}{\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

and by Lemma 6 $\epsilon/\pi < \underline{\mu}(1 - \overline{B})/4$ this forces $\alpha_2(1) = 1$. By the Lemma 5

$$\mu_{g|0} \leq \frac{2}{\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

so by Lemma 6 $\epsilon/\pi < \underline{\mu}\overline{B}/2$ we must have $\alpha_2(0) = 0$.

We may again apply the two Lemmas to conclude that

$$\mu_{g|N} \leq \frac{8}{\left(1 - 2\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

so that for

$$\epsilon/\pi < \max\left\{\underline{\mu}\overline{B}/16, 1/4\right\}$$

the short-run player must stay out on N as well.

All these responses are strict. □

Lemma 9. *For $\epsilon < (1/16)B$ there is no equilibrium in which $\alpha_2(0) = 1$.*

Proof. By Lemma 1 $\alpha_2(0) = 1$ implies never provide effort so by Lemma 8 $\alpha_2(0) = 0$ a contradiction. □

Lemma 10. *For $\epsilon < (1/32)B$ the unique equilibrium response to always provide effort is to enter only on $z = 1$ and do so with probability 1.*

Proof. This is basically the opposite of Lemma 8. Now at $z = 1$ there are mainly good and normal types so it is optimal for the short-run player to enter. While at $z = 0$ there are mainly bad types so it is optimal for the short-run player to stay out. Hence no-signal is generated by bad types from $z = 0$ so it is optimal for the short-run player to stay out there too.

Turning to the details, from Lemma 5

$$\mu_{g|0}, \mu_{n|0} \leq \frac{2}{\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

so

$$\mu_{b|0} \geq 1 - \frac{4}{\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

so by Lemma 6 $\epsilon/\pi < \underline{\mu}(1 - \overline{B})/4$ implies $a_2(0) = 0$.

Apply the two Lemmas again to see that

$$\mu_{b|1} \leq \frac{2}{\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

so for $\epsilon/\pi < \underline{\mu}\overline{B}/2$ we have $a_2(1) = 1$.

Apply the two Lemmas a third time to see that

$$\mu_{g|N}, \mu_{n|N} \leq \frac{8}{\left(1 - 4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

so that $\epsilon/\pi < \max\left\{\underline{\mu}\overline{B}/32, 1/8\right\}$ implying $a_2(N) = 0$.

All these responses are strict. □

Lemma 11. *If $\epsilon < (1/2)B$ and for some a_1 we have $\alpha_1(a_1) = a_1$ then $\alpha_2(a_1) = a_1$.*

Proof. If $\alpha_1(0) = 0$ then from Lemmas 3 and 2 $\mu^1(0) = \mu_{0|g}\mu_g/\mu_0 = \epsilon H_{0g}\mu_g/\mu_0 \leq 2\epsilon/(\pi\underline{\mu})$. If $\alpha_1(1) = 1$ then $1 - \mu^1(1) = \mu_{1|b}\mu_b/\mu_1 = \epsilon H_{1b}\mu_b/\mu_1 \leq 2\epsilon/(\pi\underline{\mu})$. Hence for $\epsilon/\pi < \overline{B}\underline{\mu}/2$ it follows that $\alpha_2(a_1) = a_1$. □

Uniqueness of Short-Run Pure Equilibria

We define an *equilibrium response* of the short-run player to a strategy of the long-run player to be a best response to $\mu_{z\tau}$ induced by the long-run player strategy and itself.

Proposition 1. *There exists an $\underline{\epsilon} > 0$ depending only on V such that for any ϵ satisfying*

$$\underline{\epsilon} > \frac{\epsilon}{\underline{\mu} \min\{\pi, 1-\pi\}} > 0$$

in any short-run pure equilibrium the short-run player must enter on the good signal and only on the good signal. Moreover this is a strict equilibrium response.

Proof. We rule out all other possibilities:

(a) A*lways enter $a_2(z) = 1$ for all z is not an equilibrium.* By Lemma 7

(b) *The unique equilibrium response to never provide effort is to enter only on $z = 1$.* From Lemma 7.

(c) *A equilibrium response requires $a_2(1) = 1, a_2(0) = 0$.* Any other strategy satisfies $a_2(0) \geq a_2(1)$. From Lemma 1 this implies no effort by the long-run player. Part (b) then forces $0 = a_2(0) < a_2(1) = 1$ a contradiction.

(d) *The unique equilibrium response to always provide effort is to enter only on $z = 1$.* From Lemma 10.

This leaves only the strategy $\tilde{a}$ in which the long-run player plays $a_1 = 1$ on entry and $a_1 = 0$ if the short-run player stays out. As we know that $\alpha_2(1) = 1, \alpha_2(0) = 0$ there are two possibilities $\alpha_2(N) = 1$ and $\alpha_2(N) = 0$. The former is ruled out because it leads to primarily bad types at $z = N$, and the latter is a strict best response by the short-run player because there are few good types at $z = N$.

Turning to the details, there is entry at $N, 1$ and not on 0 consequently there is effort on $N, 1$ and not on 0. From Lemma 3 we find:

$$\mu_{0|n} = \sum_y (1 - \alpha_1(n, y)) \left[\alpha_2(y) + \pi(1 - \alpha_2(y))\right] \mu_{y|n} + \epsilon H_{0n} = \pi \mu_{0|n} + \epsilon H_{0n},$$

$$\mu_{N|n} = (1-\pi) \left(\sum_y (1 - \alpha_2(y)) \mu_{y|n} + \epsilon H_{Nn} \right) = (1-\pi)\mu_{0|n} + (1-\pi)\epsilon H_{Nn}.$$

The former implies

$$\mu_{0|n} \leq \frac{2\epsilon}{1-\pi}$$

so that the second implies

$$\mu_{N|n} \leq 2\epsilon + (1-\pi)\epsilon H_{Nn} \leq 4\epsilon.$$

From Lemma 4

$$\mu_N \geq \frac{1-\pi}{2} \left(1 - \frac{4\epsilon}{\pi}\right) \underline{\mu}$$

so Bayes law gives

$$\mu_{n|N} \leq \frac{\pi}{1-\pi} \frac{8}{\left(1-4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}} \left(\frac{\epsilon}{\pi}\right).$$

Also by Lemma 5

$$\mu_{g|N} \leq \frac{8}{\left(1-4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}} \left(\frac{\epsilon}{\pi}\right).$$

Hence

$$\mu_{b|N} \geq 1 - \frac{16}{\left(1-4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}} \left(\frac{\epsilon}{1-\pi}\right)$$

from which the result follows. Note that it is only for this result that we require $\epsilon/(1-\pi)$ to be small as well as ϵ/π.

Finally we must show that $\alpha_2(N) = 0$ is in fact a strict equilibrium response for the short-run player. We have

$$\mu_{b|1}, \mu_{g|0} \leq \frac{2}{\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

$\mu_{b|1} = 0$ and $\mu_{g|0} = 0$ so is a strict best response to stay out in the former and enter in the latter. Finally Lemma 3 gives

$$\mu_{g|N} \leq \frac{8}{\left(1-4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}} \left(\frac{\epsilon}{\pi}\right)$$

implying for small ϵ/π it is strictly optimal for the short-run player to stay out on N. □

Mixing

Recall that all of the Lemmas concerning short-run optimality hold for $\epsilon \leq B/32$ (and the remaining Lemmas do not place restrictions on ϵ) where $B = \underline{\mu}\min\{\pi, 1-\pi\}\min\{\overline{B}, 1-\overline{B}\}$. Recall also the notion of a fundamental bound: it may depend on the fundamentals of the game π, V, δ, c but not on the type dynamics Q, ϵ. Define the fundamental bound $\overline{A} \equiv \pi^2(1-\pi)\min\{\overline{B}, 1-\overline{B}\}$ and observe that if $\epsilon \leq \underline{\mu}\overline{A}/32$ then also $\epsilon \leq B/32$. We shall assume $\epsilon \leq \underline{\mu}\overline{A}/32$ hereafter.

Lemma 12. *There is no non-pure equilibrium with $\alpha_1(1) = 1$.*

Proof. By Lemma 2 $\mu_{1|b} = \epsilon H_{1b} \leq 2\epsilon$. Hence for $\epsilon < \overline{B}/2$ by Lemma 6 $\alpha_2(1) = 1$. Then by Lemma 2 $\mu_{1|n} = \mu_{1|n} + \sum_{y\in\{0,N\}} \alpha_1(y)\left[\alpha_2(y) + \pi(1-\alpha_2(y))\right]\mu_{y|n} + \epsilon H_{z\tau}$. It follows that

$$\sum_{y\in\{0,N\}} \alpha_1(y)\mu_{y|n} \leq 2(\epsilon/\pi) \text{ so } \max_{y\in\{0,N\}} \alpha_1(y)\mu_{y|n} \leq 2(\epsilon/\pi).$$

Moreover for $z \in \{0, N\}$ we have $\mu_{z|g} = \epsilon H_{zg} \leq 2\epsilon$. Hence

$$\mu^1(0) = \frac{\mu_{0|g}\mu_g + \alpha_1(0)\mu_{0|n}\mu_n}{\mu_0} \leq 2(\epsilon/\pi)(\mu_g + \mu_n)/(\pi\underline{\mu}) \leq 2(\epsilon/\pi)/(\pi\underline{\mu}).$$

So for $\epsilon/\pi^2 < \overline{B}\underline{\mu}/2$ (this is why π^2 appears in $\overline{A}$) by Lemma 6 we have $\alpha_2(0) = 0$. This implies by Lemma 4 that

$$\mu^1(N) = \frac{\mu_{N|g}\mu_g + \alpha_1(N)\mu_{N|n}\mu_n}{\mu_N} \leq 2(\epsilon/\pi)(\mu_g + \mu_n)/\mu_N$$
$$\leq \frac{8(\epsilon/\pi)}{(1-\pi)\left(1 - \frac{4\epsilon}{\pi}\right)\underline{\mu}}.$$

So when this is less than or equal $\overline{B}$ by Lemma 6 we have $\alpha_2(N) = 0$. For $\epsilon \leq \overline{A}/8$ this is

$$\frac{16\epsilon}{\pi(1-\pi)\underline{\mu}} \leq \overline{B}$$

so holds for $\epsilon < \underline{\mu}\overline{A}/16$ which was assumed. □

Lemma 13. *In any equilibrium* $\alpha_1(0) = \alpha_2(0) = 0$.

Proof. We already know this to be true in any pure equilibrium, so we may assume the equilibrium is not pure. From Lemma 11 if $\alpha_1(0) = 0$ then $\alpha_2(0) = 0$ so we may assume this is not the case, that is $\alpha_1(0) > 0$. From Lemma 12 we know that $\alpha_1(1) < 1$. It cannot be that the normal type is indifferent at both $z = 0, 1$ for then by Lemma 1 it must be that $\alpha_2(1) = \alpha_2(0) = \tilde{\alpha}_2$ so that $V_1 = V(\tilde{\alpha}) = V_0$ and that the normal type never provides effort in which case by Lemma 8 we would have a pure strategy equilibrium. Hence either the normal type strictly prefers to provide no effort at $z = 1$ and is willing to provide effort at $z = 0$ or the normal type is indifferent at $z = 1$ and strictly prefers to provide effort at $z = 0$. In either case from Lemma 1 we must have $\alpha_2(1) < \alpha_2(0)$.

The key point is that having the short-run player enter when there is no effort is kind of like winning the lottery - you get something for nothing. If that happens in the state 0 it is particularly good because you are guaranteed that you get to play again. Since $\alpha_2(1) < \alpha_2(0)$ we can write $\alpha_2(0) = \beta + (1-\beta)\alpha_2(1)$ where $\beta > 0$ meaning that in the state $z = 0$ there is a better chance of winning the lottery. We will use this to show that $V(\alpha_2(0)) \geq V(\alpha_2(1))$ so that never provide effort is optimal and the equilibrium must be pure by Lemma 8.

Specifically we compare $V(\alpha_2(0))$ to $V(\alpha_2(1))$. We may compute $V(\alpha_2(1))$ under the assumption that the normal type does not provide effort since this is optimal at $z = 1$. This gives

$$V(\alpha_1(1)) = (1-\delta)\alpha_2(1) + \delta\left[\pi V(\alpha_2(0)) + (1-\pi)\left(\alpha_2(1)V(\alpha_2(0)) + (1-\alpha_2(1))V(\alpha_2(N))\right)\right]$$

We may compute a lower bound $V(\alpha_2(0))$ under the assumption that the normal type does not provide effort in the first period and optimizes afterwards. In this case

$$V(\alpha_2(0)) \geq (1-\delta)\alpha_2(0)+\delta\left[\pi V(\alpha_2(0)) + (1-\pi)\left(\alpha_2(0)V(\alpha_2(0)) + (1-\alpha_2(0))V(\alpha_2(N))\right)\right]$$

$$= (1-\delta)\left(\beta + (1-\beta)\alpha_2(1)\right)$$

$$+\delta\left[\pi V(\alpha_2(0)) + (1-\pi)\left(\left(\beta + (1-\beta)\alpha_2(1)\right)V(\alpha_2(0)) + (1-(\beta+(1-\beta)\alpha_2(1)))V(\alpha_2(N))\right)\right]$$

$$= (1-\delta)\left(\beta + (1-\beta)\alpha_2(1)\right)$$

$$+\delta\left[\pi V(\alpha_2(0)) + (1-\pi)\left(\left(\beta + (1-\beta)\alpha_2(1)\right)V(\alpha_2(0)) + (1-\beta)\left(1-\alpha_2(1)\right)V(\alpha_2(N))\right)\right]$$

$$= (1-\delta)\beta + \delta\beta V(\alpha_2(0)) + (1-\beta)\alpha_2(1)$$

$$+\delta\left[(1-\beta)\pi V(\alpha_2(0)) + (1-\pi)\left(\left((1-\beta)\alpha_2(1)\right)V(\alpha_2(0)) + (1-\beta)\left(1-\alpha_2(1)\right)V(\alpha_2(N)\right)\right]$$

Using the expression for $V(\alpha_1(1))$ from above this gives

$$V(\alpha_2(0)) \geq (1-\delta)\beta + \delta\beta V(\alpha_2(0)) + (1-\beta)V(\alpha_1(1)).$$

Hence

$$V(\alpha_2(0)) \geq \frac{(1-\delta)\beta}{1-\delta\beta} + \frac{1-\beta}{1-\delta\beta}V(\alpha_1(1)),$$

Since $V(\alpha_1(1)) \leq 1$ this then implies $V(\alpha_1(0)) \geq V(\alpha_1(1))$ as advertised, $\square$

Lemma 14. *In any non-pure equilibrium* $0 < \alpha_2(1) < 1$, $\alpha_1(N) > 0$, *and* $\alpha_2(N) \geq \alpha_2(1)$.

Proof. First suppose that $\alpha_2(1) = 1$. Since the short-run player must be mixing and by Lemma 13 is not doing so at $z = 0$ the short-run player must be mixing at $z = N$, that is, that $0 < \alpha_2(N) < 1$. Lemma 12 implies that at $z = 1$ the normal type does not strictly prefer to provide effort. Since $\alpha_2(N) < \alpha_2(1)$ Lemma 1 implies that at $z = N$ normal type strictly prefers to provide no effort, so $\alpha_1(N) = 0$. Hence $\mu^1(N) = \mu_{N|g}\mu_g/\mu_N = \epsilon H_{0g}\mu_g/\mu_N$. As $\alpha_2(0) = 0$ by Lemma 13 it follows from Lemma 4 that

$$\mu^1(N) \leq \frac{4\epsilon}{(1-\pi)\left(1-\frac{4\epsilon}{\pi}\right)\underline{\mu}}$$

as the RHS this is less than $\overline{B}$ by assumption we have $\alpha_2(N) = 0$ a contradiction.

Next suppose that $\alpha_2(1) = 0$. By Lemma 13 we also have $\alpha_2(0) = 0$ so by Lemma 1 the long run player never provides effort. Hence $\alpha_2(1) > 0$ follows from Lemma 8, a contradiction. We have now shown strict mixing the the short-run player at $z = 1$.

Now we show that since the short-run player is strictly mixing at $z = 1$ then $\alpha_1(N) > 0$. Strict mixing by the short-run player at $z = 1$ implies from Lemma 6 $1 - \overline{B} = 1 - \mu^1(1) = \left([1-\alpha_1(1)]\mu_{1|n}\mu_n + \mu_{1|b}\mu_b\right)/\mu_1$. From Lemma 3 and Lemma 13 if $\alpha_1(N) = 0$ we have $\mu_{1|n} \leq \alpha_1(1)\mu_{1|n} + 2\epsilon$ and $\mu_{1|b} \leq 2\epsilon$. Hence by Lemma 2 $1 - \mu^1(1) \leq 2\epsilon/(\pi\underline{\mu})$, so for $2\epsilon/(\pi\underline{\mu}) < 1 - \overline{B}$ this is a contradiction.

Since $\alpha_2(N) > 0$ the normal type weakly prefers to provide effort at $z = N$. If $\alpha_2(1) > \alpha_2(N)$ by Lemma 1 this implies the normal type would strictly prefer to provide effort at $z = 1$ contradicting Lemma 12. □

Signal Jamming

Define the *auxiliary system* with respect to $0 \le \lambda, \gamma \le 1$ as

$$V_1 = (1-\delta)\tilde{\alpha}_2 + \delta\left[(\tilde{\alpha}_2 + (1-\tilde{\alpha}_2)\pi) V_0 + (1-\tilde{\alpha}_2)(1-\pi)V_N\right]$$

$$V_N = (1-\gamma)(\lambda - c) + \gamma V_1$$

$$V_0 = \frac{\delta(1-\pi)}{1-\delta\pi} V_N.$$

Since in a mixed equilibrium we know from Lemma 12 that $\alpha_1(1) < 1$ so that at $z = 1$ the long-run player must be willing not to provide effort. This system corresponds to providing no effort at $z = 0, 1$. From the contraction mapping fixed point theorem this has a unique solution V_1, V_N, V_0. Define the function $\underline{\Delta}(\tilde{\alpha}_2) \equiv V_1 - V_0$.

Lemma 15. *We have*

$$V_1 = \frac{\delta(1-\pi)(1-\gamma)(\lambda-c) + (1-\delta)\left[1-\delta\pi-\delta(1-\pi)(1-\gamma)(\lambda-c)\right]\tilde{\alpha}_2}{(1-\delta\pi-\gamma\delta(1-\pi)) + \gamma\delta(1-\pi)(1-\delta)\tilde{\alpha}_2}$$

strictly increasing in $\tilde{\alpha}_2$.

Proof. Here we simply solve the linear system and determine the sign of the derivative of V_1.

Plugging V_0 into V_1

$$V_1 = (1-\delta)\tilde{\alpha}_2 + \delta\left[(\tilde{\alpha}_2 + (1-\tilde{\alpha}_2)\pi)\frac{\delta(1-\pi)}{1-\delta\pi} + (1-\tilde{\alpha}_2)(1-\pi)\right]V_N$$

Plugging in V_N

$$V_1 = (1-\delta)\tilde{\alpha}_2 + \delta\left[(\tilde{\alpha}_2 + (1-\tilde{\alpha}_2)\pi)\frac{\delta(1-\pi)}{1-\delta\pi} + (1-\tilde{\alpha}_2)(1-\pi)\right]((1-\gamma)(\lambda-c) + \gamma V_1)$$

from which

$$V_1 = \frac{(1-\delta)\tilde{\alpha}_2 + \delta\left[(\tilde{\alpha}_2 + (1-\tilde{\alpha}_2)\pi)\frac{\delta(1-\pi)}{1-\delta\pi} + (1-\tilde{\alpha}_2)(1-\pi)\right](1-\gamma)(\lambda-c)}{1-\gamma\delta\left[(\tilde{\alpha}_2 + (1-\tilde{\alpha}_2)\pi)\frac{\delta(1-\pi)}{1-\delta\pi} + (1-\tilde{\alpha}_2)(1-\pi)\right]}$$

$$= \frac{(1-\delta)\tilde{\alpha}_2 + \delta\left[\Sigma\right](1-\gamma)(\lambda-c)}{1-\gamma\delta\left[\Sigma\right]}.$$

We have

$$\Sigma = (\tilde{\alpha}_2 + (1-\tilde{\alpha}_2)\pi)\,\frac{\delta(1-\pi)}{1-\delta\pi} + (1-\tilde{\alpha}_2)(1-\pi)$$
$$= \tilde{\alpha}_2 \frac{\delta(1-\pi)}{1-\delta\pi} + (1-\tilde{\alpha}_2)\frac{1-\pi}{1-\delta\pi}$$
$$= \frac{1-\pi}{1-\delta\pi}\,(\delta\tilde{\alpha}_2 + 1 - \tilde{\alpha}_2)$$
$$= \frac{1-\pi}{1-\delta\pi}\,(1-(1-\delta)\tilde{\alpha}_2)$$

Plug back into $V_1$to find

$$V_1 = \frac{(1-\delta)\tilde{\alpha}_2 + \delta\left[\frac{1-\pi}{1-\delta\pi}\,(1-(1-\delta)\tilde{\alpha}_2)\right](1-\gamma)(\lambda-c)}{1-\gamma\delta\left[\frac{1-\pi}{1-\delta\pi}\,(1-(1-\delta)\tilde{\alpha}_2)\right]}$$

$$= \frac{(1-\delta)(1-\delta\pi)\tilde{\alpha}_2 + \delta\left[(1-\pi)\,(1-(1-\delta)\tilde{\alpha}_2)\right](1-\gamma)(\lambda-c)}{1-\delta\pi-\gamma\delta\left[(1-\pi)\,(1-(1-\delta)\tilde{\alpha}_2)\right]}$$

$$= \frac{\delta(1-\pi)(1-\gamma)(\lambda-c) + (1-\delta)\left[1-\delta\pi-\delta(1-\pi)(1-\gamma)(\lambda-c)\right]\tilde{\alpha}_2}{(1-\delta\pi-\gamma\delta(1-\pi)) + \gamma\delta(1-\pi)(1-\delta)\tilde{\alpha}_2}.$$

The derivative $dV_1/D\tilde{\alpha}_2$ has the same sign as

$$\sigma = [1-\delta\pi-\delta(1-\pi)(1-\gamma)(\lambda-c)]\,(1-\delta\pi-\gamma\delta(1-\pi)) - \delta^2(1-\pi)^2\gamma(1-\gamma)(\lambda-c)$$
$$= (1-\delta\pi)\,(1-\delta\pi-\gamma\delta(1-\pi)) - \delta(1-\pi)(1-\gamma)(\lambda-c)\,(1-\delta\pi-\gamma\delta(1-\pi)) - \delta^2(1-\pi)^2\gamma(1-\gamma)(\lambda-c)$$
$$= (1-\delta\pi)\,(1-\delta\pi-\gamma\delta(1-\pi)) - \delta(1-\pi)(1-\gamma)(\lambda-c)\,(1-\delta\pi) + \delta(1-\pi)(1-\gamma)(\lambda-c)\gamma\delta(1-\pi)$$
$$-\delta^2(1-\pi)^2\gamma(1-\gamma)(\lambda-c)$$
$$= (1-\delta\pi)\left[1-\delta\pi-(1-\pi)\delta\left(\gamma+(1-\gamma)(\lambda-c)\right)\right] > 0.$$

□

Lemma 16. $\underline{\Delta}(\tilde{\alpha}_2)$ *is strictly increasing. There is a solution* $0 < \hat{\alpha}_2 < 1$ *to*

$$\underline{\Delta}(\tilde{\alpha}_2) = \overline{\Delta}(\tilde{\alpha}_2) \equiv \frac{1-\delta}{\delta\left(\tilde{\alpha}_2 + (1-\tilde{\alpha}_2)\pi\right)}c,$$

it and only if

$$c < \delta\frac{(1-\delta\pi-\delta(1-\pi)\left[\gamma+\lambda(1-\gamma)\right])}{1-\delta\pi-\delta^2(1-\pi)},$$

in which case it is unique.

Proof. Here solve V_0 as a function of V_1 from the system. We subtract this from V_1 and find that $\underline{\Delta}(\tilde{\alpha}_2)$ is strictly increasing in V_1. Hence we may apply

Lemma 15. Since $\overline{\Delta}(\tilde{\alpha}_2)$ is decreasing there will be a unique intersection if and only if $\overline{\Delta}(0) > \underline{\Delta}(0)$ and $\overline{\Delta}(1) < \underline{\Delta}(1)$. By computation we show that the first condition is always satisfied and the second is the condition on c given as the result.

Turning to the details, we first find V_0

$$V_0 = \frac{\delta(1-\pi)}{1-\delta\pi} V_N = \frac{\delta(1-\pi)}{1-\delta\pi} \left((1-\gamma)(\lambda - c) + \gamma V_1\right).$$

Hence

$$\underline{\Delta}(\tilde{\alpha}_2) \equiv V_1 - V_0 = \left(1 - \gamma \frac{\delta(1-\pi)}{1-\delta\pi}\right) V_1 - \frac{\delta(1-\pi)}{1-\delta\pi} \left((1-\gamma)(\lambda - c)\right)$$

$$\underline{\Delta}(\tilde{\alpha}_2) \equiv V_1 - V_0 = \frac{1}{1-\delta\pi} \left[\left(1 - \delta\pi - \gamma\delta(1-\pi)\right) V_1 - \delta(1-\pi)(1-\gamma)(\lambda - c)\right]$$

is strictly increasing in V_1 hence by Lemma 15 in $\tilde{\alpha}_2$.

The function

$$\overline{\Delta}(\tilde{\alpha}_1) = \frac{1-\delta}{\delta\left(\tilde{\alpha}_2 + (1-\tilde{\alpha}_2)\pi\right)} c$$

strictly decreasing in $\tilde{\alpha}_2$. Hence there is a solution $0 < \hat{\alpha}_2 < 1$ to $\underline{\Delta}(\tilde{\alpha}_2) = \overline{\Delta}(\tilde{\alpha}_2)$ if and only if $\overline{\Delta}(0) > \underline{\Delta}(0)$ and $\overline{\Delta}(1) < \underline{\Delta}(1)$ in which case it is unique. This gives the first result.

From Lemma 15 at $\tilde{\alpha}_2 = 0$ we have

$$V_1 = \frac{\delta(1-\pi)(1-\gamma)(\lambda - c)}{1 - \delta\pi - \gamma\delta(1-\pi)}$$

so

$$\underline{\Delta}(0) = \frac{1}{1-\delta\pi} \left(\delta(1-\pi)(1-\gamma)(\lambda - c) - \delta(1-\pi)(1-\gamma)(\lambda - c)\right) = 0 < \frac{(1-\delta)c}{\delta\pi} = \overline{\Delta}(0).$$

Finally we study $\overline{\Delta}(1) < \underline{\Delta}(1)$. From Lemma 15

$$V_1 = \frac{\delta(1-\pi)(1-\gamma)(\lambda - c) + (1-\delta)\left[1 - \delta\pi - \delta(1-\pi)(1-\gamma)(\lambda - c)\right]}{1 - \delta\pi - \gamma\delta(1-\pi) + \gamma\delta(1-\pi)(1-\delta)}$$

$$= \frac{(1-\delta)(1-\delta\pi) + \delta^2(1-\pi)(1-\gamma)(\lambda - c)}{1 - \delta\pi - \gamma\delta^2(1-\pi)},$$

so that

$$\underline{\Delta}(1) = \frac{1}{1-\delta\pi} \left[\frac{1 - \delta\pi - \gamma\delta(1-\pi)}{1 - \delta\pi - \gamma\delta^2(1-\pi)} \left[(1-\delta)(1-\delta\pi) + \delta^2(1-\pi)(\lambda - \gamma)(1-c)\right] - \delta(1-\pi)(\lambda - \gamma)(1-c)\right]$$

$$= \frac{1}{1-\delta\pi} \left[\left(\frac{(1-\delta)(1-\delta\pi)\left(1 - \delta\pi - \gamma\delta(1-\pi)\right)}{1 - \delta\pi - \gamma\delta^2(1-\pi)}\right) + \left(\frac{\delta - \delta^2\pi - \gamma\delta^2(1-\pi)}{1 - \delta\pi - \gamma\delta^2(1-\pi)} - 1\right) \delta(1-\pi)\left((1-\gamma)(\lambda - c)\right)\right]$$

$$= \frac{1}{1-\delta\pi}\left[\left(\frac{(1-\delta)(1-\delta\pi)\,(1-\delta\pi-\gamma\delta(1-\pi))}{1-\delta\pi-\gamma\delta^2(1-\pi)}\right)+\left(\frac{\delta-\delta^2\pi-1+\delta\pi}{1-\delta\pi-\gamma\delta^2(1-\pi)}\right)\delta(1-\pi)\,((1-\gamma)(\lambda-c))\right]$$

$$= \frac{1}{1-\delta\pi}\left[\left(\frac{(1-\delta)(1-\delta\pi)\,(1-\delta\pi-\gamma\delta(1-\pi))}{1-\delta\pi-\gamma\delta^2(1-\pi)}\right)+\left(\frac{-(1-\delta)+\delta\pi(1-\delta)}{1-\delta\pi-\gamma\delta^2(1-\pi)}\right)\delta(1-\pi)\,((1-\gamma)(\lambda-c))\right]$$

$$= \frac{(1-\delta)}{1-\delta\pi-\gamma\delta^2(1-\pi)}\left[(1-\delta\pi-\gamma\delta(1-\pi))-\delta(1-\pi)(1-\gamma)(\lambda-c)\right]$$

$$= \frac{(1-\delta)\left[1-\delta\pi-\gamma\delta+\gamma\delta\pi-\delta(1-\pi)(1-\gamma)(\lambda-c)\right]}{1-\delta\pi-\gamma\delta^2(1-\pi)}.$$

Hence $\overline{\Delta}(1) < \underline{\Delta}(1)$ if and only if

$$\frac{(1-\delta)\left[1-\delta\pi-\gamma\delta+\gamma\delta\pi-\delta(1-\pi)(1-\gamma)(\lambda-c)\right]}{1-\delta\pi-\gamma\delta^2(1-\pi)} > \underline{\Delta}(1) = (1-\delta)c/\delta.$$

We rewrite this inequality

$$\delta\,(1-\delta\pi-\gamma\delta+\gamma\delta\pi-\delta(1-\pi)(1-\gamma)(\lambda-c)) > \big(1-\delta\pi-\gamma\delta^2(1-\pi)\big)\,c$$

$$\delta\,(1-\delta\pi-\gamma\delta(1-\pi)-\lambda\delta(1-\pi)(1-\gamma)) > \big(1-\delta\pi-\gamma\delta^2(1-\pi)-\delta^2(1-\pi)(1-\gamma)\big)\,c$$

$$\delta\,(1-\delta\pi-\gamma\delta(1-\pi)-\lambda\delta(1-\pi)(1-\gamma)) > \big(1-\delta\pi-\delta^2(1-\pi)\big)\,c$$

$$c < \delta\frac{(1-\delta\pi-\gamma\delta(1-\pi)-\lambda\delta(1-\pi)(1-\gamma))}{1-\delta\pi-\delta^2(1-\pi)}$$

$$c < \delta\frac{(1-\delta\pi-\delta(1-\pi)\left[\gamma+\lambda(1-\gamma)\right])}{1-\delta\pi-\delta^2(1-\pi)}.$$

This gives the final result. □

Proposition 2. *If* $\epsilon < \underline{\mu}\pi^2(1-\pi)\min\{\overline{B}, 1-\overline{B}\}/32$ *and*

$$c \geq \delta\frac{1}{1+\delta(1-\pi)}.$$

all equilibria are in pure strategies.

Proof. Suppose that $\alpha_1(z), \alpha_2(z)$ is a non-pure equilibrium. If the normal type is willing to provide effort at $z=1$ we take $\hat{\alpha}_2 = \alpha_2(1)$. If the long-run player strictly prefers to provide no effort at $z=1$ we show how to construct a $1 > \hat{\alpha}_2 > \alpha_2(1)$ for which the long-run player is indifferent at $z=1$ and strictly prefers to provide effort at $z=N$. We show that $1-c \geq V(\alpha_2(N)) \geq V(\hat{\alpha}_2)$ and use this to show that at $\hat{\alpha}_2$ we must have $\underline{\Delta}(\hat{\alpha}_2) = \overline{\Delta}(\hat{\alpha}_2)$ for $\lambda = 1$. Applying Lemma 16 then yields the desired condition.

Turning to the details, from Lemmas 12, 13, and 14 we know that $\alpha_1(0) = \alpha_2(0) = 0$, $\alpha_1(N) > 0$, $\alpha_2(N) \geq \alpha_2(1)$, $\alpha_1(1) < 1$, and $0 < \alpha_2(1) < 1$.

If the long-run player strictly prefers not to provide effort at $z=1$ then $\alpha_1(1) = 0$. Moreover we must have $\alpha_2(N) > \alpha_2(1)$ since if the two are equal and effort is weakly preferred at $\alpha_2(N)$ it would be at $\alpha_1(1)$ as well. For $V(\alpha_2(0))$ we solve the Bellman system to find

$$V_0 = \frac{\delta(1-\pi)}{1-\delta\pi} V_N$$

and for $V(\alpha_2(N))$ we solve

$$V(\alpha_2(N)) = \frac{1-\delta}{1-\delta(1-\alpha_2(N))(1-\pi)} [\alpha_2(N) - c] + \frac{\delta(\alpha_2(N) + (1-\alpha_2(N))\pi)}{1-\delta(1-\alpha_2(N))(1-\pi)} V(\alpha_2(1)).$$

Hence if hold fixed $\alpha_2(N)$ and take $\lambda = \alpha_2(N)$ and

$$\gamma = \frac{\delta(\alpha_2(N) + (1-\alpha_2(N))\pi)}{1-\delta(1-\alpha_2(N))(1-\pi)}$$

the Bellman system corresponds to the auxiliary system, so $V(\alpha_2(1)) - V(\alpha_2(0)) = \underline{\Delta}(\alpha_2(1))$. From Lemma 16 this is strictly increasing.

As earlier we may define the gain to providing no effort at z as

$$G(\alpha_2(z), \alpha_2(1)) = (1-\delta)c - \delta(\pi + \alpha_2(z)(1-\pi))[V(\alpha_2(1)) - V(\alpha_2(0))]$$

and it follows that G is strictly decreasing in both arguments. Hence as we increase $\alpha_2(1)$ to α_2 the gain $G(\alpha_2, \alpha_2)$ to providing no effort at $z = 1$ and the gain $G(\alpha_2(N), \alpha_2)$ to no effort at $z = N$ both strictly decline. Initially at $z = N$ the long-run player weakly preferred to provide effort, hence as we increase α_2 the long-run player strictly prefers to provide effort. At $z = 1$ the long-run player strictly preferred no effortbut when α_2 reaches $\alpha_2(N)$ effort is strictly preferred, and as G is continuous, this implies for some $\hat{\alpha}_2 < 1$ the long-run player is indifferent at $z = 1$.

To summarize: in all cases with the original value of $\alpha_2(0), \alpha_2(N)$ and the short-run player using $\hat{\alpha}_2 \leq \alpha_2(N)$ in state $z = 1$ the strategy for the long-run player of providing no effort in states $z = 0, 1$, providing effort in state N is optimal and the long-run player is indifferent in state $z = 1$. We next show that with respect to this (possibly modified) strategy by player 2 the long-run player has $1 - c \geq V(\alpha_2(N)) \geq V(\hat{\alpha}_2)$ and use this to show that at $\hat{\alpha}_2$ we must have $\underline{\Delta}(\hat{\alpha}_2) = \overline{\Delta}(\hat{\alpha}_2)$ for $\lambda = 1$. Applying Lemma 16 then yields the desired condition.

Since it is also optimal for the long-run payer to provide effort, the Bellman system may be written as

$$V(\hat{\alpha}_2) = (1-\delta)[\hat{\alpha}_2 - c] + \delta[(\hat{\alpha}_2 + (1-\hat{\alpha}_2)\pi) V_1(\hat{\alpha}_2) + (1-\hat{\alpha}_2)(1-\pi)V(\alpha_2(N))]$$

$$V(\alpha_2(N)) = (1-\delta)[\alpha_2(N) - c] + \delta[(\alpha_2(N) + (1-\alpha_2(N))\pi) V(\hat{\alpha}_2) + (1-\alpha_2(N))(1-\pi)V(\alpha_2(N))].$$

This implies that for some $0 \leq \lambda \leq 1$ we have $V(\hat{\alpha}_2) = (1-\lambda)[\hat{\alpha}_2 - c] + \lambda[\alpha_2(N) - c]$ that is, a weighted average of the period payoffs in the two states. Yet $V(\hat{\alpha}_2) > V(\alpha_2(N))$ gives $V(\alpha_2(N)) > (1-\delta)[\alpha_2(N) - c] + \delta V(\alpha_2(N))$

so $V(\alpha_2(N)) > \alpha_2(N) - c \geq \hat{\alpha}_2 - c$. which implies $V(\alpha_2(N)) > V(\hat{\alpha}_2)$ a contradiction. Hence $V(\alpha_2(N)) \geq V(\hat{\alpha}_2)$.

For some $0 \leq \lambda' \leq 1$ we also have $V(\alpha_2(N)) = (1-\lambda')\left[\hat{\alpha}_2 - c\right] + \lambda'\left[\alpha_2(N) - c\right] \leq 1 - c$. This establishes the target $1 - c \geq V(\alpha_2(N)) \geq V(\hat{\alpha}_2)$.

From $1 - c \geq V(\alpha_2(N)) \geq V(\hat{\alpha}_2)$ we see that for some $0 \leq \gamma \leq 1$ we have $V(\alpha_2(N)) = \gamma(1-c) + (1-\gamma)V(\hat{\alpha}_2)$. It follows from this and indifference of the long-run player at $\hat{\alpha}_2$that for this value of γ and $\lambda = 1$ that at $\hat{\alpha}_2$ we must have $\underline{\Delta}(\hat{\alpha}_2) = \overline{\Delta}(\hat{\alpha}_2)$. From Lemma 16 this means that

$$c < \delta \frac{(1 - \delta\pi - \delta(1-\pi)\left[\gamma + \lambda(1-\gamma)\right])}{1 - \delta\pi - \delta^2(1-\pi)}$$

$$= \delta \frac{1 - \delta\pi - \delta(1-\pi)}{1 - \delta\pi - \delta^2(1-\pi)}$$

$$= \delta \frac{1 - \delta}{(1-\delta)(1 + \delta(1-\pi))}$$

the desired result. □

Role of Types

We turn now to a converse of Proposition 2: that is when

$$c < \delta \frac{1}{1 + \delta(1-\pi)}$$

are there equilibria that are not pure? Intuitively this cannot be the case for all Q. If there are very few normal types then basically the short-run player ignores them and plays a best response to the behavioral types - which is to say the pure strategy of staying out on a bad or no signal and entering on a good signal. This we know leads the normal type to best-respond with a pure strategy as given in Proposition 2.

Our first result is precise result: it shows if there are enough good types there is necessarily a pure strategy equilibrium.

Proposition 3. *For any Q with*

$$\mu_g > \frac{\overline{B}}{\overline{B} + (1-\overline{B})(\pi/2)}$$

if $\epsilon \leq \underline{\mu}\overline{A}/32$ then all equilibria are pure.

Proof. From Bayes Law

$$\mu^1(1) \geq \mu_{g|1} = \frac{\mu_{1|g}\mu_g}{\mu_{1|g}\mu_g + \sum_{\tau \neq g} \mu_{1|\tau}\mu_\tau} \geq \frac{1}{1 + (1-\mu_g)/(\mu_{1|g}\mu_g)}.$$

From Lemma 3

$$\mu_{1|g} \geq \left[\alpha_2(1) + \pi(1 - \alpha_2(1))\right]\mu_{1|g} + \left[\alpha_2(N) + \pi(1 - \alpha_2(N))\right]\mu_{N|g} - 2\epsilon.$$

The same Lemma implies $\mu_{0|g} \leq 2\epsilon$, so

$$\mu_{1|g} \geq [\alpha_2(1) + \pi(1 - \alpha_2(1)) - \pi]\, \mu_{1|g} + \pi - 4\epsilon \geq \pi - 4\epsilon.$$

Combining the two

$$\mu^1(1) \geq \frac{1}{1 + (1 - \mu_g)/((\pi - 4\epsilon)\mu_g)}.$$

By Lemma 5 if

$$\frac{1}{1 + (1 - \mu_g)/((\pi - 4\epsilon)\mu_g)} > \overline{B}$$

or equivalently

$$\mu_g > \frac{\overline{B}}{\overline{B} + (1 - \overline{B})(\pi - 4\epsilon)}$$

then $\alpha_2(1) = 1$ so the result follows from Lemma 14 and the assumption that $\epsilon < \underline{\mu}\overline{A}/32 \leq \pi/2$. □

This is not terribly interesting in itself: the case of interest is when they are many normal types, but it does show that there is no converse to Proposition 2 without an assumption on Q. Hence we investigate the interesting case of many normal types.

In addition to showing that there are mixed equilibria, we can say what they look like. There are two types, *single mixing* and *double mixing*. In both types of equilibrium in the bad state $z = 0$ there is no effort and the short-run player stays out: $\alpha_1(0) = 0$, $\alpha_2(0) = 0$. In the good state $z = 1$ both players strictly mix: $0 < \alpha_1(1) < 1$, $0 < \alpha_2(1) < 1$. In the single mixing equilibrium this is the only mixing: in the state $z = N$ the normal type provides effort and the short-run player enters $\alpha_1(N) = 1, \alpha_2(N) = 1$. In the double mixing case equilibrium mixing takes place also at $z = N$: the short-run player mixes exactly as in the state $z = 1$,that is $\alpha_2(N) = \alpha_2(1)$, while normal type provides effort with a positive probability $\alpha_1(N) > 0$.

To state a precise result and also be clear about the order of limits, it is useful to define the notion of a *fundamental bound*. This is a number that may depend on the fundamentals of the game π, V, δ, c but not on the type dynamics Q, ϵ. Recall that $\overline{B}$ is the probability of effort that makes the short-run player indifferent to entry.

Lemma 17. *There exists a fundamental bound $\overline{\mu} < 1$ such that for any Q with $\mu_n \geq \overline{\mu}$ if for $\epsilon \leq \underline{\mu}\overline{A}/32$ a non-pure equilibrium is either a single- or double-mixing profile.*

Proof. The only things not covered in Lemmas 12, 13, and 14 are $\alpha_1(1) \neq 0$ and the result that $\alpha_2(N) > \alpha_2(1)$ implies $\alpha_1(N) = 1, \alpha_2(N) = 1$.

For the first result, the idea is since μ_n is large there must be many more normal types at N than good types. Since since $\alpha_2(N) > 0$ this means that $\alpha_1(N)$ cannot be too small, and this in turn implies that even though $\alpha_1(1) = 0$

there must be many more normal types at 1 then good types. If they provide no effort then the short-run player should stay out contradicting the fact that we already know $\alpha_2(1) > 0$.

For the second result we leverage the first to see that we must have $\alpha_1(N) = 1$. Moreover, since $\alpha_1(1) < 1$ there must be many normal types at $z = 0$, and so at $z = N$. As these are all providing effort, it is optimal for the short-run player to enter.

Turning to the details, suppose in fact $\alpha_1(1) = 0$. Since $\alpha_2(N) > 0$ we must have $(1-\mu_n)V + \alpha_1(N)\mu_{N|n}\mu_n V \geq (1-\alpha_1(N))\mu_{N|n}\mu_n$. We may rewrite this as

$$\alpha_1(N)\mu_{N|n} \geq \frac{1}{1+V}\left(\mu_{N|n} - \frac{(1-\mu_n)}{\mu_n}V\right).$$

From Lemma 3 we have $\mu_{1|n} \geq \pi\alpha_1(N)\mu_{N|n} - 2\epsilon$ so

$$\mu_{1|n} \geq \frac{\pi}{1+V}\left(\mu_{N|n} - \frac{1-\mu_n}{\mu_n}V\right) - 2\epsilon.$$

Also from Lemma 3 we have $\mu_{N|n} \geq (1-\pi)(1-\mu_{N|n}-\mu_{1|n}) - 2\epsilon$ so that $\mu_{N|n} \geq (1-\mu_{1|n})(1-\pi)/\pi - 2\epsilon/\pi$ implying

$$\mu_{1|n} \geq \frac{1}{1+V}\left(\left[(1-\mu_{1|n})(1-\pi) - 2\epsilon\right] - \pi\frac{1-\mu_n}{\mu_n}V\right) - 2\epsilon$$

or

$$\mu_{1|n} \geq \frac{\left([(1-\pi) - 2\epsilon] - \pi\frac{1-\mu_n}{\mu_n}V\right) - 2(1+V)\epsilon}{2+V-\pi}.$$

Since $\alpha_2(1) > 0$ we must have $(1-\mu_n)V \geq \mu_{1|n}\mu_n$, so

$$\frac{1-\mu_n}{\mu_n}V \geq \frac{\left([(1-\pi) - 2\epsilon] - \pi\frac{1-\mu_n}{\mu_n}V\right) - 2(1+V)\epsilon}{2+V-\pi}$$

or

$$\frac{1-\mu_n}{\mu_n}V \geq \frac{1-\pi}{2+V} - 2\epsilon.$$

Our assumption implies $\epsilon < (1-\pi)\overline{B}/4$ which means that $\epsilon \leq (1-\pi)/(4+2V)$. Hence

$$\frac{1-\mu_n}{\mu_n} \geq \frac{1-\pi}{V(4+2V)}$$

or

$$\mu_n \leq \frac{V(4+2V)}{V(4+2V)+1-\pi} < 1.$$

Hence $\alpha_1(1) = 0$ is ruled out by large μ_n.

Next suppose that $\alpha_2(N) > \alpha_2(1)$. Since $\alpha_1(1) > 0$ Lemma 1 implies $\alpha_1(N) = 1$. It remains to show that this in turn forces $\alpha_2(N) = 1$.

From Lemma 2

$$\mu_{N|n} \geq (1-\pi)\left((1-\alpha_2(1))\mu_{1|n} + \mu_{0|n} - 2\epsilon\right).$$

Suppose that $\mu_{N|n} \leq 1/3$. Then either $\mu_{0|n} \geq 1/3$ or $\mu_{1|n} \geq 1/3$.

In the former case we have $\mu_{N|n} \geq (1-\pi)/3 - 2\epsilon$.

In the latter case strict mixing by the short-run player at $z = 1$ implies from Lemma 6 that

$$1-\overline{B} = 1-\mu^1(1) = \frac{[1-\alpha_1(1)]\mu_{1|n}\mu_n + \mu_{1|b}\mu_b}{\mu_1} \leq \frac{[1-\alpha_1(1)]\mu_{1|n}\mu_n + \mu_{1|b}\mu_b}{\mu_{1|n}\mu_n + \mu_{1|b}\mu_b} = \gamma(1-\alpha_1(1))+(1-\gamma).$$

If $1-\alpha_1(1) < (1-\overline{B})/2$ then

$$1-\overline{B} \leq \gamma(1-\overline{B})/2 + (1-\gamma)$$

or

$$\gamma \leq \frac{2\overline{B}}{1+\overline{B}}.$$

Lemma 3 gives $\mu_{1|b} \leq 2\epsilon$ so since $\mu_{1|n} \geq 1/3$

$$\gamma = \frac{\mu_{1|n}\mu_n}{\mu_{1|n}\mu_n + \mu_{1|b}\mu_b} \geq \frac{\mu_n}{\mu_n + 6\epsilon(1-\mu_n)}.$$

Since $\epsilon \leq 1/6$ it follows that $\gamma \geq \mu_n$ so if

$$\mu_n > \frac{2\overline{B}}{1+\overline{B}}$$

we have a contradiction, so $1-\alpha_1(1) \geq (1-\overline{B})/2$. Hence by Lemma 3 $\mu_{0|n} \geq \pi(1-\overline{B})/2 - 2\epsilon$ from which $\mu_{N|n} \geq \pi(1-\pi)(1-\overline{B})/2 - 4\epsilon$.

In all cases then $\mu_{N|n} \geq \pi(1-\pi)(1-\overline{B})/2 - 4\epsilon$. As

$$\epsilon \leq \pi(1-\pi)(1-\overline{B})/16$$

this is $\mu_{N|n} \geq \pi(1-\pi)(1-\overline{B})/4$.

The short-run player must enter if

$$\mu_{N|n}\mu_n V > \mu_N - \mu_{N|n}\mu_n$$

while $\mu_N \leq \mu_{N|n}\mu_n + (1-\mu_n)$ so entry must occur if

$$\mu_{N|n}\mu_n V > 1-\mu_n$$

or

$$\mu_n > \frac{1}{1+V\mu_{N|n}} \geq \frac{4}{4+\pi(1-\pi)(1-\overline{B})},$$

□

Lemma 18. *In any single- or double-mixing profile if $\mu_n \geq 1/2$ and $\epsilon < (1 - a_2(1))(1-\pi)/12$ then*

$$\mu_{n|N} \geq 1 - \frac{1-\mu_n}{(1-\alpha_2(1))(1-\pi)/12}.$$

If in addition $\epsilon < \alpha_1(N)\pi(1-a_2(1))(1-\pi)/24$ then

$$\mu_{n|1} \geq 1 - \frac{1-\mu_n}{\alpha_1(N)\pi(1-\alpha_2(1))(1-\pi)/24}.$$

Proof. The first result says that if $\alpha_2(1)$ is less than 1 and if there are many normal types there must be many normal types at $z = N$, as they are flowing there from both $z = 0$ and $z = 1$. The second result leverages this to say that if there are many normal types at $z = N$ and $\alpha_1(N)$ is large then there must be many normal types at $z = 1$.

Turning to the details, we start with an inequality that follows from Bayes law:

$$\begin{aligned}
\mu_{n|z} = \frac{\mu_{z|n}\mu_n}{\mu_z} &\geq \frac{\mu_{z|n}\mu_n}{\mu_{z|n}\mu_n + (1-\mu_n)} \\
&= 1 - \left(1 - \frac{\mu_{z|n}\mu_n}{\mu_{z|n}\mu_n + (1-\mu_n)}\right) \\
&= 1 - \left(\frac{1-\mu_n}{\mu_{z|n}\mu_n + (1-\mu_n)}\right) \\
&\geq 1 - \frac{1-\mu_n}{\mu_{z|n}\mu_n}.
\end{aligned}$$

Since $\mu_n \geq 1/2$ this implies

$$\mu_{n|z} \geq 1 - \frac{1-\mu_n}{\mu_{z|n}/2}.$$

To get the required bounds, it then suffices to get a lower bound on $\mu_{z|n}$. Take first $z = N$. Suppose that $\mu_{N|n} \leq 1/3$. Then either $\mu_{0|n} \geq 1/3$ or $\mu_{1|n} \geq 1/3$. If $\mu_{0|n} \geq 1/3$ then by Lemma 3 $\mu_{N|n} \geq (1-\pi)/3 - 2\epsilon$. If $\mu_{1|n} \geq 1/3$ then by the same Lemma $\mu_{N|n} \geq (1-a_2(1))(1-\pi)/3 - 2\epsilon$. For $\epsilon < (1-a_2(1))(1-\pi)/12$ this is $\mu_{N|n} \geq (1-a_2(1))(1-\pi)/6$ giving the first bound.

From Lemma 3$\mu_{1|n} \geq \alpha_1(N)\pi\mu_{N!n} - 2\epsilon$ so that the first bound implies $\mu_{1|n} \geq \alpha_1(N)\pi(1-a_2(1))(1-\pi)/6 - 2\epsilon$. For $\epsilon < \alpha_1(N)\pi(1-a_2(1))(1-\pi)/24$ this is $\mu_{1|n} \geq \alpha_1(N)\pi(1-a_2(1))(1-\pi)/12$ giving the second bound. □

The next Lemma is simply an observation:

Lemma 19. *A single mixing equilibrium corresponds to the auxiliary system with $\lambda = 1$ and $\gamma = \delta$ and a double mixing equilibrium corresponds to the auxil-*

iary system with $\lambda = 1$ *and* $\gamma = 1$. *In particular in a single mixing equilibrium*

$$V(\alpha_2(1)) = \frac{(1-\delta\pi)\alpha_2(1)}{1+\delta(1-\pi)\alpha_2(1)}$$

which is increasing in $\alpha_2(1)$.

Proof. In the single mixing case this is just the Bellman equation. In the double mixing case we use the fact that $V(\alpha_2(N)) = V(\alpha_2(1))$. The value $V(\tilde{\alpha}_2)$ follows from plugging into the expression for V_1 in Lemma 15; that Lemma gives the result that it is increasing. □

Proposition 4. *There exists a fundamental bound* $\overline{\mu} < 1$ *such that for any* Q *with* $\mu_n \geq \overline{\mu}$ *if* $\epsilon \leq \underline{\mu}\overline{A}/32$ *and*

$$c < \delta \frac{1}{1+\delta(1-\pi)}$$

there is at least one single-mixing and one double-mixing equilibrium and no other type of mixed equilibrium. In both cases the equilibrium value of $\alpha_2(1)$ *is the unique solution of* $\underline{\Delta}(\alpha_2(1)) = \overline{\Delta}(\alpha_2(1))$ *where* $\lambda = 1$ *and in the single-mixing case* $\gamma = \delta$ *and in the double-mixing case* $\gamma = 1$. *Moreover, the equilibrium value of* $\alpha_1(z)$ *satisfies*

$$|\alpha_1(z) - \overline{B}| \leq \frac{1-\mu_n}{1-\overline{\mu}}$$

for $z = 1$ *in the single mixing case and* $z \in \{N, 1\}$ *in the double-mixing case.*

Proof. From Lemma 17 we know there can be no other kind of equilibrium. From Lemma 16 we know that

$$c < \delta \frac{(1-\delta\pi-\delta(1-\pi)\left[\gamma+\lambda(1-\gamma)\right])}{1-\delta\pi-\delta^2(1-\pi)}$$

and from Lemma 19 with $\lambda = 1$ and $\gamma = \delta$ is a necessary condition for the existence of single-mixing equilibrium and with $\lambda = 1$ and $\gamma = 1$ for the existence of a double-mixing equilibrium. When $\lambda = 1$ the RHS is independent of γ and given as the expression in the Theorem.This gives us a unique solution $0 < \tilde{\alpha}_2 < 1$ for the equilibrium value of $\alpha_2(1)$. The crucial fact is that $\tilde{a}_2$ arising from the optimization problem of the normal type is itself a fundamental bound.

We must now show the existence of an $\alpha_1(1)$ so that the short-run player is indifferent when $z = 1$ and weakly prefers to enter when $z = N$, and in the double mixing case the existence of $\alpha_1(1), \alpha_1(N)$ so that the short-run player is indifferent in both $z = N, 1$, and that any such strategic components satisfy the required bound.

Recall that $\mu^1(z)$ are the beliefs of the short-run player about the probability the long-run player will provide effort. This is given as $\mu^1(z) = \mu_{g|z} + \mu_{n|z}\alpha_1(z)$. Define $\tilde{A}(z, \alpha_1(z)) = \mu_1(z) - \overline{B}$. Hence the equilibrium requirement is that

$\tilde{A}(1,\alpha_1(1)) = 0$ and that in the single mixing case $\tilde{A}(N,\alpha_1(N)) = 0$ and in the double-mixing case $\tilde{A}(N,1) \geq 0$. The complication is that $\mu_{g|z}$ and $\mu_{n|z}$ for $z \in \{N,1\}$both depend upon $\alpha_1(1)$ and $\alpha_1(N)$. As by the ergodic theorem the ergodic distribution is continuous in $\alpha_1(1)$ and $\alpha_1(N)$ so are $\tilde{A}(z,\alpha_1(z))$ and we will be able to apply fixed point argument.

Write $\tilde{A}(z,\alpha_1) = \mu_{g|z} - (1-\mu_{n|z})\alpha_1 + \alpha_1 - \overline{B}$ and observe that $\mu_{g|z} \leq (1-\mu_{n|z})$. Hence $\tilde{A}(z,\alpha_1) = \alpha_1 - \overline{B} + \tilde{A}_1(1-\mu_{n|z})$ with $|\tilde{A}_1| \leq 2$.

We now apply the first bound from Lemma 18. We know that $\alpha_2(1) = \tilde{\alpha}_2$ a fundamental bound so we have $\tilde{A}(N,\alpha_1) = \alpha_1 - \overline{B} + \tilde{A}_2(1-\mu_n)$ where $|\tilde{A}_2| \leq A_2$ and A_2 is a fundamental bound. Hence for $\alpha_1 - \overline{B} \leq -A_2(1-\mu_n)$ we have $\tilde{A}(N,\alpha_1) < 0$. Taking $A_2(1-\mu_n) \leq \overline{B}/2$ for $\alpha_1 \leq \overline{B}/2$ we also have $\tilde{A}(N,\alpha_1) < 0$. We may restrict attention then to the region where $\alpha_1(N) \geq \overline{B}/2$ since there can be no equilibrium outside this region.

In the region $\alpha_1(N) \geq \overline{B}/2$ we may now apply the second bound from Lemma 18 and find that $\tilde{A}(1,\alpha_1) = \alpha_1 - \overline{B} + \tilde{A}_3(1-\mu_n)$ where $|\tilde{A}_3| \leq A_3$ and A_3 is a fundamental bound.

Take first the single-mixing case. Here if we take $A_2(1-\mu_n) \leq (1-\overline{B})/2$ we have $\tilde{A}(N,1) > 0$ and we have $\tilde{A}(1,\alpha_1)$ negative for $\alpha_1 - \overline{B} < -A_3(1-\mu_n)$ and positive for $\alpha_1 - \overline{B} > A_3(1-\mu_n)$ implying at least one solution $\tilde{A}(1,\alpha_1) = 0$ in the interval $|\alpha_1 - \overline{B}| \leq A_3(1-\mu_n)$ and none elsewhere. That is the first required result.

In the double mixing case we take the rectangle $|\alpha_1(1) - \overline{B}| \leq A_3(1-\mu_n)$ and $|\alpha_1(N) - \overline{B}| \leq A_2(1-\mu_n)$ and observe that $\tilde{A}(1,\alpha_1), \tilde{A}(N,\alpha_1)$ are not both zero outside this region. Moreover, the vector field $(\tilde{A}(1,\alpha_1(1)), \tilde{A}(N,\alpha_1(N))$ points outwards on the boundary of the rectangle. By the continuous vector field version of the Brouwer fixed point theorem there is at least one point inside the rectangle where they both vanish. □

Note that we do not guarantee a unique equilibrium of each type, but show that if there are enough normal types then all equilibria of a given type are similar and the mixing by the long-run normal type is approximately the value that makes the short-run player indifferent. The reason this is only approximate is because the short-run player also faces an endogenous number of good and bad types who are either providing effort or not.

How do the mixed equilibria differ from the pure equilibrium? Roughly speaking we can describe the pure equilibria as having three properties: the signal is informative for the short-run player, reputation is valuable, and the normal type of long-run player remains stuck in either a good or bad situation. The mixed equilibria are quite different: the signal is uninformative for the short-run player, reputation is not valuable, and the normal type of long-run player transitions back and forth between all the states.

Specifically, with the mixed equilibrium we have the following situation. In every state the short-run player is facing mostly normal types. The normal type, starting in state $z = 0$ will eventually have some luck, the short-run player will not observe the long-run player, and the state will move to N. Here the normal type provides effort with positive probability and the short-run player observes this with positive probability so there is a chance of getting to the state $z = 1$. Once there both players are mixing, so there is a chance of moving to either state $z = 0$ or state $z = N$. Indeed, the only transitions that are not seen are moving directly from $z = 0$ to $z = 1$ and in the single mixing case moving directly from $z = N$ to $z = 0$. The normal type transitions back and forth between all the states. Because of this mixing the behavioral types play no role in the inferences of the short-run player. This is similar to the cheap talk literature:[3] the mixing of the long-run player effectively jams the signal of the behavioral types, and reputation plays no role in equilibrium. These equilibria also have the property that $\alpha_2(N) \geq \alpha_2(1)$: the short-run player is no more likely to enter when there is a favorable signal than when there is no signal. This represents a precise sense in which the "signal is jammed."

Finally, we emphasize that for very low c there are always signal jamming equilibria: low c does not guarantee a good equilibrium.

Welfare

Is a mixed equilibrium good or bad for the long-run player? This is irrelevant in the bad equilibrium case where $c > \delta$ as there is no mixed equilibrium there. If $\pi < (1-\delta)/\delta$ and

$$\delta \frac{\pi}{1-\delta+\delta\pi} < c < \delta \frac{1}{1+\delta(1-\pi)}$$

then there is both a trap equilibrium and mixed equilibrium. The mixed equilibrium is clearly good for a long-run normal type who is trapped with no reputation - that type gets 0 while receives a positive payoff in the mixed equilibria. In this sense signal jamming is potentially good because it can alleviate a reputation trap.

On the other hand, a long-run normal type with a good reputation gets $1 - c$. The next result shows that in this case a double-mixing equilibrium is unambiguously bad: expected average present value starting in the good state is strictly less.

Proposition 5. *In a double mixing equilibrium*

$$V(\alpha_2(1)) < \frac{1-\delta\pi}{1+\delta(1-\pi)} \leq 1 - c.$$

Proof. From Lemma 15

$$V(\alpha_2(1)) = \frac{(1-\delta\pi)\alpha_2(1)}{1+\delta(1-\pi)\alpha_2(1)}$$

which is strictly increasing in $\alpha_2(1)$, so the first bound follows from $\alpha_2(1) < 1$. The final inequality is a restatement of the condition for the existence of a double

[3] See, for example, Crawford and Sobel [1982].

mixing equilibrium from Proposition 4. □

This has the following additional consequence. As $\delta \to 1$ regardless of initial condition utility in the good equilibrium approaches $1-c$. On the other hand the Theorem shows that $\limsup V(\alpha_2(1))$ is bounded above by $(1-\pi)/(2-\pi)$ which does not depend upon c. Hence for small enough c starting in the good state the normal long-run player does strictly worse in the double-mixing equilibrium than in the always provide effort equilibrium even as $\delta \to 1$. This result appears quite different than the long memory case analyzed in Fudenberg and Levine [1989] and Ekmekci, Gossner and Wilson [2012].

To understand why this is, observe that with sufficiently long memory by the short-run player the long-run player can foil a signal jamming equilibrium: if the long-run player persists in effort provision Fudenberg and Levine [1992] show that when there is a good type the short-run player must come to believe that the long-run player will provide effort. To understand how the conflict between the conclusions for $\delta \to 1$ arises, observe that for any fixed length of time the Fudenberg and Levine [1992] bound requires the prior probability of the good type to be sufficiently high. Here the length of time is indeed fixed - the long-run player has only one period to convince the short-run player that there will be effort. Hence, as Proposition 12 shows, and as the Fudenberg and Levine [1992] result suggests, signal jamming is ruled out if the prior probability of the good type is sufficiently high. Hence the result here that equilibrium payoffs remain bounded away from the Stackelberg payoff of $1-c$ when the probability of the good type is too low is an example confirming that the Fudenberg and Levine [1992] bound must depend on the strength of prior belief in the good type.

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