Supplement to "Strategically Simple Mechanisms"
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Proof of Proposition 3

Let $A = \{a, b, c\}$. Suppose that a mechanism is a strategically simple mechanism of type 2. We shall analyze properties of such a mechanism that ultimately imply that, up to relabeling of the agents and the alternatives, only the two mechanisms listed in Proposition 3 are candidates for type 2 strategically simple mechanisms. The analysis of these two mechanisms in the main text shows that these mechanisms are indeed type 2 strategically simple.

Throughout this proof, we shall denote the ordinal preference R_i that satisfies aR_ib and bR_ic by "abc," and we shall use analogous notation for any other ordinal preference over the three alternatives.

Claim **1.** *There is at least one preference profile* $(\hat{R}_1, \hat{R}_2)$ *such that both* $UD_1(\hat{R}_1)$ *and* $UD_2(\hat{R}_2)$ *have at least two elements.*

Proof. At a preference profile at which agent 1 is the unique local dictator, agent 1 must have at least two undominated strategies. At a preference profile at which agent 2 is the unique local dictator, agent 2 must have at least two undominated strategies. □

Claim **2.** *If for some preference profile* $(\hat{R}_1, \hat{R}_2)$ *both* $UD_1(\hat{R}_1)$ *and* $UD_2(\hat{R}_2)$ *have at least two elements, then the set* $g(UD_1(\hat{R}_1), UD_2(\hat{R}_2))$ *has no more than two elements.*

Proof. By Theorem 1, there must be a local dictator at $(\hat{R}_1, \hat{R}_2)$. Without loss of generality, assume that agent 2 is a local dictator. If the set $g(UD_1(\hat{R}_1), UD_2(\hat{R}_2))$ contains three elements, then agent 2, as a local dictator, could enforce each of them. Therefore, each of agent 1's undominated strategies would have to offer the same menu that contains all three elements. But this contradicts Corollary 2. Therefore, $g(UD_1(\hat{R}_1), UD(\hat{R}_2))$ has only one or two elements. □

We now distinguish the two cases. Case 1 is the case in which there is at least one preference profile such that both agents have multiple undominated strategies, and such that exactly two outcomes may result if both agents with these preferences choose from their sets of undominated strategies. For this case, we show that the 4×4 mechanism in Proposition 3 is the unique strategically simple mechanism, up to relabeling of the agents and the alternatives. Case 2 is the case in which for all preference profiles such that both agents have multiple undominated strategies, exactly one outcome may result if both agents with these preferences choose from their sets of undominated strategies. For this case, we show that the 5×5 mechanism in Proposition 3 is the unique strategically simple mechanism, up to relabeling of the agents and the alternatives.

Case 1: There is at least one preference profile, say $(\hat{R}_1, \hat{R}_2)$, such that both agents have multiple undominated strategies, and such that exactly two outcomes may result, say $g(UD_1(\hat{R}_1), UD(\hat{R}_2)) = \{a, b\}$, if both agents with these preferences choose from their sets of undominated strategies.

Figure 3 illustrates the proof for the first case. We shall refer to Figure 3 while presenting the proof.

$$UD_1(\hat{R}_1) \left\{ \begin{array}{ccc} & \hat{s}_2 & \hat{\hat{s}}_2 \\ \hat{s}_1 & a & b \\ \hat{\hat{s}}_1 & a & b \end{array} \right. \Rightarrow UD_1(bca) \left\{ \begin{array}{ccccc} & \hat{s}_2 & \hat{\hat{s}}_2 & s_2^{cab} & s_2^{cba} \\ \hat{s}_1 & a & b & c & c \\ \hat{\hat{s}}_1 & a & b & a & b \end{array} \right. \Rightarrow UD_1(bca) \left\{ \begin{array}{ccccc} & \hat{s}_2 & \hat{\hat{s}}_2 & s_2^{cab} & s_2^{cba} \\ \hat{s}_1 & a & b & c & c \\ \hat{\hat{s}}_1 & a & b & a & b \\ & a & & & \\ & \vdots & & & \end{array} \right.$$

$$\Rightarrow UD_1(bca) \left\{ \begin{array}{ccccc} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{UD_2(bac)}} & & \\ & \hat{s}_2 & \hat{\hat{s}}_2 & s_2^{cab} & s_2^{cba} \\ \hat{s}_1 & a & b & c & c \\ \hat{\hat{s}}_1 & a & b & a & b \\ & a & & & \\ & \vdots & & & \end{array} \right. \Rightarrow UD_1(bca) \left\{ \begin{array}{cccccc} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{UD_2(bac)}} & & & \\ & \hat{s}_2 & \hat{\hat{s}}_2 & s_2^{cab} & s_2^{cba} & s_2^{bca} \\ \hat{s}_1 & a & b & c & c & b \\ \hat{\hat{s}}_1 & a & b & a & b & b \\ & a & & & & \\ & \vdots & & & & \end{array} \right.$$

$$\Rightarrow UD_1(bca) \left\{ \begin{array}{cccccc} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{UD_2(bac)}} & & & \\ & \hat{s}_2 & \hat{\hat{s}}_2 & s_2^{cab} & s_2^{cba} & s_2^{bca} \\ \hat{s}_1 & a & b & c & c & b \\ \hat{\hat{s}}_1 & a & b & a & b & b \\ & a & c & c & c & c \\ & a & a & a & a & a \end{array} \right. \Rightarrow UD_1(bca) \left\{ \begin{array}{ccccc} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{UD_2(bac)}} & & \\ & \hat{s}_2 & \hat{\hat{s}}_2 & s_2^{cab} & s_2^{cba} \\ \hat{s}_1 & a & b & c & c \\ \hat{\hat{s}}_1 & a & b & a & b \\ & a & c & c & c \\ & a & a & a & a \end{array} \right.$$

Figure 3. There is a unique type 2 strategically simple mechanism (up to relabeling) in Case 1.

We begin our analysis of this case with the observation that with preference $\hat{R}_1$ agent 1 has only two undominated strategies.

Claim **3.** *$UD_1(\hat{R}_1)$ has exactly two elements, one, which we shall denote by $\hat{s}_1$, satisfies $M_2(\hat{s}_1) = \{a, b, c\}$, and the other one, which we shall denote by $\hat{\hat{s}}_1$, satisfies $M_2(\hat{\hat{s}}_1) = \{a, b\}$.*

Proof. Because agent 2 is a local dictator at $(\hat{R}_1, \hat{R}_2)$, every undominated strategy of agent 1 has to offer a menu that includes both a and b. There are only two such menus: $\{a, b, c\}$ and $\{a, b\}$. Because agent 1 has multiple undominated strategies and each such strategy by Corollary 2 has to offer a different menu, she has exactly two undominated strategies with one strategy offering menu $\{a, b, c\}$ and the other strategy offering menu $\{a, b\}$. □

Next, we investigate agent 2's strategy set, and for each of her strategies the outcome that results if agent 1 chooses $\hat{s}_1$ or $\hat{\hat{s}}_1$. Define:

$$S_2^a = \{s_2 \in S_2 : g(s_1, s_2) = a \text{ for all } s_1 \in UD_1(\hat{R}_1)\}, \text{ and}$$
$$S_2^b = \{s_2 \in S_2 : g(s_1, s_2) = b \text{ for all } s_1 \in UD_1(\hat{R}_1)\}.$$

Because, by assumption, in Case 1: $g(UD_1(\hat{R}_1), UD_2(\hat{R}_2)) = \{a, b\}$, and because, also by assumption, agent 2 is the local dictator at $(\hat{R}_1, \hat{R}_2)$, there must be at least one strategy in $UD_2(\hat{R}_2)$ that is in S_2^a, and also at least one strategy in $UD_2(\hat{R}_2)$ that is in S_2^b. Let us denote the former strategy by $\hat{s}_2$ and the latter by $\hat{\hat{s}}_2$. We also know that *all* strategies in $UD_2(\hat{R}_2)$ are contained in $S_2^a \cup S_2^b$. That is because agent 2 is the local dictator at $(\hat{R}_1, \hat{R}_2)$. The top left panel in Figure 3 represents, symbolically, what we have inferred so far about the mechanism that we are considering.

The focus of Claims 4 and 6 will be strategies of agent 2 that are *not* in $S_2^a \cup S_2^b$. We shall conclude that there are exactly two such strategies, and we shall show which outcomes they yield against $\hat{s}_1$ and $\hat{\hat{s}}_1$.

Claim **4.** *If $s_2 \in S_2 \backslash (S_2^a \cup S_2^b)$, then either:*

$$g(\hat{s}_1, s_2) = c \textit{ and } g(\hat{\hat{s}}_1, s_2) = a,$$

or

$$g(\hat{s}_1, s_2) = c \textit{ and } g(\hat{\hat{s}}_1, s_2) = b.$$

Proof. Recall that we have assumed that for every strategy of agent i there is some preference for which it is undominated. Suppose that R_2 ranks a top. Then part (2) of Lemma 2 implies that $UD_2(R_2) \subseteq S_2^a \cup S_2^b$. Analogously, if R_2 ranks b top, then $UD_2(R_2) \subseteq S_2^a \cup S_2^b$. By part (1) of Lemma 2, any $s_2 \in UD_2(cab)$ satisfies:

$$g(\hat{s}_1, s_2) = c \text{ and } g(\hat{\hat{s}}_1, s_2) = a,$$

and any $s_2 \in UD_2(cba)$ satisfies:

$$g(\hat{s}_1, s_2) = c \text{ and } g(\hat{\hat{s}}_1, s_2) = b.$$

□

Before we proceed with our analysis of agent 2's strategies, we observe that the conclusions of Claim 4 allows us to narrow down the set of possible candidates for the preference $\hat{R}_1$.

CLAIM **5.** *$\hat{R}_1$ is either acb or bca.*

Proof. f $\hat{R}_1$ ranks c top, then $\hat{s}_1$ would weakly dominate $\hat{\hat{s}}_1$, contradicting that $\hat{\hat{s}}_1 \in UD_1(\hat{R}_1)$. If $\hat{R}_1$ ranks c bottom, then $\hat{\hat{s}}_1$ would weakly dominate $\hat{s}_1$, contradicting that $\hat{s}_1 \in UD_1(\hat{R}_1)$. □

Without loss of generality, we assume that $\hat{R}_1 = bca$. We now return to our analysis of agent 2's strategy set.

CLAIM **6.** *There are exactly two strategies in S_2 that are not in $S_2^a \cup S_2^b$. One of these, which we shall denote by s_2^{cab}, satisfies*

$$g(\hat{s}_1, s_2^{cab}) = c \textit{ and } g(\hat{\hat{s}}_1, s_2^{cab}) = a,$$

and the other one, which we shall denote by s_2^{cba}, satisfies

$$g(\hat{s}_1, s_2^{cba}) = c \textit{ and } g(\hat{\hat{s}}_1, s_2^{cba}) = b.$$

Moreover, $UD_2(cab) = \{s_2^{cab}\}$ and $UD_2(cba) = \{s_2^{cba}\}$.

Proof. The argument in the proof of Claim 4 shows that it suffices to prove that $UD_2(cab)$ and $UD_2(cab)$ each have no more than one element. Without loss of generality we show this only for $UD_2(cab)$. Suppose that $UD_2(cab)$ had more than one element. By part (1) of Lemma 2, any $s_2 \in UD_2(cab)$ satisfies:

$$g(\hat{s}_1, s_2) = c \text{ and } g(\hat{\hat{s}}_1, s_2) = a.$$

Now consider the preference pair consisting of $\hat{R}_1$ and of cab. We could apply to this preference profile the same reasoning as we applied above to the preference profile $\hat{R}_1$ and $\hat{R}_2$, with the roles of agents 1 and 2 swapped. We could infer, as we did above in Claim 5, that agent 2's preference must be such that b is ranked in the middle. But this contradicts that agent 2's preference is cab. □

What we have inferred so far is symbolically represented by the middle panel in the top row of Figure 3. After we have pinned down the strategies that are not in $S_2^a \cup S_2^b$, we now return to the strategies of agent 2 that are in this set.

Claim **7.** *$s_2 \in S_2^a$ implies $g(s_1, s_2) = a$ for all $s_1 \in S_1$. Moreover, S_2^a has only one element, and $UD_2(abc) = UD_2(acb) = S_2^a$.*

Proof. The second sentence is an immediate implication of the first sentence, the assumption that there are no duplicate strategies, and the definition of weak dominance. For an indirect proof of the first sentence, suppose that for some $s_2 \in S_2^a$, we have $g(s_1, s_2) \neq a$ for some $s_1 \in S_1$. Then the preference $\hat{R}_1 = bca$ ranks $g(s_1, s_2)$ strictly above a. But then by Lemma 1, there would have to be a strategy $s_1' \in UD_1(\hat{R}_1)$ such that $g(s_1', s_2)$ is ranked above a. This contradicts that $s_2 \in S_2^a$. □

The right panel in the top row of Figure 3 symbolizes what we have concluded so far. Next, we can pin down the preference $\hat{R}_2$.

Claim **8.** *$\hat{R}_2 = bac$.*

Proof. It cannot be that $\hat{R}_2$ ranks a top, because then $\hat{s}_2$ would be a dominant strategy, and therefore would contradict with our assumption that $\hat{R}_2$ has at least two undominated strategies. It cannot be that $\hat{R}_2$ ranks a bottom, because then $\hat{s}_2$ would be weakly dominated. Finally, it cannot be that $\hat{R}_2$ ranks c top, because then, by Lemma 1, $UD_2(\hat{R}_2)$ would have to include a strategy that yields c against $\hat{s}_1$, which contradicts that $UD_2(\hat{R}_2) \subseteq S_2^a \cup S_2^b$. It follows that $\hat{R}_2 = bac$. □

Claim **9.** *$UD_2(bac) = \{\hat{s}_2, \hat{\hat{s}}_2\}$.*

Proof. From Claim 7, we know that $\hat{s}_2$ is the unique element in S_2^a. We now show that $M_1(s_2) = \{a, b, c\}$ for all $s_2 \in S_2^b \cap UD_2(\hat{R}_2)$. It then follows from Corollary 2 that $\hat{\hat{s}}_2$ is the unique element in $S_2^b \cap UD_2(\hat{R}_2)$. The claim follows since $UD_2(\hat{R}_2) \subseteq S_2^a \cup S_2^b$.

We proceed by elimination. It cannot be that $M_1(s_2) = \{b\}$, nor that $M_1(s_2) = \{a, b\}$, because in both cases $\hat{s}_2$ would be weakly dominated given $\hat{R}_2$. It remains to eliminate the possibility that $M_1(s_2) = \{b, c\}$.

Suppose that for some $s_2 \in S_2^b \cap UD_2(\hat{R}_2)$, $M_1(s_2) = \{b, c\}$. First consider the set of undominated strategies of agent 1 when she has preference acb. Part (1) of Lemma 2 implies that $g(s_1, s_2) = c$ for all $s_1 \in UD_1(acb)$. Next we consider agent 2 when he has preference cab. Recall from Claim 6 that agent 2 with this preference has a dominant strategy s_2^{cab}. We can then conclude that $g(s_1, s_2^{cab}) = c$ for all $s_1 \in UD_1(acb)$. But Lemma 1, combined with $g(\hat{\hat{s}}_1, s_2^{cab}) = a$, which we established in Claim 6, implies that there must exist some $s_1' \in UD_1(acb)$ such that $g(s_1', s_2^{cab}) = a$. We have thus obtained a contradiction, and the only remaining possibility is that $M_1(s_2) = \{a, b, c\}$ for all $s_2 \in S_2^b \cap UD_2(\hat{R}_2)$, which is what we wanted to show. □

By now, we know that agent 2, if he ranks a top, has a dominant strategy $\hat{s}_2$. We also know that for every preference that ranks c top, agent 2 has a dominant strategy, as described in Claim 6. Finally, we know that agent 2 with preference bac has two undominated strategies: $\hat{s}_2$ and $\hat{\hat{s}}_2$. The left panel in the middle row of Figure 3 symbolically represents what we have obtained so far. In the next step, we shall investigate agent 2's undominated strategies if he has preference bca.

Claim **10.** $|UD_2(bca)| = 1$.

Proof. We first show that $UD_2(bca) \subseteq S_2^b$. By part (2) of Lemma 2, and by the results that we have so far obtained for agent 2's strategy set, we have to have: $UD_2(bca) \subseteq S_2^a \cup S_2^b$. If there exists a strategy $s_2 \in UD_2(bca)$ but $s_2 \notin S_2^b$, then what we have established so far implies that it must be the strategy $\hat{s}_2$. But $\hat{s}_2$ is weakly dominated if agent 2 has preference bca. Therefore, we conclude $UD_2(bca) \subseteq S_2^b$.

Strategies in $UD_2(bca)$ cannot offer the menu $\{b\}$ or $\{a, b\}$, because then the strategy corresponding to this menu would weakly dominate $\hat{s}_2$ for agent 2 with preference bac, which contracts with Claim 8. Thus, strategies in $UD_2(bca)$ must either offer $\{b, c\}$ or $\{a, b, c\}$.

Suppose that $UD_2(bca)$ has at least two elements. Then Corollary 2 implies that there are exactly two strategies in $UD_2(bca)$, with one strategy offering the menu $\{b, c\}$ and the other strategy offering the menu $\{a, b, c\}$. In what follows, we show that this leads to a contradiction.

First consider agent 1 with preference acb. By part (1) of Lemma 2, each of her undominated strategies $s_1 \in UD_1(acb)$ must satisfy (1) $g(s_1, s_2) = c$ if $s_2 \in UD_2(bca)$ and $M_1(s_2) = \{b, c\}$; and (2) $g(s_1, s_2) = a$ if $s_2 \in UD_2(bca)$ and $M_1(s_2) = \{a, b, c\}$. Now consider agent 2 with preference cab. Claim 6 showed that agent 2 with this preference has a dominant strategy s_2^{cab}. Because the strategy is dominant, we have to have: $g(s_1, s_2^{cab}) = c$ for all $s_1 \in UD_1(acb)$. Claim 6 also showed that $g(\hat{\hat{s}}_1, s_2^{cab}) = a$. But Lemma 1 then implies that $g(s_1, s_2^{cab}) = a$ for at least one $s_1 \in UD_1(acb)$. We have found a contradiction. $\square$

Since $|UD_2(bca)| = 1$, agent 2 with preference bca also has a dominant strategy. We denote this strategy by s_2^{bca}. Our discussion of agent 2's strategy set so far says that agent 2 has either four (if $s_2^{bca} = \hat{\hat{s}}_2$) or five (if $s_2^{bca} \neq \hat{\hat{s}}_2$) strategies. We will resolve the question whether agent 2 has four or five strategies in the last step for Case 1. For the moment, we turn to agent 1's strategies.

Claim **11.** *For all* $s_1 \in S_1 \backslash UD_1(bca)$ *we have* $b \notin M_2(s_1)$.

Proof. The proof is indirect. Suppose that there exists some $s_1 \in S_1 \backslash UD_1(bca)$ such that $b \in M_2(s_1)$. We are going to show that s_1 is a duplicate of one of the strategies in $UD_1(bca)$, which contradicts our assumption that there are no duplicate strategies. We distinguish two cases. The first is that $M_2(s_1) = \{a, b\}$, and the second case is that $M_2(s_1) = \{a, b, c\}$. The arguments for the two cases are completely analogous. Therefore, here we only deal with the case that $M_2(s_1) = \{a, b\}$. Applying Lemma 1 to agent 2 with preference bac, we can conclude that $g(s_1, \hat{\hat{s}}_2) = b$. Because for all other preferences agent 2 has dominant strategies that we have already identified, we can conclude that:

$$g(s_1, \hat{s}_2) = g(s_1, s_2^{cab}) = a \text{ and } g(s_1, s_2^{cba}) = g(s_1, s_2^{bca}) = b.$$

This implies that s_1 is a duplicate strategy of $\hat{\hat{s}}_1$. □

This claim implies that strategies that are not in $UD_1(bca)$ must yield either a or c against any other strategy of agent 2. Let us focus on the alternative that they yield when agent 2 chooses $\hat{\hat{s}}_2$. The next two claims show that there is only one strategy outside of $UD_1(bca)$ that yields c against $\hat{\hat{s}}_2$, and also only one such strategy that yields a against $\hat{\hat{s}}_2$. This then implies that agent 1 has only four strategies, the two strategies in $UD_1(bca)$, and the two strategies not in $UD_1(bca)$.

Claim **12.** *There is a unique strategy $s_1 \in S_1 \backslash UD_1(bca)$ such that $g(s_1, \hat{\hat{s}}_2) = c$. Furthermore, for this strategy we have:*

$$g(s_1, s_2^{bca}) = g(s_1, s_2^{cab}) = g(s_1, s_2^{cba}) = c.$$

Proof. Recall that in the proof of Claim 9, we concluded that $M_1(\hat{\hat{s}}_2) = \{a, b, c\}$. This implies that there is at least one strategy s_1 such that $g(s_1, \hat{\hat{s}}_2) = c$. From Claims 7 and 11, we know that $M_2(s_1) = \{a, c\}$. Because we already know that agent 2 with preferences bca, cab, or cba has dominant strategies, we know that $g(s_1, s_2^{bca}) = g(s_1, s_2^{cab}) = g(s_1, s_2^{cba}) = c$. We have now pinned down for all strategies of agent 2 which outcome results if agent 1 chooses a strategy $s_1 \in S_1 \backslash UD_1(bca)$ such that $g(s_1, \hat{\hat{s}}_2) = c$. The uniqueness of such a strategy is therefore a consequence of the assumption that there are no duplicate strategies. □

Claim **13.** *There is a unique strategy s_1 such that $g(s_1, \hat{\hat{s}}_2) = a$. Furthermore, for this strategy we have:*

$$g(s_1, s_2^{bca}) = g(s_1, s_2^{cab}) = g(s_1, s_2^{cba}) = a.$$

Proof. Recall that in the proof of Claim 9, we concluded that $M_1(\hat{\hat{s}}_2) = \{a, b, c\}$. This implies that there is at least one strategy s_1 such that $g(s_1, \hat{\hat{s}}_2) = a$. From Claim 11 we can then infer that: $M_2(s_1)$ is either $\{a\}$ or $\{a, c\}$. For ease of notation, let:

$$S_1^a = \{s_1 \in S_1 : g(s_1, \hat{s}_2) = g(s_1, \hat{\hat{s}}_2) = a\}.$$

We first show that there exists at least one strategy $s_1 \in S_1^a$ that offers the menu $\{a\}$. The proof is indirect. Suppose that $M_2(s_1) = \{a, c\}$ for all $s_1 \in S_1^a$. We must have:

$$g(s_1, s_2^{bca}) = g(s_1, s_2^{cab}) = g(s_1, s_2^{cba}) = c$$

for all $s_1 \in S_1^a$. This is because all the strategies of agent 2 that we are referring to are dominant strategies. Because there are no duplicate strategies, we obtain that there is a unique element s_1 in S_1^a, and that for this strategy

$$g(s_1, s_2^{bca}) = g(s_1, s_2^{cab}) = g(s_1, s_2^{cba}) = c.$$

Now consider agent 1 who ranks a top. The unique element in S_1^a cannot be weakly dominated, because this is the only strategy that yields outcome a against strategy $\hat{\hat{s}}_2$. But since $g(\hat{\hat{s}}_1, s_2^{cab}) = a$, by Lemma 1, she must have another undominated strategy that yields a against s_2^{cab}. But then, if agent 1 has a preference that ranks a top, and agent 2 has preference bac, there is no local dictator. Thus we have obtained a contradiction.

Therefore, there must exist at least one strategy $s_1 \in S_1^a$ such that $M_2(s_1) = \{a\}$. Because there are no duplicate strategies, there can only be one such strategy. But now suppose there is also a strategy $s_1' \in S_1^a$ with $M_2(s_1) = \{a, c\}$. As before, it follows that

$$g(s_1', s_2^{cba}) = g(s_1', s_2^{bca}) = g(s_1', s_2^{cab}) = c.$$

But note that s_1' cannot be undominated for any preference, and we have ruled out that strategies that are not dominated for all preferences are included in the mechanism. The claim follows. $\square$

What we have found so far establishes that agent 1 has four strategies and agent 2 has either four (if $s_2^{bca} = \hat{\hat{s}}_2$) or five (if $s_2^{bca} \neq \hat{\hat{s}}_2$) strategies. Moreover, for any strategy combination, we know which outcome results. If agent 2 has five strategies, then the mechanism must take the form shown in the left panel in the bottom row of Figure 3. But note that in that panel $\hat{\hat{s}}_2$ and s_2^{bca} are duplicate strategies. Because we have assumed that there are no duplicate strategies, we can conclude that agent 2 has four strategies and the mechanism is the one shown in the right panel in the bottom row of Figure 3. This completes the proof for Case 1.

Case 2: For all preference profiles such that both agents have multiple undominated strategies, exactly one outcome may result if agents choose from the strategies that are undominated for these preference profiles.

Let us denote by $(\tilde{R}_1, \tilde{R}_2)$ a preference profile for which both agents have more than one undominated strategies. Without loss of generality, let us assume that $g(UD_1(\tilde{R}_1), UD_2(\tilde{R}_2)) = \{a\}$.

Figure 4 illustrates the proof for the second case. We shall refer to Figure 4 while presenting the proof. The left panel in the top row shows the starting point of the proof. We begin with an analysis of the sets $UD_i(\tilde{R}_i)$ for each agent and of the menus offered by the strategies in these sets.

$$UD_1(\tilde{R}_1)\left\{\begin{array}{ccc} \multicolumn{3}{c}{\overbrace{}^{UD_2(\tilde{R}_2)}} \\ a & \cdots & a \\ \vdots & & \vdots \\ a & \cdots & a \end{array}\right. \quad \Rightarrow \quad \begin{array}{c|cccc} & \tilde{s}_2 & \tilde{\tilde{s}}_2 & s_2^{cab} & s_2^{cba} \\ \hline UD_1(\tilde{R}_1)\{\ \tilde{s}_1 & a & a & c & c \\ \tilde{\tilde{s}}_1 & a & a & a & b \\ s_1^{cab} & c & a & & \\ s_1^{cba} & c & b & & \end{array} \quad (UD_2(\tilde{R}_2) = \{\tilde{s}_2, \tilde{\tilde{s}}_2\})$$

$$\Rightarrow \quad \begin{array}{c|ccccc} & \tilde{s}_2 & \tilde{\tilde{s}}_2 & s_2^{b} & s_2^{cab} & s_2^{cba} \\ \hline UD_1(acb)\{\ \tilde{s}_1 & a & a & b & c & c \\ \tilde{\tilde{s}}_1 & a & a & b & a & b \\ s_1^{b} & b & b & b & b & b \\ s_1^{cab} & c & a & b & & \\ s_1^{cba} & c & b & b & & \end{array} \quad \Rightarrow \quad \begin{array}{c|ccccc} & \tilde{s}_2 & \tilde{\tilde{s}}_2 & s_2^{b} & s_2^{cab} & s_2^{cba} \\ \hline UD_1(acb)\{\ \tilde{s}_1 & a & a & b & c & c \\ \tilde{\tilde{s}}_1 & a & a & b & a & b \\ s_1^{b} & b & b & b & b & b \\ s_1^{cab} & c & a & b & c & c \\ s_1^{cba} & c & b & b & c & c \end{array}$$

$UD_2(acb)$

Figure 4. There is a unique type 2 strategically simple mechanism (up to relabeling) in the second case.

Claim **14.** *If* $s_1 \in UD_1(\tilde{R}_1)$ *then* $M_2(s_1) \neq \{a\}$. *(The analogous statement for agent 2 can be proved in the same way.)*

Proof. The proof is indirect. Suppose that $M_2(s_1) = \{a\}$ for some $s_1 \in UD_1(\tilde{R}_1)$. Let s_1' be another element of $UD_1(\tilde{R}_1)$. First observe that $M_2(s_1')$ has to be $\{a, b, c\}$, because in all other cases, for every preference of agent 1, either s_1 weakly dominates s_1' or the other way round.

For both s_1 and s_1' to be undominated for agent 1 with preference $\tilde{R}_1$, it must be that $\tilde{R}_1$ ranks a in the middle. Without loss of generality, we assume that $\tilde{R}_1 = bac$. By Lemma 1, we conclude that $b \notin M_1(s_2)$ for any $s_2 \in UD_2(\tilde{R}_2)$.

Now let s_2 and s_2' denote two different elements of $UD_2(\tilde{R}_2)$. We just concluded that neither strategy offers a menu that includes b. By Corollary 1, they have to offer different menus, and therefore, without loss of generality, we can write that $M_1(s_2) = \{a\}$ and $M_1(s_2') = \{a, c\}$. But then there is no preference of agent 2 under which both s_2 and s_2' are undominated. □

Claim **15.** *The set $UD_1(\tilde{R}_1)$ has exactly two elements, say $\tilde{s}_1$ and $\tilde{\tilde{s}}_1$. Moreover, for one of these two strategies, say $\tilde{s}_1$, we have: $M_2(\tilde{s}_1) = \{a, b, c\}$. For the other strategy, either $M_2(\tilde{\tilde{s}}_1) = \{a, b\}$ or $M_2(\tilde{\tilde{s}}_1) = \{a, c\}$. (The analogous claim is true for agent 2.)*

Proof. The claim follows from Claim 14 and Corollary 2 once we rule out the case in which there are simultaneously a strategy in $UD_1(\tilde{R}_1)$ that offers menu $\{a, b\}$ and another strategy in $UD_1(\tilde{R}_1)$ that offers menu $\{a, c\}$. We prove indirectly that this cannot be the case.

Thus we assume that there is a strategy $s_1 \in UD_1(\tilde{R}_1)$ with $M_2(s_1) = \{a, b\}$ and another strategy $s_1' \in UD_1(\tilde{R}_1)$ with $M_2(s_1') = \{a, c\}$. By Lemma 1, it would have to be the case that for agent 2 with preference bca, there is an undominated strategy that yields b against s_1 and also an undominated strategy that yields c against s_1'. Therefore, we would conclude that $g(UD_1(\tilde{R}_1), UD_2(bca)) = 2$. By the definition of case 2, it has to be that $UD_2(bca)$ has just one element. In other words, agent 2 with preference bca has a dominant strategy s_2^{bca}. Using the same arguments as above, we can conclude that agent 2 with preference cba has a dominant strategy s_2^{cba}.

Now consider any two different strategies $s_2, s_2' \in UD_2(\tilde{R}_2)$. By assumption, both strategies' menus include a, and by Claim 14 cannot only include a. Therefore, at least one of these menus must contain exactly two elements, one of which is a. Without loss of generality let the other one be c. Thus, we consider: $M_1(s_2) = \{a, c\}$. By Corollary 2, s_2' has to offer a different menu, and this implies: $b \in M_1(s_2')$.

Using the same argument as in the second paragraph of the current proof, we can conclude that agent 1 with preference bca has a dominant strategy, say s_1^{bca}, and that $g(s_1^{bca}, s_2) = c$, and that $g(s_1^{bca}, s_2') = b$.

Now consider $g(s_1^{bca}, s_2^{cba})$. Because s_1^{bca} is a dominant strategy for agent 1 with preference bca, and because $g(s_1, s_2^{cba}) = b$, it follows that $g(s_1^{bca}, s_2^{cba}) = b$. But similarly, because s_2^{cba} is a dominant strategy for agent 2 with preference cba, and because $g(s_1^{bca}, s_2) = c$, it follows that $g(s_1^{bca}, s_2^{cba}) = c$. We have obtained a contradiction. □

Without loss of generality, we now assume that $M_2(\tilde{\tilde{s}}_1) = \{a, b\}$. Next, we show that, as a consequence, we have to have that $M_1(\tilde{\tilde{s}}_2) = \{a, b\}$.

Claim **16.** $M_1(\tilde{\tilde{s}}_2) = \{a, b\}$.

Proof. The proof is indirect. Suppose that $M_1(\tilde{\tilde{s}}_2) = \{a, c\}$. As in the proof of Claim 15, we can then infer that agent 1 with preference bca has a dominant strategy s_1^{bca}. Furthermore, $g(s_1^{bca}, \tilde{s}_2) = b$ and $g(s_1^{bca}, \tilde{\tilde{s}}_2) = c$. Similarly, the assumption that $M_2(\tilde{\tilde{s}}_1) = \{a, b\}$ implies that agent 2 with preference cba has a dominant strategy s_2^{cba}. Furthermore, $g(\tilde{s}_1, s_2^{cba}) = c$ and $g(\tilde{\tilde{s}}_1, s_2^{cba}) = b$. A contradiction is then reached as in the proof of Claim 15 by showing that $g(s_1^{bca}, s_2^{cba})$ has to be simultaneously b and c. □

Claim **17.** *Agent 1 with preference cab has a dominant strategy s_1^{cab}, and $g(s_1^{cab}, \tilde{s}_2) = c$ and $g(s_1^{cab}, \tilde{\tilde{s}}_2) = a$. Agent 1 with preference cba has a dominant strategy s_1^{cba}, and $g(s_1^{cba}, \tilde{s}_2) = c$ and $g(s_1^{cba}, \tilde{\tilde{s}}_2) = b$. (The analogous claims are true for agent 2.)*

Proof. This follows from the arguments used in the second paragraph of the proof of Claim 15. □

At this point we have a good understanding of the sets $UD_i(\tilde{R}_i)$ and of the menus offered by the strategies in these sets. What we have obtained so far is symbolically represented in the right panel in the top row in Figure 4. (Observe that the strategies s_i^{cab} and s_i^{cba} are not contained in $UD(\tilde{R}_i)$.)

Claim **18.** *If agent 1 ranks a top, then every undominated strategy s_1 of agent 1 satisfies $g(s_1, \tilde{s}_2) = g(s_1, \tilde{\tilde{s}}_2) = a$. If agent 1 ranks b top, then every undominated strategy s_1 of agent 1 satisfies $g(s_1, \tilde{s}_2) = g(s_1, \tilde{\tilde{s}}_2) = b$. (The analogous claims are true for agent 2.)*

Proof. This follows from part (2) of Lemma 2 and from the definition of Case 2. □

Claim **19.** $\tilde{R}_1 = \tilde{R}_2 = acb$.

Proof. Claims 17 and 18 show that an agent with multiple undominated strategies must rank a top. This leaves just two possible preferences: abc and acb. But if $\tilde{R}_1 = abc$, then clearly $\tilde{\tilde{s}}_1$ would weakly dominate $\tilde{s}_1$. □

Claim **20.** *There is a unique strategy, say s_1^b, such that, if agent 1 ranks b top, then this strategy is dominant. Moreover, $g(s_1^b, s_2) = b$ for all $s_2 \in S_2$. (The analogous statement is true for agent 2.)*

Proof. By Claim 18, if agent 1 ranks b top, every undominated strategy s_1 of agent 1 satisfies: $g(s_1, \tilde{s}_2) = g(s_1, \tilde{\tilde{s}}_2) = b$. Claim 19 showed that $\tilde{R}_2 = acb$, which ranks b bottom. Therefore, by Lemma 1, we have to have that $g(s_1, s_2) = b$ for all $s_2 \in S_2$. There can only be one such strategy, because there are no duplicate strategies. Moreover, this strategy is dominant whenever agent 1 ranks b top. $\square$

The left panel in the bottom row of Figure 4 shows what we have inferred so far about the mechanism.

Claim **21.** *For agent 1 with preference abc, strategy $\tilde{\tilde{s}}_1$ is dominant. (The analogous statement is true for agent 2.)*

Proof. Consider agent 1 with preference abc. Whenever agent 2's strategy is undominated for a preference that puts a top, then, by Claim 18, if agent 1 chooses $\tilde{\tilde{s}}_1$, the outcome is a, which is agent 1's most preferred outcome. If agent 2's strategy is undominated for a preference that puts b at the top, by Claim 20, all strategies of agent 1 yield the same outcome b. Finally, if agent 2 chooses an undominated strategy for preference cab, then, by Claim 17, the outcome that results if agent 1 chooses $\tilde{\tilde{s}}_1$ is a.

The only remaining case is that agent 2 has preference cba and chooses his dominant strategy s_2^{cba}. By Claim 17, $g(\tilde{\tilde{s}}_1, s_2^{cba}) = b$. Thus we have to show that $a \notin M_1(s_2^{cba})$. If $a \in M_1(s_2^{cba})$, by Lemma 1, there would have to be an undominated strategy of agent 1 with preference acb that yields a against s_2^{cba}. In Claim 17 we showed that no such strategy exists. $\square$

We can now wrap up the analysis of the second case. For five of the six possible preferences of each agent, we have established that they have dominant strategies. Moreover, for agents with preference acb, we have established that they have only two undominated strategies. Moreover, the dominant strategy of agents with preference abc is one of the undominated strategies of agents with preference acb, and agents with preferences that put b top have the same dominant strategy. A short calculation reveals that every agent has exactly 5 strategies. The results that we have obtained so far show for most strategy combinations which outcome results. What remains to be filled in is are the outcomes that result when both agents choose their strategies s_i^{cab} and s_i^{cba}. But because these are dominant strategies, and because we already know that each agent has a strategy available that achieves outcome c against these two strategies of the other agent, it must be that:

$$g(s_1^{cab}, s_2^{cab}) = g(s_1^{cab}, s_2^{cba}) = g(s_1^{cba}, s_2^{cab}) = g(s_1^{cba}, s_2^{cba}) = c.$$

Thus, there is a unique type 2 strategically simple mechanism (up to relabeling) in the second case as shown in the right panel in the bottom row of Figure 4.