Online Appendix for *Asset Pricing with Endogenously Uninsurable Tail Risk*

A Additional details for restrictions on the space of contracts

In this section, we show that given the equilibrium we construct in the main text, there are no incentives for firms to offer any insurance to workers that are not currently matched with the firm. We build the argument in several steps. We first show that firms do not have incentives to offer any unemployment insurance to workers after separation. Then we show that the restricted employment contract that we construct in the main text of the paper is in fact optimal in a larger contracting space where all firms are allowed to offer insurance to all workers.

A.1 Insurance provision to unemployed workers

First, consider an optimal contracting problem of a firm that offers payments $\left\{\widetilde{C}_{t+s}\right\}_{s=0}^{\infty}$ to an unemployed worker subject to two-sided limited commitment. Let $\widetilde{V}(U,y,S)$ be the value of the insurance contract to a firm as a function of worker output y, promised utility U for a given aggregate state S. Following the steps in the main text, the above contracting problem can be expressed as

$$\widetilde{V}(U,y,S) = max_{\widetilde{C},\left\{\widetilde{U}'(g')\right\}_{g'}}\left\{-\widetilde{C}+\kappa\sum_{g'}\pi\left(g'|g\right)\Lambda\left(S',S\right)(1-\chi)\widetilde{V}\left(\widetilde{U}'\left(g'\right),\lambda y,S'\right)\right\}$$

subject to

$$\left\{(1-\beta)\left[by+\widetilde{C}\right]^{1-\frac{1}{\psi}}+\beta\left(\kappa\sum_{g'}\pi\left(g'|g\right)\left[(1-\chi)\widetilde{U}'\left(g'\right)^{1-\gamma}+\chi\left(u^{*}\left(S'\right)\lambda y\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right)^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}}\geq U, \tag{36}$$

$$\widetilde{U}'\left(g'\right)\geq\bar{u}\left(S'\right)\lambda y,\ \ for\ \ all\ \ g', \tag{37}$$

$$\widetilde{V}\left(\widetilde{U}'\left(g'\right),\lambda y,S'\right)\geq 0,\ \ for\ \ all\ \ g'. \tag{38}$$

where functions $u^{*}(S)$ and $\bar{u}(S)$ are defined in equations (13) and equation (12) respectively.

The optimal contract chooses the current period payment to the worker $\widetilde{C}$ and a menu of continuation utilities $\left\{\widetilde{U}'\left(g'\right)\right\}_{g'}$ to maximize the net present value of the contract to the firm. The human capital of an unemployed worker depreciates deterministically at rate $1-\lambda$; therefore, in the absence of idiosyncratic shocks, the continuation utility is only a function of aggregate shock g'. To understand the expression for the continuation payoff, note that in the next period, with probability χ, the worker stay unemployed, in which case the value of the contract is $\widetilde{V}\left(\widetilde{U}'(g'), \lambda y, S'\right)$. With probability $1-\chi$, the worker receives an opportunity to match with a firm, which can be a different firm or the same firm who is providing the insurance. In either case, because of competition, the value of the continuation contract after the worker finds an employment opportunity give the worker a continuation value of $u^*(S)\lambda y$ and a value of zero to the firm.

Inequality (36) is the promise keeping constraint. The worker receives by as unemployment benefit and a transfer of $\widetilde{C}$ from the insurance firm. In the next period, with probability $1-\chi$, the worker stays unemployed and receives promised utility $\widetilde{U}'(g')$. With probability χ, the worker has an opportunity to match with another firm and receives $u^*(S')\lambda y$. Inequality (37) is the limited commitment constraint for workers: promised utility under the insurance contract has to be higher than the utility associated with consuming unemployment benefit as workers always have an option to default on the contract offered by the insurance firm and to consume the unemployment benefit thereafter. Because workers' human capital depreciate at rate $1-\lambda$, the utility associated with consuming unemployment benefit is $\bar{u}(S')\lambda y$ in the next period. Finally, inequality (37) is the limited commitment constraint for the firm which requires the net present value of the insurance contract to be non-negative for the firm.

The lemma below provides a sufficient condition for the absence of unemployment insurance offered by firms.

Lemma 1. *Suppose λ is small enough, in particular, for all S and S',*

$$\lambda \leq \left[\frac{x(S')}{x(S)}\right]\left[\frac{w(S')}{n(S)}\right]^{\psi\gamma-1}. \tag{39}$$

Then, at $U = \bar{u}(S)y$, we must have $\widetilde{C}(U,y,S) = 0$, $\widetilde{U}'(U,y,S,g') = \bar{u}(S')\lambda y$, and $\widetilde{V}(U,y,S) = 0$.

Proof. As in the main text, the optimal contracting problem can be normalized. Homogeneity of the problem implies $\widetilde{V}(U,y,S) = \widetilde{v}(u,S)y$ for some $\widetilde{v}$, where $u = \frac{U}{y}$. Using normalized value and policy functions, we can write the optimal contracting problem in the normalized form. Define the T operator as

$$T\widetilde{v}\left(u,S\right)=max_{c,\left\{u'\left(g'\right)\right\}_{g'}}\left\{-\widetilde{c}+\kappa\sum_{g'}\pi\left(g'|g\right)\Lambda\left(S',S\right)\chi e^{g'}\widetilde{v}\left(u'\left(g'\right),S'\right)\right\} \tag{40}$$

$$s.t.\left\{(1-\beta)\left[b+\widetilde{c}\right]^{1-\frac{1}{\psi}}+\beta\left(\lambda m\right)^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-1/\psi}}\geq u \tag{41}$$

$$m=\left\{\kappa\sum_{g'}\pi\left(g'|g\right)\left[(1-\chi)\,u'\left(g'\right)^{1-\gamma}+\chi u^{*}\left(S'\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right\} \tag{42}$$

$$u'\left(g'\right)\geq\bar{u}\left(S'\right),for\quad all\quad g' \tag{43}$$

$$\widetilde{v}\left(u'\left(g'\right),S'\right)\geq 0,for\quad all\quad g'. \tag{44}$$

Under standard discounting assumptions, T is a contraction on the set of bounded continuous functions and $\widetilde{v}\left(u,S\right)$ is the unique fixed point of the T operator. The conclusion of the above lemma is therefore equivalent to the following property of the normalized optimal contracting problem, that is, for $u=\bar{u}\left(S\right)$,

$$\widetilde{c}\left(u,S\right)=0;\;u'\left(u,S,g'\right)=\bar{u}\left(S'\left(g'\right)\right)\;\;for\;all\;g';\;and\;\widetilde{v}\left(u,S\right)=0. \tag{45}$$

Consider the constrained maximization problem (40). Let μ be the Lagrangian multiplier for the constraint (41). The first order conditions are:

$$1=\mu\left(1-\beta\right)\left(\frac{c}{u}\right)^{-\frac{1}{\psi}},$$

$$\Lambda\left(S',S\right)\frac{d}{du}\widetilde{v}\left(u'\left(g'\right),S'\right)+\mu\beta e^{-\gamma g'}\left(\frac{\lambda m}{u}\right)^{-\frac{1}{\psi}}\left(\frac{u'\left(g'\right)}{m}\right)^{-\gamma}\geq 0,\;\;for\;all\;g' \tag{46}$$

and "=" holds if $\widetilde{v}\left(u'\left(g'\right)|S'\right)>0$. The envelope condition implies $\frac{d}{du}\widetilde{v}\left(u,S\right)=\mu$. Combining the above conditions, and using the expression for the stochastic discount factor in (20), the optimality condition (46) can be written as

$$\left[\frac{x\left(S'\right)}{x\left(S\right)}\right]^{-\frac{1}{\psi}}\left[\frac{w\left(S'\right)}{n\left(S\right)}\right]^{\frac{1}{\psi}-\gamma}\leq\lambda^{-\frac{1}{\psi}}\left[\frac{b+\widetilde{c}\left(\bar{u}\left(g'\right),S'\right)}{b+\widetilde{c}\left(u,S\right)}\right]^{-\frac{1}{\psi}}\left[\frac{u'\left(g'\right)}{m}\right]^{\frac{1}{\psi}-\gamma}, \tag{47}$$

and "=" holds if $\widetilde{v}\left(u'\left(g'\right)|S'\right)>0$. Because (44) is a standard convex programming problem, (47) is both necessary and sufficient for optimality.

To prove (45), note that set of functions that are concave in its first argument and satisfy $\widetilde{v}\left(\bar{u}\left(S\right),S\right)=0$ is a closed subset in the set of bounded continuous functions. To prove that the unique fixed of T satisfies $\widetilde{v}\left(\bar{u}\left(S\right),S\right)=0$, we start by assuming $\widetilde{v}\left(u,S\right)$ is concave in the first argument and satisfies $\widetilde{v}(\bar{u}(S), S) = 0$, and we need to show that $T\widetilde{v}$ satisfies the same properties.

Because $\widetilde{v}(u, S)$ is concave in its first argument, condition (47) together with the promise keeping constraint (41) are sufficient for optimality. Under assumption (39), the proposed policy functions in (45) satisfy the first order condition (47). In addition, the promise keeping constraint is satisfied by the definition of $\bar{u}(S)$ in (13). Therefore, the policy functions (45) are optimal. Clearly, under the proposed policy functions, $T\widetilde{v}(\bar{u}(S), S) = 0$. The fact that $T\widetilde{v}(\bar{u}(S), S)$ must be concave follows from standard argument (see, for example, Ai and Li (2015)). □

The above lemma has two implications. First, under condition (39), in equilibrium, a firm cannot earn a positive profit by offering a non-trivial insurance contract to any unemployed worker. To see this, note that the value function $\widetilde{V}(U, y, S)$ must be strictly decreasing in U. Because the utility provided by the unemployment benefit is the lower bound of the utility that an unemployed worker can achieve, we must have $U \geq \bar{u}(s) y$ under any non-trivial insurance contract. Therefore, $\widetilde{V}(U, y, S) \leq 0$, i.e. no firm can make a positive profit by deviating from the trivial insurance contract.

Second, employer firms cannot offer any severance pay to a worker upon unemployment. To see this, consider an augmented contract space $\mathscr{C} \cup \widetilde{\mathscr{C}}$ with $\widetilde{\mathscr{C}}$ specifying payments to worker after separation. From the history at which the worker is unemployed, the firm's value under any contract with non-trivial payment to the worker cannot exceed $\widetilde{V}(U, y, S)$ defined in (38). An augmented contract with non-trivial severance pay will give unemployed workers a value higher than the autarky value of consuming the unemployment benefits $\bar{u}(S') y'$. Thus by the same argument as in the previous paragraph, any such arrangement will imply $\widetilde{V}(U, y, S) < 0$, which violates the firm-side limited commitment.

Intuitively, it is the joint assumption of two-sided limited commitment and perfect competition on firm side that rule out unemployment insurance in equilibrium. The income of unemployed workers are front loaded. In our model, as human capital depreciates, so does the unemployment benefit. To provide any non-trivial intertemporal consumption smoothing to unemployed workers, a firm would need to backload its payment. The limited commitment on firm side ($\widetilde{v}(\bar{u}(S), S) \geq 0$) implies that firms cannot commit to backloaded payments unless they can expect some profit in the future. However, there is no profit to be made in an insurance contract with an unemployed worker: the worker will need continued payment as long as he is unemployed; once the worker is employed, limited commitment on worker side and perfect competition between firms mean that the worker will extract all surplus in the new match and cannot commit to pay back the unemployment insurance provider. The fact that workers extract all surplus in a new employment contract is the key feature of our model that rules out equilibrium private unemployment insurance.

Finally, from a quantitative point of view, condition (39) is a fairly weak assumption on λ. In an economy without aggregate risk, it is equivalent to $\lambda \leq 1$. In our calibration, $\lambda = 0.96$ at the quarterly level, and (39) is certainty satisfied.

A.2 Insurance provision to other workers

Here we show that the employment contract that we construct in the main text of the paper is in fact optimal in a larger contracting space where all firms are allowed to offer insurance to all workers. To do so, we follow several steps. In step 1, we describe a dynamic game in which firms compete for workers by offering long-term contracts where all firms are allowed to pay all workers subject to incentive compatibility. In step 2, we describe an equilibrium strategy in the above game where contracts only involve non-trivial payments from firms to their employees. In step 3, we show that the proposed contract is optimal in a Subgame Perfect Nash Equilibrium (SPNE) of the game.

Step 1: Here, we describe a game where all firms are allowed to offer contracts to all workers. We first introduce some terminologies and notations. We define $\{\iota_{i,t}\}_{t=0}^{\infty}$ to be the stochastic process that records the birth, death, and unemployment shocks experienced by worker i.[25] In addition, upon receiving an opportunity to match, a worker randomizes among all firms that offer the most favorable contract. We use $\upsilon_{i,t}$ to denote the outcome of the randomization device, with $\upsilon_{i,t} = j$ if firm j is chosen by worker i in period t. We use $\zeta_{i,j}(t) = (g_t, \eta_{j,t}, \varepsilon_{i,t}, \iota_{i,t}, \upsilon_{i,t})$ to denote time t shocks for a firm-worker pair and $\zeta_{i,j}^t = \left(g^t, \eta_j^t, \varepsilon_i^t, \iota_i^t, \upsilon_i^t\right) = \{g_s, \eta_{j,s}, \varepsilon_{i,s}, \iota_{i,s}, \upsilon_{i,s}\}_{s=0}^{t}$ to denote the history of shocks of a firm-worker pair up to time t.[26] Because all workers are endowed with one unit of human capital at birth, given the history of shocks, $\zeta_{i,j}^t$, we can recover $h_{i,t}$ and $y_{i,t}$ for all t using equations (1) and (2), which we denote as $h_t(\zeta_{i,j}^t)$.

A contract offered by firm j to worker i specifies the net transfers from the firm to the

[25] To keep the convention that $\{\iota_{i,t}\}_{t=0}^{\infty}$ are exogenous shocks not influenced by agents' actions, we can assume that they are i.i.d. random variables uniformly distributed on $[0,1]$. If worker i is employed in period t, the outcome of the match with the employer firm is decribed by $I_{\{\iota_{i,t+1} \leq \theta_{i,j,t}\}}$, where I is the indicator function. That is, $I_{\{\iota_{i,t+1} \leq \theta_{i,j,t}\}} = 0$ if the worker separates from the current firm and become unemployed in period $t+1$ and $I_{\{\iota_{i,t+1} \leq \theta_{i,j,t}\}} = 1$ if the worker continues the match with his current employer in period $t+1$. Consistent with the setup of our model, the probability of the survival of the match is $\theta_{i,j,t}$. Similarly, for worker i who is unemployed in period t, $I_{\{\iota_{i,t+1} \leq 1-\chi\}} = 0$ if the worker continue to stay unemployed in period $t+1$,and $I_{\{\iota_{i,t+1} \leq 1-\chi\}} = 1$ if the worker receives an employment opportunity to match with a firm in period $t+1$.

[26] The usage of ζ is consistent with the main text in the sense that it is a vector of aggregate and idiosyncratic shocks in period t. However, unlike in the main text of the paper, here, worker i and firm j do not necessarily have an employment relationship. In addition, the shock structure in this more general setup is richer — for example, it contains unemployment shocks, $\iota_{i,t}$ — in order to allow for a larger contracting space.

worker and the retention effort , $\mathscr{C}_{i,j} \equiv \left\{C_{i,j,t}\left(\zeta_{i,j}^t\right), \theta_{i,j,t}\left(\zeta_{i,j}^t\right)\right\}_{t=0}^{\infty}$, as functions of the history of shocks.[27] Clearly, our setup implies that $\theta_{i,j,t}\left(\zeta_{i,j}^t\right) = 0$ unless worker i is matched with firm j at history $\zeta_{i,j}^t$. We use $\boldsymbol{\mathscr{C}} = \langle \mathscr{C}_{i,j} \rangle_{i,j}$ as the collection of contracts offered by all firms to all workers. We suppress the decision for keeping the firm-worker match $\delta_{i,j,t}$ and assume from the outset that a firm-worker match is never voluntarily separated. As we have shown in proposition 1 in the main text of the paper, this is without loss of generality.

We consider a repeated game in which in each period t, all firms offer longer term contracts to all workers and denote a contract offered by firm j as $\langle \mathscr{C}_{i,j} \rangle_i$. After all firms make offers, workers make decisions on which contract(s) to accept. A worker is free to default on previous contracts at any history. Default on the contract offered by the employer firm results in a separation of the match and termination of all future cash transfers. Default on a contract offered by an unrelated firm results in termination of all future cash flows and nothing else. In addition, a worker who has an employment opportunity can choose the firm that offers the most attractive employment contract to match. If indifferent, he randomizes. Finally, firms are free to default on their contract at any point in time. A default on the firm side results in a separation of the match (if this is a contract with an employee) and termination of all future cash transfers.

In a typical period t, given the action of all firms, $\boldsymbol{\mathscr{C}}$, a firm's payoff is calculated as the present value of all cash flows generated by contracts with all workers. A worker's payoff is the present value of utility that the worker receives under all accepted contracts with all firms.

Step 2: Here we construct a SPNE of the game specified above using the optimal employment contract that we describe in the main text of the paper. To decribe an equilibrium contract, we can without loss of generality focus on contracts that will be accepted at all times and all histories, because if part of the contract is not accepted at some history in equilibrium, we can simply rename the contract so that it prescribes zero transfer between the firm and the worker after that history. As a result, even though firms are allowed to offer a different contract in every period, in the construction of the SPNE of the game, we can focus on the case where the same contract (that will be accepted at all future histories and all times) is offered in every period.

In every period, firms offer contracts to three different types of workers: workers employed by them, workers employed by other firms, and unemployed workers. To decribe an SPNE strategy, we first specify the contract offered to an employee based on the optimal employment contract we describe in the paper. Given the pricing kernel $\Lambda(S', S)$ and the equilibrium value of workers with a job opportunity, $yu^*(S)$, firms' value function defined in (11) is the unique

[27] Here, we use the same notation $\mathscr{C}_{i,j}$ as in the main text to denote contracts in a larger space.

fixed point of the following T operator:

$$TV(U,y,S)=max_{C,\theta,\{U'(\zeta')\}}\left\{(y-C)+\kappa\theta\int\Lambda(S',S)\,V\left(U'(\zeta'),ye^{g'+\eta'+\varepsilon'},S'\right)\Omega\left(d\zeta'|g\right)\right\}$$

$$s.t.\left\{(1-\beta)\,C^{1-\frac{1}{\psi}}+\beta\left(\kappa\int\left[\theta U'(\zeta')^{1-\gamma}+(1-\theta)\left[\bar{u}(S')\,ye^{g'+\eta'+\varepsilon'}\right]^{1-\gamma}\right]\Omega(d\zeta'|g)\right)^{\frac{1}{1-\gamma}}\right\}\geq U$$

$$U'(\zeta')\geq\bar{u}(S')\,ye^{g'+\eta'+\varepsilon'} \tag{48}$$

$$V\left(U'(\zeta'),ye^{g'+\eta'+\varepsilon'},S'\right)\geq 0. \tag{49}$$

We denote the policy functions associated with the above dynamic programming program as $C(U,y,S)$ and $\{U'(U,y,S,\zeta')\}_{\zeta'}$.

As is standard in the dynamic contracting literature, in any period t, given a vector of initial state variables, (U,y,S), the continuation contract from period t that specifies payment to employed workers in all future dates and states can be constructed recursively from the policy functions of the above dynamic contracting problem. We denote an employment contract with initial condition (U,y,S) as $\mathscr{C}(U,y,S)$. To specify the continuation contract from any history $\zeta^t_{i,j}$, we only need a procedure to construct the initial state variables $\left(U_{i,t}\left(\zeta^t_{i,j}\right),y_{i,t}\left(\zeta^t_{i,j}\right),S_t\left(g^t\right)\right)$ at that history. The construction of the exogenous state variables $y_{i,t}\left(\zeta^t_{i,j}\right)$ and $S_t\left(g^t\right)$ are straightforward and are described in the main text of the paper. We use the following procedure to construct the promised uitility at history $\zeta^t_{i,j}$. Let $\zeta^\tau_{i,j}<\zeta^t_{i,j}$ be the closest history that preceeds $\zeta^t_{i,j}$ such that at $\zeta^\tau_{i,j}$, the worker has an employment opportunity to match with a firm. Set $U_{i,\tau}\left(\zeta^\tau_{i,j}\right)=u^*(S_\tau)\,y_\tau\left(\zeta^\tau_{i,j}\right)$. Given this initial promised utility at history $\zeta^\tau_{i,j}$, we use the history of shocks between $\zeta^\tau_{i,j}$ and $\zeta^t_{i,j}$ and the policy function from (49) to construct the promised utility at $\zeta^t_{i,j}$, $U_{i,t}\left(\zeta^t_{i,j}\right)$. Below is our proposed SPNE strategy.

- Offer the contract $\mathscr{C}\left(U_{i,t}\left(\zeta^t_{i,j}\right),y_{i,t}\left(\zeta^t_{i,j}\right),S_t\left(g^t\right)\right)$ to worker i if worker i is currently an employee.
- Promise to offer $\mathscr{C}\left(U_{i,\tau}\left(\zeta^\tau_{i,j}\right),y_{i,\tau}\left(\zeta^\tau_{i,j}\right),S_\tau\left(g^\tau\right)\right)$ at any future history $\zeta^\tau_{i,j}$ where the worker has an employment opportunity, where $U_{i,\tau}\left(\zeta^\tau_{i,j}\right)=u^*(S_\tau)\,y_\tau\left(\zeta^\tau_{i,j}\right)$.
- Offer a trivial contract, that is, a contract with zero transfers between the firm and the worker at all future contingencies, if worker i is not currently anemployee. Here the worker can either be unemployed or employed by another firm.

With a slight abuse of terminology, we will call the above contracts employment contracts.[28]

[28]The notion of employment contract here is the same as the one defined in the main text of the paper but

Step 3: Here we provide a formal proof that the above described employment contracts constitute an SPNE in the game we describe in Step 1. We first summarize our results in the following lemma.

Lemma 2. *The employment contracts described above is an SPNE.*

Proof. Because at any history $\zeta_{i,j}^{t}$, a worker can either be employed by a firm, or unemployed but have an employment opportunity to match with a firm, or unemployed and do not have an employment opportuntiy in the current period, to establish that the employment contract is an SPNE, we need to show that given all other firms' strategy, none of the following deviations can yield a higher profit for the firm without violating any of the incentive compatibility constraints: (i) a different contract to an employed worker; (ii) a different contract to a worker who is employed by another firm; (iii) a different contract to a worker who is previously unemployed but has an employment opportunity in the current period; (iv) a different contract to an unemployed worker who remains unemployed in the current period; (v) a combination of the above.

First, because the employment contact solves the optimal contracting problem (49), firms cannot obtain a higher profit by offering a different contract to an employee.

Second, no firm can obtain a higher profit by offering a non-trivial insurance contract to a workers who is currently working for another firm. We prove this claim by contradiction. Suppose at $\zeta_{i,j}^{t}$, a firm has a profitable deviation by offering a nontrivial insurance contract to a worker who is currently employed by another firm, the policy functions and the associated value functions for the insurance contract must solve the following optimal contracting problem:

$$\widetilde{V}\left(U,y,S\right)=max_{\widetilde{C},\left\{\widetilde{U}'(\zeta')\right\}_{z'}}\left\{-\widetilde{C}+\kappa\int\Lambda\left(S',S\right)\widetilde{V}\left(\widetilde{U}'\left(\zeta'\right),y'\left(\zeta'\right)|S'\right)\Omega\left(d\zeta'|S\right)\right\}$$

$$subject\ to\left\{(1-\beta)\left[C\left(U,y\right)+\widetilde{C}\right]^{1-\frac{1}{\psi}}+\beta\left(\mathbb{E}\left[\widetilde{U}'\left(\zeta'\right)^{1-\gamma}|S\right]\right)^{\frac{1}{1-\gamma}}\right\}^{1-\frac{1}{\psi}}\geq U, \tag{50}$$

$$\widetilde{U}'\left(\zeta'\right)\geq U'\left(U,y,\zeta',S'\right), \tag{51}$$

$$\widetilde{V}\left(\widetilde{U}'\left(\zeta'\right),y'\left(\zeta'\right)|S'\right)\geq 0. \tag{52}$$

where $C\left(U,y\right)$ and $\left\{U'\left(U,y,\zeta',S'\right)\right\}$ are the policy functions of the optimal contracting problem in (49). In the objective function of the optimal contracting problem, (52), $\widetilde{C}$ is the net payment from the firm to the unrelated worker. Inequality (50) is the promise keep constraint. If the worker accept the contract, his utility is given by

extended to a larger contracting space that allows the specification of payment between firms and unrelated workers.

$\left\{(1-\beta)\left[C(U,y)+\widetilde{C}\right]^{1-\frac{1}{\psi}}+\beta\left(\mathbb{E}\left[\widetilde{U}'(\zeta')\,|S\right]\right)^{\frac{1}{1-\gamma}}\right\}$, where the current period consumption includes the payment from the current employer, $C(U,y)$ as well as the transfer from the unrelated firm, $\widetilde{C}$. Equation (51) is the limited commitment constraint for the worker. Because the worker can always default on the contract offered by the unrelated firm and obtain the utility under the employment contract $U'(U,y,\zeta',S')$, in order to prevent the worker from default, the promised utility for the next period, $\widetilde{U}'(\zeta')$, must be at least as high as what the worker can obtain under the employment contract, $U'(U,y,\zeta',S')$. Inequality (52) in the firm-side limited commitment constraint. Because the insurance contract is a profitable deviation, we must have $\widetilde{V}(U,y,S)>0$. To arrive at a contradiction, we define $\breve{C}(U,y,S)=C(U,y,S)+\widetilde{C}(U,y,S)$ and $\breve{U}(U,y,S,\zeta')=\widetilde{U}(U,y,S,\zeta')$. Note that $\left\{\breve{C}(U,y,S),\left[\breve{U}(U,y,S,\zeta')\right]\right\}$ is a feasible policy for (49) with $\breve{V}(U,y,S)=V(U,y,S)+\widetilde{V}(U,y,S)$ as the value function. However, $\breve{V}(U,y,S)>V(U,y,S)$, which contracts $V(U,y,S)$ being the optimal solution to (49).

Third, given that all firms offer the optimal contract that provides the highest utility to workers when worker obtains an employment opportunity, no firm can make a higher profit by deviating from this strategy. Fourth, as we show in lemma 1, firms cannot make a positive profit by offering a non-trivial insurance contract to unemployed workers. Finally, combining all of the above arguments, it is clear that a combinations of deviations will not be profitable either. □

B Proof for propositions 1 and 2

B.1 Characterization of equilibrium

In this section, to prepare for the proofs for propositions 1, and 2, we provide a set of necessary and sufficient conditions that characterize the equilibrium. We first state a lemma that establishes that the equality constraint (18) can be replaced by an inequality constraint so that the optimal contracting problem *P1* is a standard convex programming problem.

Lemma 3. *Suppose $A'(\theta)$, $A''(\theta)$, and $A'''(\theta)>0$ for all $\theta\in(0,1)$. The policy functions for the optimal contracting problem P1 in the main text can be constructed from the solution* *to the a convex programming problem described below*

$$v(u,S)=\max_{c,\theta,\{u'(\zeta'),\delta'(\zeta')\}_{\zeta'}}\left\{\begin{array}{c}1-c-A(\theta)+\\ \kappa\theta\int\Lambda(S',S)\,e^{g'+\eta'+\varepsilon'}\left[\delta'(\zeta')\,v(u'(\zeta'),S')\right]\Omega(d\zeta'|g),\end{array}\right\} \tag{53}$$

$$s.t:\quad u\leq\left[(1-\beta)\,c^{1-\frac{1}{\psi}}+\beta m^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}, \tag{54}$$

$$\delta'(\zeta')\,v\left(u'(\zeta'),S'\right)\geq 0,\ \ for\ \ all\ \ \zeta', \tag{55}$$

$$\delta'(\zeta')\left[u'(\zeta')-\lambda\overline{u}(S')\right]\geq 0,\ \ for\ \ all\ \ \zeta', \tag{56}$$

$$A'(\theta)\leq\kappa\int\Lambda(S',S)\,e^{g'+\eta'+\varepsilon'}\delta'(\zeta')\,v\left(u'(\zeta'),S'\right)\Omega(d\zeta'|g), \tag{57}$$

where $m=\left\{\kappa\int e^{(1-\gamma)(g'+\eta'+\varepsilon')}\left[\theta\delta'(\zeta')\,u'(\zeta')^{1-\gamma}+(1-\theta\delta'(\zeta'))\,\lambda\overline{u}(S')^{1-\gamma}\right]\Omega(d\zeta'|g)\right\}^{\frac{1}{1-\gamma}}$.

Proof. We label the above-stated maximization problem as *P2*. The assumption that $A'(\theta)$ is strictly convex means that (54)–(57) describe a convex set with a nonempty interior and the objective function (53) is concave. Thus, problem *P2* is a convex programming problem. Suppose the stochastic discount factor and the law of motion of the aggregate state variables jointly satisfy the following condition:

Assumption 6. *For some $\epsilon>0$, and for all (S),*

$$\sum\pi\left(g'\middle|\,g\right)\Lambda\left(S',S\right)e^{g'}<1-\epsilon. \tag{58}$$

Given assumption 6, standard arguments from Stokey et. al (1989) imply that there is a unique v in the space of bounded continuous functions that satisfies (53). In addition, v is continuous, strictly decreasing, strictly concave and differentiable in the interior. We denote the optimal policy functions for *P2* by $\left\{c(u,S),\theta(u,S),\{\delta'(u,S,\zeta'),u'(u,S,\zeta')\}_{\zeta'}\right\}$. We first show that policy function for separations satisfies $\delta'(u,S,\zeta')=1$ for all ζ'.

Suppose there exists some (u,S) such that with strictly positive probability, $\delta'\left(\tilde{\zeta}'\right)=0$. Consider an alternative set of policy functions denoted by hats:

$$\hat{c}(u,S)=c(u,S),\quad\hat{\theta}(u,S)=\theta(u,S),\quad\hat{\delta}'\left(u,S,\zeta'\right)=1\ for\ \ all\ \ \zeta'$$

$$\hat{u}'\left(u,S,\zeta'\right)=I_{\{\delta'(u,S,\zeta')=1\}}\times u\left(u,S,\zeta'\right)+I_{\{\delta'(u,S,\zeta')=0\}}\times\left(\lambda\overline{u}\left(S'\right)+\epsilon\right)$$

for some $\epsilon>0$ such that $\lambda\overline{u}(S')+\epsilon<u^*(S')$, where $u^*(S)$ is such that $v(u^*(S),S)=0$. Because the value function is strictly decreasing, we have $v\left(\hat{u}'\left(u,S,\tilde{\zeta}'\right),S'\right)>0$ for $\tilde{\zeta}'$ where $\delta'\left(\tilde{\zeta}'\right)=0$ Then it is easy to verify that see that the hat policy functions satisfy (54)–(57) and achieve a higher value for the objective in equation (53) and therefore cannot be optimal.

Thus,

$$\delta'\left(u,S,\zeta'\right)=1\ for\ \ all\ \ \zeta' \tag{59}$$

We next show that optimal choices for *P2* are feasible for problem *P1*. Optimal policies for *P2* satisfy a set of first-order necessary conditions. In particular, let $\iota\geq 0$ be the Lagrange multiplier of the constraint (57), first-order conditions with respect to θ after imposing (59) implies

$$\iota A''\left(\theta\right)=\frac{\beta}{1-\beta}c^{\frac{1}{\psi}}m^{\gamma-\frac{1}{\psi}}\frac{1}{1-\gamma}\int e^{(1-\gamma)(\eta'+\varepsilon')}\left\{u'\left(\zeta'\right)^{1-\gamma}-\lambda\bar{u}\left(S'\right)^{1-\gamma}\right\}\Omega(d\zeta'|g). \tag{60}$$

The limited commitment constraint on worker side, equation (56) along with (59) implies that right-hand side of (60) must be strictly positive. Therefore, $\iota>0$ and (57) must holds with equality at the optimum.

Let ι_u be the Lagrange multiplier of the promise keeping constraint (54), the first-order condition with respect to c implies

$$\iota_u=\frac{1}{1-\beta}\left(\frac{c}{u}\right)^{\frac{1}{\psi}}>0. \tag{61}$$

Thus, inequality (54) must also hold with equality at the optimum. As a result, the optimal policy for *P2* satisfy all of the constraints for *P1* and as the constraint set for *P2* larger, the optimal policies to *P2* also attain the maximum for *P1*. □

The first-order necessary conditions for *P2* imply that the above policy functions must satisfy

1. $\forall\eta'+\varepsilon'\in[\underline{\varepsilon}(u,S,g'),\ \bar{\varepsilon}(u,S,g')]$, $u'\left(u,S,\zeta'\right)$ satisfy

$$\Lambda\left(S',S\right)=\frac{\beta e^{-\gamma(g'+\eta'+\varepsilon')}}{1+\frac{\iota(u,S)}{\theta(u,S)}}\left[\frac{c\left(u'\left(u,S,\zeta'\right),S'\right)}{c\left(u,S\right)}\right]^{-\frac{1}{\psi}}\left[\frac{u'\left(u,S,\zeta'\right)}{m\left(u,S\right)}\right]^{\frac{1}{\psi}-\gamma}. \tag{62}$$

2. $\forall\eta'+\varepsilon'\geq\bar{\varepsilon}\left(u,S,g'\right)$ and $\forall\eta'+\varepsilon'\leq\underline{\varepsilon}\left(u,S,g'\right)$,

$$u'\left(u,S,\zeta'\right)=\lambda\bar{u}\left(S'\right), \tag{63}$$

$$u'\left(u,S,\zeta'\right)=u^*\left(S'\right). \tag{64}$$

3. The Lagrange multiplier $\iota(u,S)$ satisfies

$$\iota(u,S) = \frac{1}{A''(\theta(u,S))}\frac{\beta}{1-\beta}c(u,S)\,m(u,S) \times \frac{1}{1-\gamma}$$
$$\times \left\{ \int e^{(1-\gamma)(\eta'+\varepsilon')} \left\{ u'(u,S,\zeta')^{1-\gamma} - \lambda \bar{u}(S')^{1-\gamma} \right\} \Omega(d\zeta' | g) \right\}. \quad (65)$$

The policy functions must satisfy the equality constraints of the problem *P1*

$$A'(\theta(u,S)) = \kappa \int \Lambda(S',S)\, e^{g'+\eta'+\varepsilon'} v(u'(s'),S')\, \Omega(\zeta'), \quad (66)$$

$$u = \left[(1-\beta)\, c^{1-\frac{1}{\psi}} + \beta m(u,S)^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}. \quad (67)$$

The following lemma states that conditions (62) - (67) are both necessary and sufficient for optimality.

Lemma 4. *Suppose there exist an SDF $\Lambda(S',S)$, a worker's value from unemployment, $\bar{u}(S)$, and a law motion for aggregate state variables that satisfy assumption 6. Suppose that given $\Lambda(S',S)$, $\bar{u}(S)$, and the law of motion for state variables, policy functions for problem P2 satisfy (59), the optimality conditions (62)-(65), and the equality constraints (66)-(67). In addition, $\frac{c(u,S)}{u}$ is nondecreasing in u for all S. Let $v(u,S)$ be the unique fixed point of the operator T:*

$$Tv(u,S) = \begin{array}{c} 1 - c(u,S) - A(\theta(u,S)) + \\ \kappa\theta(u,S) \int \Lambda(S',S)\, e^{g'+\eta'+\varepsilon'} v(u'(u,S,\zeta'),S')\, \Omega(d\zeta' | g). \end{array} \quad (68)$$

Then, the policy functions together with the value function $v(u,S)$ solve the problem P2.

Proof. Suppose there exists a set of policy functions that satisfy conditions (62)-(67). Given condition (6), the operator defined in (68) is a contraction, and we can construct the value function $v(u,S)$ from the policy functions as the unique fixed point of (68). The first-order conditions (62)-(64) imply that the value function constructed above must satisfy

$$\frac{\partial}{\partial u} v(u,S) = -\frac{1}{1-\beta}\left(\frac{c(u,S)}{u}\right)^{\frac{1}{\psi}}. \quad (69)$$

Because $\frac{c(u,S)}{u}$ is nondecreasing in u, $\frac{\partial}{\partial u} v(u,S)$ must be nonincreasing, that is, $v(u,S)$ is a concave function of u. As a result, given $v(u,S)$, the first-order conditions, (62)-(67) can be shown to be equivalent to the set of first-order conditions for the programming problem *P2*, which is necessary and sufficient for optimality. Therefore, the above constructed value functions and policy functions must solve the optimal contracting problem *P2*, as needed. □

Given the above discussion, it is straightforward to provide a characterization for the equilibrium price and quantities using optimality conditions. We summarize these conditions in the following lemma. The proof is omitted as it follows directly from lemma 3 and lemma 4.

Lemma 5. *The equilibrium prices and quantities can be summarized as a set of functions:* $x(S)$,$c(u,S)$,$\theta(u,S)$,$\iota(u,S)$,$\{\bar{\varepsilon}(u,S,g'),\underline{\varepsilon}(u,S,g')\}_{g'}$, $\{\delta(u,S,\zeta'), u'(u,S,\zeta')\}_{\zeta'}$*;worker' outside option* $\bar{u}(S)$ *and initial utility at employment* $u^*(S)$*; a law of motion of* ϕ *and* B*; a SDF and a firm value function* $v(u,S)$*, such that the SDF is consistent with capital owner's consumption, that is,* $\Lambda(S',S)$ *and* $x(S)$ *satisfy equation (20), where the capital owner's utility,* $w(S)$ *is constructed from* $x(S)$ *using equation (19); the value function and policy functions satisfy (59), and the optimality conditions (62)-(67); the outside option* $\bar{u}(S)$ *satisfies (13),* $u^*(S)$ *satisfies* $v(u^*(S),S)=0$ *for all* S*, and the law of motion of the aggregate state variables satisfy (23) and (25).*

We now prove proposition 1 and 2.

B.2 Proofs of propositions 1 and 2

In lemma 3, we have already proved that $\delta(u,S,\zeta')=1$ for all ζ' is optimal for problem *P2* and lemma 5 asserts that the same policy rule is optimal for problem *P1* too.

We next provide the characterization for the policy functions $u'(u,S,\zeta')$ and then $\theta(u,S)$. Given assumption 6 and lemma 3, standard arguments from Stokey et. al (1989) imply that the value function v for the optimal contracting problem (14) is continuous, strictly decreasing, strictly concave and differentiable in the interior. Because the value function is strictly decreasing, the limited commitment constraint (16) can be written as $u'(s')\leq u^*(S')$ for all s', where $u^*(S)$ is defined by equation (12). Therefore, the first-order condition with respect to continuation utility and the envelope condition for the programming problem (53) together imply that one of the following three cases have to true:

1. In the interior, equation (31) holds.
2. The worker-side limited commitment constraint binds, $u'(u,S,\zeta')=\lambda\bar{u}(S')$, and,

$$\left[\tfrac{x(S')}{x(S)}\right]^{-\frac{1}{\psi}}\left[\tfrac{w(S')}{n(S)}\right]^{\frac{1}{\psi}-\gamma}\left(1+\tfrac{\iota(u,S)}{\theta(u,S)}\right)\geq\quad e^{-\gamma(\eta'+\varepsilon')}\left[\frac{c(u'(u,S,\zeta'),\phi',B')}{c(u,S)}\right]^{-\frac{1}{\psi}}\left[\frac{u'(u,S,\zeta')}{m(u,S)}\right]^{\frac{1}{\psi}-\gamma} \tag{70}$$

3. The firm-side limited commitment costraint binds, $u'(s')=u^*(S')$,

$$\left[\tfrac{x(S')}{x(S)}\right]^{-\frac{1}{\psi}}\left[\tfrac{w(S')}{n(S)}\right]^{\frac{1}{\psi}-\gamma}\left(1+\tfrac{\iota(u,S)}{\theta(u,S)}\right)\leq\quad e^{-\gamma(\eta'+\varepsilon')}\left[\frac{c(u'(u,S,\zeta'),\phi',B')}{c(u,S)}\right]^{-\frac{1}{\psi}}\left[\frac{u'(u,S,\zeta')}{m(u,S)}\right]^{\frac{1}{\psi}-\gamma} \tag{71}$$

Define $\mathcal{E} = \{\eta' + \varepsilon' : \text{ equation (31) holds}\}$. Also, let

$$\underline{\varepsilon}\left(u, S, g'\right) = \inf \mathcal{E}, \quad \overline{\varepsilon}\left(u, S, g'\right) = \sup \mathcal{E}. \tag{72}$$

Let $l_u(u, S)$ be the Lagrange multiplier for the promise-keeping constraint of the programing problem (53), then

$$\frac{\partial}{\partial u} v(u, S) = l_u(u, S) = \frac{1}{1-\beta}\left(\frac{c(u, S)}{u}\right)^{\frac{1}{\psi}}, \tag{73}$$

where the first equality is the envelope theorem, and the second equality is the first-order condition, (61). Because v is concave, the above condition implies that $c(u, S)$ must be strictly increasing in u. Thereore, the optimality condition (31) implies that on $\mathcal{E}$, $u'(u, S, \zeta')$ must be strictly decreasing in $\eta' + \varepsilon'$. Clearly, the strict monotonicity of $u'(u, S, \zeta')$ implies that $u'(u, S, \zeta') = \lambda \bar{u}(S')$ if $\eta' + \varepsilon' = \underline{\varepsilon}(u, S, g')$ and $u'(u, S, \zeta') = u^*(S')$ if $\eta' + \varepsilon' = \overline{\varepsilon}(u, S, g')$.

First, $\forall \eta' + \varepsilon' > \underline{\varepsilon}(u, S, g')$, we must have $u'(u, S, \zeta') = \lambda \bar{u}(S')$. Otherwise, none of the equations, (31), (70), or (71) can hold. Similarly, $\forall \eta' + \varepsilon' < \overline{\varepsilon}(u, S, g')$, we must have $u'(u, S, \zeta') = u^*(S')$.

Second, to complete the proof of part 1 and 2 of proposition 2, we need to show that $\forall \eta' + \varepsilon' \in (\underline{\varepsilon}(u, S, g'), \overline{\varepsilon}(u, S, g'))$, condition (31) must hold. It is enough to show $u'(u, S, \zeta') \in (\lambda \bar{u}(S'), u^*(S'))$. This can be proved by contradiction. Suppose $\eta' + \varepsilon' \in (\underline{\varepsilon}(u, S, g'), \overline{\varepsilon}(u, S, g'))$ and $u'(u, S, \zeta') = \lambda \bar{u}(S')$, then the fact that equation (31) holds at $\overline{\varepsilon}(u, S, g')$ implies that (note that $\eta' + \varepsilon' < \overline{\varepsilon}(u, S, g')$)

$$\left[\tfrac{x(S')}{x(S)}\right]^{-\frac{1}{\psi}} \left[\tfrac{w(S')}{n(S)}\right]^{\frac{1}{\psi}-\gamma} \left(1 + \tfrac{\iota(u,S)}{\theta(u,S)}\right) < \quad e^{-\gamma(\eta'+\varepsilon')} \left[\frac{c(\lambda \bar{u}(S'), \phi', B')}{c(u, S)}\right]^{-\frac{1}{\psi}} \left[\frac{\lambda \bar{u}(S')}{m(u, S)}\right]^{\frac{1}{\psi}-\gamma},$$

which is a contradiction to condition (70). Similarly, one can show that $u'(u, S, \zeta') = u^*(S')$ cannot be true either.

To prove the second part of proposition 1, note that because the value function is strictly concave in u, the Lagrange multiplier $\iota_u(u, S)$ must be strictly increasing in u. The first-order condition with respect to $u'(u, S, \zeta')$ in the programming problem (53) then implies that $u'(u, S, \zeta')$ must be strictly increasing in u as well. Given constraint (18), the monotonicity of $\theta(u, S)$ with respect to u then follows directly from the $u'(u, S, \zeta')$ is increasing with respect to u and the fact that $v(u', S')$ is strictly decreasing in u'.

C Proofs of propositions 4 and 5

This section provides the proofs for propositions 4 and 5. In subsection C.1, we provide closed-form solutions for the equilibrium prices and quantities for the simple economy. We prove proposition 4 in subsection C.2 and proposition 5 in subsection C.3.

C.1 Equilibrium in the simple economy

Notation We first introduce some notation. In the simple model in Section 4, we assume that the worker-specific shock follows a negative exponential distribution. The density of a negative exponential distribution is given by $f(\varepsilon|\, g_L) = \xi e^{\xi(\varepsilon-\varepsilon_{MAX})}$, for $\varepsilon \le \varepsilon_{MAX}$, and $f(\varepsilon|\, g_L) = 0$ otherwise, where ξ and ε_{MAX} are the parameters of the distribution. For later reference, we note that the moments of $f(\varepsilon|\, g_L)$ can be easily computed as $\int_{-\infty}^{\varepsilon} e^{\theta t} f(t|\, g_L)\, dt = \frac{\xi}{\xi+\theta} e^{-\xi\varepsilon_{MAX}+(\theta+\xi)\varepsilon}$, for $\xi+\theta>0$. Clearly, the assumption $\mathbb{E}[e^{\varepsilon}] = 1$ amounts to a parameter restriction that $\varepsilon_{MAX} = \ln\frac{1+\xi}{\xi}$. We will impose this restriction and call the above distribution a negative exponential distribution with parameter ξ.

As explained in the main text of the paper, we represent policy functions and value functions as functions of the period-0 promised utility u_0. For an arbitrary u_0, we use $u_H(u_0) \equiv u'(u_0, g_H)$, and $u_L(u_0, \varepsilon') \equiv u'(u_0, g_L, \varepsilon')$ to denote the normalized promised utility for a worker with initial promised utility u_0 at nodes H and L, respectively. We use $c_0(u_0)$ for workers' consumption policy at nodes 0. The rest of the policy and value functions are the same as defined in the main text. We also denote $\underline{\varepsilon}_L(u_0) \equiv \underline{\varepsilon}(u_0, g_L)$ as the lowest level of realization of the ε_1 shock such that the limited commitment constraint does not bind at node L. In addition, let u_H^{FB} and u_L^{FB} denote the utility-to-consumption ratio of an agent who consumes the aggregate consumption in state g_H and g_L, respectively. That is, they are the normalized utility associated with full risk sharing. The first best levels, u_H^{FB} and u_L^{FB} are determined by $u_H^{FB} = \left(e^{g_H} u_H^{FB}\right)^{\beta}$ and $u_L^{FB} = \left(e^{g_L} u_L^{FB}\right)^{\beta}$. We use u_L^{CD} to denote the normalized utility of an agent in an economy without risk sharing. That is, it is utility-consumption ratio of an agent who consumes y_t every period:

$$u_L^{CD} = \left(\int \left[e^{\{\varepsilon'+g_L\}} u_L^{CD}\right]^{1-\gamma} f(\varepsilon'|g_L) d\varepsilon\right)^{\frac{\beta}{1-\gamma}}. \tag{74}$$

It is straightforward to show that as $\gamma \to 1+\xi$, $u_L^{CD} \to 0$. We solve the general equilibrium in the simple economy by backward induction.

Below, we first solve the value functions and policy functions at nodes H and L in period 1. In the second step, we analyze the optimal contracting problem in period 0 for an arbitrary promised utility u_0. Finally, we impose market clearing to solve for the equilibrium stochastic discount factor.

Value functions at nodes H and L To solve the optimal contracting problem in period 1, note that our assumption that from period 2 and on, all workers consume an α fraction of their output implies that firm value functions in period 2, after normalized by worker output, take a simple form: $v_2(u', g_H) = v_2(u', g_L) = \frac{1-\alpha}{1-\beta}$. This allows us to derive a closed-form solution for the value functions and consumption policies for period 1 at node H and L, respectively. We summarize our results in the following lemma and refer readers to section B of the online appendix (not for publication) for the detailed derivation.

Lemma 6. *(Value function in period 1)*

The firm's value function at nodes H and L are give by

$$v(u, g_H) = 1 - c(u, g_H) + \frac{\beta}{1-\beta} x_H - a \ln \left[1 + \frac{\beta x_H}{a(1-\beta)} \right], \text{ and} \tag{75}$$

$$v(u, g_L) = 1 - c(u, g_L) + \frac{\beta}{1-\beta} x_L - a \ln \left[1 + \frac{\beta x_L}{a(1-\beta)} \right], \tag{76}$$

respectively, where the consumption policies are given by $c(u, g_H) = \left(\alpha e^{g_H} u_H^{FB} \right)^{-\frac{\beta}{1-\beta}} u^{\frac{1}{1-\beta}}$, *and* $c(u, g_L) = \left(\alpha \Upsilon e^{g_L} u^{CD} \right)^{-\frac{\beta}{1-\beta}} u^{\frac{1}{1-\beta}}$, *where the parameter* Υ *is defined as*

$$\Upsilon = \left\{ \int_{-\infty}^{\infty} e^{(1-\gamma)\varepsilon'} f\left(\varepsilon' \middle| g_L \right) d\varepsilon' \right\}^{\frac{1}{1-\gamma}}. \tag{77}$$

The policy functions for effort choice do not depend on u. We denote $\theta_H = \theta(u, g_H)$, *and* $\theta_L = \theta(u, g_L)$, *and*

$$\theta_H = 1 - \frac{a}{a + \frac{\beta}{1-\beta} x_H}, \quad \theta_L = 1 - \frac{a}{a + \frac{\beta}{1-\beta} x_L}. \tag{78}$$

At node L, limited commitment on firm side requires that $v_L(u) \geq 0$. Therefore, by equation (76), the maximum amount of consumption that the firm can promise to deliver to a worker at node L is $1 - A(\theta_L) + \theta_L \frac{\beta}{1-\beta} x_L$, which we will denote as c_L^{MAX}. Recall that for a worker with initial promised utility u_0, $\underline{\varepsilon}_L(u_0)$ is the lowest level of realization of the ε' shock such that the limited commmitment constraint does not bind at node L. We must have, for all u_0,

$$c_L(u_0, \underline{\varepsilon}_L(u_0)) = 1 + \frac{\beta}{1-\beta} x_L - a \ln \left[1 + \frac{\beta x_L}{a(1-\beta)} \right]. \tag{79}$$

We now turn to the optimal contracting problem as node 0.

Optimal contracting at node 0 We develop our results in several lemmas. The key to characterize the policy functions for the optimal contracting problem at node 0 is the consumption and promised utility for the marginal worker in period 1 at node L. Here, the marginal worker is defined as the one with the lowest level of realization of ε_1 shock such that the limited commitment constraint does not bind at node L, i.e., $\varepsilon_1 = \underline{\varepsilon}_L(u_0)$. Our first lemma uses the optimal risk sharing condition (31) to relate the marginal rate of substitution of a marginal worker to that of the capital owners.

Lemma 7. *(FOC for the marginal agent)*

Given the consumption share of the capital owners, x_H and x_L, for all u_0, the normalized consumption of the marginal worker with $\varepsilon_1 = \underline{\varepsilon}_L(u_0)$ must satisfy:

$$\frac{c_H(u_0)}{e^{(1+\tau)\underline{\varepsilon}_L(u_0)} c_L(u_0, \underline{\varepsilon}_L(u_0))} \left[\frac{u_L^{FB} k(\theta_H)}{\Upsilon u_L^{CD} k(\theta_L)} \right]^\tau = \frac{x_H}{x_L}, \tag{80}$$

where we denote $\tau = \frac{\beta(\gamma-1)}{1+(1-\beta)(\gamma-1)}$, and $k(\theta) = \left[\theta + (1-\theta)\lambda^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$.

Next, we provide a lemma that links the consumption of a marginal worker to the expected consumption of an average worker at node L.

Lemma 8. *(Expected worker consumption at node L)*

Given the consumption share of the capital owners, x_H and x_L, the expected consumption of a worker with promised utility u_0 at node L is given by:

$$E\left[e^{\varepsilon'} c_L(u_0, \varepsilon')\right] = e^{(1+\tau)\underline{\varepsilon}_L(u_0)} c_L(u_0, \underline{\varepsilon}_L(u_0)) \Phi(\underline{\varepsilon}_L(u_0)), \tag{81}$$

for all u_0, where the function $\Phi(\varepsilon)$ is defined as

$$\Phi(\varepsilon) = \frac{\xi}{\xi - \tau} e^{-\tau \varepsilon_{MAX}} - \frac{\xi(1+\tau)}{(1+\xi)(\xi-\tau)} e^{-\xi \varepsilon_{MAX} + (\xi - \tau)\varepsilon}. \tag{82}$$

lemma 7 is the optimal risk sharing condition that equalizes the marginal rate of substitution of workers and capital owners across the two states in period 1. The next lemma provides another first-order condition that links the marginal rate of substitution of capital owners and workers across time. Together lemma 7 and lemma 9 below completely characterize optimal risk sharing conditions.

Lemma 9. *(Optimal risk sharing)*

Optimal risk sharing requires that for all u_0,

$$\left[\frac{x_H}{c_H(u_0)}\right]^{1+(1-\beta)(\gamma-1)} = \left[\frac{x_0}{c_0(u_0)}\right] \left[\frac{\bar{n}_0(x_H, x_L)}{\bar{m}_0(u_0)}\right]^{\gamma-1}, \text{ where} \tag{83}$$

$$\bar{n}_0(x_H, x_L) = \left[\pi\left(e^{(1+\beta)g_H} x_H^{(1-\beta)} \left(u_H^{FB}\right)^\beta\right)^{1-\gamma} + (1-\pi)\left(e^{(1-\gamma)g_L} x_L^{(1-\beta)} \left(u_L^{FB}\right)^\beta\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}, \text{ and,}$$

$$\bar{m}_0(u_0) = \left[\begin{array}{c} \pi\left(e^{(1+\beta)g_H} c_H^{1-\beta} \left(u_H^{FB} k(\theta_H)\right)^\beta\right)^{1-\gamma} + \\ (1-\pi) e^{(1-\gamma)g_L} \left[e^{(1+\tau)\underline{\varepsilon}_L(u_0)} c_L(u_0,\ \underline{\varepsilon}_L(u_0))\right]^{(1-\beta)(1-\gamma)} \left\{\frac{1}{\alpha} m_L\right\}^{\beta(1-\gamma)} \Psi(\underline{\varepsilon}_L(u_0)) \end{array} \right]^{\frac{1}{1-\gamma}}, \tag{84}$$

where $\Psi(\varepsilon)$ *is given by:*

$$\Psi(\varepsilon)=\left\{\frac{\xi}{\xi-\tau}e^{-\tau\varepsilon_{MAX}}-\frac{\xi(1-\gamma+\tau)}{(\xi-\tau)(\xi+1-\gamma)}e^{-\xi\varepsilon_{MAX}+(\xi-\tau)\underline{\varepsilon}_L(u_0)}\right\}. \tag{85}$$

General Equilibrium A unit measure of a single type of workers and market clearing at node 0, node H, and node L implies that u_0^* solves $c_0(u_0^*)=1-x_0$, $c_H(u_0^*)=1-x_H$, and $E\left[e^{\varepsilon'}c_L(u_0^*,\varepsilon')\right]=1-x_L$,respectively. Note that equation (79) implies

$$c_L(u_0^*,\underline{\varepsilon}_L(u_0^*))=1+\frac{\beta}{1-\beta}x_L-a\ln\left[1+\frac{\beta x_L}{a(1-\beta)}\right]. \tag{86}$$

Using the market clearing at node L and equation (81) in lemma 8, we have $1-x_L=e^{(1+\tau)\underline{\varepsilon}_L(u_0^*)}c_L(u_0,\underline{\varepsilon}_L(u_0^*))\,\Phi(\underline{\varepsilon}_L(u_0^*))$, which, after combining with (86), gives

$$e^{(1+\tau)\underline{\varepsilon}_L(u_0^*)}\Phi(\underline{\varepsilon}_L(u_0^*))=\frac{1-x_L}{1+\frac{\beta}{1-\beta}x_L-a\ln\left[1+\frac{\beta x_L}{a(1-\beta)}\right]}. \tag{87}$$

Equations (86) and (87) together define $c_L(u_0,\underline{\varepsilon}_L(u_0^*))$ and $\underline{\varepsilon}_L(u_0^*)$ as functions of x_L. With a bit abuse of notation, we denote these functions as $c_L(x_L)$ and $\underline{\varepsilon}(x_L)$.

Focusing
on type-u_0^* agents, using lemma 8, we can replace the term $e^{(1+\tau)\underline{\varepsilon}_L(u_0^*)}c_L(u_0^*,\underline{\varepsilon}_L(u_0^*))$ in equation (80) by the following

$$e^{(1+\tau)\underline{\varepsilon}_L(u_0^*)}c_L(u_0^*,\underline{\varepsilon}_L(u_0^*))=(1-x_L)\,\Phi(\underline{\varepsilon}(x_L))^{-1}. \tag{88}$$

Therefore, the first order condition (80) can be written as

$$\Phi(\underline{\varepsilon}(x_L))\left[\frac{u_L^{FB}k(\theta_H)}{\Upsilon u_L^{CD}k(\theta_L)}\right]^{\tau}=\frac{x_H}{x_L}\frac{1-x_L}{1-x_H}. \tag{89}$$

Also, we use the marketing clearing condition to replace c_H by $1-x_H$ and use (88) to replace $e^{(1+\tau)\underline{\varepsilon}_L(u_0^*)}c_L(u_0^*,\underline{\varepsilon}_L(u_0^*))$. We define workers' certainty equivalent as a function of x_H, x_L, and ε using (84)

$$\bar{m}_0(x_H,x_L,\varepsilon)=\left\{\begin{array}{c}\pi\left[e^{(1+\beta)g_H}(1-x_H)^{(1-\beta)}\left(u_H^{FB}k(\theta_H)\right)^{\beta}\right]^{1-\gamma}\\+(1-\pi)\left[e^{(1+\beta)g_L}\left[\frac{1-x_L}{\Phi(\varepsilon)}\right]^{(1-\beta)}\left[\Upsilon u_L^{CD}k(\theta_L)\right]^{\beta}\right]^{1-\gamma}\Psi(\varepsilon)\end{array}\right\}^{\frac{1}{1-\gamma}}. \tag{90}$$

This allows us to write the first order condition (83) as functions of x_H and x_L:

$$\left[\frac{x_H}{1-x_H}\right]^{1+(1-\beta)(\gamma-1)} = \left[\frac{x_0}{1-x_0}\right]\left[\frac{\bar{n}_0\left(x_H, x_L\right)}{\bar{m}_0\left(x_H, x_L, \underline{\varepsilon}\left(x_L\right)\right)}\right]^{\gamma-1}. \tag{91}$$

Give an initial condition of x_0, equations (89) and (91) can be jointed solved for x_H and x_L. Other equilibrium quantities can then be constructed analogously.

C.2 Proof of proposition 4

1. From the definition of u_L^{CD} in (74), it is clear that as $\gamma \to 1+\xi$, $u_L^{CD} \to 0$. Consider equation (89). It is straightforward to verify that $\Phi(\varepsilon)$ is strictly positive and bounded (see equation (82)). Also, both $k(\theta_H)$ and $k(\theta_L)$ are bounded. Therefore, as $\gamma \to 1+\xi$ the left hand side converges to ∞, and we must have $\frac{x_H}{x_L} \to \infty$. By continuity, there exists $\hat{\gamma} \in (1, 1+\xi)$ such that $\frac{x_H}{x_L} > 1$ if and only if $\gamma > \hat{\gamma}$, as needed.

   In addition, if $\gamma = 1$, then $\tau = 0$. Using the definition of $\Phi(\varepsilon)$, $\Phi(\varepsilon) = 1 - \frac{1}{(1+\xi)} e^{-\xi(\varepsilon_{MAX}-\varepsilon)} < 1$. Therefore, we must have $\frac{x_H}{x_L} < 1$.

2. The economy without moral hazard is a special case in which the parameter for cost of effort, $a = 0$. We use $\theta_H(a)$ and $\theta_L(a)$ to denote policy functions with the understanding that they are policy functions of the moral hazard economy if $a > 0$, and they stand for policy functions in the economy without moral hazard if $a = 0$. Using our result from Part 1 of the proof, as $\gamma \to 1+\xi$, $\frac{x_H}{x_L} \to \infty$. Because both x_H and x_L are bounded between $[0, 1]$, we must have $x_L \to 0$. Therefore, $\theta_L(a) \to 0$ by equation (78). Also, equation (91) implies that as $\gamma \to 1+\xi$, $\bar{m}_0\left(x_H(a), x_L(a), \underline{\varepsilon}\left(x_L(a)\right)\right) \to 0$; therefore, $x_H(a) \to 0$ as well. Therefore, as $\gamma \to 1+\xi$, $\theta_H(a) \to 1 - \frac{a}{a+\frac{\beta}{1-\beta}x_H^*}$. Consider equation (89), for an arbitrary a, $\left[\frac{k(\theta_H(a))}{k(\theta_L(a))}\right]^{\tau} = \left[\frac{\theta_H(a)+(1-\theta_H(a))\lambda^{1-\gamma}}{\theta_L(a)+(1-\theta_L(a))\lambda^{1-\gamma}}\right]^{\frac{\tau}{1-\gamma}}$. Suppose $a > 0$, then as $\gamma \to 1+\xi$, there exist $\epsilon > 0$ such that

$$\left[\frac{\theta_H(a)+(1-\theta_H(a))\lambda^{1-\gamma}}{\theta_L(a)+(1-\theta_L(a))\lambda^{1-\gamma}}\right]^{\frac{\tau}{1-\gamma}} \to \left[\frac{1-\frac{a}{a+\frac{\beta}{1-\beta}x_H^*(a)}+\left(1-\frac{a}{a+\frac{\beta}{1-\beta}x_H^*(a)}\right)\lambda^{-\xi}}{\lambda^{-\xi}}\right]^{-\frac{1}{\xi}\frac{\beta\xi}{1+\xi(1-\beta)}} > 1+\epsilon.$$

   In addition, equation (87) implies that as $\gamma \to 1+\xi$, $x_L \to 0$, and therefore, $\underline{\varepsilon}_L(a) \to \varepsilon^*$ for all a, where ε^* is such that $e^{(1+\tau)\varepsilon^*}\Phi(\varepsilon^*) = 1$. Therefore, with $a > 0$, for γ close enough to $1+\xi$, we must have $\Phi\left(\underline{\varepsilon}_L(a)\right)\left[\frac{u_L^{FB}k(\theta_H(a))}{\Upsilon u_L^{CD}k(\theta_L(a))}\right]^{\tau} > \Phi\left(\underline{\varepsilon}_L(0)\right)\left[\frac{u_L^{FB}}{\Upsilon u_L^{CD}}\right]^{\tau}$. Equation (89) implies that for γ close enough to $1+\xi$, $\frac{x_H}{x_L} > \frac{x_H}{x_L}$ because as $\gamma \to 1+\xi$, $x_L \to 0$ and $x_H \to x_H^*$ has a limit.

3. By Part 1 of the proposition, for γ large enough, $x_H > x_L$. The fact that $\theta_H > \theta_L$ follows from equation (78).

C.3 Proof of proposition 5

Firm risk pass through Fixing u_0, equation (109) implies that $\forall \varepsilon' \geq \underline{\varepsilon}(u_0)$, $\frac{d\ln[e^{\varepsilon}c_L(u_0,\varepsilon)]}{d\varepsilon} = -\tau$. For $\varepsilon' < \underline{\varepsilon}(u_0)$, the limited commitment constraint binds, and $e^{\varepsilon}c_L(u_0,\varepsilon) = e^{\varepsilon}c_L(u_0,\underline{\varepsilon}(u_0))$. Therefore, $\frac{d\ln[e^{\varepsilon}c_L(u_0,\varepsilon)]}{d\varepsilon} = 1$. Combining the above two equations, we have $E\left[\frac{\partial \ln[e^{\varepsilon}c_L(u_0,\varepsilon)]}{\partial\varepsilon}\right] = \int_{-\infty}^{\underline{\varepsilon}_L(u_0)} f(\varepsilon'|\, g_L)\, d\varepsilon'\tau + \int_{\underline{\varepsilon}_L(u_0)}^{\varepsilon_{MAX}} f(\varepsilon'|\, g_L)\, d\varepsilon' = e^{-\xi(\varepsilon_{MAX}-\underline{\varepsilon}_L(u_0))} - \tau\left[1-e^{-\xi(\varepsilon_{MAX}-\underline{\varepsilon}_L(u_0))}\right]$. Clearly, the average elasticity is increasing in $\underline{\varepsilon}_L(u_0)$. Using the optimal risk sharing conditions (99) and (83), we can show that $\underline{\varepsilon}_L(u_0)$ is an increasing function of u_0.

Cross section of expected returns To characterize the dependence of $\frac{v_H(u_0)}{E[e^{\varepsilon}v_L(u_0,\varepsilon)]}$, note that in general, $c_H(u_0) = \frac{x_H}{x_L}\left[\frac{\xi u_L^{CD}k(\theta_L)}{u_L^{FB}k(\theta_L)}\right]^{\tau} e^{(1+\tau)\underline{\varepsilon}_L(u_0)}c_L\left(u_0,\underline{\varepsilon}_L(u_0)\right)$ by lemma 7, and $E\left[e^{\varepsilon}c_L(u_0,\varepsilon)\right] = e^{(1+\tau)\underline{\varepsilon}_L(u_0)}c_L(u_0,\underline{\varepsilon}_L(u_0))\,\Phi(\underline{\varepsilon}_L(u_0))$ by lemma 8. Because at $\varepsilon = \underline{\varepsilon}_L(u_0)$, the limited commitment constraint, $v_L(u_0,\varepsilon) = 0$ binds, $c_L(u_0,\underline{\varepsilon}_L(u_0)) = 1 + \frac{\beta}{1-\beta}x_L - a\ln\left[1+\frac{\beta x_L}{a(1-\beta)}\right]$ by (79). To simplify notation, we denote $\Gamma_H = 1+\frac{\beta}{1-\beta}x_H - a\ln\left[1+\frac{\beta x_H}{a(1-\beta)}\right]$ and $\Gamma_L = 1+\frac{\beta}{1-\beta}x_L - a\ln\left[1+\frac{\beta x_L}{a(1-\beta)}\right]$. We then write $\frac{v_H(u_0)}{E[e^{\varepsilon}v_L(u_0,\varepsilon)]}$ as $\frac{v_H(u_0)}{E[e^{\varepsilon}v_L(u_0,\varepsilon)]} = \frac{\Gamma_H-\phi e^{(1+\tau)\underline{\varepsilon}_L(u_0)}}{\Gamma_L\left\{1-e^{(1+\tau)\underline{\varepsilon}_L(u_0)}\Phi(\underline{\varepsilon}(u_0))\right\}}$, where we denote $\phi = \frac{x_H}{x_L}\left[\frac{\xi u_L^{CD}k(\theta_L)}{u_L^{FB}k(\theta_L)}\right]^{\tau}\Gamma_L$ to simplify notation. By proposition 2, $\underline{\varepsilon}(u_0)$ is a strictly increasing function of u_0. Therefore, we complete the proof for proposition 5 by noticing that

$$\frac{\partial}{\partial\underline{\varepsilon}}\frac{\Gamma_H-\phi e^{(1+\tau)\underline{\varepsilon}}}{\left\{1-e^{(1+\tau)\underline{\varepsilon}}\Phi(\underline{\varepsilon})\right\}} > 0. \tag{92}$$

It is possible to show that the above inequality holds for γ large enough (but smaller than $1+\xi$ so that worker utility is well defined). We refer the readers to lemma 11 in section B of the online appendix (not for publication) for the details of proof.

D Computational Algorithm

We describe our computation algorithm. The algorithm consists of an "outer loop", in which we iterate over the law of motion for aggregate states and an associated stochastic discount factor, and an "inner loop", in which we solve for the optimal contract. Below are the steps of our numerical procedure.

1. Initialize the law of motion of x, $\Gamma_x(g, x, g')$. We use a log-linear functional form:

$$\log x' = a(g, g') + b(g, g')\log x. \tag{93}$$

Given the law of motion of x, the SDF $\Lambda(x, g, g')$ is calculated using

$$\Lambda\left(x, g, g'\right) = \beta\left[\frac{x'\left(g'|g, x\right)e^{g'}}{x}\right]^{-\frac{1}{\psi}}\left[\frac{w\left(x', g'\right)e^{g'}}{n(g, x)}\right]^{\frac{1}{\psi}-\gamma},$$

where $w(g, x)$ and $n(g, x)$ are derived from equation (19).

2. The inner loop consists of using $\Gamma_x(g, x, g')$ and $\Lambda(x, g, g')$, to solve the value function $v(u, g, x)$, the worker-outside value $\overline{u}(g, x)$ and value of a new job $u^*(g, x)$ along with the policy functions $c(u, g, x)$, $\theta(u, g, x)$ and $u'(u, g, x, \zeta')$ that solve the optimal contracting problem *P1*. We solve Bellman equation by a modified value function iteration as applying a standard value function iteration is complicated by the presence of the occasionally binding constraints (16) and (17). Our procedure borrows elements from endogenous grid method of Carroll (2006). Please see below "Details of Inner Loop."

3. To check the accuracy in computing the optimal contract, we compute Euler equation errors. Fixing u, x, g and the aggregate state next period g', we draw 1000 idiosyncratic shocks $\varepsilon' + \eta'$ such that both agent and firm-side limited commitment constraints are not binding. We then use the maximum absolute log10 ratio of worker's MRS to owners' MRS across these shocks as our measure of Euler Equation Error. We repeat this procedure for different (u, x, g) and g' combinations with values of (u, x) that are not on the grid points where the value function is solved. The Euler equation errors computed this way range between -3 and -4, which suggests that our approximation is reasonable.

4. We now describe the outer loop where we use optimal policies to simulate the model and update Γ_x. Please see below the paragraph "Details of the simulation procedure"

5. Up to now, we have described a procedure to simulate forward the economy. This allows us to compute the market clearing $\left\{x_{t+1}^{MC}\right\}_{t=0}^{\infty}$ as follows:

$$x_{t+1}^{MC} = \sum_{m=1}^{N+2}\phi_{t+1}[m] - \sum_{m=1}^{N+2}c(\hat{u}[m](t+1)|\, g_{t+1}, x_{t+1})\phi_{t+1}[m] - B_{t+1}. \tag{94}$$

Given the sequence of $\{g_t\}_{t=1}^{T}$, we simulate the economy forward for T periods to obtain $\left\{x_t^{MC}\right\}_{t=0}^{T}$. We divide the sample into four cases: $g_H \to g_H$, $g_H \to g_L$, $g_L \to g_H$, $g_L \to g_L$ and use regression to update the law of motion of x. We go back to step

1 to iterate. Note the under the above procedure, given the sequence of $\{g_t\}_{t=1}^{T}$, the sequence of x_{t+1} that is used for computing decision rules is complete determined by (94). In the simulation, we assume that x_{t+1} follows the perceived law of motion, based on which agent make their decisions. We use the market clearing condition to update the actual law of motion of x and iterate.

6. We divide the sample into four cases: $g_H \to g_H$, $g_H \to g_L$, $g_L \to g_H$, $g_L \to g_L$ and use regressions (93) to update the law of motion of x. We go back to step 1 to iterate until the unconditional R^2 approaches 99.9%.

Details of the Inner Loop

1. Guess $v(u,g,x)$ and $c(u,g,x)$. These imply functions $u^*(g,x)$ and $\overline{u}(g,x)$ using equations (12) and (13). We denote $c(u^*(g,x)|g,x)$ and $c(\lambda\overline{u}(g,x),g,x)$ by $c^*(g,x)$ and $\overline{c}(g,x)$.

2. Let $\{\underline{\varepsilon}(u,S,g'),\overline{\varepsilon}(u,S,g')\}_{g'}$ be the thresholds for $\eta'+\varepsilon'$ such that constraint (16) and (17) bind for a worker with state u, aggregate states (S) and next period for aggregate shock $g'=g_L$. Define a grid $\underline{\mathcal{E}}_L \times \mathcal{X} \equiv \left\{(\underline{\varepsilon}_{L,0},x_0),\left(\underline{\varepsilon}_{L,1},x_0\right),\ldots,\left(\underline{\varepsilon}_{L,n\mathcal{E}},x_{n\mathcal{X}}\right)\right\}$ with the understanding that $\underline{\varepsilon}_L(j)$ and $x(j)$ are the entries in the j^{th} element of the grid $\underline{\mathcal{E}}_L \times \mathcal{X}$ with $j \in \{1,2,\ldots,n\mathcal{E}\times n\mathcal{X}\}$.

3. For all $j \in \{1,2,\ldots,n\mathcal{E}\times n\mathcal{X}\}$, we solve for $\left\{\underline{\varepsilon}_{g'}(j),\overline{\varepsilon}_{g'}(j)\right\}_{g'}$ that are consistent with $\underline{\varepsilon}_L(j)$ and the guess for functions v and c in step (a) using the following equations that need to hold for all g'

$$\frac{\Lambda\left(x(j),g,g'\right)}{\Lambda\left(g_L,g,x(j)\right)}=\frac{e^{-\gamma\left(\underline{\varepsilon}_{g'}(j)\right)}}{e^{-\gamma\left(\underline{\varepsilon}_{g_L}(j)\right)}}\left[\frac{c^*\left(g',\Gamma_x\left(x(j),g,g'\right)\right)}{c^*\left(g_L,\Gamma_x\left(x(j),g,g'\right)\right)}\right]^{-\frac{1}{\psi}}\left[\frac{u^*\left(g',\Gamma_x\left(x(j),g,g'\right)\right)}{u^*\left(g_L,\Gamma_x\left(x(j),g,g'\right)\right)}\right]^{\frac{1}{\psi}-\gamma}$$

and

$$\frac{\Lambda\left(x(j),g,g'\right)}{\Lambda\left(g_L,g,x(j)\right)}=\frac{e^{-\gamma\left(\overline{\varepsilon}_{g'}(j)\right)}}{e^{-\gamma\left(\overline{\varepsilon}_{g_L}(j)\right)}}\left[\frac{\overline{c}\left(g',\Gamma_x\left(x(j),g,g'\right)\right)}{\overline{c}\left(g_L,\Gamma_x\left(x(j),g,g'\right)\right)}\right]^{-\frac{1}{\psi}}\left[\frac{\overline{u}\left(g',\Gamma_x\left(x(j),g,g'\right)\right)}{\overline{u}\left(g_L,\Gamma_x\left(x(j),g,g'\right)\right)}\right]^{\frac{1}{\psi}-\gamma}.$$

4. Now we construct the policy function $u'\left(\zeta',j\right)$. First, $\forall\eta'+\varepsilon'<\underline{\varepsilon}_{g'}(j)$ use (63) and $\forall\eta'+\varepsilon'>\overline{\varepsilon}_{g'}(j)$ use (64); for $\eta'+\varepsilon'\in\left(\underline{\varepsilon}_{g'}(j),\overline{\varepsilon}_{g'}(j)\right)$ use

$$\frac{e^{-\gamma\left(\underline{\varepsilon}_{g'}(j)\right)}}{e^{-\gamma\left(\eta'+\varepsilon'\right)}}\left[\frac{c^*\left(g',\Gamma_x\left(x(j),g,g'\right)\right)}{c\left(u'\right)}\right]^{-\frac{1}{\psi}}\left[\frac{u^*\left(g',\Gamma_x\left(x(j),g,g'\right)\right)}{u'}\right]^{\frac{1}{\psi}-\gamma}=1$$

to solve out for u'.

5. We compute $c(j)$, $\theta(j)$ and $\iota(j)$ using equations (62), 65, and (66), where certainty equivalent $m(j)$ only depends on $\{u'(s',g)\}_{s'}$ and $\{\overline{u}(g',\Gamma_x(x(j),g,g'))\}_{g'}$.

6. Finally, we use the promise keeping constraint (15) to back out $u(j)$ that is consistent with $c(j)$ and $\{u'(s',g)\}_{s'}$ and we use the objective function of the firm, the right hand side of (14) to obtain v_j

7. The guess for $v(u,g,x)$ and $c(u,g,x)$ are updated by interpolating values $\{u_j,v_j\}$ and $\{u_j,c_j\}$. We then iterate until the value function and consumption functions both converge with a tolerance of $1e-7$ under a sup norm.

Details for the simulation procedure: Let $\phi(t)$ denote the summary measure at time t. In simulations, we approximate the continuous distribution $\phi(t)$ by a finite-state distribution as follows. We choose $u_1^{(t)},u_2^{(t)},\cdots,u_{N+1}^{(t)}$, where $u_1^{(t)}=\lambda\overline{u}(g_t,x_t)$ and $u_{N+1}^{(t)}=u^*(g_t,x_t)$. A density ϕ is characterized by a set of grid points $\{\hat{u}[n](t)\}_{n=1}^{N+3}$ and corresponding weights $\{\phi[n](t)\}_{n=1}^{N+3}$ such that (i) $\hat{u}[1]$ and $\hat{u}[N+1]$ are the boundaries where the limited commitment constraint binds: $\hat{u}[1]=\lambda\overline{u}(g_t,x_t)$ and $\hat{u}[N+1]=u^*(g_t,x_t)$; $\hat{u}[N+2]=u^*(g_t,x_t)$ is the restarting utility; (ii) $\{\hat{u}[n]\}_{n=2,3,\cdots N}$ are the interior points: $\hat{u}[j]\in(u_{j-1},u_j)$, for $j=2,3\cdots N$, are chosen appropriately to minimize the approximation error.; (iii) $\phi[1]$ and $\phi[N+1]$ are the total amount of human capital owned by agents with a binding limited commitment constraint at $\hat{u}[1]$ and $\hat{u}[N+1]$, respectively; (iv) $\{\phi[n]\}_{n=2,3,\cdots N}$ are the human capital owned by agents in the interior; (iv) the mass on $\phi[N+2]$ is the human capital of agents who (re)start at $u^*(g,x)$, this include both the newly employed and the new born. (v) the mass $\phi[N+3]$, which is the total human capital owned by workers in the the unemployed pool.

1. Start with an initial distribution of u, denoted $\{\phi_0(u)\}$. Having solved x_0, use the law of motion of $u'(u,g,x,\zeta')$ to compute ϕ_1. Here we describe a general procedure to solve for $\{\phi[n](t+1);\ \hat{u}[n](t+1);\ x_{t+1}\}_{n=1}^{N+3}$ and B_{t+1} given $\{\phi[n](t);\ \hat{u}[n](t);\ x_t\}_{n=1}^{N+3}$ and B_t. Note that the assumed law of motion gives a natural candidate for x_{t+1}. We denote $x_{t+1}=\Gamma(x_t,g_t,g_{t+1})$.

2. First, we approximate the distribution $s\sim f(\varepsilon+\eta|\,g)$ by a finite dimensional distribution such that $\sum_k^K f_g[j]=1$ and $\sum_k^K e^{s_k}f_g[j]=1$, for $g=g_H,g_L$.

3. Given $\{\phi[n](t),\hat{u}[n](t)\}_{n=1}^{N+3}$ for period t, conditioning on the realization of aggregate state g_{t+1}, for each $n=1,2,\cdots,N+2$, we compute $\{\phi_{t+1}[n,k]\}_{n,k}$.

$$\phi_{t+1}[n,k]=\kappa\theta(\hat{u}[n](t),\ g_t,x_t)\,f_{g_{t+1}}[k]\,\phi_t[n]\,e^{s_k},\quad k=1,2\ldots,K.$$

4. We now compute $\{\phi_{t+1}[m]\}_m$ for the next period.

$$\phi_{t+1}[1] = \sum_{n=1}^{N+2}\sum_{k=1}^{K}\phi_{t+1}[n,j]\, I_{\{u'(\hat{u}[n](t),g_t,x_t,g_{t+1},s_k)\leq\lambda\overline{u}(g_{t+1},x_{t+1})\}},$$

$$\phi_{t+1}[2] = \sum_{n=1}^{N+2}\sum_{k=1}^{K}\phi_{t+1}[n,k]\, I_{\left\{u'(\hat{u}[n](t),g_t,x_t,g_{t+1},s_k)\in\left(u_1^{(t+1)},u_2^{(t+1)}\right)\right\}},$$

$$\phi_{t+1}[m] = \sum_{n=1}^{N+2}\sum_{k=1}^{K}\phi_{t+1}[n,k]\, I_{\left\{u'(\hat{u}[n](t),g_t,x_t,g_{t+1},s_k)\in\left[u_{m-1}^{(t+1)},u_m^{(t+1)}\right)\right\}},\quad m=3,\ldots N$$

$$\phi_{t+1}[N+1] = \sum_{n=1}^{N+2}\sum_{k=1}^{K}\phi_{t+1}[n,k]\, I_{\{u'(\hat{u}[n](t),g_t,x_t,g_{t+1},s_k)\geq u^*(g_{t+1},x_{t+1})\}},$$

$$\phi_{t+1}[N+2] = 1-\kappa+\kappa\lambda\chi\phi_t[N+3]$$

$$\phi_{t+1}[N+3] = \kappa\lambda\left\{\sum_{n=1}^{N+2}\left[1-\theta\left(\hat{u}[n](t),g_t,x_t\right)\right]\phi_t[n] + [1-\chi]\,\phi_t[N+3]\right\}$$

5. We need to update the vector normalized utilities $\{\hat{u}[n](t+1)\}_{n=1}^{N+2}$. Clearly, we should have $\hat{u}[1](t+1) = \lambda\overline{u}(g_{t+1},x_{t+1})$, $\hat{u}[N+1](t+1) = u^*(g_{t+1},x_{t+1})$ and $\hat{u}[N+2](t+1) = u^*(g_{t+1},x_{t+1})$. For $m = 2,\ldots,N$, we choose $\hat{u}[m](t+1) \in \left[u_{m-1}^{(t+1)},u_m^{(t+1)}\right)$ such that the resource constraint holds exactly for $u\in\left[u_{m-1}^{(t+1)},u_m^{(t+1)}\right)$. That is, we pick $\hat{u}[m](t+1)$ to be the solution (denoted $\hat{u}$) to

$$\begin{aligned}&\sum_{n=1}^{N+2}\sum_{k=1}^{K}\phi_{t+1}[n,k]\, c\left(u'\left(\hat{u}[n](t),g_t,x_t,g_{t+1},\varepsilon_j\right),g_{t+1},x_{t+1}\right) I_{\left\{u'(\hat{u}[n](t),g_t,x_t,g_{t+1},\varepsilon_k)\in\left[u_{m-1}^{(t+1)},u_m^{(t+1)}\right)\right\}}\\ =\;& c(\hat{u},g_{t+1},x_{t+1})\phi_{t+1}[m].\end{aligned}$$

6. Finally, the total unemployment benefit consumed by all unemployed workers is $B_{t+1} = b\phi_{t+1}[N+3]$.