SUPPLEMENT TO “SEMIPARAMETRIC POWER ENVELOPES FOR TESTS OF THE UNIT ROOT HYPOTHESIS”  

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PROOF OF LEMMA 2: Suppose $f$ satisfies Assumption DQM.  
The result $\ell_f \in \mathcal{L}_f$ follows from standard arguments. Specifically, $E[\ell_f(\varepsilon)] = 0$ and $E[\ell_f(\varepsilon)^2] < \infty$ by van der Vaart (2002, Lemma 1.8). Furthermore, using van der Vaart (2002, Example 1.15), the property $E[\varepsilon \ell_f(\varepsilon)] = 1$ can be deduced from the fact that the functional $\int_{\infty}^{-\infty} f(\varepsilon - \theta) d\varepsilon = \theta$ is differentiable in the ordinary sense and the sense of van der Vaart (2002, Definition 1.14).  
Finally, by the Cauchy–Schwarz inequality, $E[\ell_f(\varepsilon)^2] \geq E[\varepsilon^2] / E[\varepsilon \ell_f(\varepsilon)^2] = 1$.  

To establish the locally asymptotically quadratic (LAQ) property, let $c_T$ be a bounded sequence. The log likelihood ratio $L_T^f(c_T)$ admits the expansion

$$L_T^f(c_T) = \frac{c_T}{T} \sum_{t=2}^{T} y_{t-1} \ell_f(\Delta y_t) + \sum_{t=2}^{T} R_{Tt}$$

$$- \frac{1}{4} \sum_{t=2}^{T} \left[ \frac{c_T}{T} y_{t-1} \ell_f(\Delta y_t) + R_{Tt} \right]^2 (1 + \beta_{Tt}),$$

where $R_{Tt} := R_f(\Delta y_t, c_T y_{t-1} / T)$, $\beta_{Tt} := \beta[c_T y_{t-1} \ell_f(\Delta y_t) / T + R_{Tt}]$, and the defining properties of $R_f(\cdot)$ and $\beta(\cdot)$ are

$$\sqrt{f(\varepsilon - \theta)} / f(\varepsilon) = 1 + \frac{1}{2} \theta \ell_f(\varepsilon) + \frac{1}{2} R_f(\varepsilon, \theta),$$

$$\log(1 + r) = r - \frac{1}{2} r^2 [1 + \beta(2r)].$$

The proof of Lemma 2 will be completed by showing that

(S1)  
$$\sum_{t=2}^{T} R_{Tt} = - \frac{1}{4} c_T^2 T_{ff} \sum_{t=2}^{T} y_{t-1}^2 + o_{p_0,f}(1),$$

(S2)  
$$\sum_{t=2}^{T} \left[ \frac{c_T}{T} y_{t-1} \ell_f(\Delta y_t) + R_{Tt} \right]^2 (1 + \beta_{Tt}) = c_T^2 T_{ff} \sum_{t=2}^{T} y_{t-1}^2 + o_{p_0,f}(1).$$

In the rest of the proof, suppose $H_0$ holds and let $\vartheta_T$ be any positive sequence for which $\vartheta_T \to 0$ and $\sqrt{T} \vartheta_T \to \infty$ (as $T \to \infty$).
Equation (S1). Let \( \tilde{R}_{Tt} := 1(\lvert c_T y_{t-1} / T \rvert \leq \vartheta_T)R_{Tt} \) denote a truncated version of \( R_{Tt} \). Because \( \max_{2 \leq t \leq T} |c_T y_{t-1} / \sqrt{T}| = O_p(1) \) and \( \sqrt{T} \vartheta_T \to \infty \), the sequences \( \tilde{R}_{Tt} \) and \( R_{Tt} \) are asymptotically equivalent in the sense that \( \sum_{t=2}^{T} R_{Tt} = \sum_{t=2}^{T} \tilde{R}_{Tt} + o_p(1) \).

Now

\[
E_{t-1}(\tilde{R}_{Tt}^2) = 1 \left( \frac{\lvert c_T y_{t-1} \rvert}{T} \leq \vartheta_T \right) E_{t-1} \left[ R_f \left( \varepsilon_t, \frac{c_T y_{t-1}}{T} \right)^2 \right] 
\leq V_f(\vartheta_T) \frac{c_T^2}{T^2} y_{t-1}^2,
\]

where \( V_f(\vartheta) := \sup_{\theta | \vartheta, \theta \neq 0} \theta^{-2} E[R_f(\varepsilon, \theta)^2] \) and \( E_{t-1}[\cdot] \) denotes conditional expectation given \( \{ \varepsilon_1, \ldots, \varepsilon_{t-1} \} \). By Assumption DQM, \( \lim_{\vartheta \downarrow 0} V_f(\vartheta) = 0 \). As a consequence, using \( \vartheta_T = o(1) \) and \( E(y_{t-1}^2) = t - 1 \),

\[
\sum_{t=2}^{T} E_{t-1}(\tilde{R}_{Tt}^2) \leq V_f(\vartheta_T) E \left( \frac{c_T^2}{T^2} \sum_{t=2}^{T} y_{t-1}^2 \right) = V_f(\vartheta_T) O(1) = o(1),
\]

implying that \( \sum_{t=2}^{T} \tilde{R}_{Tt} = \sum_{t=2}^{T} E_{t-1}(\tilde{R}_{Tt}) + o_p(1) \). Moreover,

\[
\sum_{t=2}^{T} E_{t-1}(\tilde{R}_{Tt}) = -\frac{1}{4} I_{ff} c_T^2 \frac{2}{T^2} \sum_{t=2}^{T} 1 \left( \lvert \frac{c_T y_{t-1}}{T} \rvert \leq \vartheta_T \right) y_{t-1}^2 + \sum_{t=2}^{T} 1 \left( \lvert \frac{c_T y_{t-1}}{T} \rvert \leq \vartheta_T \right) r_f \left( \frac{c_T y_{t-1}}{T} \right),
\]

where \( r_f(\theta) := \frac{1}{2} I_{ff} \theta^2 + E[R_f(\varepsilon, \theta)] \) and

\[
\frac{1}{T^2} \sum_{t=2}^{T} 1 \left( \lvert \frac{c_T y_{t-1}}{T} \rvert \leq \vartheta_T \right) y_{t-1}^2 = \frac{1}{T^2} \sum_{t=2}^{T} y_{t-1}^2 + o_p(1)
\]

because \( \max_{2 \leq t \leq T} |c_T y_{t-1} / \sqrt{T}| = O_p(1) \) and \( \sqrt{T} \vartheta_T \to \infty \). The proof of (S1) can therefore be completed by showing that

\[
\sum_{t=2}^{T} 1 \left( \lvert \frac{c_T y_{t-1}}{T} \rvert \leq \vartheta_T \right) r_f \left( \frac{c_T y_{t-1}}{T} \right) = o_p(1).
\]

The relationship in the preceding display follows from \( \vartheta_T = o(1) \) and the fact that

\[
\sum_{t=2}^{T} 1 \left( \lvert \frac{c_T y_{t-1}}{T} \rvert \leq \vartheta_T \right) r_f' \left( \frac{c_T y_{t-1}}{T} \right) \leq v_f(\vartheta_T) \frac{c_T^2}{T^2} \sum_{t=2}^{T} y_{t-1}^2
\]
\[ v_f(\theta_T) = v_f(\theta) = \sup_{|\theta| \leq \vartheta, \theta \neq 0} \theta^{-2} |r_f(\theta)| = o(1) \text{ as } \theta \downarrow 0 \text{ (Pollard (1997, Lemma 1)).} \]

Equation (S2). To prove (S2), it suffices to show that
\[
\sum_{t=2}^{T} \left[ \frac{c_T}{T} y_{t-1} \ell_f(\varepsilon_t) + R_{T_t} \right]^2 = c_T^2 \frac{T_{ff}}{T^2} \sum_{t=2}^{T} y_{t-1}^2 + o_p(1)
\]
and
\[
\max_{2 \leq t \leq T} |\beta(c_T T^{-1} y_{t-1} \ell_f(\varepsilon_t) + R_{T_t})| = o_p(1).
\]

By Taylor’s theorem, \( \beta(r) \rightarrow 0 \) as \( |r| \rightarrow 0 \). Moreover,
\[
\max_{2 \leq t \leq T} \left| \frac{y_{t-1}}{T} \ell_f(\varepsilon_t) \right| \leq \max_{2 \leq t \leq T} \left| \frac{y_{t-1}}{\sqrt{T}} \right| \max_{2 \leq t \leq T} \left| \frac{\ell_f(\varepsilon_t)}{\sqrt{T}} \right| = O_p(1) o_p(1) = o_p(1)
\]
and \( \max_{2 \leq t \leq T} |R_{T_t}| \leq \sqrt{\sum_{t=2}^{T} R_{T_t}^2} \). Therefore, the desired result will follow from

(S3) \[
\frac{1}{T^2} \sum_{t=2}^{T} y_{t-1}^2 \ell_f(\varepsilon_t)^2 = \frac{T_{ff}}{T^2} \sum_{t=2}^{T} y_{t-1}^2 + o_p(1)
\]
and

(S4) \[
\sum_{t=2}^{T} R_{T_t}^2 = o_p(1).
\]

As noted by Jeganathan (1995, Lemma 24), (S3) can be deduced with the help of the proof of Hall and Heyde (1980, Theorem 2.23) if it can be shown that
\[
\frac{1}{T^2} \sum_{t=2}^{T} E_{t-1} \left[ y_{t-1}^2 \ell_f(\varepsilon_t)^2 1\left( \left| \frac{y_{t-1}}{T} \ell_f(\varepsilon_t) \right| > \varrho \right) \right] = o_p(1) \quad \forall \varrho > 0.
\]

To do so, let \( \varrho > 0 \) be given and define \( Q_f(r) := E[\ell_f(\varepsilon)^2 1(|\ell_f(\varepsilon)| > r)] \). Because \( Q_f \) is nonincreasing and \( \lim_{r \rightarrow \infty} Q_f(r) = 0 \),
\[
\frac{1}{T^2} \sum_{t=2}^{T} E_{t-1} \left[ y_{t-1}^2 \ell_f(\varepsilon_t)^2 1\left( \left| \frac{y_{t-1}}{T} \ell_f(\varepsilon_t) \right| > \varrho \right) \right]
\]
\[
= \frac{1}{T^2} \sum_{t=2}^{T} y_{t-1}^2 Q_f\left( \frac{\sqrt{T} \varrho}{|y_{t-1}/\sqrt{T}|} \right)
\]
\[
\leq \left( \frac{1}{T^2} \sum_{t=2}^{T} y_{t-1}^2 \right) \max_{2 \leq t \leq T} Q_f \left( \frac{\sqrt{T} \varrho}{|y_{t-1}/\sqrt{T}|} \right)
= O_p(1) \sigma_p(1) = o_p(1),
\]
where the penultimate equality uses \( \max_{2 \leq t \leq T} |y_{t-1}/\sqrt{T}| = O_p(1) \).

It can be shown that \( \sum_{t=2}^{T} R_{tT}^2 = \sum_{t=2}^{T} \tilde{R}_{tT}^2 + o_p(1) \). Moreover,

\[
\sum_{t=2}^{T} E_{t-1} \left[ \tilde{R}_{tT}^2 | |R_{tT}| > \varrho \right] \leq \sum_{t=2}^{T} E_{t-1} (\tilde{R}_{tT}^2) = o_p(1) \quad \forall \varrho > 0,
\]

where the equality was established in the proof of (S1). A second application of the proof of Hall and Heyde (1980, Theorem 2.23) therefore establishes (S4).

**Q.E.D.**

**PROOF OF LEMMA 7:** For any \( b \), any \( c < 0 \), any \( \alpha \in (0, 1) \), and any symmetric \( 2 \times 2 \) matrix \( \mathcal{I}_F \) for which

\[
\text{Var} \left( \begin{pmatrix} W(1) \\ B_F(1) \end{pmatrix} \right) = \begin{pmatrix} 1 & e'_1 \\ e_1 & \mathcal{I}_F \end{pmatrix}
\]

is positive semidefinite, let \( K^S_{\alpha}(b, c; \mathcal{I}_F) \) be the \( 1 - \alpha \) quantile of

\[
G(W, Z, b, c; \mathcal{I}_F)
:= c \left[ \int_0^1 W(r) \, dW(r) + \frac{\mathcal{H}_{f\eta} b}{\mathcal{H}_{\eta\eta}} + \sqrt{\mathcal{H}_{ff\eta} - \int_0^1 W(r)^2 \, dr} \, Z \right]
- \frac{1}{2} e^2 \mathcal{H}_{ff},
\]

where \( Z \sim \mathcal{N}(0, 1) \) is independent of \( W \) and \( \mathcal{H}_{f\eta}, \mathcal{H}_{\eta\eta} \), etc. are as in Section 4.

The function \( K^S_{\alpha} \) satisfies \( E[\psi^S_{\alpha}(S_F, \mathcal{H}_F|c, \alpha)|S_n] = \alpha \) because it follows from elementary facts about Brownian motions that

\[
\frac{S_{f\eta} - \int_0^1 W(r) \, dW(r)}{\sqrt{\mathcal{H}_{ff\eta} - \int_0^1 W(r)^2 \, dr}} \sim \mathcal{N}(0, 1)
\]

independent of \( W \) and \( S_n \), where \( S_{f\eta} \) and \( S_n \) are as in Section 4.

Continuity of \( K^S_{\alpha} \) follows from the fact that \( G(W, Z, b_n, c_n; \mathcal{I}_{F,n}) \) converges in distribution to a continuous random variable whenever the sequence \( (b_n, c_n, \mathcal{I}_{F,n}) \) is convergent (and \( G(W, Z, b_n, c_n; \mathcal{I}_{F,n}) \) is well defined for each \( n \)).

**Q.E.D.**
PROOF OF (27): Let \( f \in \mathcal{F}_{\text{DOM}} \) and \( c < 0 \) be given, suppose \( F \) satisfies Assumption DQM*, and let \((S^F_T, H^F_T, S^S_{f_T}, S^S_{\eta_T})\) → \( (S_f, \mathcal{H}_{ff}, S^S_f, S^S_{\eta}) \), etc. be as in Section 4. Because \( K^S_\alpha \) is continuous (Lemma 7) and

\[
(S^F_T, H^F_T, S^S_{f_T}, S^S_{\eta_T}) \to_{d_0} (S_f, \mathcal{H}_{ff}, S^S_f, S^S_{\eta}),
\]

the sequence \( \phi^{S}_{f,T}(\cdot|c, \alpha) \) satisfies

\[
\phi^{S}_{f,T}(Y_T|c, \alpha) \to_{d_0} \psi^{S}_{f}(S_f, \mathcal{H}_{ff}, S^S_f|c, \alpha).
\]

It follows from these convergence results, Le Cam’s third lemma, and the result

\[
L^F_T(c, h) \to_{d_0} A_F(c, h) := (c, h)S_F - \frac{1}{2}(c, h)\mathcal{H}_F(c, h) \quad \forall (c, h)
\]

that

\[
\lim_{T \to \infty} E_{\rho_T(c'), \eta_T(h)} \phi^{S}_{f,T}(Y_T|c, \alpha; f) = E[\psi^{S}_{f}(S_f, \mathcal{H}_{ff}, S^S_f|c, \alpha) \exp(A_F(c', h))]
\]

for every \((c', h)\). In particular, \( \lim_{T \to \infty} E_{\rho_T(c), \eta_T(h)} \phi^{S}_{f,T}(Y_T|c, \alpha; f) = \Psi^{S}_{f}(c, \alpha) \), implying that the proof of (27) can be completed by showing that \( \phi^{S}_{f,T}(\cdot|c, \alpha) \) is locally asymptotically \( \alpha \)-similar in \( F \).

To do so, it suffices to show that \( E[\psi^{S}_{f}(S_f, \mathcal{H}_{ff}, S^S_f|c, \alpha)|S_\eta] = \alpha \). Let

\[
S^\perp_\eta := S_\eta - \frac{I_{_{ff}}}{I_{_{ff}} - 1} S^S_f.
\]

Because \( B_\eta - I_{_{ff}}(I_{_{ff}} - 1)^{-1}(B_f - W) \) and \((W, B_f)\) are independent, \( S^\perp_\eta \) is independent of \((S_f, \mathcal{H}_{ff}, S^S_f)\) and

\[
E[\psi^{S}_{f}(S_f, \mathcal{H}_{ff}, S^S_f|c, \alpha)|S^\perp_\eta, S^S_f] = E[\psi^{S}_{f}(S_f, \mathcal{H}_{ff}, S^S_f|c, \alpha)|S^S_f] = \alpha,
\]

where the second equality is the defining property of \( K^S_\alpha \). Because \( S_\eta \) is a function of \((S^S_f, S^\perp_\eta)\), it therefore follows from the law of iterated expectations that

\[
E[\psi^{S}_{f}(S_f, \mathcal{H}_{ff}, S^S_f|c, \alpha)|S_\eta] = E(E[\psi^{S}_{f}(S_f, \mathcal{H}_{ff}, S^S_f|c, \alpha)|S^S_f, S^\perp_\eta]|S_\eta) = \alpha,
\]

as was to be shown.

\( Q.E.D. \)
REFERENCES


