Corrigendum to "On the Existence of Pure and Mixed Strategy Nash Equilibrium in Discontinuous Games,"

by

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Reny (1999) contains two errors. Neither error affects the main results of the paper, namely Theorem 3.1 and Corollary 5.2. The only results that are affected are those in Sections 4 and 5 that pertain to quasi-symmetric games.[4] Even then, the main symmetric equilibrium existence result for quasi-symmetric games, Theorem 4.1, is not affected.

The first error concerns the definition of diagonal payoff security on p.1041 in Section 4.[5] There are two "y's" that are missing from that definition and so it should be corrected to say the following.

Definition: $G = (X_i, u_i)_{i=1}^N$ is *diagonally payoff secure* if for every $x, y \in X$ and for every $\varepsilon > 0$, each player i can secure a payoff of $u_i(x, ..., y, ..., x) - \varepsilon$ along the diagonal at $(x, ..., x)$.[6]

This corrected definition is in fact the definition originally given in Reny (1996).[7] Ewerhart (2022) points out the error in Reny (1999) and independently provides the correction.[8]

With this correction to the definition of diagonal payoff security, all of the affected results in Section 4 of Reny (1999) are rendered correct as stated. In particular, the guidance on p.1041 for proving Proposition 4.2 is now correct and easily followed.[9] We note that a sufficient condition for a game to be diagonally payoff secure as above is that for every $y \in X$, each player's payoff when he plays y and all the others play x is lower semicontinuous in x on X.

To see that the missing "y's" in the original definition of diagonal payoff security given in Reny (1999) are needed, we exhibit a game, let us call it G^0, that satisfies the (pre-correction) hypotheses of Reny's Corollary 4.3 but does not possess a symmetric pure strategy Nash equilibrium.[10] Consider the two-person game on the unit square in which a player's payoff, $u(x, y)$, when he chooses x, is x if the other player chooses $y < 1/2$ but is $1 - x$ if the other player chooses $y \geq 1/2$. Then each player's payoff is continuous in x when both players choose x because, in that case, their payoffs are each $\min\{x, 1 - x\}$. The game G^0

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[4] Specifically, Proposition 4.2, Corollary 4.3, and Example 4.1 in Section 4, and the second part of Corollary 5.3 in Section 5.

[5] All notation here is as in Reny (1999, Sections 4 and 5).

[6] See Reny (1999, p.1040) for the definition of securing a payoff along the diagonal.

[7] Reny (1996) was the initial submission to *Econometrica* for Reny (1999). One of the present authors, Reny, evidently dropped the "y's" during the process of revision and accepts all responsibility.

[8] Ewerhart (2022) uses the term "*strong* diagonal payoff security" for the corrected definition. We drop "strong" here.

[9] Once Proposition 4.2 is rendered correct as stated, all of the other affected results become correct as stated because they follow from Proposition 4.2 and (the unaffected) Theorem 4.1.

[10] This game is due to Ewerhart (2022).

is diagonally quasiconcave (Reny 1999, p.1040) because if x is a convex combination of two or more pure strategies, then two of those strategies, y and z say, must satisfy $y \leq x \leq z$, in which case $u(x, x) \geq \min\{u(y, x), u(z, x)\}$ (because $u(x, x) = x \geq y = u(y, x)$ if $x < 1/2$, and $u(x, x) = 1 - x \geq 1 - z = u(z, x)$ if $x \geq 1/2$). Moreover, G^0 is diagonally payoff secure according to the original definition in Reny (1999) since for any pure strategy x, the pure strategy $\bar{x} = 1/2$ secures the payoff $u(x, x)$ along the diagonal at (x, x) because $u(1/2, y) = 1/2 \geq \min\{x, 1 - x\} = u(x, x)$ holds for every y. Hence, G^0 satisfies the (pre-correction) hypotheses of Corollary 4.3 in Reny (1999). However, G^0 does not possess a symmetric pure strategy Nash equilibrium, contradicting the conclusion of that corollary, because for every strategy profile in which both players choose x, each player has a profitable deviation to either $y = 0$ or to $y = 1$.

But observe that, after the correction, G^0 no longer contradicts Corollary 4.3 (which becomes correct as stated) because G^0 does not satisfy the corrected definition of diagonal payoff security. Indeed, consider $x = 1/2$ and $y = 0$. Then $u(y, x) = 1$ and the player choosing $y = 0$ cannot secure a payoff of more than $1/2$, let alone $1 - \varepsilon$ for small ε, along the diagonal at $(x, x) = (1/2, 1/2)$ because for any $\bar{x}$, there is x' arbitrarily near $x = 1/2$ such that $u(\bar{x}, x') = \min\{\bar{x}, 1 - \bar{x}\} \leq 1/2$.

The second error in Reny (1999), independently pointed out by John Duggan and Asaf Plan, is in Section 5. In the statement of Corollary 5.3 on p.1047, instead of assuming merely that the game G is quasi-symmetric, it should be assumed that $\bar{G}$, the mixed extension of G, is quasi-symmetric. Plan (2017) provides a novel symmetry condition on a game that ensures that its mixed extension is quasi-symmetric.

References

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