

AFFIRMATIVE ACTION IN INDIA VIA VERTICAL, HORIZONTAL, AND OVERLAPPING RESERVATIONS

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ABSTRACT. Sanctioned by its constitution, India is home to the world’s most comprehensive affirmative action program, where historically discriminated groups are protected with vertical reservations implemented as “set asides,” and other disadvantaged groups are protected with horizontal reservations implemented as “minimum guarantees.” A mechanism mandated by the Supreme Court in 1995 suffers from important anomalies, triggering countless litigations in India. Foretelling a recent reform correcting the flawed mechanism, we propose the 2SMG mechanism that resolves all anomalies, and characterize it with desiderata reflecting laws of India. Subsequently rediscovered with a high court judgment and enforced in Gujarat, 2SMG is also endorsed by *Saurav Yadav v. State of UP (2020)*, in a Supreme Court ruling that rescinded the flawed mechanism. While not explicitly enforced, 2SMG is indirectly enforced for an important subclass of applications in India, because no other mechanism satisfies the new mandates of the Supreme Court.

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1. Introduction

Sanctioned by its constitution, India is home to one of the world's largest affirmative action programs. Allocation of homogeneous government positions is directed by a series of Supreme Court mandates discussed in Section 1.2. Under these mandates, a mechanism that otherwise allocates positions based on a merit ranking of individuals is amended to implement two types of affirmative action policies known as *vertical reservations* (VR) and *horizontal reservations* (HR). Under both policies, a fraction of positions are "reserved" for each of a number of protected groups. The key conceptual distinction between the two policies lies within the answer to the following question:

If a member of a protected group is "entitled" to receive an unreserved position based on her merit ranking, then is she awarded an unreserved position or a reserved position?

For the case of the VR policy, any such individual is awarded an unreserved position, thereby making it easier for the remaining (lower merit-ranking) members of the VR-protected group to secure a reserved position. For the case of the HR policy, in contrast, any such individual (up to capacity) is awarded a reserved position, thereby using up an HR-protected position for a member of the protected group who is in no need of positive discrimination, and thus effectively rendering the protected status of the position redundant.¹

When used as a stand-alone policy to provide positive discrimination for a single protected group, the VR policy can be implemented with the following procedure:

Over-and-Above Choice Rule (Dur et al., 2018)

Step 1. Allocate the unreserved positions to the highest merit-ranking individuals.

Step 2. Allocate the reserved positions to the highest merit-ranking members of the VR-protected group who have not received an unreserved position in Step 1.

Similarly, the HR policy can be implemented with the following procedure when it is used as a stand-alone policy that provides positive discrimination for a single group:

Minimum Guarantee Choice Rule (Echenique and Yenmez, 2015)

Step 1. Allocate the reserved positions to the highest merit-ranking members of the HR-protected group.

¹This important distinction between the two policies is intentional, and it captures the different roles they are meant to serve. VR policy is devised as a higher-level protection that provides an additional boost to the share of positions awarded to the historically oppressed groups in excess of their share based on merit rankings only. HR policy, on the other hand, is devised as a lower-level protection that provides a "minimum guarantee" to the other disadvantaged groups. VR-protected groups in India include *Scheduled Castes*, *Scheduled Tribes*, and *Other Backward Classes*, which collectively account for about 70 percent of the population, and HR-protected groups include persons with disabilities.

Step 2. Allocate any remaining positions to the highest merit-ranking individuals who have not received a reserved position in Step 1.

Observe that the two procedures differ mainly in the processing sequence of the reserved and unreserved positions. Assuming that the protected groups do not overlap, a generalized procedure which processes the HR-protected positions prior to unreserved positions and the VR-protected positions subsequent to unreserved positions can be used to implement the two policies concurrently for multiple protected groups. However, while the VR-protected groups do not overlap with each other in India, they overlap with the HR-protected groups. Moreover, in some field applications, the HR-protected groups also overlap with each other. This additional complexity due to the overlapping structure of protected groups not only introduces a number of theoretical challenges which are not addressed in the literature, but also triggers large scale implementation challenges observed in the field. Most notably, a poorly designed choice rule may generate “frictions” between the two policies. Importantly, this theoretical possibility has materialized for decades in India due to an especially flawed choice rule of the Supreme Court where individuals risk losing their HR protections upon claiming their VR protections. This crisis, documented in detail in Online Appendix C, is the starting point of our analysis.

In this paper we study a previously unexplored version of the problem where the HR-protected groups overlap both with the VR-protected groups and also with each other. As our main theoretical contributions, we formulate the Supreme Court’s mandates on joint implementation of the VR and HR policies as four formal axioms, and characterize the unique choice rule that satisfies all four. As our contributions to the field of market design, we relate our analysis to a series of court rulings in the last 30 years.

1.1. Contributions to Theory. Parallel to the practice in India, throughout the paper we assume that the VR-protected groups do not overlap with each other. We refer to positions that are reserved for a VR-protected category c of individuals as *category- c* positions, and the remaining positions as *open-category* positions. While any individual is eligible to receive an open-category position, only members of a VR-protected category c are eligible to receive a category- c position. HR protections are implemented within each vertical category of positions (including the open category). For example, for the case of a single VR-protected category *Scheduled Castes* (SC) and a single HR-protected group *women*, minimum guarantee constraints for women are specified for the open-category positions and for the category-SC positions, separately. Unlike the VR policy, the HR policy is a “soft” reserve policy. This means, if there are more HR-protected positions reserved for any group than the size of the group, then any excess positions can be awarded to individuals who are not members of the HR-protected group (subject to category eligibility).

An *outcome* selects a distinct set of individuals for each category of positions subject to capacity and eligibility constraints, and a *choice rule* selects an outcome for each problem. The following concept is central in our analysis.

Given a category v of positions and a set J of eligible individuals for category v (who can be thought as the set of individuals tentatively holding the category- v positions), a category- v eligible individual $i \notin J$ *increases the HR utilization at category v* , if assignment of one of the category- v positions to individual i (possibly replacing an individual in J) strictly increases the number of HR-protected positions at category v which are awarded to members of the intended HR-protected groups.

Our approach is axiomatic. We formulate the following four axioms, each defined both for outcomes and choice rules, as desiderata reflecting the Indian legislation on concurrent implementation of VR and HR policies:

1. **Non-wastefulness:** A position at any given category v can remain idle *only if* no individual who remains unassigned is eligible for a category- v position.
2. **Maximal accommodation of HR protections:** An individual can remain unassigned *only if* she does not increase the HR utilization at any category for which she has eligibility.
3. **No justified envy:** A lower merit-ranking individual i can receive a position at any given category v at the expense of an eligible higher merit-ranking individual j who remains unassigned *only if* replacing individual j with individual i increases the HR utilization at category v .
4. **Compliance with VR protections:** A VR-protected position can be awarded to an eligible individual i (rather than an open-category position) *only if* all of the following three conditions hold:
 - a. No open-category position remains idle.
 - b. For any lower merit-ranking individual j (than individual i) who is assigned an open category position, replacing individual j with individual i decreases the HR utilization at the open category.
 - c. Individual i does not increase the HR utilization at open category.

When the HR-protected groups do not overlap with each other, the unique choice rule that satisfies the four axioms is the *two-step minimum guarantee* (2SMG) choice rule which first allocates the open-category positions with the minimum guarantee choice rule, and next allocates the positions at each VR-protected category to its remaining members with the same choice rule (Theorem 1).

Analysis of the problem with overlapping HR protections is more involved. Assuming *women* and *persons with disabilities* are two HR-protected groups, consider a disabled

woman. Legislation in India is silent on whether this individual accommodates (upon admission) the minimum guarantee constraints both for women and persons with disabilities (*one-to-all HR matching*) or only for one of these HR-protected groups (*one-to-one HR matching*). Adopting the one-to-one HR matching convention,² we first focus on the stand-alone implementation of the HR policy, and introduce a generalization of the minimum guarantee choice rule building on an insight from the following simple example.

There are three individuals i_1, i_2, i_3 who are merit ranked based on their index. There are two positions, one that is HR-protected for persons with disabilities and one that is HR-protected for women. Individual i_1 is a disabled woman, individual i_2 is a disabled man, and individual i_3 is a woman with no disability. Hence i_1 can accommodate the minimum guarantee constraint for either one of the HR-protected groups, i_2 can accommodate the minimum guarantee constraint only for persons with disabilities, and i_3 can accommodate the minimum guarantee constraint only for women. If we follow the traditional approach in the literature and process the HR-protected positions in a mechanical way for a given fixed sequence of protected groups, say first for persons with disabilities and then for women, the HR-protected position for persons with disabilities is awarded to i_1 and the HR-protected position for women is awarded to the only remaining woman i_3 . But this outcome is implausible, because the highest merit-ranking individual i_1 could have been instead awarded the HR-protected position for women, which would then enable the second highest merit-ranking individual i_2 to receive the HR-protected position for persons with disabilities. Under this alternative outcome, it would have been the lowest merit-ranking individual i_3 who remains unassigned rather than the second highest merit-ranking individual i_2 . This example highlights the need for a “smart” processing of reserved positions, which is the basis for the *meritorious horizontal choice rule* introduced in Section 4.2.3. In the absence of the VR policy, the *meritorious horizontal choice rule* is the only choice rule that satisfies *non-wastefulness*, *maximal accommodation of HR protections*, and *no justified envy* (Theorem 2).

Starting with Kominers and Sönmez (2016), earlier literature on reserve systems restricted attention to choice rules where positions are allocated sequentially for a given processing sequence of positions and using a position-specific priority order of individuals at each position. We consider the meritorious horizontal choice rule as one of our

²As explained in detail in Section 4.1, we adopt the one-to-one HR matching convention for two reasons. The first reason is technical. Adoption of the alternative one-to-all HR matching convention introduces complementarities between individuals, a complexity that not only generates multiplicities, but also renders the problem to be computationally hard. In the above example, the admission of a man with no disability may be tied to the admission of a woman with disability. The second reason is practical. Often the number of positions is announced for pairs of VR-protected and HR-protected groups in India, which automatically embeds the one-to-one HR matching convention into the problem.

main conceptual contributions, since it is the first choice rule that instead utilizes the above-described “smart reserve processing” approach.

For the most general version of the problem with both VR and overlapping HR protections, we introduce the *two-step meritorious horizontal* (2SMH) choice rule which first allocates the open-category positions with the meritorious horizontal rule, and next allocates the positions at each VR-protected category to its remaining members with the same choice rule. As our most general theoretical result, in Theorem 3 we characterize the 2SMH choice rule as the unique choice rule that satisfies all four axioms.

1.2. Relation to Legislation in India and Policy Implications. The distinction between the VR and HR policies is first identified by the landmark Supreme Court judgment *Indra Sawhney and others v. Union of India* (1992), widely known as the *Mandal Commission Case*.³ While this judgement clearly laid out the principles that guide the implementation of these policies when either policy is implemented by itself, it has not provided detailed guidance on their concurrent implementation. This gap has been later filled in *Anil Kumar Gupta v. State of U.P.* (1995),⁴ another judgment of the Supreme Court, where a procedure was devised and enforced in India.⁵ We refer to this procedure as the *SCI-AKG choice rule*. For the past quarter century, this judgment has served as a main reference for numerous litigations on concurrent implementation of VR and HR policies. At the time of the initial submission of our paper to this journal, the SCI-AKG choice rule was still in effect in India.

The SCI-AKG choice rule derives its outcome in several steps as follows: The procedure first ignores the HR protections and derives a tentative outcome using the over-and-above choice rule, then it makes any necessary replacements for the open-category positions to accommodate HR protections within open-category positions, and finally it makes any necessary replacements for the VR-protected positions to accommodate HR protections within VR-protected positions. One critical aspect of this procedure, however, has introduced a highly consequential anomaly into the procedure, often generating unintuitive outcomes at odds with the philosophy of affirmative action, and thereby sparking thousands of litigations in India for the next 25 years.⁶ To present the scale of the resulting disarray, some of the key litigations triggered by the flawed procedure are documented in detail in Online Appendix C.

³The case is available at <https://indiankanoon.org/doc/1363234/> (last accessed on 10/02/2021).

⁴The case is available at <https://indiankanoon.org/doc/1055016/> (last accessed on 10/02/2021).

⁵The procedure is uniquely defined only for the case of non-overlapping HR protections.

⁶A search of the phrase “horizontal reservation” via Indian Kanoon, a free search engine for Indian Law, reveals the scale of the litigations relating to this concept. Excluding cases at lower courts, as of 10/02/2021 there are 2128 cases at the Supreme Court and State High Courts on implementation of HR policy.

The root cause of the failure of the SCI-AKG choice rule boils down to its exclusion of the members of the VR-protected groups from any replacements necessary to accommodate the HR protections within open-category positions. This restriction creates situations where higher merit-ranking individuals from VR-protected groups lose their positions to lower merit-ranking individuals from the higher-privilege general category, thus resulting in failure of the axiom *no justified envy*.⁷ Therefore, a simple and intuitive solution lies in the removal of this restriction. For the case of non-overlapping HR protections, this simple adjustment results in the 2SMG choice rule. As already discussed in Section 1.1, not only the 2SMG choice rule satisfies *no justified envy*, but it is also the unique choice rule that satisfies this axiom along with three additional axioms (Theorem 1).

SCI-AKG choice rule, in contrast, not only fails *no justified envy*, but also *maximal accommodation of HR protections* and *compliance with VR protections*. While all our axioms are formulated to capture the principles outlined in *Indra Sawhney (1992)*, only the *non-wastefulness* axiom was mandated in the country until recently. In a December 2020 judgement of the Supreme Court that parallels our analysis and policy recommendations, this situation has completely changed with *Saurav Yadav & Ors v. State of Uttar Pradesh & Ors (2020)*.⁸ With this recent judgement, (i) all four axioms became federally mandated, (ii) the SCI-AKG choice rule is rescinded, and (iii) the 2SMG choice rule is endorsed. Critically, while the 2SMG choice rule is not explicitly mandated by *Saurav Yadav (2020)*, Theorem 1 implies that it is indirectly enforced for the case of non-overlapping HR protections.

1.3. Organization of the Rest of the Paper. After the model is introduced in Section 2, an analysis for the simpler version of the problem with non-overlapping HR protections is presented in Section 3. We present an analysis for the more general version of the model with overlapping HR protections in Section 4. A discussion of the related theoretical literature is also presented in this section. We conclude with an epilogue in Section 5 which presents an in depth discussion of *Saurav Yadav (2020)*, the judgement that reconciles our policy recommendations and the legislation in India. We present all proofs in the Appendix. Finally, we relegate the institutional background on VR and HR policies, extensive evidence from Indian court rulings on the disruption caused by the SCI-AKG choice rule, and the equivalence of our formulation of the SCI-AKG choice rule with its original formulation in *Anil Kumar Gupta (1995)* to the Online Appendix.

⁷The same restriction also creates a conflict for individuals who qualify for both types of protections, since for these individuals claiming their VR protections would mean giving up their HR protections for open-category positions, an anomaly we refer to as a failure of *incentive compatibility*. Both types of failures have been originally formulated in Aygün and Bó (2021) in the context of Brazilian college admissions, although our paper is the first one to document their disruptive implications through numerous litigations and interrupted recruitment processes.

⁸The case is available at <https://indiankanoon.org/doc/27820739/> (last accessed on 10/04/2021).

2. Model and Vertical/Horizontal Reservations

There exists a finite set of individuals \mathcal{I} competing for q identical positions. Each individual $i \in \mathcal{I}$ is in need of a single position, and has a distinct merit score $\sigma(i) \in \mathbb{R}_+$.⁹ While individuals with higher merit scores have higher claims for a position in the absence of affirmative action policies, disadvantaged populations are protected through two types of affirmative action policies, (i) **vertical reservation (VR)** policies providing “higher level” **VR protections**, and (ii) **horizontal reservation (HR)** policies providing “lower level” **HR protections**.

2.1. Vertical Reservations. There exists a set of **reserve-eligible categories** \mathcal{R} and a **general category** $g \notin \mathcal{R}$. Each individual belongs to a single category in $\mathcal{R} \cup \{g\}$. Define the (reserve-eligible) category membership function $\rho : \mathcal{I} \rightarrow \mathcal{R} \cup \{\emptyset\}$ such that, for any individual $i \in \mathcal{I}$,

$$\begin{aligned} \rho(i) = c & \text{ indicates that } i \text{ is a member of the reserve-eligible category } c \in \mathcal{R}, \text{ and} \\ \rho(i) = \emptyset & \text{ indicates that } i \text{ is a member of the general category } g. \end{aligned}$$

Given a set of individuals $I \subseteq \mathcal{I}$ and a reserve-eligible category $c \in \mathcal{R}$, define

$$I^c = \{i \in I : \rho(i) = c\}$$

as the set of individuals in I who are members of the reserve-eligible category $c \in \mathcal{R}$. Given a set of individuals $I \subseteq \mathcal{I}$, define

$$I^g = \{i \in I : \rho(i) = \emptyset\}$$

as the set of individuals in I who are members of the general category g .

There are q^c positions exclusively set aside for the members of category $c \in \mathcal{R}$. We refer to these positions as **category-c positions**. In contrast, members of the general category do not receive any special provisions under the VR policies. Therefore,

$$q^o = q - \sum_{c \in \mathcal{R}} q^c$$

positions are open for all individuals. We refer to these positions as **open-category positions** (or **category-o positions**). Let $\mathcal{V} = \mathcal{R} \cup \{o\}$ denote the set of **vertical categories for positions**.

⁹While students can have the same merit score in practice, tie-breaking rules are used to strictly rank them. For example, the Union Public Service Commission uses age and exam scores to break ties. See <https://www.upsc.gov.in/sites/default/files/TiePrinciplesEngl-26022020-R.pdf> (last accessed on 6/7/2020).

It is important to emphasize that, in contrast to category- c positions that are exclusively reserved for the members of category $c \in \mathcal{R}$, open-category positions are available for all, and hence they are not exclusively reserved for the members of the general category g .

Definition 1. Given a reserve-eligible category $c \in \mathcal{R}$, an individual $i \in \mathcal{I}$ is *eligible for category- c positions* if,

$$\rho(i) = c.$$

Any individual $i \in \mathcal{I}$ is *eligible for open-category positions*.

Given a category $v \in \mathcal{V}$, let $\mathcal{I}^v \subseteq \mathcal{I}$ denote the set of individuals who are eligible for category- v positions.

VR protections have one important property that makes them the “higher level” affirmative action policy. Positions that are earned by the members of reserve-eligible categories without invoking the VR protections, and thus on the basis of their merit scores only, do not count against the VR-protected positions. In this sense, VR protections are implemented on an “over-and-above” basis.

2.2. Single-Category Choice Rule, Choice Rule, and Aggregate Choice Rule. We next formulate the solution concepts used in our paper.

Definition 2. Given a category $v \in \mathcal{V}$, a *single-category choice rule* is a function $C^v : 2^{\mathcal{I}} \rightarrow 2^{\mathcal{I}^v}$ such that, for any $I \subseteq \mathcal{I}$,

$$C^v(I) \subseteq I \cap \mathcal{I}^v \quad \text{and} \quad |C^v(I)| \leq q^v.$$

Definition 3. A *choice rule* is a multidimensional function $C = (C^v)_{v \in \mathcal{V}} : 2^{\mathcal{I}} \rightarrow \prod_{v \in \mathcal{V}} 2^{\mathcal{I}^v}$ such that, for any $I \subseteq \mathcal{I}$,

(1) for any category $v \in \mathcal{V}$,

$$C^v(I) \subseteq I \cap \mathcal{I}^v \quad \text{and} \quad |C^v(I)| \leq q^v,$$

(2) for any two distinct categories $v, v' \in \mathcal{V}$,

$$C^v(I) \cap C^{v'}(I) = \emptyset.$$

In addition to specifying the recipients, our formulation of a choice rule also specifies the categories of their positions.

Definition 4. For any choice rule $C = (C^v)_{v \in \mathcal{V}}$, the resulting *aggregate choice rule* $\hat{C} : 2^{\mathcal{I}} \rightarrow 2^{\mathcal{I}}$ is given as

$$\hat{C}(I) = \bigcup_{v \in \mathcal{V}} C^v(I) \quad \text{for any } I \subseteq \mathcal{I}.$$

For any set of individuals, the aggregate choice rule yields the set of chosen individuals across all categories.

In the absence of horizontal reservations, which will be introduced in Section 2.3, the following three principles mandated in *Indra Sawhney (1992)* uniquely define a choice rule, thus making the implementation of VR policies straightforward. First, an allocation must respect *inter se* merit: Given two individuals from the same category, if the lower merit-score individual is awarded a position, then the higher merit-score individual must also be awarded a position. Next, VR protections must be allocated on an “over-and-above” basis; i.e., positions that can be received without invoking the VR protections do not count against VR-protected positions. Finally, subject to eligibility requirements, all positions have to be filled without contradicting the two principles above. It is easy to see that these three principles uniquely imply the following choice rule: First, individuals with the highest merit scores are assigned the open-category positions. Next, positions reserved for the reserve-eligible categories are assigned to the remaining members of these categories, again based on their merit scores. We refer to this choice rule as the *over-and-above choice rule*.

2.3. Horizontal Reservations within Vertical Categories. In addition to the reserve-eligible categories in \mathcal{R} that are associated with the higher level VR protections, there is a finite set of traits \mathcal{T} associated with the lower level HR protections. Each individual has a (possibly empty) subset of traits, given by the trait function $\tau : \mathcal{I} \rightarrow 2^{\mathcal{T}}$. Each trait represents a societal disadvantage, and individuals who have this trait are provided with easier access to positions through a second type of affirmative action policy.

HR protections are provided within each vertical category.¹⁰ For any reserve-eligible category $c \in \mathcal{R}$ and trait $t \in \mathcal{T}$, subject to the availability of qualified individuals, a minimum of q_t^c category- c positions are to be assigned to individuals from category c with trait t . We refer to these positions as **category- c HR-protected positions for trait t** . Similarly, for any trait $t \in \mathcal{T}$ and subject to the availability of individuals with trait t , a minimum of q_t^o open-category positions are to be assigned to individuals with trait t . We refer to these positions as **open-category HR-protected positions for trait t** .

For each vertical category $v \in \mathcal{V}$, we assume that the total number of category- v HR-protected positions is no more than the number of positions in category v . That is, for each category $v \in \mathcal{V}$,

$$\sum_{t \in \mathcal{T}} q_t^v \leq q^v.$$

¹⁰This is not a federal mandate in India but rather a formal recommendation by the Supreme Court judgment *Anil Kumar Gupta (1995)*. The vast majority of the institutions in India follow this recommendation in implementing HR policies in this form, also called *compartmentalized horizontal reservations*.

We refer to HR policies where an individual can have at most one trait as **non-overlapping HR protections**, and HR policies where an individual can have multiple traits as **overlapping HR protections**. In many field applications in India, HR protections are non-overlapping.¹¹ Unlike this version of the problem which is relatively less complex, analysis of the problem with overlapping HR protections introduces a number of subtleties.

In contrast to VR protections, which are provided on an “over-and-above” basis, HR protections are provided within each vertical category on a “minimum guarantee” basis. This means that positions obtained without invoking any HR protection still accommodate the HR protections.¹²

Given a category $v \in \mathcal{V}$ and assuming that HR policies are non-overlapping, category- v HR protections can be implemented with the following (category- v) *minimum guarantee choice rule* C_{mg}^v (Echenique and Yenmez, 2015).

Minimum Guarantee Choice Rule C_{mg}^v

Given a set of individuals $I \subseteq \mathcal{I}^v$,

Step 1: for each trait $t \in \mathcal{T}$, choose all individuals in I with trait t if the number of trait- t individuals in I is less than or equal to q_t^v , and q_t^v highest merit-score individuals in I with trait t otherwise.

Step 2: For positions unfilled in Step 1, choose unassigned individuals in I with highest merit scores.

The reason for restricting attention to problems with non-overlapping HR protections in defining this choice rule is technical. It is easy to see that the processing sequence of traits in Step 1 of the procedure becomes immaterial for this case. In contrast, the processing sequence of traits can affect the outcome under overlapping HR protections. Moreover, in this more general case, and even with the additional specification of a trait processing sequence, it is not clear whether the resulting choice rule is equally plausible for implementing the HR protections. Indeed, in Section 4.2.3 we advocate for an alternative approach in extending the *minimum guarantee choice rule* for problems with overlapping HR protections.

¹¹That is in part because *persons with disabilities* are the only group that is explicitly granted HR protections at the federal level.

¹²The official language used for the distinction between HR protections and VR protections is given in Section B.2 of the Online Appendix.

3. Analysis and Policy Implications with Non-Overlapping HR Protections

In this section, we present an analysis of concurrent implementation of VR and non-overlapping HR protections. Therefore, throughout this section, each individual is assumed to have at most one trait. While Indian judgments and legislation on VR and HR policies more broadly apply to applications with overlapping HR protections as well, as we show in this section they have sharper implications for field applications with non-overlapping HR protections. Moreover, choice rules that have been either mandated or endorsed by the Supreme Court since *Indra Sawhney (1992)* all abstract away from any details pertaining to overlapping HR protections. Therefore, our analysis of this more restrictive version of the model in this section has more direct policy implications in India.

3.1. SCI-AKG Choice Rule and Its Flaws. We start our analysis by introducing the *SCI-AKG choice rule* that was mandated in India for 25 years until December 2020. The following definition simplifies the description of the SCI-AKG choice rule.

Definition 5. *A member of a reserve-eligible category $i \in \cup_{c \in \mathcal{R}} \mathcal{I}^c$ is a **meritorious reserved candidate** if she has one of the q^0 highest merit scores among all individuals in I .*

Let I^m denote the set of meritorious reserved candidates.

We are ready to formulate the *SCI-AKG choice rule*, originally introduced in the Supreme Court judgment *Anil Kumar Gupta (1995)* for the case of a single trait.

SCI-AKG Choice Rule $C^{SCI} = (C^{SCI,v})_{v \in \mathcal{V}}$

Given a set of individuals $I \subseteq \mathcal{I}$,

$$C^{SCI,o}(I) = C_{mg}^o(I^m \cup I^g), \text{ and}$$

$$C^{SCI,c}(I) = C_{mg}^c(I^c \setminus C_{mg}^o(I^m \cup I^g)) \quad \text{for any } c \in \mathcal{R}.$$

It is important to emphasize that the formulation of the SCI-AKG choice rule given above is not the original formulation presented in *Anil Kumar Gupta (1995)*. The original formulation is based on first tentatively allocating the positions based on the over-and-above choice rule presented in Section 2.1, and subsequently carrying out any necessary adjustments to accommodate the HR protections. We instead present a simpler formulation of the SCI-AKG choice rule, using its relation to the *minimum guarantee choice rule* introduced in Section 2.3.¹³ It is also important to note that, while the justices formally introduced the SCI-AKG choice rule only for the case of a single trait, their formulation

¹³The original description of the SCI-AKG choice rule in the Supreme Court judgments *Anil Kumar Gupta (1995)* and *Rajesh Kumar Daria vs Rajasthan Public Service (2007)*, and the result that shows the outcome equivalence of this formulation can be seen in Section B.3 of the Online Appendix.

immediately extends to multiple traits assuming HR protections are non-overlapping. Later in Section 4.2, we show that extending the SCI-AKG choice rule to the more general version of problem with overlapping HR protections introduces a number of subtleties, allowing for multiple generalizations of this rule.

We next show that the SCI-AKG choice rule has two important flaws even for the simple case with a single trait.

Example 1. There are VR protections for members of a reserve-eligible category $c \in \mathcal{R}$ and HR protections for women. The set of individuals $I = \{m_1^s, m_2^s, w_1^s, m_1^c, w_1^c\}$ consists of two general category men m_1^s, m_2^s , one general-category woman w_1^s , one category- c man m_1^c , and one category- c woman w_1^c . There are two open-category positions and one VR-protected position for category c . One of the open-category positions is HR-protected for women. Individuals have the following merit ranking:

$$\sigma(m_1^s) > \sigma(m_2^s) > \sigma(m_1^c) > \sigma(w_1^c) > \sigma(w_1^s).$$

Since there are two open-category positions and neither of the two highest merit score individuals are from the reserve-eligible category c , the set of meritorious reserved candidates is $I^m = \emptyset$. Therefore, the set of individuals under consideration for open positions is $I^m \cup I^s = I^s = \{m_1^s, m_2^s, w_1^s\}$. Since w_1^s is the only woman in the set $I^m \cup I^s$, she is awarded the open-category HR-protected position for women despite having the lowest merit score. Woman w_1^c is not eligible for this position, although she would be had she not declared her category membership for the reserve-eligible category c . The other open-category position is awarded to the highest merit score individual m_1^s . Hence, $C^{SCI,o}(I) = C_{mg}^o(I^m \cup I^s) = \{m_1^s, w_1^s\}$.

Since there is no category- c HR-protected position for women, the highest merit score category- c individual receives the only category- c position, and hence $C^{SCI,c}(I) = \{m_1^c\}$. Therefore, the set of individuals who are each awarded a position under the SCI-AKG choice rule is $\widehat{C}^{SCI}(I) = \{m_1^s, w_1^s, m_1^c\}$.

There are two troubling aspects of this outcome. The first issue is that, even though the category- c woman w_1^c has a higher merit score than the general category woman w_1^s , the latter receives a position while the former does not. That is, contrary to the philosophy of affirmative action, a lower merit score individual from the (unprotected) general category receives a position at the expense of a higher merit score individual from a protected category. The second issue is that, since she is the highest merit score woman among all applicants, woman w_1^c can receive the open-category HR-protected position for women simply by not declaring her eligibility for the VR-protected position for category- c . \square

The shortcomings of the SCI-AKG choice rule presented in Example 1 are not merely abstract possibilities, but rather are highly visible flaws that have been responsible for thousands of litigations that disrupt recruitment processes throughout India, as documented in Section C.1 of the Online Appendix. The root cause of both anomalies is the restriction of the open-category HR protections to general category individuals only. This restriction creates an immediate (and rather obvious) conflict for individuals who qualify for both VR and HR protections: With the exception of meritorious reserved candidates, any such individual loses her qualifications for open-category HR protections by claiming her VR protections. Consequently, this conflict reflects itself in the following two deficiencies that go against the philosophy of affirmative action:

- (1) *Possibility of a higher-merit protected individual losing a position to a lower-merit unprotected individual:* For example, a woman from the VR-protected category Scheduled Castes may remain unassigned while a lower merit-score woman from the higher-privilege general category receives a position through open-category HR protections for women.
- (2) *Necessity to give up VR protections to claim open-category HR protections:* For example, a woman from Scheduled Castes may remain unassigned by declaring her membership for Scheduled Castes, but she can receive an open-category HR-protected position for women by withholding her Scheduled Castes membership, and thus she benefits from not declaring this information.

These deficiencies motivate our axioms of *no justified envy* and *incentive compatibility*.

The following *HR-maximality function* plays a key role not only in our formulation of the axiom of *no justified envy*, but also in our formulation of two additional axioms introduced later in this section. Moreover, the extension of our analysis to the more general model with overlapping HR protections later presented in Section 4 also critically depends on the extension of this function.

Definition 6. *Given a vertical category $v \in \mathcal{V}$, the (category- v) HR-maximality function $n^v : 2^{\mathcal{I}^v} \rightarrow \mathbb{N}$ is defined as, for any $I \subseteq \mathcal{I}^v$,*

$$n^v(I) = \sum_{t \in \mathcal{T}} \min \left\{ |\{i \in I : t \in \tau(i)\}|, q_t^v \right\}.$$

Observe that, for any set of individuals I who are eligible for category- v positions, the category- v HR-maximality function n^v gives the maximum number of category- v HR-protected positions that can be awarded.¹⁴

¹⁴One way this maximum can be obtained is via the category- v minimum guarantee choice rule C_{mg}^v .

Definition 7. A choice rule $C = (C^v)_{v \in \mathcal{V}}$ satisfies **no justified envy** if, for every $I \subseteq \mathcal{I}$, $v \in \mathcal{V}$, $i \in C^v(I)$, and $j \in (I \cap \mathcal{I}^v) \setminus \widehat{C}(I)$,

$$\sigma(j) > \sigma(i) \implies n^v\left((C^v(I) \setminus \{i\}) \cup \{j\}\right) < n^v(C^v(I)).$$

This axiom requires that, given two individuals who are both eligible for a position in a category, the lower merit-score individual can receive a position at the expense of the higher merit-score individual only if not doing so strictly decreases the number of HR protections that are accommodated in that category. Therefore, under this axiom, increasing the utilization of HR protections in a category can be the only reason to award a position at this category to a lower merit-score individual at the expense of an unassigned higher merit-score eligible individual.

We next formulate the axiom of *incentive compatibility*, first introduced by Aygün and Bó (2021) in their analysis of the affirmative action policies in Brazilian college admissions:

Definition 8. An individual *withholds some of her reserve-eligible privileges* if she does not declare either her reserve-eligible category membership or some of her traits.

In India, individuals are not required to declare their reserve-eligible privileges.

Definition 9. A choice rule C is **incentive compatible** if, for every $I \subseteq \mathcal{I}$, any individual $i \in I$ who is selected from I under the aggregate choice rule \widehat{C} by withholding some of her reserve-eligible privileges is also selected from I under \widehat{C} by declaring all her reserve-eligible privileges.

Under a choice rule that satisfies this axiom, privileges that are meant to provide positive discrimination would never produce the opposite effect and thus hurt an individual upon declaring eligibility. Failure of incentive compatibility is implausible both from a normative perspective, since it is against the philosophy of affirmative action, and also from a strategic perspective, since it may force individuals to withhold their privileges. As we document clear evidence in Section C.1.2 of the Online Appendix, it also creates one additional difficulty in India.

Eligibility for VR protections typically depends on an individual's caste membership. While this information is supposed to be private information, it can often be inferred by the central planner due to various indications such as the individual's last name. A central planner can also obtain this information through documents such as a diploma. Hence, eligibility for VR protections may not be truly private information, and the lack of incentive compatibility of a choice rule may enable a malicious central planner to exploit this information to deny an applicant her open-category HR protections. As documented in Section C.1.2 of the Online Appendix, this type of misconduct not only has

been widespread in parts of India, but it even appears to be centrally organized by the local governing bodies in some of its jurisdictions.

3.2. An Easy Fix: 2SMG Choice Rule. Apart from its simplicity, an additional advantage of formulating the SCI-AKG choice rule using its relation to the *minimum guarantee choice rule* is that, unlike its original formulation that obscures a possible remedy, our equivalent formulation suggests an easy fix. Both anomalies of the SCI-AKG choice rule are caused by the exclusive access given to the general-category individuals for open-category HR protections. This restriction reflects itself in our formulation of the SCI-AKG choice rule during the derivation of the open-category assignments through the formula

$$C^{SCI,o}(I) = C_{mg}^o(I^m \cup I^g).$$

Observe that, instead of running the choice rule C_{mg}^o for the set of individuals $I^m \cup I^g$, running it for the set of all individuals I provides us with an immediate and intuitive fix. We refer to this alternative mechanism as the *two-step minimum guarantee (2SMG) choice rule*.

Two-Step Minimum Guarantee (2SMG) Choice Rule $C_{mg}^{2s} = (C_{mg}^{2s,v})_{v \in \mathcal{V}}$

Given a set of individuals $I \subseteq \mathcal{I}$,

$$\begin{aligned} C_{mg}^{2s,o}(I) &= C_{mg}^o(I), \text{ and} \\ C_{mg}^{2s,c}(I) &= C_{mg}^c(I^c \setminus C_{mg}^o(I)) \quad \text{for any } c \in \mathcal{R}. \end{aligned}$$

Since the SCI-AKG choice rule is formally introduced in *Anil Kumar Gupta (1995)* for the case of a single trait, and in particular when HR protections are non-overlapping, it is best to consider the 2SMG choice rule for the model with non-overlapping HR protections only.¹⁵

As one would naturally expect, replacing the SCI-AKG choice rule with the 2SMG choice rule results in a weakly less favorable outcome for members of the general category. The comparison for members of reserve-eligible categories is less straightforward, because in addition to the VR-protected positions, these individuals also compete for the open positions. However, assuming sufficient demand at each reserve-eligible category, replacing the SCI-AKG choice rule with the 2SMG choice rule results in a weakly more favorable outcome in aggregate for members of the reserve-eligible categories.

Proposition 1. *For every $I \subseteq \mathcal{I}$,*

$$\widehat{C}_{mg}^{2s}(I) \cap I^g \subseteq \widehat{C}^{SCI}(I) \cap I^g,$$

¹⁵When HR protections are overlapping, the outcome of the 2SMG choice rule depends on the processing sequence of traits at each vertical category of positions.

and assuming $|I^c| \geq q^0 + q^c$ for each reserve-eligible category $c \in \mathcal{R}$,

$$\sum_{c \in \mathcal{R}} |\widehat{C}_{mg}^{2s}(I) \cap I^c| \geq \sum_{c \in \mathcal{R}} |\widehat{C}^{SCI}(I) \cap I^c|.$$

3.3. The Demise of the SCI-AKG Choice Rule and the Rise of the 2SMG Choice Rule.

In a rather unexpected development and while this paper was under revision for this journal, in *Saurav Yadav (2020)* a three-judge bench of the Supreme Court declared that the SCI-AKG choice rule is a product of misinterpretation of the Court's earlier judgments. Referring to the failure of the SCI-AKG choice rule to satisfy *no justified envy* as an "incongruity," the justices annulled this mechanism, since it can result in "irrational" results. Importantly, the same judgment also endorsed the 2SMG choice rule as a possible replacement for the abandoned SCI-AKG choice rule. While the justices have not mandated the 2SMG choice rule in *Saurav Yadav (2020)*, they mandated that any choice rule adopted in India satisfy the axiom of *no justified envy* and further brought clarity for one additional subtle aspect of the HR protections presented in Section 3.4. Importantly, the 2SMG choice rule is the only mechanism that satisfies these new mandates together with those from *Indra Sawhney (1992)* in applications with non-overlapping HR protections. We next present this significant implication of *Saurav Yadav (2020)*, which is not observed in this important judgment.

3.4. The Implicit Mandate of the 2SMG Choice Rule Under *Saurav Yadav (2020)*.

We next formulate three additional axioms, the first of which is originally mandated by *Indra Sawhney (1992)* and maintained by *Saurav Yadav (2020)*, whereas the latter two are only recently mandated by *Saurav Yadav (2020)* (as in the case of the *no justified envy* axiom formulated in Section 3.1) at their strength formulated below.

Definition 10. A choice rule $C = (C^v)_{v \in \mathcal{V}}$ is *non-wasteful* if, for every $I \subseteq \mathcal{I}$, $v \in \mathcal{V}$, and $j \in I$,

$$j \notin \widehat{C}(I) \text{ and } |C^v(I)| < q^v \implies j \notin \mathcal{I}^v.$$

That is, if an individual j is declined a position from each one of the categories (thus remaining unmatched) while there is an idle position at some category $v \in \mathcal{V}$, then it must be the case that individual j is not eligible for a position at category v . This mild efficiency axiom has been mandated in India since *Indra Sawhney (1992)*.

Definition 11. A choice rule $C = (C^v)_{v \in \mathcal{V}}$ *maximally accommodates HR protections*, if for every $I \subseteq \mathcal{I}$, $v \in \mathcal{V}$, and $j \in (I \cap \mathcal{I}^v) \setminus \widehat{C}(I)$,

$$n^v(C^v(I)) = n^v(C^v(I) \cup \{j\}).$$

In words, an individual who remains unassigned should not be able to increase the utilization of HR protections at any category where she has eligibility, if she were to be instead assigned a position in this category. The only reason this axiom was not mandated in India prior to *Saurav Yadav (2020)* is that under the previous interpretation of *Anil Kumar Gupta (1995)* members of reserve-eligible categories were considered ineligible for open-category HR protections. This restriction, which has been the root cause of the controversies involving the SCI-AMG choice rule, has been revoked by *Saurav Yadav (2020)*, and consequently the axiom of *maximum accommodation of HR protections* is mandated in its stronger form as formulated above.

Definition 12. A choice rule $C = (C^v)_{v \in \mathcal{V}}$ *complies with VR protections* if, for every $I \subseteq \mathcal{I}$, $c \in \mathcal{R}$, and $i \in C^c(I)$,

$$(1) |C^o(I)| = q^o,$$

$$(2) \text{ for every } j \in C^o(I),$$

$$\sigma(j) < \sigma(i) \implies n^o(C^o(I)) > n^o((C^o(I) \setminus \{j\}) \cup \{i\}), \text{ and}$$

$$(3) n^o(C^o(I) \cup \{i\}) = n^o(C^o(I)).$$

Here the first two conditions formulate the idea of a vertical reservation à la Indra Sawhney (1992), and they are directly implied by the concept of “over-and-above.” For an individual i to receive a position set aside for a reserve-eligible category (thereby not receiving an open position), it must be the case that each open position is either assigned to a higher merit-score individual j , or to an individual j whose selection instead of i increases the utilization of open-category HR protections. The third condition additionally requires that a member of a reserve-eligible category who can improve the utilization of open-category HR protections shall not use up a VR-protected position. Importantly, this third condition is an implication of another mandate in *Saurav Yadav (2020)*, and therefore this judgment enforces the axiom of *compliance with VR protections* in its stronger form as formulated above.¹⁶

We are ready to present our first main result.

Theorem 1. *Suppose each individual has at most one trait. A choice rule*

(1) *maximally accommodates HR protections,*

(2) *satisfies no justified envy,*

(3) *is non-wasteful, and*

(4) *complies with VR protections*

if, and only if, it is the 2SMG choice rule C_{mg}^{2s} .

¹⁶See Section B.4 of the Online Appendix for this important mandate in *Saurav Yadav (2020)*.

Prior to its endorsement by the three-judge bench of the Supreme Court in *Saurav Yadav (2020)*, the 2SMG choice rule has been introduced by the justices of the High Court of Gujarat in *Tamannaben Ashokbhai Desai v. Shital Amrutlal Nishar (2020)*,¹⁷ an August 2020 judgment which also mandated the 2SMG choice rule in the state of Gujarat.¹⁸ However, while this choice rule is merely endorsed and not explicitly mandated by *Saurav Yadav (2020)* throughout India, our first main result in Theorem 1 implies that this important judgment has implicitly mandated this mechanism in field applications with non-overlapping HR protections.

4. General Analysis and Policy Recommendations with Overlapping HR Protections

To the best of our knowledge, the judgments on the implementation of HR policies in India largely abstract away from any technical complications due to overlapping HR protections. Since this more general version of the problem is fairly common in the field, in this section we extend our analysis to the model with overlapping HR protections. This version of the problem, however, introduces a subtle but critical technical consideration that allows for at least two approaches to generalize our model. Hence, before presenting an analysis of concurrent implementation of VR and overlapping HR protections, we first elaborate on this consideration and justify the modeling choice we make for our generalization.

4.1. One-to One vs One-to-All HR Matching. Whether horizontal reservations are overlapping or not, an individual loses her open-category HR protections upon declaring her VR protections under the Supreme Court judgment *Anil Kumar Gupta (1995)*. Therefore, the main flaws of the SCI-AMG choice rule, originally defined for a single trait, carry over to any possible generalization with overlapping HR protections. For this more general and complex case, however, one technical and subtle aspect of implementation of HR protections has been left unlegislated and remains at the discretion of the central planner. The law is silent on whether the admission of an individual with multiple traits accommodates the minimum guarantee requirements for all her traits or only for one of her traits. For example, suppose there is one HR-protected position for women and one HR-protected position for persons with disabilities. If a woman with a disability is admitted, the law does not specify whether she is to accommodate the minimum guarantee requirements both for women and also for persons with disabilities, or only for one of these two protected groups.

¹⁷The case available at https://www.livelaw.in/pdf_upload/pdf_upload-380856.pdf (last accessed on 06/10/2021).

¹⁸Our introduction and advocacy of the 2SMG choice rule predates both of these important judgments.

In our extension, we focus on the second convention of implementing the HR protections, and thus assume that an individual counts toward the minimum guarantee requirement for only one of her traits upon admission. We refer to this convention of implementing HR protections as **one-to-one HR matching**, and the alternative convention (where an individual counts toward the minimum guarantee requirements for all her traits upon admission) as **one-to-all HR matching**. There are two reasons for this important modeling choice.

The first reason is technical. The alternative convention of one-to-all HR matching introduces complementarities between individuals, making their admissions potentially contingent on each other. For example, if there is one HR-protected position for women and one HR-protected position for persons with disabilities, the admission of a man without a disability may depend on the admission of a woman with a disability who can accommodate the HR protections for both protected groups. This complementarity, in turn, not only renders the derivation of feasible groups of individuals computationally hard, but it also makes any possible solution technically less elegant.¹⁹ In contrast, our adopted convention of one-to-one HR matching enables a fairly clean and computationally simple solution, as we present later in this section.

The second reason is practical. While either convention appears to be allowed by the Indian judgments and legislation, we have been unable to find any application with overlapping HR protections where the allocation rules clearly specify (or imply) the adoption of the one-to-all HR matching convention. In contrast, in many field applications, the central planner announces the number of positions for each category-trait pair,²⁰ which implicitly implies that they adopt the one-to-one HR matching convention.²¹

4.2. Single-Category Analysis with Overlapping HR Protections. Since HR policies are implemented within vertical categories, we start our analysis with the simple case of a single category. This version of the problem also relates to practical applications other than our main application in India, such as the allocation of K-12 public school seats in Chile where there are overlapping HR protections (Correa et al., 2019).

Throughout Section 4.2, we fix a category $v \in \mathcal{V}$.

4.2.1. The Case Against a Fixed Processing Sequence of Traits. The 2SMG choice rule, introduced in Section 3.2, is not well-defined in problems with overlapping HR protections,

¹⁹See Section 4 in Sönmez and Yenmez (2020) for an analysis under the one-to-all HR matching convention with two traits.

²⁰See for example Table 2 in Saurav Yadav (2020).

²¹We are also able to find a field application, where the allocation rules explicitly specify the adoption of the one-to-one HR matching convention.

because, for any vertical category $v \in \mathcal{V}$, the outcome of the category- v minimum guarantee choice rule may depend on the processing sequence of traits. Therefore, it may be compelling to resolve this multiplicity by simply specifying a processing sequence of traits for each vertical category as additional list of parameters of the choice rule. However, we caution against this (admittedly compelling) generalization for it may introduce additional flaws in the system.

Example 2. There is one category (say open category), three individuals i_1, i_2, i_3 , and two positions. There are two traits t_1, t_2 , with one HR-protected position each. Individual i_1 has both traits, individual i_2 has no trait, and individual i_3 has trait t_1 only. Individuals are merit ranked as

$$\sigma(i_1) > \sigma(i_2) > \sigma(i_3).$$

We next generate the outcome of the (open category) *minimum guarantee* choice rule for both processing sequences of the two traits, first by processing the trait- t_1 minimum guarantee prior to the trait- t_2 minimum guarantee, and subsequently by processing them in the reverse order.

Trait t_1 first, trait t_2 next: The highest merit score individual with trait t_1 is i_1 ; she receives a position, accommodating the minimum guarantee for trait t_1 . No remaining individual has trait t_2 ; therefore only individual i_1 receives a position in Step 1. The highest merit score remaining individual i_2 receives the second position in Step 2. The set of selected individuals is $\{i_1, i_2\}$, and only the trait t_1 minimum guarantee is accommodated under the first processing sequence of traits.

Trait t_2 first, trait t_1 next: The highest merit score individual with trait t_2 is i_1 ; she receives a position, accommodating the minimum guarantee for trait t_2 . Among the remaining individuals, the highest merit score individual with trait t_1 is i_3 ; she receives a position, accommodating the minimum guarantee for trait t_1 . No position remains, and therefore the set of selected individuals is $\{i_1, i_3\}$. Minimum guarantees for both traits are accommodated under the second processing sequence of traits. \square

Example 2 shows that;

- (1) the outcome of the minimum guarantee choice rule, in general, depends on the processing sequence of traits, and
- (2) for some processing sequences of traits, it may accommodate fewer than the maximum possible HR protections.

Essentially, Example 2 shows that a fixed processing sequence of traits may result in denial of HR protections which can be avoided.

Our next example reveals another problematic implication of implementing the minimum guarantee choice rule under a fixed processing sequence of traits.

Example 3. There is one category (say the open category), four individuals i_1, i_2, i_3, i_4 , and three positions. There are two traits t_1, t_2 , with one HR-protected position each. Individual i_1 has both traits, individual i_2 has no trait, individual i_3 has only trait t_1 , and individual i_4 has only trait t_2 . Individuals are merit ranked as

$$\sigma(i_1) > \sigma(i_2) > \sigma(i_3) > \sigma(i_4).$$

We next generate the outcome of the (open category) *minimum guarantee* choice rule for both processing sequences of the two traits, first by processing the trait- t_1 minimum guarantee prior to the trait- t_2 minimum guarantee, and subsequently by processing them in the reverse order.

Trait t_1 first, trait t_2 next: The highest merit score individual with trait t_1 is i_1 ; she receives a position, accommodating the minimum guarantee for trait t_1 . Among the remaining individuals, the one with the highest merit score with trait t_2 is i_4 ; she receives a position, accommodating the minimum guarantee for trait t_2 and finalizing Step 1. The last position is assigned in Step 2 to the highest merit score remaining individual i_2 , and therefore the set of selected individuals is $\{i_1, i_2, i_4\}$ under the first processing sequence of traits.

Trait t_2 first, trait t_1 next: The highest merit score individual with trait t_2 is i_1 ; she receives a position, accommodating the minimum guarantee for trait t_2 . Among the remaining individuals, the one with the highest merit score with trait t_1 is i_3 ; she receives a position, accommodating the minimum guarantee for trait t_1 and finalizing Step 1. The last position is assigned in Step 2 to the highest merit score remaining individual i_2 , and therefore the set of selected individuals is $\{i_1, i_2, i_3\}$ under the second processing sequence of traits. \square

Example 3 reveals that, depending on the processing sequence of traits, the outcome of the minimum guarantee choice rule may admit lower merit score individuals at the expense of higher merit score ones without affecting adherence to the horizontal reservation policies. In Example 3, when the merit based outcome of $\{i_1, i_2, i_3\}$ already accommodates the HR protections, there is clearly no reason to select a less meritorious group.

These two examples not only guide us on adjustments of our axioms to account for overlapping HR protections, they also motivate the *meritorious horizontal choice rule*, introduced in Section 4.2.3, as a natural extension of the 2SMG choice rule.

4.2.2. HR Graph and the Generalized HR-maximality Function. In contrast to the version of our model with non-overlapping HR protections where maximizing the accommodation

of HR protections is a straightforward task, doing the same for the general version of the model with overlapping HR protections requires embedding a *maximum trait matching* procedure within each category. Therefore, we rely on the following construction to generalize our HR-maximality function, which we will use

- (1) to extend our axioms initially presented in Section 3 for the model with non-overlapping HR protections, and
- (2) to generalize the 2SMG choice rule for the model with overlapping HR protections in a way that escapes the shortcomings presented in Examples 2 and 3.

Given a category $v \in \mathcal{V}$ and a set of individuals $I \subseteq \mathcal{I}^v$, construct the following two-sided **category- v HR graph**. On one side of the graph, there are individuals in I . On the other side, there are HR-protected positions for category v . Let H_t^v denote the set of trait- t HR-protected positions for category v and let $H^v = \bigcup_{t \in \mathcal{T}} H_t^v$. There are q_t^v positions in H_t^v and $\sum_{t \in \mathcal{T}} q_t^v$ positions in H^v . An individual $i \in I$ and a position $p \in H_t^v$ are **connected** in this graph if and only if individual i has trait t .

Definition 13. *Given a category $v \in \mathcal{V}$ and a set of individuals $I \subseteq \mathcal{I}^v$, a **trait-matching** of individuals in I with HR-protected positions in H^v is a function $\mu : I \rightarrow H^v \cup \{\emptyset\}$ such that,*

- (1) for any $i \in I, t \in \mathcal{T}$,

$$\mu(i) \in H_t^v \implies t \in \tau(i),$$

- (2) for any $i, j \in I$,

$$\mu(i) = \mu(j) \neq \emptyset \implies i = j.$$

Definition 14. *Given a category $v \in \mathcal{V}$ and a set of individuals $I \subseteq \mathcal{I}^v$, a **trait-matching** of individuals in I with HR-protected positions in H^v has **maximum cardinality in a (category- v) HR graph** if there exists no other trait-matching that assigns a strictly higher number of HR-protected positions to individuals.*

Let $n^v(I)$ denote the maximum number of category- v HR-protected positions that can be assigned to individuals in I .²² Observe that function n^v generalizes the category- v HR-maximality function presented in Definition 6 for the model with non-overlapping HR protections to the more general version of the model with overlapping HR protections (under the convention of one-to-one HR matching).

Remark 1. All our axioms in Section 3 are extended for our more general model with overlapping HR protections by simply replacing the simpler version of the HR-maximality function given in Definition 6 with the generalized version.

²²This number can be found through several polynomial time algorithms such as *Edmonds' Blossom Algorithm* (Edmonds, 1965).

The following terminology is useful for our generalization of the 2SMG choice rule.

Definition 15. *Given a category $v \in \mathcal{V}$ and a set of individuals $I \subseteq \mathcal{I}^v$, an individual $i \in \mathcal{I}^v \setminus I$ increases the (category- v) HR utilization of I if*

$$n^v(I \cup \{i\}) = n^v(I) + 1.$$

4.2.3. Meritorious Horizontal Choice Rule. We are ready to introduce a single-category choice rule that escapes the shortcomings presented in Examples 2 and 3. The main innovation in this choice rule is the optimization it carries out to determine who is to account for each minimum guarantee when some of the individuals can account for one or another due to multiple traits they have. Intuitively, this choice rule exploits the flexibility in trait-matching in order to accommodate the HR protections with higher merit-score individuals.

Given a category $v \in \mathcal{V}$ and a set of individuals $I \subseteq \mathcal{I}^v$, the outcome of this choice rule is obtained using the following procedure.

Meritorious Horizontal Choice Rule $C_{\mathbb{M}}^v$

Step 1.1: Choose the highest merit-score individual in I with a trait for an HR-protected position. Denote this individual by i_1 and let $I_1 = \{i_1\}$. If no such individual exists, proceed to Step 2.

Step 1.k ($k \in \{2, \dots, \sum_{t \in \mathcal{T}} q_t^v\}$): Assuming such an individual exists, choose the highest merit-score individual in $I \setminus I_{k-1}$ who increases the HR utilization of I_{k-1} .²³ Denote this individual by i_k and let $I_k = I_{k-1} \cup \{i_k\}$. If no such individual exists, proceed to Step 2.

Step 2: For unfilled positions, choose unassigned individuals with highest merit scores until either all positions are filled or all individuals are selected.

When the number of individuals is less than q^v , this procedure selects all individuals. Otherwise, if there are more than q^v individuals, then it chooses a set with q^v individuals.

4.2.4. Single-Category Results with Overlapping HR Protections. We next present two single-category results under overlapping HR protections, which suggest that the case for the meritorious horizontal choice rule is especially strong in this framework.

Justifying the naming of this choice rule, our next result shows that the meritorious horizontal choice rule $C_{\mathbb{M}}^v$ always selects higher merit-score individuals compared to other choice rules that maximally accommodate HR protections.

²³This can be done with various computationally efficient algorithms; see, for example, the bipartite cardinality matching algorithm (Lawler, 2001, Page 195).

Proposition 2. *Given a category $v \in \mathcal{V}$, let C^v be any single-category choice rule that maximally accommodates HR protections. Then, for every set of individuals $I \subseteq \mathcal{I}^v$,*

- (1) $|C^v(I)| \leq |C_{\mathbb{M}}^v(I)|$, and
- (2) for every $k \leq |C^v(I)|$, if i is the k -th highest merit-score individual in $C_{\mathbb{M}}^v(I)$ and j is the k -th highest merit-score individual in $C^v(I)$, then

$$i = j \quad \text{or} \quad \sigma(i) > \sigma(j).$$

We next present a characterization of the meritorious horizontal choice rule $C_{\mathbb{M}}^v$.

Theorem 2. *Given a category $v \in \mathcal{V}$, a single-category choice rule*

- (1) *maximally accommodates HR protections,*
- (2) *satisfies no justified envy, and*
- (3) *is non-wasteful*

if, and only if, it is the meritorious horizontal choice rule $C_{\mathbb{M}}^v$.

4.3. Two-Step Meritorious Horizontal Choice Rule & Its Characterization. We are ready to formulate and propose a choice rule for our model in its full generality. The following choice rule uses the meritorious horizontal choice rule multiple times, first to allocate open-category positions, and next for each reserve-eligible category to allocate VR-protected positions.

Two-Step Meritorious Horizontal (2SMH) Choice Rule $C_{\mathbb{M}}^{2s} = (C_{\mathbb{M}}^{2s,v})_{v \in \mathcal{V}}$

For every $I \subseteq \mathcal{I}$,

$$\begin{aligned} C_{\mathbb{M}}^{2s,o}(I) &= C_{\mathbb{M}}^o(I), \text{ and} \\ C_{\mathbb{M}}^{2s,c}(I) &= C_{\mathbb{M}}^c(I^c \setminus C_{\mathbb{M}}^o(I)) \quad \text{for any } c \in \mathcal{R}. \end{aligned}$$

We next present our main characterization result, extending our analogous characterization of the 2SMG choice rule under non-overlapping HR protections to its generalization the 2SMH choice rule under overlapping HR protections.

Theorem 3. *A choice rule*

- (1) *maximally accommodates HR protections,*
- (2) *satisfies no justified envy,*
- (3) *is non-wasteful, and*
- (4) *complies with VR protections*

if, and only if, it is the 2SMH choice rule $C_{\mathbb{M}}^{2s}$.

In addition to being the only choice rule that satisfies each of the four axioms in Theorem 3, our proposed 2SMH choice rule C_{M}^{2s} also satisfies the axiom of incentive compatibility defined in Section 3.1.

Proposition 3. *The 2SMH choice rule C_{M}^{2s} satisfies incentive compatibility.*

4.4. Related Literature. Our theoretical analysis of reservation policies differs from its predecessors in two ways:

- (1) *concurrent* implementation of VR and HR protections, and
- (2) potentially *overlapping* structure of HR protections.

While there is a rich literature on affirmative action policies in India and elsewhere, our paper is the first one to formally analyze vertical and horizontal reservation policies when they are implemented concurrently.

There are a number of recent papers on reservation policies, most in the context of school choice. Abdulkadiroğlu and Sönmez (2003) study affirmative action policies that limit the number of admitted students of a given type through hard quotas. Kojima (2012) shows that a policy of limiting the number of majority students through hard quotas can hurt minority students, the intended beneficiaries. To overcome the detrimental effect of affirmative action policies based on majority quotas, Hafalir et al. (2013) introduce policies based on *minority reserves*. In the absence of overlapping reservations, Echenique and Yenmez (2015) present an axiomatic characterization of the minimum guarantee choice rule. Most recently, Pathak et al. (2020b) consider a general model of reservation policies to balance various ethical principles for pandemic medical resource allocation, although their model is not equipped to analyze concurrent implementation of vertical and overlapping horizontal reservation policies.

A few papers study the implementation of vertical or (non-overlapping) horizontal reservations individually in various real-life applications. These include Dur et al. (2018) for school choice in Boston, Dur et al. (2020) for school choice in Chicago, and Pathak et al. (2020a) for H-1B visa allocation in the US. All these models are applications of the more general model in Kominers and Sönmez (2016), where the authors introduce a matching model with *slot-specific priorities*. In contrast, our model is independent from Kominers and Sönmez (2016). Three additional papers on reservation policies include Aygün and Turhan (2017, 2020), where the authors study admissions to engineering colleges in India, and Aygün and Bó (2021), where the authors study admissions to Brazilian public universities. While the application in Aygün and Turhan (2017, 2020) is closely related to ours, their analysis is independent because not only horizontal reservations are assumed away altogether in these papers, but also analyses in these papers largely abstract away from

the legal requirements in India.²⁴ In contrast, the presence of horizontal reservations is of key importance for our analysis that is build on Indian legislation. The Brazilian affirmative action application studied by Aygün and Bó (2021) relates to ours in that it also includes multi-dimensional reservation policies, but unlike our models their application is a special case of Kominers and Sönmez (2016). There is, however, one important element in our paper that directly builds on Aygün and Bó (2021). The two desiderata that play an important role in our proposed reform in India, *no justified envy* and *incentive compatibility* are originally introduced by Aygün and Bó (2021). Evidence from aggregate data suggesting that the presence of justified envy is widespread in Brazil is also presented in this paper. As in Aygün and Bó (2021), we also present extensive evidence of justified envy in the field, but in addition we also document the large scale disruption this anomaly creates in the field. Other less related papers on reservation policies include Westkamp (2013), Ehlers et al. (2014), Kamada and Kojima (2015), and Fragiadakis and Troyan (2017).

In the absence of vertical reservations, analysis of overlapping horizontal reservations has received some attention in the literature (Kurata et al. (2017)), albeit for a different variant of the problem where individuals have strict preferences for whether and which protection is invoked in securing a position. When applied in an environment where individuals are indifferent between all positions, choice rules recommended in Kurata et al. (2017) result in the limitations presented in Section 4.2. Building on the literature in matroid theory, we overcome these difficulties with the meritorious horizontal choice rule. More specifically, Proposition 2 and Theorem 2 are conceptually related to abstract results in matroid theory. Proposition 2 can be seen as a generalization of a result in Gale (1968) which shows that the outcome of the Greedy algorithm “dominates” any independent set of a matroid. In our appendix, we refer to this domination relation as “Gale domination.” The first step of our meritorious horizontal choice rule corresponds to the Greedy algorithm defined on an adequately defined matroid, and Proposition 2 shows that this choice rule Gale dominates any choice rule that maximally complies with HR protections. The proof uses mathematical induction on the number of individuals chosen at the second step of our choice rule and uses Gale’s result for the base case. Parts of the proof of Theorem 2 uses the properties of the Greedy algorithm.

More broadly, our paper contributes to the field of market design, where economists are increasingly taking advantage of advances in technology to design new or improved allocation mechanisms in applications as diverse as entry-level labor markets (Roth and Peranson, 1999), school choice (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez,

²⁴See also the discussion of Indian college admissions in (Echenique and Yenmez, 2015, Appendix C.1).

2003), spectrum auctions (Milgrom, 2000), kidney exchange (Roth et al., 2004, 2005), internet auctions (Edelman et al., 2007; Varian, 2007), course allocation (Sönmez and Ünver, 2010; Budish, 2011), cadet-branch matching (Sönmez and Switzer, 2013; Sönmez, 2013), assignment of airline arrival slots (Schummer and Vohra, 2013; Schummer and Abizada, 2017), and refugee matching (Jones and Teytelboym, 2017; Delacrétaz et al., 2016; Andersson, 2017).

5. Epilogue: Life Imitates Science with the December 2020 Supreme Court judgment *Saurav Yadav v State of Uttar Pradesh (2020)*

As our paper was under revision for this journal, a December 2020 Supreme Court judgment in *Saurav Yadav v State of Uttar Pradesh (2020)* became headline news in India.²⁵ Using arguments parallel to our analysis presented in Section 3 and the evidence we documented from high court cases presented in Section C.1.1 of the Online Appendix, a three-judge bench of the highest court reached much of the same conclusions we had reached earlier in the March 2019 working version of this paper in Sönmez and Yenmez (2019). Most notably, similar to our policy recommendations, with this judgment

- (1) all allocation rules for public recruitment are federally mandated to satisfy *no justified envy*, and thereby
- (2) the SCI-AKG choice rule, mandated for 25 years, becomes rescinded.

Using several of the same judgments we present in Section C.1.1 of the Online Appendix, the justices also highlighted the inconsistencies between several high court judgments in relation to desiderata we formulated as the axiom of *no justified envy*. The justices also declared that while the “first view” that enforces *no justified envy* by the high court judgments of Rajasthan, Bombay, Gujarat, and Uttarakhand is “correct and rational,” the “second view” that allows for *justified envy* by the high court judgments of Allahabad and Madhya Pradesh is not.²⁶

While the axiom of *no justified envy* becomes federally enforced with *Saurav Yadav (2020)*, unlike in *Anil Kumar Gupta (1995)* no explicit procedure is mandated with this new Supreme Court ruling. Two points, however, are important to emphasize in this

²⁵See, for example, *The Indian Express* opinion dated 12/26/2020 “SC verdict exposes fallacy of using general category as reservation for upper castes,” available in <https://indianexpress.com/article/opinion/columns/casteism-supreme-court-saurav-yadav-reservation-7120348/>, and *The Wire* analysis dated 12/23/2020 “How the Supreme Court Blocked Attempts to Dilute Merit Under the Open Category,” available in <https://thewire.in/law/supreme-court-reservation-merit>. Both links last accessed on 06/03/2021.

²⁶It is important to emphasize that, prior to this ruling, the second view—now deemed incorrect and irrational—was in line with the SCI-AKG choice rule, whereas the first view—now deemed correct and rational—deviated from the previously mandated choice rule.

regard. The first one is that prior to *Saurav Yadav (2020)*, the 2SMG choice rule became mandated in the state of Gujarat with the August 2020 high court judgment *Tamannaben Ashokbhai Desai (2020)*.²⁷ While the justices of the Supreme Court have not enforced any specific rule in their December 2020 judgment, they endorsed the 2SMG choice rule given in *Tamannaben Ashokbhai Desai (2020)*:

36. Finally, we must say that the steps indicated by the High Court of Gujarat in para 56 of its judgment in *Tamannaben Ashokbhai Desai* contemplate the correct and appropriate procedure for considering and giving effect to both vertical and horizontal reservations. The illustration given by us deals with only one possible dimension. There could be multiple such possibilities. Even going by the present illustration, the first female candidate allocated in the vertical column for Scheduled Tribes may have secured higher position than the candidate at Serial No.64. In that event said candidate must be shifted from the category of Scheduled Tribes to Open / General category causing a resultant vacancy in the vertical column of Scheduled Tribes. Such vacancy must then enure to the benefit of the candidate in the Waiting List for Scheduled Tribes - Female.

The steps indicated by Gujarat High Court will take care of every such possibility. It is true that the exercise of laying down a procedure must necessarily be left to the concerned authorities but we may observe that one set out in said judgment will certainly satisfy all claims and will not lead to any incongruity as highlighted by us in the preceding paragraphs.

Since both the Supreme Court's and the Gujarati High Court's judgments abstract away from any issues in relation to overlapping horizontal reservations, these rulings are parallel to our recommendations in Section 3. There is, however, a potentially misleading aspect in the last sentence of the above quote in *Saurav Yadav (2020)*, which brings us to our second point.

Apart from enforcing the axiom of *no justified envy* and rescinding the SCI-AKG choice rule, *Saurav Yadav (2020)* also brought clarity to a subtle aspect of implementation of vertical reservations in the presence of horizontal reservations. When the concept of vertical reservations was originally introduced in *Indra Sawhney (1992)*, positions awarded to individuals selected in the open competition on the basis of their merit were prohibited from counting against vertically reserved positions. Since then, this aspect of vertical reservations has been used as its key defining characteristic in India. However, no judgment of

²⁷The mandated choice rule in Gujarat is described for a single group of beneficiaries (women) for horizontal reservations under this High Court ruling. See Section B.5 in the Online Appendix for the description of the procedure in *Tamannaben Ashokbhai Desai (2020)*.

the Supreme Court prior to *Saurav Yadav (2020)* explicitly formulated what it means to get selected in the open competition on the basis of merit in the presence of horizontal reservations. To a large extent, much of the disarray in India in relation to concurrent implementation of VR and HR policies boils down to this ambiguity. This important gap is now clarified under *Saurav Yadav (2020)*, where an individual who qualifies for an open-category HR-protected position on the basis of her merit is explicitly considered as an individual who gets selected in the open competition on the basis of merit. This clarification is of key importance, because with the resolution of this ambiguity the 2SMG choice rule remains the only choice rule by Theorem 1 that satisfies all mandates of the Supreme Court for applications in the field with non-overlapping horizontal reservations. Therefore, while the justices have not explicitly mandated the 2SMG choice rule with *Saurav Yadav (2020)* and they merely endorsed it emphasizing that “the exercise of laying down a procedure must necessarily be left to the concerned authorities,” they have indirectly enforced it when individuals have at most one trait.

Finally, while the judgments of the Supreme Court offer some flexibility for the more general case of overlapping horizontal reservations, we have advocated in Section 4 for a specific choice rule, the *two-step meritorious horizontal* choice rule, for this more general case, and characterized it in Theorem 3 with axioms which can be considered natural extensions of the simpler versions mandated by the Supreme Court.

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Appendix

Appendix A. Proofs

In this Appendix, we present the proofs of our results. Some of our results for the more general version of the model in Section 4, most notably Proposition 2 and Theorem 2, are conceptually related to abstract results in *matroid theory*. Although these results have more direct proofs that rely on the literature on *maximum matchings in bipartite graphs*, we present proofs that highlight the conceptual connection between our results and the literature on matroid theory.

Before we present the proofs of our results in Section A.3, we present preliminaries in matroid theory in Sections A.1 and A.2.

A.1. Preliminary Definitions and Results in Matroid Theory. In this section we provide some basic definitions and results in matroid theory. We follow Oxley (2006).

A *matroid* is a pair (E, \mathcal{M}) where E is a finite set and \mathcal{M} is a collection of subsets of E that satisfies the following three properties:

M1: $\emptyset \in \mathcal{M}$.

M2: If $M \in \mathcal{M}$ and $M' \subseteq M$, then $M' \in \mathcal{M}$.

M3: If $M_1, M_2 \in \mathcal{M}$ and $|M_1| < |M_2|$, then there is $m \in M_2 \setminus M_1$ such that $M_1 \cup \{m\} \in \mathcal{M}$.

Set E is called the *ground set* of the matroid. Each set in \mathcal{M} is called an *independent set*. An independent set M is *maximal* if there is no proper superset of M that is independent. A maximal independent set of a matroid is called a *base*. All bases of a matroid have the same cardinality by M3. The set of bases \mathcal{B} satisfies the following two properties:

B1: \mathcal{B} is non-empty.

B2: If B_1 and B_2 are in \mathcal{B} and $e_1 \in B_1 \setminus B_2$, then there exists an element e_2 of $B_2 \setminus B_1$ such that $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$.

The stronger version of B2 where the implication is $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$ and $(B_2 \setminus \{e_2\}) \cup \{e_1\} \in \mathcal{B}$ also holds (Brualdi, 1969). An analogous statement holds when instead of individual elements in $B_1 \setminus B_2$ and $B_2 \setminus B_1$, we consider sets of elements (Brylawski, 1973; Greene, 1973; Woodall, 1974):

B2': If B_1 and B_2 are in \mathcal{B} and $E_1 \subseteq B_1 \setminus B_2$, then there exists $E_2 \subseteq B_2 \setminus B_1$ such that $(B_1 \setminus E_1) \cup E_2 \in \mathcal{B}$ and $(B_2 \setminus E_2) \cup E_1 \in \mathcal{B}$.

The *restriction* of matroid (E, \mathcal{M}) to $E' \subseteq E$ is a matroid (E', \mathcal{M}') where $\mathcal{M}' = \{X \subseteq E' : X \in \mathcal{M}\}$. The *rank* of $X \subseteq E$ is defined as the cardinality of a maximal independent set in the restriction of (E, \mathcal{M}) to X . Since all maximal independent sets have the same cardinality, the rank of a set is well-defined. The rank of $X \subseteq E$ is denoted by $r(X)$. The rank function satisfies the following properties:

R1: If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.

R2: If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

R3: If $X, Y \subseteq E$, then

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

A.2. Greedy Choice Rule and Its Properties. For a given weight function $w : E \rightarrow \mathbb{R}_+$ that takes distinct values, the *greedy algorithm* chooses the element with the highest weight subject to the constraint that the chosen set of elements is independent.

Greedy Algorithm

Step 1: Set $X_0 = \emptyset$ and $i = 0$.

Step 2: If there exists $e \in E \setminus X_i$ such that $X_i \cup \{e\} \in \mathcal{M}$, then choose such an element e_{i+1} of maximum weight, let $X_{i+1} = X_i \cup \{e_{i+1}\}$, and go to Step 3; otherwise let $B = X_i$ and go to Step 4.

Step 3: Add 1 to i and go to Step 2.

Step 4: Stop.

The textbook definition of the Greedy algorithm takes w to be any weight function that can take same values for different elements of E . In this case, the Greedy algorithm can select different sets depending on how elements are chosen when they have the same weight. To avoid this issue, we assume that distinct elements of E have different weights.

The greedy algorithm is defined on matroid (E, \mathcal{M}) . However, it can be applied to any restriction of this matroid. Therefore, the greedy algorithm can be viewed as a single-category choice rule on 2^E (Fleiner, 2001). For the rest of the paper, we view it as a single-category choice rule and refer to it as the *greedy choice rule*.

The greedy algorithm chooses an independent set that has the maximum weight, where the weight of a set is the sum of weights of individual elements. Before we introduce a stronger property of the greedy algorithm, we need the following definition.

Let elements of the sets $X, Y \subseteq E$ be enumerated such that,

$$\begin{aligned} \text{for every } i, j \in \{1, \dots, |X|\}, \quad i \leq j &\implies w(x_i) \geq w(x_j), \text{ and} \\ \text{for every } i, j \in \{1, \dots, |Y|\}, \quad i \leq j &\implies w(y_i) \geq w(y_j). \end{aligned}$$

Then, the set $X = \{x_1, \dots, x_{|X|}\} \subseteq E$ *Gale dominates* the set $Y = \{y_1, \dots, y_{|Y|}\} \subseteq E$ if $|X| \geq |Y|$ and, for every $i \in \{1, \dots, |Y|\}$,

$$w(x_i) \geq w(y_i).$$

We use the notation $X \succeq^G Y$ to denote set X Gale dominates set Y .

The following property of the greedy choice rule is the driving force for a similar property of the meritorious horizontal choice rule that is presented in Proposition 2.

Lemma 1. (Gale, 1968) *For every $E' \subseteq E$, the outcome of the greedy choice rule for E' Gale dominates any independent subset of E' .*

The following property of choice rules plays an important role in market design.

Definition 16. (Kelso and Crawford, 1982) *A choice rule $C : 2^E \rightarrow 2^E$ satisfies the **substitutes condition**, if, for every $E' \subseteq E$,*

$$e \in C(E') \text{ and } e' \in E' \setminus \{e\} \implies e \in C(E' \setminus \{e'\}).$$

We use the following result in some of our proofs.

Lemma 2. (Fleiner, 2001) *The greedy choice rule satisfies the substitutes condition.*

Fix a matroid (E, \mathcal{M}) with rank function r . We next formulate some properties of choice rules. The first two properties extend the notion of independence for sets and the maximality for independent sets to choice rules.

Definition 17. A choice rule $C : 2^E \rightarrow 2^E$ is *independent* if, for every $E' \subseteq E$, $C(E')$ is an independent set.

Definition 18. A choice rule $C : 2^E \rightarrow 2^E$ is *rank maximal* if, for every $E' \subseteq E$,

$$r(C(E')) = r(E').$$

The next property is a reformulation of our axiom *no justified envy* in the abstract context of matroids.

Definition 19. A choice rule $C : 2^E \rightarrow 2^E$ satisfies *no justified envy* if, for every $E' \subseteq E$, $e \in C(E')$, and $e' \in E' \setminus C(E')$

$$w(e') > w(e) \implies r((C(E') \setminus \{e\}) \cup \{e'\}) < r(C(E')).$$

The following result follows from the well-known properties of the greedy algorithm. We will rely on it extensively in the proof of Theorem 2 to highlight the conceptual similarities between our characterization of the meritorious horizontal choice rule and some of the key properties of the greedy algorithm.

Lemma 3. A choice rule $C : 2^E \rightarrow 2^E$ is independent, rank maximal, and satisfies no justified envy if, and only if, it is the greedy choice rule.

Proof. Let C be the greedy choice rule. Then by construction it is independent. Rank maximality follows easily because if $C(E')$ is not rank maximal for some $E' \subseteq E$, then there exists $e \in E$ such that $C(E') \cup \{e\}$ is independent. Thus, the greedy choice rule cannot produce $C(E')$.

Suppose, for contradiction, that C fails to satisfy no justified envy. Then there exists $E' \subseteq E$, $e \in C(E')$, and $e' \in E' \setminus C(E')$ with $w(e') > w(e)$ such that

$$r((C(E') \setminus \{e\}) \cup \{e'\}) \geq r(C(E')).$$

Since C is rank maximal, $r(C(E')) = r(E')$. By R2, $r((C(E') \setminus \{e\}) \cup \{e'\}) \leq r(E')$. Therefore, $r(C(E') \setminus \{e\}) \cup \{e'\} = r(E')$. Furthermore, since $|C(E')| = r(C(E'))$ because $C(E')$ is independent, we get $r(C(E') \setminus \{e\}) \cup \{e'\} = |C(E') \setminus \{e\} \cup \{e'\}|$, so $(C(E') \setminus \{e\}) \cup \{e'\}$ is also an independent set. By Lemma 1, $C(E')$ Gale dominates $(C(E') \setminus \{e\}) \cup \{e'\}$, so we get $w(e) > w(e')$, which is a contradiction.

We next show that any choice rule satisfying the properties has to be the greedy choice rule. Let D be a choice rule that satisfies the three axioms. Suppose, for contradiction, that $D(E') \neq C(E')$ for some $E' \subseteq E$. Since both D and C are independent and rank maximal, $D(E')$ and $C(E')$ are bases in the matroid restriction of (E, \mathcal{M}) to E' and so $|D(E')| = |C(E')|$. Therefore, there exists e_1 in $D(E') \setminus C(E')$. Then, by B2', there exists

$e_2 \in C(E') \setminus D(E')$ such that $(D(E') \setminus \{e_1\}) \cup \{e_2\}$ and $(C(E') \setminus \{e_2\}) \cup \{e_1\}$ are also bases. By Lemma 1, $C(E')$ Gale dominates $(C(E') \setminus \{e_2\}) \cup \{e_1\}$, which implies that $w(e_2) > w(e_1)$. Since D satisfies no justified envy, $e_1 \in D(E')$, $e_2 \in E' \setminus D(E')$, and $w(e_2) > w(e_1)$, we get

$$r((D(E') \setminus \{e_1\}) \cup \{e_2\}) < r(D(E')).$$

This is a contradiction because $(D(E') \setminus \{e_1\}) \cup \{e_2\}$ and $D(E')$ are both bases and, therefore, they have the same rank since they have the same cardinality. \square

A.3. Proofs of Main Results. Using the HR graph for a category $v \in \mathcal{V}$, we can study the *transversal matroid* with the ground set \mathcal{I}^v (Edmonds and Fulkerson, 1965). In this matroid, a set of individuals is independent if they can be matched with distinct positions and therefore the rank of a set of individuals is equal to the maximum number of distinct positions they can be matched with, which is the n^v function that we have defined for a category $v \in \mathcal{V}$. Furthermore, the weight of an individual can be defined as their merit score. With this setup, Step 1 of the meritorious horizontal choice rule is the same as the greedy choice rule for the transversal matroid. We use this important observation in proofs of Proposition 2 and Theorem 2 presented below.

Proof of Proposition 1. Let $I \subseteq \mathcal{I}$ be a set of individuals and $I^m \subseteq I$ be the set of reserve-eligible individuals considered at Step 1 of \widehat{C}^{SCI} when I is the set of applicants.

Let $i \in \widehat{C}_{mg}^{2s}(I) \cap I^g$. Then $i \in C_{mg}^o(I) \cap I^g$ because $\widehat{C}_{mg}^{2s}(I) \cap I^g = C_{mg}^o(I) \cap I^g$. Since C_{mg}^o satisfies the substitutes condition (Echenique and Yenmez, 2015), $i \in C_{mg}^o(I^m \cup I^g)$ because $i \in I^g$ and $i \in C_{mg}^o(I)$. Therefore, $i \in C_{mg}^o(I^m \cup I^g) \cap I^g$, which implies $i \in \widehat{C}^{SCI}(I) \cap I^g$ because $\widehat{C}^{SCI}(I) \cap I^g = C_{mg}^o(I^m \cup I^g) \cap I^g$. Therefore, we conclude that $\widehat{C}_{mg}^{2s}(I) \cap I^g \subseteq \widehat{C}^{SCI}(I) \cap I^g$.

The assumption that $|I^c| \geq q^o + q^c$ for each reserve-eligible category $c \in \mathcal{R}$ implies that all category- c positions are filled under both C_{mg}^{2s} and C^{SCI} . In addition, the first part of the proposition implies that there are weakly more individuals with reserved categories assigned to open-category positions under C_{mg}^{2s} than under C^{SCI} . Therefore,

$$\sum_{c \in \mathcal{R}} |\widehat{C}_{mg}^{2s}(I) \cap I^c| \geq \sum_{c \in \mathcal{R}} |\widehat{C}^{SCI}(I) \cap I^c|.$$

\square

Proof of Theorem 1. Since the 2SMH choice rule reduces to the 2SMG choice rule in the absence of overlapping horizontal reservations, the result is a direct corollary of Theorem 3. \square

Proof of Proposition 2. Let $I \subseteq \mathcal{I}^v$ be a set of individuals. To show part (1), note that $|C_{\mathbb{M}}^v(I)| = \min\{q^v, |I|\}$ since $C_{\mathbb{M}}^v$ is non-wasteful. Furthermore, for single-category choice rule C^v , $C^v(I) \subseteq I$ and $|C^v(I)| \leq q^v$ imply

$$|C^v(I)| \leq \min\{q^v, |I|\} = |C_{\mathbb{M}}^v(I)|.$$

We show the second part using mathematical induction on the number of individuals chosen at the second step of $C_{\mathbb{M}}^v$. For the inductive step, we decrease q^v by one so one less individual is chosen at the second step of $C_{\mathbb{M}}^v$ and we also consider a subset of I .

For the base case, when no individuals are chosen at the second step of $C_{\mathbb{M}}^v$, Lemma 1 states $C_{\mathbb{M}}^v(I) \succeq^G C^v(I)$ since the first step of $C_{\mathbb{M}}^v$ is the greedy choice rule for the transversal matroid and $C^v(I)$ is another base since C^v maximally accommodates HR protections and $|C_{\mathbb{M}}^v(I)| = |C^v(I)|$ in this case.

Now assume that the claim holds when the number of individuals chosen at the second step of $C_{\mathbb{M}}^v$ is less than $k > 0$. Let C_R^v be the choice rule corresponding to the second step of $C_{\mathbb{M}}^v$. Consider a set of individuals I such that $|C_R^v(I)| = k$. Let $J = C_{\mathbb{M}}^v(I)$, $K = C_R^v(I)$, $J' = C_{\mathbb{M}}^v(C^v(I))$, and $K' = C^v(I) \setminus J'$. If $|K'| = 0$, then the proof is complete as in the base case using Lemma 1. For the rest of the proof suppose that $|K'| > 0$.

Lemma 4. *There exists $j \in K$ and $j' \in K'$ such that $\sigma(j) \geq \sigma(j')$.*

Proof. Suppose, for contradiction, that for every $j \in K$ and $j' \in K'$ we have $\sigma(j) < \sigma(j')$. Since $j \in K = C_R^v(I)$, $j' \notin K$, and $\sigma(j') > \sigma(j)$, we must have $j' \in J$. Therefore, every individual in K' is also in J , which means $K' \subseteq J$. Since $K' \cap J' = \emptyset$ we have $K' \subseteq J \setminus J'$. Therefore, by B2' there exists $K'' \subseteq J \setminus J'$ such that $(J \setminus K') \cup K''$ and $(J' \setminus K'') \cup K'$ are also bases. By Lemma 1 we get

$$J \succeq^G (J \setminus K') \cup K''$$

and

$$J' \succeq^G (J' \setminus K'') \cup K'.$$

These are equivalent to $K' \succeq^G K''$ and $K'' \succeq^G K'$, respectively. Therefore, we must have $K' = K''$, which is a contradiction because $K' \neq \emptyset$ and $K' \cap K'' = \emptyset$. ■

Now we use Lemma 4 to finish the proof. Consider $j \in K$ and $j' \in K'$ such that $\sigma(j) \geq \sigma(j')$. If $\sigma(j) = \sigma(j')$, then $j = j'$ since different individuals have distinct merit scores. In this case, consider the set of individuals $I \setminus \{j\}$ and reduce the capacity q^v to $\min\{q^v, |C_{\mathbb{M}}^v(I)|\} - 1$. For $C_{\mathbb{M}}^v$, the same set of individuals is chosen at Step 1 by the greedy choice rule and the same set of individuals except j is chosen at the second step. Consider a choice rule D^v such that $D^v(I \setminus \{j\}) = C^v(I) \setminus \{j\}$ and, for other $I' \neq I$, let $D^v(I')$ be such that $n^v(D^v(I')) = n^v(I')$. Then D^v maximally accommodates HR protections. By the

mathematical induction hypothesis, we get that $C_{\mathbb{M}}^v(I) \setminus \{j\} \succeq^G D^v(I \setminus \{j\}) = C^v(I) \setminus \{j\}$. Therefore, $C_{\mathbb{M}}^v(I) \succeq^G C^v(I)$ because $j \in C_{\mathbb{M}}^v(I)$ and $j \in C^v(I)$.

Next consider the case when $\sigma(j) > \sigma(j')$. We need the following result.

Lemma 5. *Individual j' is not a member of J .*

Proof. Suppose, for contradiction, that $j' \in J$. Since $j' \in K'$, we have $j' \notin J'$. Therefore, $j' \in J \setminus J'$. By B2', there exists $j'' \in J' \setminus J$ such that both $(J \setminus \{j'\}) \cup \{j''\}$ and $(J' \setminus \{j''\}) \cup \{j'\}$ are bases. By Lemma 1, $J \succeq^G (J \setminus \{j'\}) \cup \{j''\}$ and $J' \succeq^G (J' \setminus \{j''\}) \cup \{j'\}$, which are equivalent to $\{j'\} \succeq^G \{j''\}$ and $\{j''\} \succeq^G \{j'\}$, respectively. The last two inequalities can only hold when $j' = j''$, which is a contradiction because $j' \in K'$, $j'' \in J'$, and $J' \cap J'' = \emptyset$. \square

We apply the inductive hypothesis to the market with the set of individuals $I \setminus \{j, j'\}$ and the capacity $\min\{q^v, |C_{\mathbb{M}}^v(I)|\} - 1$ as in the previous case (when $j = j'$). In this reduced market, at the first step of $C_{\mathbb{M}}^v$, the greedy choice rule selects the same set of individuals as in the original market and the responsive choice rule selects the same set of individuals except j . Construct choice rule D^v such that $D^v(I \setminus \{j, j'\}) = C^v(I) \setminus \{j'\}$. For any other $I' \neq I$ let $D^v(I')$ be such that $n^v(D^v(I')) = n^v(I')$. Since $D^v(I \setminus \{j, j'\}) \supseteq J'$, $n^v(D^v(I \setminus \{j, j'\})) \geq n^v(J') = n^v(I)$, where the inequality follows from monotonicity of n^v and the equality follows since C^v maximally satisfies HR protections. Therefore, $n^v(D^v(I \setminus \{j, j'\})) = n^v(I \setminus \{j, j'\})$, and hence D^v maximally accommodates HR protections. By the mathematical induction hypothesis, $C_{\mathbb{M}}^v(I) \setminus \{j\} \succeq^G C^v(I) \setminus \{j'\}$. Furthermore, since $\sigma(j) > \sigma(j')$, we conclude that $C_{\mathbb{M}}^v(I) \succeq^G C^v(I)$. \square

Proof of Theorem 2. We first show $C_{\mathbb{M}}^v$ satisfies the stated properties in several lemmas and then show that the unique category- v choice rule satisfying these properties is $C_{\mathbb{M}}^v$. Let C_G^v be the greedy choice rule that corresponds to the first step of $C_{\mathbb{M}}^v$. Let C_R^v be the choice rule that corresponds to the second step of $C_{\mathbb{M}}^v$: For every $I \subseteq \mathcal{I}^v$, $C_R^v(I)$ consists of $\min\{q^v - |C_G^v(I)|, |I| - |C_G^v(I)|\}$ individuals in $I \setminus C_G^v(I)$ with the highest merit scores. Therefore, we have

$$C_{\mathbb{M}}^v(I) = C_G^v(I) \cup C_R^v(I).$$

Lemma 6. $C_{\mathbb{M}}^v$ maximally accommodates HR protections.

Proof. For every $I \subseteq \mathcal{I}^v$, by Lemma 3, $n^v(C_G^v(I)) = n^v(I)$. Furthermore, by monotonicity of n^v , $n^v(C_{\mathbb{M}}^v(I)) \geq n^v(C_G^v(I)) = n^v(I)$, which implies $n^v(C_{\mathbb{M}}^v(I)) = n^v(I)$. We conclude that $C_{\mathbb{M}}^v$ maximally accommodates HR protections. \blacksquare

Lemma 7. $C_{\mathbb{M}}^v$ satisfies no justified envy.

Proof. Suppose, for contradiction, that $C_{\mathbb{M}}^v$ fails no justified envy. Therefore, there exist a set of individuals $I \subseteq \mathcal{I}^v$, individuals $i \in C_{\mathbb{M}}^v(I)$, $j \in I \setminus C_{\mathbb{M}}^v(I)$ with $\sigma(j) > \sigma(i)$ and $n^v((C_{\mathbb{M}}^v(I) \setminus \{i\}) \cup \{j\}) \geq n^v(C_{\mathbb{M}}^v(I))$. Since $C_{\mathbb{M}}^v(I)$ maximally accommodates HR protections, the last inequality implies $n^v((C_{\mathbb{M}}^v(I) \setminus \{i\}) \cup \{j\}) = n^v(I)$.

Since every individual in $C_R^v(I)$ has a higher merit score than j , we must have $i \in C_G^v(I)$. Furthermore, every individual in $C_R^v(I)$ has a higher merit score than i as well.

Let $I_1 = C_G^v((C_{\mathbb{M}}^v(I) \setminus \{i\}) \cup \{j\})$. Since C_G^v is rank maximal (Lemma 3), $n^v(I_1) = n^v(I)$. Furthermore, since C_G^v is independent (Lemma 3), I_1 is independent. Therefore, we get $|I_1| = n^v(I)$. In addition, since C_G^v satisfies the substitutes condition (Lemma 2), $I_1 \supseteq C_G^v(I) \setminus \{i\}$. Therefore, $I_1 = (C_G^v(I) \setminus \{i\}) \cup \{k\}$ where $k \in C_R^v(I) \cup \{j\}$. This gives us a contradiction since I_1 is an independent set, so by Lemma 1, $C_G^v(I) \succeq^G I_1$ which is equivalent to $\sigma(i) > \sigma(k)$ but every individual in $C_R^v(I) \cup \{j\}$ has a higher merit score than i . ■

Lemma 8. $C_{\mathbb{M}}^v$ is non-wasteful.

Proof. $C_{\mathbb{M}}^v$ is non-wasteful because at the second step all the unfilled positions are filled with the unmatched individuals until all positions are filled or all individuals are assigned to positions. ■

Lemma 9. If a category- v choice rule maximally accommodates HR protections, satisfies no justified envy, and is non-wasteful, then it has to be $C_{\mathbb{M}}^v$.

Proof. Let C^v be a category- v choice rule that maximally accommodates HR protections, satisfies no justified envy, and is non-wasteful. Construct the following choice rule D , where for any $I \subseteq \mathcal{I}^v$,

$$D^v(I) = C_G^v(C^v(I)).$$

We show that D^v is the greedy choice rule by showing that D^v is independent, rank maximal, and satisfies no justified envy.

First, D^v is independent because C_G^v is independent (Lemma 3).

Second, since C^v maximally accommodates HR protections, $n^v(C^v(I)) = n^v(I)$. In addition, since the greedy choice rule is rank maximal $n^v(D^v(I)) = n^v(C^v(I))$. Therefore, $n^v(D^v(I)) = n^v(I)$, which means that D^v is rank maximal.

Finally, suppose for contradiction that D^v fails to satisfy no justified envy. Then there exist $I \subseteq \mathcal{I}$, $i \in D^v(I)$, and $j \in I \setminus D^v(I)$ with $\sigma(j) > \sigma(i)$ such that

$$n^v((D^v(I) \setminus \{i\}) \cup \{j\}) \geq n^v(D^v(I)).$$

Since C_G^v satisfies no justified envy (Lemma 3), j has to be in $I \setminus C^v(I)$. Furthermore $n^v(D^v(I)) = n^v(I)$ by rank maximality of $D^v(I)$, so we get $n^v((D^v(I) \setminus \{i\}) \cup \{j\}) = n^v(I)$. As a result, $n^v((C^v(I) \setminus \{i\}) \cup \{j\}) = n^v(I)$ as well. This gives a contradiction to the assumption that C^v satisfies no justified envy because $i \in C^v(I)$, $j \in I \setminus C^v(I)$, $\sigma(j) > \sigma(i)$, and $n^v((C^v(I) \setminus \{i\}) \cup \{j\}) = n^v(I) = n^v(C^v(I))$.

Since $D^v(I)$ is independent, rank maximal, and satisfies no justified envy, we conclude by Lemma 3 that $D^v = C_G^v$. Now consider $C^v(I) \setminus C_G^v(I)$. Since C^v is non-wasteful, $|C^v(I) \setminus C_G^v(I)| = \min\{q^v - C_G^v(I), |I| - C_G^v(I)\}$. Furthermore, by no justified envy, there cannot be an individual in $I \setminus C^v(I)$ who has a higher merit score than any individual in $C^v(I) \setminus C_G^v(I)$. Therefore, we get

$$C^v(I) \setminus C_G^v(I) = C_R^v(I).$$

Since $C^v(I) \supseteq C_G^v(I)$, we conclude that $C^v(I) = C_G^v(I) \cup C_R^v(I)$ is the meritorious horizontal choice rule. ■

This finishes the proof of Theorem 2. □

Proof of Theorem 3. Let $C = (C^v)_{v \in \mathcal{V}}$ be a choice rule that complies with VR protections, maximally accommodates HR protections, satisfies no justified envy, and is non-wasteful. We show this result using the following lemmas.

Lemma 10. $C^o = C_{\text{M}}^{2s,o}$.

Proof. We prove that C^o maximally accommodates category- o HR protections, satisfies no justified envy, and is non-wasteful.

First, we show that C^o maximally accommodates category- o HR protections. Suppose, for contradiction, that $n^o(C^o(I)) < n^o(I)$ for some $I \subseteq \mathcal{I}$. Then there exists $i \in I \setminus C^o(I)$ such that $n^o(C^o(I) \cup \{i\}) = n^o(C^o(I)) + 1$. If $i \in I \setminus \widehat{C}(I)$, then we get a contradiction with the assumption that C maximally accommodates HR protections. Otherwise, if $i \in C^c(I)$ where $c \in \mathcal{R}$, then we get a contradiction with the assumption that C complies with VR protections. Therefore, C^o maximally accommodates category- o HR protections.

Next, we show that C^o satisfies no justified envy. Let $i \in C^o(I)$ and $j \in I \setminus C^o(I)$ such that $\sigma(j) > \sigma(i)$. If $j \in I \setminus \widehat{C}(I)$, then

$$n^o((C^o(I) \setminus \{i\}) \cup \{j\}) < n^o(C^o(I))$$

because C satisfies no justified envy. However, if $i \in C^c(I)$ for category $c \in \mathcal{R}$, then

$$n^o((C^o(I) \setminus \{i\}) \cup \{j\}) < n^o(C^o(I))$$

because C complies with VR protections. Therefore, C^o satisfies no justified envy.

Now, we show that C^o is non-wasteful, which means that $|C^o(I)| = \min\{|I|, q^o\}$ for every $I \subseteq \mathcal{I}$. If there exists an individual $i \in I$ such that $i \notin \widehat{C}(I)$, then $|C^o(I)| = q^o$ because C is non-wasteful. If there exists an individual $i \in I$ such that $i \in C^c(I)$ where $c = \rho(i) \in \mathcal{R}$, then $|C^o(I)| = q^o$ because C complies with VR protections. If these two conditions do not hold, then all the individuals are allocated open-category positions, i.e., $I = C^o(I)$. Therefore, under all possibilities, we get $|C^o(I)| = \min\{|I|, q^o\}$, which means that C^o is non-wasteful.

Since C^o maximally accommodates category- o HR protections, satisfies no justified envy, and is non-wasteful, we get $C^o = C_{\mathbb{M}}^o$ (Theorem 2), and hence $C^o = C_{\mathbb{M}}^{2s,o}$. ■

Let $c \in \mathcal{R}$, $I \subseteq \mathcal{I}$, and $\bar{I}^c = \{i \in I \setminus C_{\mathbb{M}}^o(I) \mid \rho(i) = c\}$.

Lemma 11. $C^c(I)$ maximally accommodates category- c HR protections for \bar{I}^c .

Proof. Suppose, for contradiction, that $n^c(C^c(I)) < n^c(\bar{I}^c)$. This is equivalent to

$$n^c(C^c(I)) < n^c(\bar{I}^c) = n^c\left(C^c(I) \cup \{i \in I \setminus \widehat{C}(I) \mid \rho(i) = c\}\right),$$

which implies that there exists $i \in I \setminus \widehat{C}(I)$ who is eligible for category c such that

$$n^c(C^c(I \cup \{i\})) = n^c(C^c(I)) + 1.$$

This equation contradicts the assumption that C maximally accommodates HR protections. Therefore, $C^c(I)$ maximally accommodates category- c HR protections for \bar{I}^c . ■

Lemma 12. $C^c(I)$ satisfies no justified envy for \bar{I}^c .

Proof. Let $i \in C^c(I)$ and $j \in \bar{I}^c \setminus C^c(\bar{I}^c)$ be such that $\sigma(j) > \sigma(i)$. Note that $i \in \bar{I}^c$. Since C satisfies no justified envy, we have

$$n^c(C^c(I)) > n^c((C^c(I) \setminus \{j\}) \cup \{i\}).$$

Hence, C^c satisfies no justified envy for \bar{I}^c . ■

Lemma 13. $|C^c(I)| = \min\{|\bar{I}^c|, q^c\}$.

Proof. We consider two cases. First, if $C^c(I) = \bar{I}^c$, then $|C^c(I)| = \min\{|\bar{I}^c|, q^c\}$ because $|C^c(I)| \leq q^c$. Otherwise, if $C^c(I) \neq \bar{I}^c$, then there exists $i \in \bar{I}^c \setminus C^c(I)$. Therefore, $i \in I \setminus \widehat{C}(I)$. Since C is non-wasteful, we get $|C^c(I)| = q^c$. Since $i \in \bar{I}^c \setminus C^c(I)$ and $|C^c(I)| = q^c$, $|\bar{I}^c| > q^c$. Therefore, $|C^c(I)| = q^c = \min\{|\bar{I}^c|, q^c\}$. ■

Therefore, $C^c(I)$ maximally accommodates category- c HR protections for \bar{I}^c , $C^c(I)$ satisfies no justified envy for \bar{I}^c , and $C^c(I)$ is non-wasteful for \bar{I}^c . By Theorem 2, $C^c(I) =$

$C_{\mathbb{M}}^c(\bar{I}^c)$ and, thus,

$$C^c(I) = C_{\mathbb{M}}^c(\bar{I}^c) = C_{\mathbb{M}}^c(\{i \in I \setminus C_{\mathbb{M}}^o(I) \mid \rho(i) = c\}) = C_{\mathbb{M}}^{2s,c}(I).$$

□

Proof of Proposition 3. Suppose that i is chosen by $\widehat{C}_{\mathbb{M}}^{2s}$ when she withholds some of her reserve-eligible privileges. If i is chosen by $C_{\mathbb{M}}^o$ for an open-category position, then i will still be chosen by declaring all her reserve-eligible privileges because $C_{\mathbb{M}}^o$ does not use the category information of individuals and an individual can never benefit from not declaring some of her traits under $C_{\mathbb{M}}^o$ because she will have more edges in the category- o HR graph. Otherwise, if i is chosen by $C_{\mathbb{M}}^c$ where $\rho(i) = c \in \mathcal{R}$ then she must have declared her reserve-eligible category c . In addition, by declaring all her traits she will still be chosen by $C_{\mathbb{M}}^c$ if she is not chosen before for the open-category positions because she will have more edges in the HR graph for category- c positions.

□