DYNAMIC NOISY RATIONAL EXPECTATIONS EQUILIBRIUM WITH INSIDER INFORMATION

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Abstract. We study equilibria in multi-asset and multi-agent continuous-time economies with asymmetric information and bounded rational noise traders. We establish existence of two equilibria. First, a full communication one where the informed agents' signal is disclosed to the market, and static policies are optimal. Second, a partial communication one where the signal disclosed is affine in the informed and noise traders' signals, and dynamic policies are optimal. Here, information asymmetry creates demand for two public funds, as well as a dark pool where private information trades can be implemented. Markets are endogenously complete and equilibrium returns have a three factor structure, with stochastic factors and loadings. Results are valid for constant absolute risk averse investors; general vector diffusions for fundamentals; non-linear terminal payoffs, and non-Gaussian noise trading. Asset price dynamics and public information flows are endogenous, and rational expectations equilibria are special cases of the general results.

1. Introduction

This paper studies equilibria in multi-asset, multi-agent continuous-time economies with asymmetric information and bounded rational noise traders. Following the rational expectations equilibrium paradigm of Radner (1979), we show that a dynamic, noisy, rational expectations equilibrium (NREE) exists, under very broad conditions on the fundamental processes and agents' signals. We also prove existence of a no-trade dynamic, fully revealing, rational expectations equilibrium. Asset dynamics and optimal policies in both equilibria are not necessarily linear functions of the

Date: June 4, 2020.

Key words and phrases. Rational Expectations Equilibrium; Heterogenous Information; Diffusions; Prices; Risk Premia; Volatility; Portfolios.

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We thank the Co-Editor Allessandro Lizzeri and three anonymous referees for detailed and constructive comments.

fundamental process and endogenous market signal. Equilibrium portfolios satisfy a fund separation property with two public funds, and a dark pool where dynamic trades based on private information are carried out. Markets are endogenously complete and, irrespective of the number of state variables, prices have a three factor structure, while risk premia satisfy a three beta intertemporal CAPM.

The notion that prices communicate private information goes back to Hayek (1945). Formalizations of this idea, in Grossman (1976, 1978), show that prices can be fully revealing, in that they serve as sufficient statistics for all the private information disseminated in the economy. This result, derived in a static setting, assumes (i) the joint distribution of the private information signals and asset payoff is a multivariate normal; and (ii) agents have constant absolute risk aversion (CARA). By adding noise to the asset supply, Hellwig (1980) shows the possibility of partial revelation: while prices convey information, they are not sufficient statistics for all private information disseminated. Extensions to dynamic versions of the CARA-normal setting include Wang (1993), Brennan and Cao (1996) and He and Wang (1995), where the last two references deal with finite horizon models. A static CARA model beyond the Gaussian setting is provided in Breon-Drish (2015), assuming the conditional law of the asset payoff, given the private signal, falls into an exponential family of distributions.

In this paper, we extend the dynamic single asset CARA-normal structure, to a general multi-asset diffusive setting. The fundamental process X is an arbitrary d -dimensional diffusion. There are d assets with terminal payoff vector $\Psi(X_T)$. Asset shares are in constant supply $\Pi \in \mathbb{R}^d$. By allowing for general Ψ, Π we can treat the case of derivatives (where one asset's terminal payoff is a function of another asset's terminal payoff, and has zero supply) as well as the more common linear case (where $\Psi(x) = x$ and Π is the vector of ones). There are three types of agents, informed, uninformed and bounded rational noise traders. These agents behave as if they were informed, in that both noise and informed traders use the same conditional beliefs map to take private signals into optimal policies. However, noise traders evaluate this map using the realization of an incorrect signal, in contrast to the informed traders. Thus, on the one hand, for a given signal realization noise traders behave optimally, choosing portfolios that maximize expected (CARA) utility from terminal wealth. However, by trading on an incorrect signal, rather than the true informational signal, they deviate from the informed rational traders. More specifically, we assume CARA utility functions (with potentially different risk aversions), and for the informed traders' vector of signals $G_I = X_T + Y_I$, that the noise component Y_I is a d-dimensional normal random vector independent of X_T. The noise traders' vector of signals is $G_N = \tau_N G_I + Y_N$ where Y_N is a d-dimensional normal independent of Y_I and X_T. The specification of G_N includes the extreme behavior described in Black (1986), where noise traders transact on pure noise ($\tau_N = 0$). It also includes the case of (noisy) bilateral communication with the informed where they infer and trade on a noisy version of the informed signal, mistakenly believing it is the true informed signal ($\tau_N \neq 0$). The overall framework thus retains the CARA-normal setting for the utilities and noise components of signals, but allows for non-linear payoffs and general factor processes. In Section 7, we consider non-Gaussian noises Y_I, Y_N.

We examine several concepts of equilibrium. The first is the notion of Full Communication Equilibrium (FCE) introduced in Radner (1979). A FCE is an equilibrium for a pure exchange economy in which "before market activity takes place some exogenous information about the environment is made available to all traders." In Radner's setting, the information made available is the private signal of the informed agent (or the vector of private signals if there are many informed agents). An FCE is then said to be "revealing if it is one-to-one, i.e., it maps different signals into distinct price vectors." In this case the FCE is a fully revealing rational expectations equilibrium, where private information is communicated through the price system. The second is the notion of Partial Communication Equilibrium (PCE), which to our knowledge is new. A PCE is a variation of an FCE in which the information communicated at the outset is partial, consisting of a noisy version of the private signal of the informed. A PCE is revealing if endogenous quantities such as prices and residual demands map different realizations of the communicated signal into distinct values. In this case it is a noisy rational expectations equilibrium (NREE) where a noisy version of the informed private signal is communicated through observable quantities.

We establish existence of two equilibria: an FCE where the informed agents' private signal G_I is communicated to investors; and a PCE where the signal communicated is $H = \alpha G_I + (1 - \alpha) G_N$ for a constant $\alpha \in (0, 1)$. The FCE induces static trading, while the PCE leads to dynamic trading in two "core" portfolios. In both cases, the residual demand and the equilibrium price dynamics are nonlinear functions of H and X. As we show by presenting results for a more general setting with non-Gaussian G_N in Section 7, the affine nature of the public signal is almost entirely due to the Gaussian distribution for the informed agents' noise component Y_I. Indeed, up to providing explicit formulas, the process X and nature of the noise traders' signal G_N play no role in determining the form of H. This striking result establishes the robustness of equilibria to arbitrary specifications along these two dimensions.

As classically defined, a REE (either fully revealing or noisy) requires the public filtration to coincide with the natural filtration of observable quantities: fundamentals, asset prices and residual demand (Kreps (1977)). This enforces a mechanism to transfer information across the economy. For the equilibria we establish, the public filtration is precisely that generated by fundamentals, augmented by the information conveyed through a market signal communicated to all agents at time 0 (the signal is G_I in the fully revealing case and H in the partially revealing case). We show when payoffs are linear ($\Psi(x) = x$), or monotone in the one-dimensional case, that the public and observable information coincide, and hence our equilibria are a fully revealing REE and NREE respectively. For general payoffs, the observable filtration is contained within the public filtration communicated at the outset, and in fact, even if there is a loss of information when restricting to observable quantities, we show in the FCE this loss has no effect on optimal policies.

As made precise in Section 3, proving existence of an FCE is rather straightforward. As each agent type has the same information set (fundamental filtration refined by the informed agents' signal), the economy aggregates and agents share the outstanding supply according to their weighted risk tolerance. Proving existence of a PCE is much more challenging. Here, our arguments are inspired by Amendinger (2000). However, unlike in Amendinger (2000) where prices are exogenously specified and adapted to the fundamental filtration, which the informed agent combines with her private signal, in our setup there is an intermediate step. Indeed, we seek a to-be-determined public signal which refines the fundamental filtration. The uninformed and noise traders use this filtration, while the informed agent performs a second refinement, capturing the incremental contribution of her private signal. Despite the two refinements, classic replication arguments apply, and, provided in equilibrium asset volatilities are non-degenerate, the enlarged markets are complete. We then use completeness to identify a terminal clearing condition (c.f. Theorem 7.3) which is necessary and sufficient for a candidate signal and market price of risk to yield an equilibrium. As for the non-degeneracy of volatility, our results significantly extend those in Anderson and Raimondo (2008); Hugonnier, Malamud, and Trubowitz (2012) because the volatility changes with each signal realization. Therefore, we have to show non-degeneracy for (almost) all signal realizations, and this is possible using recent results in Kramkov and Pulido (2019) on representation theorems for analytic maps.

Aside from its informational properties, the PCE also sheds light on economic structures, such as dark pools, that have emerged over the past decade. Indeed, equilibrium portfolios satisfy an informational separation property with two mutual funds and a dark pool. The first fund is the market portfolio of risky securities. It is in positive supply and held by all agents. The second one is a portfolio tied to public information. This fund is in zero net supply and enables agents to hedge fluctuations in beliefs associated with public information. The dark pool is a private exchange with endogenous participation comprised of the informed and the noise trader. It provides a venue where these agents can dynamically trade on differences in the realizations of their respective signals. The dark pool is opaque to outsiders: private information trades net out and cannot be seen from the outside.

Equilibrium asset prices have a three factor structure. The first factor is the value of the market portfolio of risky assets. The second one is the value of the mutual fund for public information trades. The last one is the value of a dynamic portfolio that hedges the impact of innovations on the future wealth of the uninformed agent. Asset risk premia are determined by the same factors, hence satisfy a three beta CAPM. These pricing relations may seem surprising in light of equilibrium portfolios. The dark pool fails to matter for pricing because participants are comprised of the informed and the noise trader and their informational trades net out. In contrast, the fund for public information trades matters, even though it is in zero net supply, because the uninformed participates, implying the fund is directly relevant to his welfare. As prices are determined by the marginal utility of the uninformed, they reflect the three factors affecting it. Lastly, we note that while it is well known CAPM does not hold in the presence of asymmetric information (Dybvig and Ross (1985)), our result explicitly identifies the missing factors that emerge endogenously when information is asymmetric.[1]

Examples in Section 6 highlight the flexibility of our methodology. We solve for the FCE and PCE in two economies with respectively (i) multiple factors, quadratic payoffs and a multivariate Gaussian distribution; and (ii) a single factor, linear payoff and non-central chi-square distribution. Prices in these economies are non-linear

[1]Note the three beta CAPM holds even though there are d stochastic state variables.

functions of the information communicated and have stochastic volatility. In the second one, the volatility of volatility (volvol) is also stochastic. Illustrations show the behaviors of prices, volatility, volvol and the uninformed trading strategy.

In Section 7.1 we study a general version of the model requiring only that the noise component in the informed agents' private signal be Gaussian, proving the public signal is affine in the informed and noise traders' signals. Theorem 7.3 establishes the non-degeneracy of the equilibrium volatility matrix. Motivated by Anderson and Raimondo (2008); Hugonnier, Malamud, and Trubowitz (2012), the proof shows that the (random) maps from signal realization to (i) the equilibrium stochastic discount factor (SDF) and (ii) the equilibrium discounted terminal payoff, are analytic with probability one. As such, using results from the recent Kramkov and Pulido (2019), as long as the volatility is non-degenerate for *one* signal realization (precisely our Assumption 4.2), we prove it is non-degenerate for Lebesgue almost all signal realizations, and (as the signal communicated is a continuous random vector) non-degenerate in the public filtration communicated at the outset.

Lastly, there are delicate measurability issues which arise throughout the analysis, as equilibrium quantities are parameterized by realizations of the respective signals. The functional forms of the parameterizations are determined through representation results with respect to parameter-dependent random variables. Thus, to use the (random) signals, rather than their (deterministic) realizations, one has to verify measurability. Though most of the measurability results may be found in the literature (see Stricker and Yor (1978); Amendinger (2000) and in particular Fontana (2018)), for completeness we state (and fill in proofs where needed) the results in the supplemental Appendix $\{4\}$ (references to the supplemental file use $\{\cdot\}$).

The rest of this paper is organized as follows. Section 2 describes the uncertainty model, fundamental processes, asset payoffs, agents, and defines equilibrium. The FCE is detailed in Section 3, and the PCE in Section 4. A discussion of the latter is in Section 5. Section 6 provides explicit solutions for specific economies and highlights their informational and dynamic properties. Section 7 considers the PCE in a general abstract setting, culminating in Theorem 7.3 which proves necessity and sufficiency of a terminal market clearing condition. Proofs and auxiliary results are in Appendices A - E, and the supplemental Appendices $\{1\}$ - $\{4\}$.

2. Uncertainty, Economic Agents, Information, and Equilibrium

We begin by describing the uncertainty model, fundamental processes and assets' terminal payoffs. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and d a positive integer. The fundamental factor process X is a diffusion, taking values in a region $E \subseteq \mathbb{R}^d$, with

$$dX_t = b(X_t)dt + a(X_t)dB_t; \qquad X_0 = x_0 \in E, \tag{2.1}$$

where B is a d-dimensional Brownian motion. The investment time horizon is $T > 0$ and the money market rate is exogenously set to zero. There are d traded assets with terminal (vector) payoff $\Psi = \Psi(X_T)$ for a given function $\Psi : E \to \mathbb{R}^d$, and fixed supplies $\Pi = (\Pi^1, ..., \Pi^d)'$ where Π^i is the number of shares of asset i outstanding.

We assume X is a strong solution to (2.1), with additional technical restrictions upon the region E, the drift vector b, and the covariance matrix $A := aa'$ given in Appendix A. We also enforce the standing assumptions

Assumption 2.1.

(1) There is an $\varepsilon_0 > 0$ such that $\mathbb{E}\left[e^{\varepsilon_0 |X_T|}\right] < \infty$.
(2) There is a $K_1 > 0$ such that $|\Psi(x)| \leq K_1(1 + |x|^2)$.
(3) There are $K_2 > 0$ and $0 < \varepsilon_1 < 2$ such that $\Pi'\Psi(x) \geq -K_2\,(1 + |x|^{2-\varepsilon_1})$.

The integrability condition (1) is used to verify both PCE market completeness, and finiteness of the agents' value functions in both equilibria. It implies the moments $\mu_X := \mathbb{E}[X_T]$ and $\Sigma_X := \mathrm{Cov}(X_T)$ are well defined, and is satisfied by most models in the literature (e.g., affine processes, see Duffie, Filipović, and Schachermayer (2003)). The quadratic growth condition (2) is standard: it ensures finite equilibrium prices. Condition (3) imposes a lower bound on terminal aggregate wealth. It trivially holds in most applications as either Ψ is linear, or $\Pi, \Psi(x)$ are non-negative.

Economic agents. There is a continuum of agents, split into three types: informed (I), uninformed (U), and noise trader (N), which are in proportion $\omega_I, \omega_U, \omega_N$. We select a representative agent from each type and assume that representative agents have exponential (CARA) utility functions with respective risk aversions $\gamma_I, \gamma_U, \gamma_N$, and derive utility from terminal wealth. The population-weighted risk-tolerances of the respective types, and the aggregate risk aversion, are

$$\alpha_i := \frac{\omega_i}{\gamma_i}; \qquad i \in \{I, U, N\}; \qquad \gamma := (\alpha_I + \alpha_U + \alpha_N)^{-1}. \tag{2.2}$$

Public information and asset dynamics. We posit the existence of a public filtration $\mathbb{F}^m := \{\mathcal{F}^m_t : t \in [0,T]\}$ which is communicated to all agents and represents their common information. $\mathbb{F}^m$ contains the fundamental information $\mathbb{F}^B$ (the filtration of B which drives the factor process), but may also include information generated by prices or the combined holdings of the noise traders and informed agents.

We assume a competitive economy. Let S be the d-dimensional vector of asset prices, σ the $d \times d$ matrix of volatility components and ν the d-dimensional vector of prices of risk. As the terminal asset price $S_T = \Psi(X_T)$ is fixed, our goal is to obtain $\mathbb{F}^m$ predictable (ν, σ, S) (see (Jacod and Shiryaev, 2003, Chapter 1) for a definition of the predictable sigma-field $\mathcal{P}(\mathbb{F}^m)$) satisfying the backwards stochastic differential equation (BSDE)

$$dS_t = \sigma_t \left(\nu_t dt + dB^m_t\right); \qquad S_T = \Psi(X_T), \tag{2.3}$$

where B^m is a $(\mathbb{P}, \mathbb{F}^m)$ Brownian motion, and such that the optimal demands of investors satisfy the market clearing condition

$$\Pi = \omega_I \hat{\pi}^I_\cdot + \omega_U \hat{\pi}^U_\cdot + \omega_N \hat{\pi}^N_\cdot . \tag{2.4}$$

The coefficient $\sigma\nu$ is the vector of risk premia, and to describe the individual optimal investment problems and optimal demands, assume $\mathbb{F}^m, \nu, \sigma, B^m$ and S are given.

Uninformed investor. The uninformed investor has time 0 information $\mathcal{F}^m_0$. He chooses a trading strategy $\pi \in \mathcal{P}(\mathbb{F}^m)$, where π^j_t are the shares of S^j owned at t. As in Huang (1987), the self-financing gains process (Harrison and Kreps (1979), Harrison and Pliska (1981)) has dynamics $\mathcal{W}^\pi_\cdot := \int_0^\cdot \pi'_u dS_u$. A precise definition of the set $\mathcal{A}^m$ of admissible trading strategies is given in Appendix A, and with this set, the uniformed investor's value function is

$$u^U := \sup_{\pi \in \mathcal{A}^m} \mathbb{E}\left[-\frac{1}{\gamma_U} e^{-\gamma_U \mathcal{W}^\pi_T} \big| \mathcal{F}^m_0 \right].$$

Informed investor. In addition to the public information $\mathbb{F}^m$, at time 0 the informed investor observes a noisy private signal about the terminal factor value X_T given by

$$G_I := X_T + Y_I, \tag{2.5}$$

where $Y_I \sim N(0, C_I)$ is a d-dimensional normal random vector independent of X (specifically, $\mathcal{F}^B_T$) with mean vector 0 and covariance matrix $C_I \in \mathbb{S}^d_{++}$, the space of positive definite symmetric $d \times d$ matrices. To make investment decisions, the informed investor uses the filtration $\mathbb{F}^I := \left\{\mathcal{F}_t^I : t \in [0,T]\right\}$ consisting of public information $\mathbb{F}^m$ refined by the information conveyed by the signal G_I.[2]

We allow for two possibilities regarding $\mathbb{F}^I$ and $\mathbb{F}^m$. The first, corresponding to full communication of G_I, is when $\mathbb{F}^I = \mathbb{F}^m$ (the informed and public filtrations coincide) so that all agents have the same perception of uncertainty. In this case $B^I = B^m$ is a $(\mathbb{P}, \mathbb{F}^I)$-Brownian motion, and the insider also believes S has dynamics in (2.3). The second case, corresponding to partial communication of G_I, is when $\mathbb{F}^I \supset \mathbb{F}^m$. Here, $\mathbb{F}^I$ is more informative than $\mathbb{F}^m$, and the informed has an informational advantage, which modifies her perception of economic uncertainty.

In this latter case, provided a certain condition on G_I and $\mathbb{F}^m$ holds (see Section 5.2 below), the Brownian motion B^m in the public filtration becomes a Brownian motion with drift μ^{G_I} in the informed filtration, so that $B^I_\cdot := B^m_\cdot - \int_0^\cdot \mu_u^{G_I} du$ is a $(\mathbb{P}, \mathbb{F}^I)$-Brownian motion. The component μ^{G_I} is the private information price of risk, i.e., the incremental value of information in the private signal relative to the market information. Substituting for B^m in (2.3) shows that S satisfies the $(\mathbb{P}, \mathbb{F}^I)$ BSDE

$$dS_t = \sigma_t \left((\nu_t + \mu_t^{G_I}) dt + dB_t^I \right); \qquad S_T = \Psi(X_T). \tag{2.6}$$

In fact, even if $\mathbb{F}^m = \mathbb{F}^I$, the above equation makes sense, with the understanding that $\mu^{G_I} \equiv 0$ if there is no informational advantage, and $B^m = B^I$. Using (2.6), the informed investor's optimal investment problem is defined similarly to the uninformed investor's. The set of admissible informed strategies is $\mathcal{A}^I$ defined in Appendix A, and the value function is

$$u^I := \sup_{\pi \in \mathcal{A}^I} \mathbb{E}\left[-\frac{1}{\gamma_I} e^{-\gamma_I \mathcal{W}_T^\pi} \middle| \mathcal{F}_0^I \right].$$

Noise trader. The noise trader is an agent with bounded rationality who receives a noisy version of the informed signal, namely

$$G_N := \tau_N G_I + Y_N \tag{2.7}$$

where $Y_N \sim N(\mu_N, C_N)$ is independent of both X and Y_I, $\mu_N \in \mathbb{R}^d$ and $C_N \in \mathbb{S}_{++}^d$. The parameters τ_N, μ_N, C_N capture moment bias of G_N relative to G_I.

This general specification enables us to incorporate several models of interest. The first one is the extreme case $\tau_N = 0$, where trading is based on pure noise as suggested

[2]Formally, $\mathbb{F}^I$ is the initial enlargement of $\mathbb{F}^m$ with respect to G_I: i.e., $\mathbb{F}^I = \mathbb{F}^m \vee \mathfrak{s}(G_I)$ where $\mathfrak{s}(G_I)$ denotes the sigma-field generated by G_I.

by Black (c.f. Black (1986)). In this instance, if $\mu_N = \mu_X$ and $C_N = C_I + \Sigma_X$, the first two moments of G_N and G_I coincide. The second is the polar opposite $\tau_N = 1, \mu_N = C_N = 0$, where the bounded rational agent is just another informed agent. The last one $\tau_N = 1, C_N \neq 0$ corresponds to a model where trades are based on a noisy version of the informed signal $G_N = G_I + Y_N$. If in addition $\mu_N = 0$, the signal G_N is an unbiased version of G_I. This specification captures the case of a bilateral noisy communication channel between the informed and the bounded rational agent, where the latter is unaware of the presence of noise. All other cases with $\tau_N \neq 0$ are informationally equivalent to $\tau_N = 1$ with appropriate re-parameterizations.

The bounded rational agent is similar to the informed agent, except that the signal received is G_N, and is misperceived for being G_I. To precisely define bounded rationality, recall (2.6). Upon a realization $G_I = g$, the informed believes that assets have instantaneous drift $\sigma_t\left(\nu_t + \mu_t^g\right)$. For any fixed $g \in \mathbb{R}^d$, the noise trader also believes the drift to be $\sigma_t(\nu_t + \mu_t^g)$. However, he does not obtain g from G_I, but from G_N. Hence, for a realization $G_N = g$, the noise trader has information $\mathbb{F}^m$, but believes S has instantaneous drift $\sigma_t\left(\nu_t + \mu_t^g\right)$. This means the noise trader has beliefs $\mathbb{P}^g$ where

$$\frac{d\mathbb{P}^g}{d\mathbb{P}} = \mathcal{E}\left(\int_0^{\cdot} (\mu_t^g)' dB_t^m\right)_T := \exp\left(\int_0^T (\mu_t^g)' dB_t^m - \frac{1}{2}\int_0^T (\mu_t^g)'\mu_t^g dt\right), \tag{2.8}$$

where $\mathcal{E}(M)$, defined by the expression on the right hand side, is the stochastic exponential of $M_\cdot = \int_0^\cdot (\mu_t^g)' dB_t^m$. Bounded rational behavior is formally defined as the use of beliefs $\mathbb{P}^g$, evaluated at $g = G_N$ instead of G_I.[3]

The noise trader's optimal investment problem under $\mathbb{P}^g$, for each g fixed, is similar to the uninformed and informed traders' problems. Let $\mathcal{A}^{N,g}$ be the set of admissible strategies for the noise trader, as defined in Appendix A. The value function is

$$u^{N,g} := \sup_{\pi \in \mathcal{A}^{N,g}} \mathbb{E}^g\left[-\frac{1}{\gamma_N} e^{-\gamma_N \mathcal{W}_T^\pi} \big| \mathcal{F}_0^m\right].$$

For now, we assume the noise trader has an optimal demand process $\hat{\pi}^{N,g}$ for each $g \in \mathbb{R}^d$, measurable in g, so that his demand $\hat{\pi}^N = \hat{\pi}^{N,G_N}$ is well defined.

[3]Conditional on the realization $G_I = g$, the beliefs of the informed about events in $\mathcal{F}_T^m$ are given by $\mathbb{P}^g$ with density in (2.8). The noise trader has the same structural beliefs as the informed, but evaluated at the incorrect signal G_N. When the informed signal is public information, informed and uninformed beliefs coincide, i.e., $\mu^g = 0$. In this case, the beliefs density (2.8) is one, therefore independent of the conditioning signal realization g.

Remark 2.2. As will be shown, the insider's optimal trading strategy takes the form $\hat{\pi}^I = (1/\gamma_I)\hat{\psi}^{G_I}$ where for each fixed $g \in \mathbb{R}^d$, $\hat{\psi}^g$ is $\mathbb{F}^m$ adapted, and is independent of the risk aversion γ_I. Our assumptions upon the noise trader imply his optimal demand is $\hat{\pi}^{N,G_N} = (1/\gamma_N)\hat{\psi}^{G_N}$. In other words, after adjusting for risk aversion, the noise trader mimics the insider, and builds the same "functional" trading strategy $\hat{\psi}^g$. However, he evaluates it at $g = G_N$ rather than $g = G_I$. In the special case $\mathbb{F}^I = \mathbb{F}^m$, the private information price of risk is null, $\mu^g = 0$, and $\hat{\psi}^g = \hat{\psi}$ independently of g. Optimal demands are then $\hat{\pi}^I = (1/\gamma_I)\hat{\psi}$ and $\hat{\pi}^{N,G_N} = (1/\gamma_N)\hat{\psi}$ independent of the initial signals G_I and G_N (c.f. Footnote 3).

Equilibrium definition.

Definition 2.3. A *communication equilibrium with public filtration* $\mathbb{F}^m$ ($\mathbb{F}^m$-CE) is a collection $\left\{\mathbb{F}^m, B^m, \nu, \sigma, S, \hat{\pi}^I, \hat{\pi}^U, \hat{\pi}^{N,G_N}\right\}$ such that (i) $\left(\hat{\pi}^I, \hat{\pi}^U, \hat{\pi}^{N,g}\right)$ are the optimal policies for the agents and (ii) the market clearing condition (2.4) holds with $\hat{\pi}^N = \hat{\pi}^{N,G_N}$ therein.

The $\mathbb{F}^m$-CE is a *rational expectations equilibrium* (REE) if additionally, the public filtration $\mathbb{F}^m$ coincides with $\hat{\mathbb{F}}^m$, the $\mathbb{P}$-augmented, right-continuous refinement of the filtration generated by $\mathbb{F}^B$, S, and the residual demand $\hat{\pi}^R := \omega_I\hat{\pi}^I + \omega_N\hat{\pi}^{N,G_N}$.

In an $\mathbb{F}^m$-CE, the public filtration $\mathbb{F}^m$ is given at the outset and may contain more information than what can be extracted from fundamentals, prices and residual demands. Investors optimize given $\mathbb{F}^m$ and their individual characteristics, and markets clear. The notion of $\mathbb{F}^m$-CE is a generalization of the notion of full communication equilibrium (FCE) introduced by Radner (1979). It differs because $\mathbb{F}^m$ need not contain all the information disseminated in the economy. In such a case the $\mathbb{F}^m$-CE is a partial communication equilibrium (PCE). If it does include all the information disseminated it is a FCE. It also differs from the notion of rational expectations equilibrium, which is a special case. In a REE, the public filtration $\mathbb{F}^m$ coincides with the filtration $\hat{\mathbb{F}}^m$ generated by the fundamentals, prices and residual demands.[4]

[4]Wilson (1978) provides a rationale for the notion of communication equilibrium. He shows, in the absence of information acquisition and transmission costs, the informed may be willing to share her information to enlarge the core of the economy. He also shows markets in an FCE without such costs can be informationally efficient. In this equilibrium, the paradox of Grossman and Stiglitz (1980) does not arise.

An Example. To conclude this section we anticipate on later results and provide a one-dimensional example when X is Gaussian and Ψ is linear-quadratic (see Section 6.1 for a more general version). We focus on the PCE of Section 4. Here, we recover dynamic versions of classical results (c.f. Grossman (1976); Hellwig (1980)) when Ψ is linear, but for quadratic Ψ we produce interesting behavior such as stochastic volatility, "instantaneous" recovery of the market signal (i.e. not at 0 but at 0+) and trading strategies which are ratios of affine functions of the factor and market signal.

In this example, X follows a univariate Ornstein-Uhlenbeck (OU) process, and the terminal payoff is $\Psi(x) = x + kx^2$ for a constant $k > 0$. In equilibrium, at time 0 the signal $H = (\alpha_I G_I + \alpha_N G_N)/(\alpha_I + \alpha_N)$ is communicated, and this is how $\mathbb{F}^m$ enlarges $\mathbb{F}^B$. The equilibrium price process takes the form

$$S_t = A_t + B_t X_t + C_t H + k\left((A_t + B_t X_t + C_t H)^2 + D_t\right),$$

for functions of time A, B, C, D given explicitly in Section 6.1. Thus, for linear payoffs ($k = 0$), prices are time-varying affine in the state X and signal H, and volatility is deterministic but time varying. Optimal policies are also (time-varying) affine in X and H. As such, we extend well-known one period results to continuous time.

When $k \neq 0$ the situation changes dramatically. First, prices are no longer linear in X, H; they are linear-quadratic. This leads to stochastic volatility, with the volatility affine in X, H, as well as time-varying volvol. Second, the time zero price S_0 does not reveal H. However, H is instantaneously recoverable, because the publicly observable covariation $\langle S, X\rangle$ is affine in H. Thus, the equilibrium is a REE. Lastly, optimal policies generically involve the ratio of two affine functions in X, H, and display a rich behavior based on the state/signal environment.

3. Full communication equilibrium

We first consider when the insider's signal is fully communicated to the market, so that $\mathbb{F}^I = \mathbb{F}^m = \mathbb{F}^B \vee \mathfrak{s}(G_I)$. Here, all agents know G_I at the outset, and the informed trader knows she has no informational advantage. Her beliefs do not depend on her initial private signal. The noise trader, by definition of bounded rationality, has the same structural beliefs as the informed, hence also independent of his initial signal (c.f. Footnote 3 and Remark 2.2). With identical public information and beliefs, there is no motive to dynamically trade: a single initial trade where agents split the outstanding supply according to their weighted risk tolerance is sufficient.

With this motivation, we obtain a static equilibrium, as asserted next.

Theorem 3.1. *Let Assumption 2.1 hold, and define the public filtration $\mathbb{F}^m = \mathbb{F}^B \vee \mathfrak{s}(G_I)$. Then an $\mathbb{F}^m$-FCE exists. Equilibrium asset holdings are*

$$\hat{\pi}_t^U = \frac{\gamma}{\gamma_U}\Pi; \qquad \hat{\pi}_t^I = \frac{\gamma}{\gamma_I}\Pi; \qquad \hat{\pi}_t^{N,g} = \frac{\gamma}{\gamma_N}\Pi, \forall\ g \in \mathbb{R}^d,$$

which are static policies. When either $\Psi(x) = x$, or $d = 1$ and Ψ is strictly monotone, the equilibrium is additionally a REE.

3.1. **Equilibrium construction.** To construct the equilibrium, we exploit the agents' homogeneous public information and beliefs. Thus, with CARA utility, the economy ought to aggregate (c.f. Gorman (1953)), with the fund separation result that each agent holds a constant fraction of the outstanding supply (c.f. Cass and Stiglitz (1970)). Therefore, we conjecture

$$S_t^{G_I} = \mathbb{E}^{\mathbb{Q}^{G_I}}\left[\Psi(X_T)\middle|\mathcal{F}_t^m\right]; \qquad \frac{d\mathbb{Q}^{G_I}}{d\mathbb{P}} = \frac{e^{-\gamma\Pi'\Psi(X_T)}}{\mathbb{E}\left[e^{-\gamma\Pi'\Psi(X_T)}\middle|\mathcal{F}_0^m\right]}, \tag{3.1}$$

where γ from (2.2) is aggregate risk aversion, $e^{-\gamma\Pi'\Psi(X_T)}$ is the marginal utility of the representative agent and $d\mathbb{Q}^{G_I}/d\mathbb{P}$ is the SDF. Indeed, using S^{G_I} and the trading strategies of Theorem 3.1, each agent has marginal utility

$$e^{-\gamma\Pi'(\psi(X_T)-S_0^{G_I})} = \lambda_0\frac{d\mathbb{Q}^{G_I}}{d\mathbb{P}}; \qquad \lambda_0 = \mathbb{E}\left[e^{-\gamma\Pi'\Psi(X_T)}\middle|\mathcal{F}_0^m\right] \times e^{\gamma\Pi' S_0^{G_I}}.$$

Therefore, the first order conditions for optimality are met. As market clearing trivially holds, equilibrium will follow provided individual optimal wealth processes are martingales under $\mathbb{Q}^{G_I}$. But this holds by construction of S^{G_I}, as policies are static.

In fact, the equilibrium of Theorem 3.1 is unique within the class where S is a $(\mathbb{P}, \mathbb{F}^m)$ semi-martingale. Here, as the set of admissible strategies is a cone (i.e. π admissible implies $\lambda\pi$ is as well for $\lambda \geq 0$), identical beliefs, information and endowments force $\gamma_U\hat{\pi}^U = \gamma_I\hat{\pi}^I = \gamma_N\hat{\pi}^{N,g}$. Market clearing forces strategies as in Theorem 3.1, and the first order condition for optimality identifies the wealth processes.

The equilibrium price process is defined in (3.1), but this does not yield the drift and volatility of S^{G_I}. To identify these components, under Assumption 2.1 we can construct the Brownian motion B^m, market price of risk vector ν and volatility matrix σ so that (2.3) holds. As the details are remarkably similar to the PCE construction, we postpone their presentation until the next section.

3.2. **Equilibrium structure.** The filtration $\mathbb{F}^m = \mathbb{F}^B \vee \mathfrak{s}(G_I)$ contains all the (relevant) information available in the economy, and, as it is communicated at the outset, all agents have identical information. The $\mathbb{F}^m$-equilibrium therefore corresponds to a FCE in the sense of Radner (1979). Moreover, given CARA preferences, the economy aggregates and the representative agent's absolute risk tolerance is the weighted average of individual risk tolerances in (2.2) (see Gorman (1953); Rubinstein (1974)). Individual policies are proportional to each other with proportionality factor given by the ratio of individual risk tolerances.

The $\mathbb{F}^m$-equilibrium need not be a REE. However, it is when the filtration $\mathbb{F}^{B,S,\hat{\pi}^R}$ generated by fundamentals, prices and residual demands coincides with $\mathbb{F}^m$. This happens, in particular, when the payoff function is the identity (or strictly monotone in the univariate setting). In this case, as shown in the proof of Theorem 3.1, the time 0 price vector reveals the signal G_I.[5]

Recall that $\hat{\mathbb{F}}^m$ is the right-continuous refinement of $\mathbb{F}^{B,S,\hat{\pi}^R}$. It may be of interest to note that there is an $\hat{\mathbb{F}}^m$-CE even if $\hat{\mathbb{F}}^m \subset \mathbb{F}^B \vee \mathfrak{s}(G_I)$ (a non-fully revealing CE). Indeed, suppose that $\hat{\mathbb{F}}^m \subset \mathbb{F}^B \vee \mathfrak{s}(G_I)$. By the law of iterated expectations

$$\mathbb{E}\left[-\frac{1}{\gamma_U} e^{-\gamma_U \mathcal{W}_T^\pi} \big| \hat{\mathcal{F}}_0^m\right] = \mathbb{E}\left[\mathbb{E}\left[-\frac{1}{\gamma_U} e^{-\gamma_U \mathcal{W}_T^\pi} \big| \mathcal{F}_0^m\right] \big| \hat{\mathcal{F}}_0^m\right] \leq \mathbb{E}\left[u^U \big| \hat{\mathcal{F}}_0^m\right],$$

with equality if the optimal $\mathbb{F}^m$-predictable strategy is also $\hat{\mathbb{F}}^m$-predictable. But, this is trivially true as the optimal policy is constant. The same holds for the informed (optimizing over $\mathbb{F}^B \vee \mathfrak{s}(G_I)$) and the noise trader. Hence, FCE prices and policies form an $\hat{\mathbb{F}}^m$-CE even though $\hat{\mathbb{F}}^m$ does not convey all the private information available.

At equilibrium, the representative agent's marginal utility from terminal wealth is $e^{-\gamma \Pi' \Psi(X_T)}$, which leads to the pricing measure (3.1). Asset prices (3.1) are present values of terminal dividends, where values are calculated using the pricing measure $\mathbb{Q}^{G_I}$. Discounted asset prices are $(\mathbb{Q}^{G_I}, \mathbb{F}^m)$-martingales, where discounting is trivial at the (null) interest rate. As such, discounted prices are their initial values augmented by the cumulative innovations to date. Asset price volatilities determine the responses

[5]Information in the FCE is impounded in prices because it is communicated at the outset. All agents know G_I from the start. The noise trader has the same beliefs as the informed by bounded rationality, so agents have homogeneous beliefs and information. Information G_I helps to assess the terminal payoff, hence it is used by all to form optimal demands. Market clearing ensures it is reflected in prices.

to innovations. Hence, standard economic properties hold in the $\mathbb{F}^m$-FCE (Harrison and Kreps (1979), Harrison and Pliska (1981)).

3.3. **The noise trader in the FCE.** In this equilibrium, the private signal G_I is communicated at the outset, hence becomes public. As private information has no incremental value, i.e., the private information price of risk is null, the beliefs of the informed are independent of the realization of G_I (cf. Footnote 3). Noise trader beliefs, by bounded rationality, are also independent of the realization of G_N. Although he can see if $G_I \neq G_N$, this fact does not contain meaningful information and does not warrant action (c.f. Remark 2.2). Note that one might also be able to embed the FCE in a model with a large number of insiders and noise traders who each receive a noisy signal, which aggregates out to G_I (insiders) and G_N (noise traders). In such a setting, individual signals need not coincide with the signal communicated, and the incremental value of individual signals is null.

4. Partial communication equilibrium

The equilibrium in Theorem 3.1 is unsatisfactory as, once the insider's signal is disclosed, investors are essentially the same (save their holding proportions and risk aversions). Hence there is no dynamic trading, a counterfactual. It is therefore of interest to identify equilibria where $\mathbb{F}^I$ is strictly larger than $\mathbb{F}^m$. Such equilibria will reflect the special position of the informed and generate dynamic trading.

Definition 4.1. A *partial communication equilibrium with public filtration* $\mathbb{F}^m$ ($\mathbb{F}^m$-*PCE*) is an equilibrium as in Definition 2.3 with $\mathbb{F}^m$ strictly contained in $\mathbb{F}^I$.

It turns out that, under very general conditions, there is an $\mathbb{F}^m$-PCE. The public filtration $\mathbb{F}^m$ is $\mathbb{F}^B \vee \mathfrak{s}(H)$ where $H = (\alpha_I G_I + \alpha_N G_N)/(\alpha_I + \alpha_N)$ is the weighted risk tolerance average of the private signals G_I, G_N (c.f. (2.2)). In such an equilibrium, a combination of the insider and noise trader signals is disclosed at the outset.

In addition to Assumption 2.1, our results require a non-degeneracy condition. To state it, for matrices $M, \tilde{M}$ and a vector ζ define the function

$$\Theta(x,y) := \Pi'\Psi(x) + \frac{1}{2}x'Mx + x'\zeta - x'\tilde{M}y. \tag{4.1}$$

As we will show, in the PCE for a given realization of the public signal $H = h$, Θ acts as the "signal adjusted" aggregate dividend, in that the price process takes the form $S^H = (S^h)|_{h=H}$ where

$$S_t^h = \mathbb{E}^{\mathbb{Q}^h}\left[\Psi(X_T)\middle|\mathcal{F}_t^B\right]; \qquad \frac{d\mathbb{Q}^h}{d\mathbb{P}}\Big|_{\mathcal{F}_T^B} = \frac{e^{-\gamma\Theta(X_T,h)}}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,h)}\right]}. \tag{4.2}$$

$M, \tilde{M}, \zeta$ are identified by first defining

$$\tilde{C} = \frac{(\alpha_I + \alpha_N\tau_N)^2 C_I + \alpha_N^2 C_N}{(\alpha_I + \alpha_N)^2}, \tag{4.3}$$

as the covariance matrix for the noise components in H, and then setting

$$\begin{cases} M = (\alpha_I + \alpha_N)C_I^{-1} + \alpha_U\left(\frac{\alpha_I+\alpha_N\tau_N}{\alpha_I+\alpha_N}\right)^2 \tilde{C}^{-1} \\ \tilde{M} = (\alpha_I + \alpha_N)C_I^{-1} + \alpha_U\left(\frac{\alpha_I+\alpha_N\tau_N}{\alpha_I+\alpha_N}\right) \tilde{C}^{-1} \\ \zeta = \frac{\alpha_U\alpha_N}{\alpha_I+\alpha_N\tau_N}\left(\frac{\alpha_I+\alpha_N\tau_N}{\alpha_I+\alpha_N}\right)^2 \tilde{C}^{-1}\mu_N \end{cases}. \tag{4.4}$$

To provide a qualitative understanding, define the conditional densities

$$u(t,x,g) := \frac{d}{dg}\mathbb{P}\left[G_I \in dg\middle|X_t = x\right], \quad \ell(t,x,h) := \frac{d}{dh}\mathbb{P}\left[H \in dh\middle|X_t = x\right]. \tag{4.5}$$

Explicit formulas for both g and ℓ (which also show they are well defined) are in (B.2) and (7.2) below. Then, Θ is constructed to satisfy

$$\frac{e^{-\gamma\Theta(X_T,h)}}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,h)}\right]} = \frac{e^{-\gamma(\Pi'\Psi(X_T) - (\alpha_I+\alpha_N)\log(u(T,X_T,h)) - \alpha_U\log(\ell(T,X_T,h)))}}{\mathbb{E}\left[e^{-\gamma(\Pi'\Psi(X_T) - (\alpha_I+\alpha_N)\log(u(T,X_T,g)) - \alpha_U\log(\ell(T,X_T,h)))}\right]}. \tag{4.6}$$

As such, there are two adjustments made due to the signal realization. First, the insider (and noise trader as he mimics the insider) identifies the density her private signal G_I takes the value h given X_T, $u(T,X_T,h)$. Second, the uninformed does the same for the public signal H, computing $\ell(T,X_T,h)$. The adjustments are then scaled by the weighted risk tolerance and added to the aggregate dividend.

For Θ from (4.1), we impose a "market completeness" assumption at $h=0$,

Assumption 4.2. For Θ from (4.1), $\mathbb{Q}^h$ from (4.2) and $M, \tilde{M}, \zeta$ as in (4.4), define σ^0 via $\mathbb{E}^{\mathbb{Q}^0}\left[\Psi(X_T)\middle|\mathcal{F}_\cdot^B\right] = \mathbb{E}^{\mathbb{Q}^0}\left[\Psi(X_T)\right] + \int_0^\cdot \sigma_u^0 dB_u^{\mathbb{Q}^0}$, where $B^{\mathbb{Q}^0}$ is a $\mathbb{Q}^0$ Brownian motion.[6] Then $\mathsf{Leb}_{[0,T]} \times \mathbb{P}$ almost surely, σ^0 has full rank.

Assumption 4.2 requires that for a *single* realization of the signal (i.e. on the set $\{H=0\}$ though in fact the set $\{H=h_0\}$ for any h_0 would suffice), the market with information set $\mathbb{F}^B$ is complete. Remarkably, $\mathbb{F}^B$- completeness for a single realization h_0 of H yields completeness in the market with refined information set

[6] $\mathbb{E}\left[e^{-\gamma\Theta(X_T,0)}\right]$ and $\mathbb{E}\left[|\Psi(X_T)|e^{-\gamma\Theta(X_T,0)}\right]$ are finite so σ^0 is well defined.

$\mathbb{F}^m$, where now H is a $\mathcal{F}_0^m$ measurable random variable. This fact is not at all obvious, and follows from very recent results (c.f. Kramkov and Pulido (2019)) on representation theorems for analytic maps. While Section 5.7 contains additional discussion, we remark that by verifying a certain analyticity condition, we are able to extend the analysis of Anderson and Raimondo (2008); Hugonnier, Malamud, and Trubowitz (2012), where agents have homogenous (Brownian) information, to when additionally, public information is refined at time 0 by H. Lastly, given $\Theta(X_T, H) = \Theta(X_T, 0) - X_T'\tilde{M}H$, the condition is quite minimal for PCE market completeness.

The verification of Assumption 4.2 is addressed in Anderson and Raimondo (2008); Hugonnier, Malamud, and Trubowitz (2012); Kramkov and Predoiu (2014); Schwarz (2017). In particular, Kramkov and Predoiu (2014) gives conditions when the Jacobian of Ψ has full rank for Lebesgue almost every x. If $\Psi(X_T)$ gives the payoff of both primary assets and derivatives, the Jacobian will not be of full rank. However, in this setting Assumption 4.2 has been verified (under certain conditions) in Schwarz (2017). Deferring to the literature, we leave Assumption 4.2 as is, rather than providing sufficient conditions on the model coefficients for it. Lastly, we again stress that our results are the first to extend Anderson and Raimondo (2008); Hugonnier, Malamud, and Trubowitz (2012) to initial information signals, and since the pricing measure changes with each signal realization, this extension is highly non-trivial.

With all the assumptions in place, the following is the main result of the paper.

Theorem 4.3. *Let Assumptions 2.1 and 4.2 hold. Then, there exists a $\mathbb{F}^m$-PCE. Furthermore, if either $\Psi(x) = x$, or $d = 1$ and Ψ is strictly monotone, then the equilibrium is a NREE.*

Theorem 4.3 is a general existence result for the class of economies under consideration. It asserts the existence of a PCE, i.e., where $\mathbb{F}^m \subset \mathbb{F}^I$. The specific information communicated is described in the next section. When payoffs are linear (or strictly monotone in the univariate case), the public information $\mathbb{F}^m$ is revealed by observables, i.e., $\mathbb{F}^m = \mathbb{F}^{B,S,\hat{\pi}^R}$. In this instance, the PCE is a NREE.

It is worth stressing that the PCE exists even though fundamentals are general diffusions satisfying mild regularity conditions and terminal dividends are general functions of fundamentals. This stands in contrast to the literature on REE which has so far relied on Gaussian processes with linear payoffs (Wang (1993), He and Wang (1995), Brennan and Cao (1996)). Theorem 4.3 shows the structure of fundamentals is not critical for the existence of a PCE. What is relevant is the Gaussian structure of the noise components in the private information signals G_I and the noise trader signals G_N. Section 6 illustrates the construction of equilibrium in that context for particular specifications of terminal payoffs and fundamental processes. As we show in Section 7, the Gaussian structure of the noise Y_N in G_N can be relaxed as well.

It is also important to note that while the linearity condition $\Psi(x) = x$ (monotonicity in the univariate case) is sufficient for the existence of a NREE, it is not necessary. As the first example in Section 6.2 shows, a NREE can exist even when the payoff function is non-linear and non-monotonic.

5. Partial communication equilibrium structure

5.1. **Public and private information.** Recalling G_I, G_N from (2.5), (2.7), define

$$H(G_I, G_N) := \frac{\alpha_I G_I + \alpha_N G_N}{\alpha_I + \alpha_N} = \frac{\alpha_I + \alpha_N \tau_N}{\alpha_I + \alpha_N} G_I + \frac{\alpha_N}{\alpha_I + \alpha_N} Y_N. \tag{5.1}$$

The signal $H = H(G_I, G_N)$ is the risk-tolerance weighted average of the informed and noise trader signals, and the public filtration is $\mathbb{F}^m = \mathbb{F} \vee \mathfrak{s}(H)$. Therefore, public information in the PCE is the refinement of $\mathbb{F}^B$ by the information in H. H is a (scaled) noisy version of the insider signal G_I, and hence $\mathbb{F}^m$ is strictly less informative than $\mathbb{F}^I = \mathbb{F}^m \vee \mathfrak{s}(G_I)$, the insider's private information. Public information in the $\mathbb{F}^m$-PCE is coarser than $\mathbb{F}^I$, but finer than $\mathbb{F}^B$. The particular form of H follows from the residual demand for terminal wealth. Because of CARA utility, individual demands are linear in the logarithm of the relevant conditional density functions. The informed and noise trader use the density function for the Gaussian dispersion Y_I, evaluated at $G_I - X_T$ and $G_N - X_T$ respectively. Aggregation implies the residual demand is affine in the combination of G_I, G_N given by H, and this dependence is consistent with informational restrictions on equilibrium (see Theorem 7.3).[7]

5.2. **Price processes and pricing measures.** We now derive the equilibrium price dynamics. As alluded to above, constructions for the FCE and PCE are very similar. We detail the PCE construction first, and then state changes to apply for the FCE.

We first recall that two measures m_1, m_2 on a measure space $(O, \mathcal{O})$ are equivalent if and only if they have the same null sets, i.e. $m_1(A) = 0 \Leftrightarrow m_2(A) = 0$ for all $A \in \mathcal{O}$.

[7]For any signal H which leads to an equilibrium, the signal $f(H)$ for an invertible function f will lead to the same equilibrium. This follows as the information contents of H and $f(H)$ coincide.

We write $m_1 \sim m_2$, and note that if $m_1 \sim m_2$, there exists a random variable p, called the density of m_1 with respect to m_2, such that for all $A \in \mathcal{O}$, $m_1(A) = \int_A p dm_2$.

Using the independence of X_T, Y_N, Y_Z one can show H from (5.1) satisfies the "Jacod" (c.f. Jacod (1985)) measure equivalence condition $\mathbb{P}\left[H \in \cdot \middle| \mathcal{F}_t^B\right] \sim \mathbb{P}\left[H \in \cdot\right]$ almost surely on $t \leq T$, with resulting density p_t^h,[8] and $B_\cdot^m := B_\cdot - \int_0^\cdot \mu_u^H du$ is a $(\mathbb{P}, \mathbb{F}^m)$ Brownian motion. In fact, p^H and μ^H are explicitly identifiable using partial differential equations (PDEs): see Lemma {3.1}.

The key object is the market information price of risk μ^H, i.e., the incremental value to the market of the information contained in H. μ^H gives rise to the density

$$\frac{d\widetilde{\mathbb{P}}^H}{d\mathbb{P}} := \frac{1}{p_T^H} = \mathcal{E}\left(-\int_0^\cdot (\mu_t^H)' dB_t^m\right)_T . \tag{5.2}$$

[9]

Crucially, as $B_\cdot = B_\cdot^m + \int_0^\cdot \mu_u^H du$, we know from Girsanov's theorem that B is a $(\widetilde{\mathbb{P}}^H, \mathbb{F}^m)$ Brownian motion as well. This is the remarkable feature of the measure $\widetilde{\mathbb{P}}^H$: it preserves the Brownian motion property for B in the enlarged filtration. As a consequence, distributional properties of X under $(\widetilde{\mathbb{P}}^H, \mathbb{F}^m)$ are the same as under $(\mathbb{P}, \mathbb{F}^B)$, and in Lemma {4.8} we show for all appropriately integrable functions χ

$$\mathbb{E}^{\widetilde{\mathbb{P}}^H}\left[\chi(X_T, H) \middle| \mathcal{F}_t^m\right] = \left(\mathbb{E}\left[\chi(X_T, h) \middle| \mathcal{F}_t^B\right]\right)\Big|_{h=H}. \tag{5.3}$$

For h fixed, the conditional expectation on the right hand side is identifiable with a PDE parameterized by h, and this identity allows us to go "back and forth" between the larger information set $\mathbb{F}^m$ where H is a random variable, and the smaller information set $\mathbb{F}^B$ where $H = h$ takes a realized value.

Evaluating Θ using (4.4), equilibrium prices and pricing measure are

$$S_t^H = \mathbb{E}^{\mathbb{Q}^H}\left[\Psi(X_T) \middle| \mathcal{F}_t^m\right]; \qquad \frac{d\mathbb{Q}^H}{d\widetilde{\mathbb{P}}^H} = \frac{e^{-\gamma\Theta(X_T, H)}}{\mathbb{E}^{\widetilde{\mathbb{P}}^H}\left[e^{-\gamma\Theta(X_T, H)} \middle| \mathcal{F}_0^m\right]}. \tag{5.4}$$

[8]More precisely, H is such that for all $t \leq T$ and with probability one, the measures on $\mathbb{R}^d$ defined by $m_1(A) = \mathbb{P}\left[H \in A \middle| \mathcal{F}_t^B\right](\omega)$ and $m_2(A) = \mathbb{P}\left[H \in A\right]$ are equivalent. The density p_t^h is such that $\mathbb{P}\left[H \in \cdot \middle| \mathcal{F}_t^B\right](\omega) = \int_{\mathbb{R}^d} p_t^h(\omega)\mathbb{P}\left[H \in dh\right]$ almost surely on $[0, T]$. Lastly, should the equivalence condition fail, there are very delicate issues relating to absence of arbitrage: see Acciaio, Fontana, and Kardaras (2016).

[9]$\widetilde{\mathbb{P}}^H$ is the "martingale preserving measure" of Pikovsky and Karatzas (1996); Amendinger (2000).

Using (5.3) we conclude $S_t^H = (S_t^h)|_{h=H}$ where S^h is given in (4.2). Additionally we obtain the process $\tilde{\nu}^h$ through

$$\frac{d\mathbb{Q}^h}{d\mathbb{P}} = \frac{e^{-\gamma\Theta(X_T,h)}}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,h)}\right]} = \mathcal{E}\left(-\int_0^{\cdot} (\check{\nu}_u^h)' dB_u\right)_T. \tag{5.5}$$

The volatility matrix process σ^h is obtained through $S_t^h = S_0^h + \int_0^t \sigma_u^h(dB_u + \check{\nu}_u^h du)$, and plugging in $h = H$ gives $\mathbb{F}^m$ - dynamics

$$S_t^H = S_0^H + \int_0^t \sigma_u^H \left(dB_u + \check{\nu}_u^H du\right) = S_0^H + \int_0^t \sigma_u^H \left(dB_u^m + \nu_u^H du\right), \tag{5.6}$$ [10]

where the market price of risk process is $\nu^H = \mu^H + \check{\nu}^H$. Lastly, both ν^H and σ^H are $\mathbb{F}^m$ predictable, and we may write the SDF as

$$\frac{d\mathbb{Q}^H}{d\mathbb{P}} = \frac{d\mathbb{Q}^H}{d\widetilde{\mathbb{P}}^H} \times \frac{1}{p_T^H}. \tag{5.7}$$

Corollary 5.1. *Consider the $\mathbb{F}^m$-PCE in Theorem 4.3. The public filtration is $\mathbb{F}^m = \mathbb{F}^B \vee \mathfrak{s}(H)$. The equilibrium price processes are given by* (5.6)*. The pricing measure is $\mathbb{Q}^H$ with SDF from* (5.7)*.*

As for the insider, in equilibrium she has private information $\mathbb{F}^I = \mathbb{F}^m \vee \mathfrak{s}(G_I)$. The signal G_I also satisfies $\mathbb{P}\left[G_I \in \cdot \middle| \mathcal{F}_t^m\right] \sim \mathbb{P}\left[G_I \in \cdot\right]$ almost surely on $t \leq T$ with density $p_t^{H,g}$, where we write H to stress the dependence upon $\mathbb{F}^m = \mathbb{F}^B \vee \mathfrak{s}(H)$. $p^{H,g}$ and the associated drift μ^{H,G_I} (which makes $B_\cdot^I := B_\cdot^m - \int_0^\cdot \mu_u^{H,G_I} du$ a $(\mathbb{P}, \mathbb{F}^I)$ Brownian motion) are explicitly given in Lemma {3.2}. μ^{H,G_I} is the private information price of risk in the PCE.

The measure $\widetilde{\mathbb{P}}^{H,G_I}$ which preserves $(\mathbb{P}, \mathbb{F}^m)$ martingales under $\mathbb{F}^I$ has density $1/p^{H,G_I}$. Therefore, the pricing measure on $\mathbb{F}^I$ for the insider is $\mathbb{Q}^I$ defined by

$$\frac{d\mathbb{Q}^I}{d\mathbb{P}} := \frac{d\mathbb{Q}^H}{d\mathbb{P}} \times \frac{1}{p_T^{H,G_I}} = \frac{d\mathbb{Q}^H}{d\widetilde{\mathbb{P}}^H} \times \frac{1}{p_T^H p_T^{H,G_I}}. \tag{5.8}$$

5.2.1. *Dynamics in the FCE.* For the FCE we need only minor adjustments to the above. Here, as $\mathbb{F}^m = \mathbb{F}^B \vee \mathfrak{s}(G_I)$ is more informative, B has drift $\tilde{\mu}^{G_I}$, and $B_\cdot - \int_0^\cdot \tilde{\mu}_u^{G_I} du$ is a $(\mathbb{P}, \mathbb{F}^m)$ Brownian motion. The equivalence condition $\mathbb{P}\left[G_I \in \cdot \middle| \mathcal{F}_t^B\right] \sim \mathbb{P}\left[G_I \in \cdot\right], a.s.\ t \leq T$ holds, and the resultant density is $\tilde{p}_t^g$. The market information price of risk is $\tilde{\mu}^{G_I}$, and the martingale preserving measure is

$$\frac{d\widetilde{\mathbb{P}}^{G_I}}{d\mathbb{P}} := \frac{1}{\tilde{p}_T^{G_I}} = \mathcal{E}\left(-\int_0^\cdot \left(\tilde{\mu}_t^{G_I}\right)' dB_t^m\right).$$

[10] This uses the identity $\int \sigma_u^H dB_u = (\int \sigma_u^h dB_u)|_{h=H}$ proved in Appendix {4}.

Using S^{G_I} from (3.1), we conclude (note that (2.2) implies $1/\gamma = (\alpha_I + \alpha_N + \alpha_U)$)

$$S_t^{G_I} = \frac{\mathbb{E}^{\widetilde{\mathbb{P}}^{G_I}}\left[\Psi(X_T)e^{-\gamma\left(\Pi'\Psi(X_T)-(\alpha_I+\alpha_N+\alpha_U)\log\left(\tilde{p}_T^{G_I}\right)\right)}\Big|\mathcal{F}_t^m\right]}{\mathbb{E}^{\widetilde{\mathbb{P}}^{G_I}}\left[e^{-\gamma\left(\Pi'\Psi(X_T)-(\alpha_I+\alpha_N+\alpha_U)\log\left(\tilde{p}_T^{G_I}\right)\right)}\Big|\mathcal{F}_t^m\right]}.$$

Recall (4.5). In our Markov setting $\tilde{p}_T^g = u(T, X_T, g)/u(0, X_0, g)$. Therefore, using the analog of (5.3), $S_t^{G_I} = (S_t^g)|_{g=G_I}$ where

$$S_t^g = \frac{\mathbb{E}\left[\Psi(X_T)e^{-\gamma(\Pi'\Psi(X_T)-(\alpha_I+\alpha_N+\alpha_U)\log(u(T,X_T,g)))}\Big|\mathcal{F}_t^B\right]}{\mathbb{E}\left[e^{-\gamma(\Pi'\Psi(X_T)-(\alpha_I+\alpha_N+\alpha_U)\log(u(T,X_T,g)))}\Big|\mathcal{F}_t^B\right]}.$$

Compare the right hand sides of the above and (4.6). While in the PCE the insider receives G_I and the uninformed receives H, in the FCE all participants receive G_I and combine, post signal realization, to adjust the aggregate dividend. The behavior is the same in both instances: it is only the signals that change. Furthermore, using that G_I given X_T is normally distributed with mean X_T and covariance C_I, we may simplify the above to obtain

$$S_t^g = \mathbb{E}^{\mathbb{Q}^g}\left[\Psi(X_T)\big|\mathcal{F}_t^B\right]; \quad \frac{d\mathbb{Q}^g}{d\mathbb{P}} = \frac{e^{-\gamma\Theta(X_T,g)}}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,g)}\right]} = \mathcal{E}\left(-\int_0^\cdot (\tilde{\nu}_u^g)' dB_u\right)_T, \tag{5.9}$$

where Θ is again from (4.1), but we now evaluate it at

$$M = \tilde{M} = (\gamma C_I)^{-1}, \qquad \zeta = 0. \tag{5.10}$$

The volatility process σ^g is found from $S_t^g = \mathbb{E}^{\mathbb{Q}^g}\left[\Psi(X_T)\big|\mathcal{F}_t^B\right] = S_0^g + \int_0^t \sigma_u^g\,(dB_u + \tilde{\nu}_u^g du)$. Plugging in $g = G_I$ gives the equilibrium dynamics

$$S_t^{G_I} = S_0^{G_I} + \int_0^t \sigma_u^{G_I}\left(dB_u + \tilde{\nu}_u^{G_I} du\right) = S_0^{G_I} + \int_0^t \sigma_u^{G_I}\left(dB_u^m + \nu_u^{G_I} du\right), \tag{5.11}$$

where $\nu^{G_I} = \tilde{\nu}^{G_I} + \tilde{\mu}^{G_I}$. Summarizing, we have established

Corollary 5.2. *Consider the $\mathbb{F}^m$-FCE equilibrium in Theorem 3.1. Equilibrium price processes are defined in* (3.1) *with dynamics given in* (5.11). *The equilibrium pricing measure is $\mathbb{Q}^{G_I}$ defined in* (3.1).

We end this section reiterating that the PCE and FCE equilibria constructions are very similar. The only differences are a) the market signal is $H = G_I$ in the fully revealing case and $H = (\alpha_I G_I + \alpha_N G_N)/(\alpha_I + \alpha_N)$ in the partially revealing case; and b) the matrices $M, \tilde{M}$ and vector ζ used in Θ are from (5.10) in the fully revealing case and from (4.4) in the partially revealing case. In other respects, the structure of prices is identical. This is a remarkable result given the generality of the model.

5.3. **Equilibrium gains from trade (wealth) distribution.** Write $Z_T^H = d\mathbb{Q}^H/d\mathbb{P}$ and recall (5.8). As the agents condition upon their respective time-0 information, the first order conditions for optimality imply the terminal gains from trade

$$\begin{aligned} \mathcal{W}_T^{\hat{\pi}^U} &= -\frac{1}{\gamma_U}\left(\log\left(Z_T^H\right) - \mathbb{E}^{\mathbb{Q}^H}\left[\log\left(Z_T^H\right)\middle|\mathcal{F}_0^m\right]\right) \\ \mathcal{W}_T^{\hat{\pi}^I} &= -\frac{1}{\gamma_I}\left(\log\left(Z_T^H\right) - \mathbb{E}^{\mathbb{Q}^H}\left[\log\left(Z_T^H\right)\middle|\mathcal{F}_0^m\right] - M^{G_I}\right) \\ \mathcal{W}_T^{\hat{\pi}^N, G_N} &= -\frac{1}{\gamma_N}\left(\log\left(Z_T^H\right) - \mathbb{E}^{\mathbb{Q}^H}\left[\log\left(Z_T^H\right)\middle|\mathcal{F}_0^m\right] - M^{G_N}\right) \end{aligned} \tag{5.12}$$

where $M^g := \log\left(p_T^{H,g}\right) - \mathbb{E}^{\mathbb{Q}^H}\left[\log\left(p_T^{H,g}\right)\middle|\mathcal{F}_0^m\right]$. Above, we explicitly identified the Lagrange multipliers for the static budget constraints and used Lemma {4.8} to conclude $\mathbb{E}^{\mathbb{Q}^I}\left[\chi(\cdot, G_I)\middle|\mathcal{F}_0^I\right] = \mathbb{E}^{\mathbb{Q}^H}\left[\chi(\cdot, g)\middle|\mathcal{F}_0^m\right]$ on the set $\{G_I = g\}$. The key feature which enables market clearing is that (c.f. (4.5) and Lemma {3.2})

$$M^g = -\frac{1}{2}X_T' C_I^{-1} X_T + X_T' C_I^{-1} g - \log(\ell(T, X_T, h)),$$

and hence with our H

$$\alpha_I M^{G_I} + \alpha_N M^{G_N} = -(\alpha_I + \alpha_N)\left(F_{1T}^H - \mathbb{E}^{\mathbb{Q}^H}\left[F_{1T}^H\middle|\mathcal{F}_0^m\right]\right), \tag{5.13}$$

where the public information factor F_{1T}^h is defined by

$$F_{1T}^h := \frac{1}{2}X_T' C_I^{-1} X_T - X_T' C_I^{-1} h + \log\left(\ell(T, X_T, h)\right). \tag{5.14}$$

Therefore, aggregating (5.12) over agents, and imposing market clearing gives

$$\begin{aligned} &-\frac{1}{\gamma}\left(\log\left(Z_T^H\right) - \mathbb{E}^{\mathbb{Q}^H}\left[\log\left(Z_T^H\right)\middle|\mathcal{F}_0^m\right]\right) \\ &\qquad = \Pi'\Psi(X_T) - \mathbb{E}^{\mathbb{Q}^H}\left[\Pi'\Psi(X_T)\middle|\mathcal{F}_0^m\right] + (\alpha_I + \alpha_N)\left(F_{1T}^H - \mathbb{E}^{\mathbb{Q}^H}\left[F_{1T}^H\middle|\mathcal{F}_0^m\right]\right). \end{aligned}$$

The last term contains the impact of (G_I, G_N) on the residual demand, which depends on (G_I, G_N) only through H. Substituting back in (5.12) gives

Corollary 5.3. *The equilibrium distribution of terminal gains decomposes as*

$$\begin{aligned}
\mathcal{W}_T^{\hat{\pi}^U} &= \frac{\gamma}{\gamma_U}\left(\Pi'\Psi(X_T) - \mathbb{E}^{\mathbb{Q}^H}\left[\Pi'\Psi(X_T)\middle|\mathcal{F}_0^m\right]\right) + \frac{\gamma(\alpha_I+\alpha_N)}{\gamma_U}\left(F_{1T}^H - \mathbb{E}^{\mathbb{Q}^H}\left[F_{1T}^H\middle|\mathcal{F}_0^m\right]\right);\\
\mathcal{W}_T^{\hat{\pi}^I} &= \frac{\gamma}{\gamma_I}\left(\Pi'\Psi(X_T) - \mathbb{E}^{\mathbb{Q}^H}\left[\Pi'\Psi(X_T)\middle|\mathcal{F}_0^m\right]\right) - \frac{\gamma\alpha_U}{\gamma_I}\left(F_{1T}^H - \mathbb{E}^{\mathbb{Q}^H}\left[F_{1T}^H\middle|\mathcal{F}_0^m\right]\right)\\
&\quad + \frac{1}{\gamma_I}\left(F_{2T}^{H,g} - \mathbb{E}^{\mathbb{Q}^H}\left[F_{2T}^{H,g}\middle|\mathcal{F}_0^m\right]\right)\Big|_{g=G_I};\\
\mathcal{W}_T^{\hat{\pi}^{N,G_N}} &= \frac{\gamma}{\gamma_N}\left(\Pi'\Psi(X_T) - \mathbb{E}^{\mathbb{Q}^H}\left[\Pi'\Psi(X_T)\middle|\mathcal{F}_0^m\right]\right) - \frac{\gamma\alpha_U}{\gamma_N}\left(F_{1T}^H - \mathbb{E}^{\mathbb{Q}^H}\left[F_{1T}^H\middle|\mathcal{F}_0^m\right]\right)\\
&\quad + \frac{1}{\gamma_N}\left(F_{2T}^{H,g} - \mathbb{E}^{\mathbb{Q}^H}\left[F_{2T}^{H,g}\middle|\mathcal{F}_0^m\right]\right)\Big|_{g=G_N},
\end{aligned}$$

where $F_{1T}^h, F_{2T}^{H,g}$ *are informational factors, with* F_{1T}^h *defined in* (5.14) *and*

$$F_{2T}^{h,g} := X_T' C_I^{-1}(g-h), \ \textit{for } g,h \in \mathbb{R}^d. \tag{5.15}$$

Corollary 5.3 shows that the gains distribution has a three factor structure. The first factor is the aggregate dividend $\Pi'\Psi(X_T)$, the second is the public information factor F_{1T}^H and the last one the private factor $F_{2T}^{H,g}$. In equilibrium, all individual gains depend on the first two factors $\Pi'\Psi(X_T), F_{1T}^H$. Uninformed gains are unrelated to the private information factor, as it should be. Informed and noise trading gains depend on it. Importantly, residual gains $\mathcal{W}_T^R := \omega^I \mathcal{W}_T^{\hat{\pi}^I} + \omega^N \mathcal{W}_T^{\hat{\pi}^{N,G_N}}$ are

$$\mathcal{W}_T^R = \gamma(\alpha_I+\alpha_N)\left(\Pi'\Psi(X_T) - \alpha_U F_{1T}^H - \mathbb{E}^{\mathbb{Q}^H}\left[\Pi'\Psi(X_T) - \alpha_U F_{1T}^H\middle|\mathcal{F}_0^m\right]\right),$$

and do not depend on $F_{2T}^{H,g}$. By market clearing, aggregate gains correspond to the aggregate dividend gains, and hence have a single factor structure.

5.4. **The representative agent.** The representative agent uses market information $\mathbb{F}^m$, and has terminal marginal utility $e^{-\gamma\Pi'\Psi(X_T)}$.[11] For this marginal utility to be optimal, she cannot have conditional beliefs $\mathbb{P}$: rather she has beliefs $\mathbb{P}^{A,H}$ which enforce the optimality condition

$$e^{-\gamma\Pi'\Psi(X_T)} = \lambda^A \frac{d\mathbb{Q}^H}{d\mathbb{P}^{A,H}} = \lambda^A \frac{d\mathbb{Q}^H}{d\mathbb{P}} \times \frac{d\mathbb{P}}{d\mathbb{P}^{A,H}} = \frac{\lambda^A}{\lambda^U} e^{-\gamma_U \mathcal{W}_T^{\hat{\pi}^U}} \frac{d\mathbb{P}}{d\mathbb{P}^{A,H}},$$

[11] This notion of representative agent is a construction permitting the characterization of equilibrium prices (see Cuoco, He, et al. (2001)). It differs from the standard notion of aggregation as in Gorman (1953).

for given $\mathcal{F}_0^m$ measurable lagrange multipliers λ^A, λ^U, and where we have used the uninformed agent's first order conditions for optimality. Therefore, as an immediate consequence of Corollary 5.3, we find

$$\frac{d\mathbb{P}^{A,H}}{d\mathbb{P}} = \frac{e^{-\gamma(\alpha_I+\alpha_N)F_{1T}^H}}{\mathbb{E}\left[e^{-\gamma(\alpha_I+\alpha_N)F_{1T}^H}\middle|\mathcal{F}_0^m\right]}, \tag{5.16}$$

Thus, in contrast to the FCE, both the conditional beliefs and the pricing measure depend on the distribution of risk tolerances across agents. To further interpret $\mathbb{P}^{A,H}$, we use the well-known connection between beliefs and random endowment for CARA preferences. Indeed, beliefs $\mathbb{P}^{A,H} \sim \mathbb{P}$ and endowment $\Xi_A = 0$ are in one-to-one correspondence with beliefs $\mathbb{P}$ and endowment $\Xi_A = -(1/\gamma)\log(d\mathbb{P}^{A,H}/d\mathbb{P})$. Next, the insider and noise trader have beliefs $d\mathbb{P}^g/d\mathbb{P} = p_T^{H,g}$ evaluated at the signals G_I, G_N. Therefore, we can view these agents as having beliefs $\mathbb{P}$ but endowments $\Xi_i = -(1/\gamma_i)\log(p_T^{H,G_I})$ for $i \in \{I, N\}$. Since the uninformed agent has beliefs $\mathbb{P}$ and $\Xi_U = 0$ endowment, the residual demand component (5.13) implies

$$\Xi_A = \omega_U\Xi_U + \omega_I\Xi_I + \omega_N\Xi_N = -(\alpha_I\log(p^{H,G_I}) + \alpha_N\log(p^{H,G_N})) = (\alpha_I+\alpha_N)F_{1T}^H + k_0^H,$$

where k_0^H is a constant. The connection $\Xi_A = -(1/\gamma)\log(d\mathbb{P}^{A,H}/d\mathbb{P})$ proves (5.16).

5.5. **Portfolio separation: mutual funds and dark pool.** The terminal gains distribution in Corollary 5.3 suggests that the equilibrium allocation can be attained in a simple market structure with a small set of mutual funds and a dark pool.[12] The next result describes this equivalent market structure.

Corollary 5.4. *Equilibrium portfolios satisfy an informational fund separation property. The equilibrium allocation is attained if the menu of* d *risky assets and* 1 *riskless asset is replaced by* 2 *risky mutual funds,* 1 *dark pool and the riskless asset. The first mutual fund is the market portfolio* $\pi^{m,H} = \Pi$ *of risky assets with terminal payoff* $\Pi'\Psi(X_T)$*. The second one is the portfolio* $\pi^{mf,H}$ *financing the public information factor* F_{1T}^H*. This fund is a dynamic portfolio strategy that is publicly traded and universally held, but in zero net supply. The dark pool is a private exchange for dynamically trading the private information factor* $F_{2T}^{H,g}$ *for the respective realizations* $g = G_I, G_N$*. Participation is endogenous and consists of the informed and the noise trader. The dark pool is opaque to outsiders and has zero net trade.*

[12] Dark pools are private exchanges characterized by complete lack of transparency. Our use of the terminology reflects this property.

The market portfolio of risky assets is in positive supply by definition. The mutual fund for trading public information is an inside asset: net trades sum to zero. All investors participate and the value of the fund is public information. The dark pool has endogenous membership consisting of the informed and the noise trader. Both trade the portfolio financing the private information factor. Their net trades sum to zero. The pool is completely opaque: neither trades nor the financing portfolio are observable to outsiders.[13]

It is important to note the fund for trading public information and the trades in the dark pool are dynamic trading strategies: they involve continuous re-balancing of the assets held in order to replicate the factors F^H_{1T}, $F^{H,g}_{2T}$. As factors are not linear in $\Psi(X_T)$ (F^H_{1T} is not linear in X_T either), they cannot be replicated via static trading. Therefore, even though signals are received at 0 and agents consume at T, dynamic trading is necessary. Illustrations are in Figure 5, based on Example 2 in Section 6.

Proposition 5.5. *(fund separation with dark pool) The population optimal fund holdings $\omega_i \times \hat{\pi}^i, i \in \{I, U, N\}$ are given in Table I. Trading in the dark pool takes the form of a pair of dynamic trading strategies that are implemented by the informed and the noise trader. The uninformed has no incentive to participate in the dark pool: his optimal policies can be implemented using the market portfolio and the mutual fund for public information trades.*

	$\pi^{m,H}$	$\pi^{mf,H}$	$\pi^{dp,g,H}$
(U)	$\alpha_U\gamma$	$\alpha_U(\alpha_I+\alpha_N)\gamma$	
(I)	$\alpha_I\gamma$	$-\alpha_I\alpha_U\gamma$	α_I at $g = G_I$
(N)	$\alpha_N\gamma$	$-\alpha_N\alpha_U\gamma$	α_N at $g = G_N$
Total	1	0	0

TABLE I. Optimal weighted fund decomposition $\omega^i \times \hat{\pi}^i, i \in \{I, U, N\}$.

Heterogeneity in the realizations of G_I, G_N and H is essential for activity in the dark pool. If $g = h$, the beliefs of the informed and the noise trader coincide so that

[13] Consider the alternate market structure in Corollary 5.4. Here, the uninformed can implement his equilibrium PCE strategy investing in the public funds. The informed and noise trader cannot. To implement their equilibrium PCE strategies they must use the dark pool. In that sense participation in the dark pool is endogenous: it reflects endogenous equilibrium strategies in the PCE.

the motive for trades tied to private information between these two parties vanishes, as $F_{2T}^{h,h} = 0$. In this instance however, there is still motive for trading in the mutual fund for public information because the common beliefs of the informed and noise trader differ from the beliefs of the uninformed. Agents seeking to hedge fluctuations in beliefs will find it convenient to trade on this venue. Absence of trade in the fund for public information only arises when all agents have identical beliefs.

The mutual fund proposition above differs from classic mutual fund theorems (e.g. Cass and Stiglitz (1970); Merton (1973); Breeden (1979)). The first main difference is the emergence of a private venue for dynamic trades based on the private information of informed agents. Such a venue attracts informed individuals and noise traders, and is opaque to outsiders. The second difference is the emergence of a mutual fund for dynamically hedging fluctuations in beliefs tied to public information. Beliefs heterogeneity and dependence on public information underpin the activity on this public exchange. Net trades on either venue aggregate to zero in equilibrium.

5.6. **Price decomposition.** The informational fund result in Proposition 5.5 suggests the price decomposition $dS_t^H = \sum_{j=1}^J \beta^{j,H} d\mathcal{W}_t^{\pi^{j,H}} + dR_t^H$, where the $\{\pi^{j.H}\}$ are specific funds with corresponding gains processes $\mathcal{W}_t^{\pi^{j,H}}$ and R^H is a) orthogonal to each of the gains processes $\mathcal{W}^{\pi^j}$ and b) both a $(\mathbb{P}, \mathbb{F}^m)$ and $(\mathbb{Q}^H, \mathbb{F}^m)$ martingale. In view of Corollary 5.3, it is natural to include both $\pi^{m,H}$ and $\pi^{mf,H}$. However, such a decomposition is not possible using only these two funds: it is necessary to include an additional fund tied to the marginal utility of the uninformed, as described next.

From (5.4) and (5.6) we find

$$\frac{d\mathbb{Q}^H}{d\mathbb{P}} = \mathcal{E}\left(-\int_0^\cdot (\nu_u^H)' dB_u^m\right)_T = e^{-\int_0^T (\nu_u^H)' dB_u^{\mathbb{Q}^H} + \frac{1}{2}\int_0^T |\nu_u^H|^2 du},$$

where $B_\cdot^{\mathbb{Q}^H} = B_\cdot^m + \int_0^\cdot \nu_u^H du$ is a $(\mathbb{Q}^H, \mathbb{F}^m)$ Brownian motion. Next, using (5.7), (5.12) and Proposition 5.5 we obtain the optimality condition for the uninformed

$$\frac{d\mathbb{Q}^H}{d\mathbb{P}} = ke^{-\gamma_U \mathcal{W}_T^{\hat{\pi}^U}} = ke^{-\gamma\int_0^T (\pi_u^{m,H})'\sigma_u^H dB_u^{\mathbb{Q}^H} - \gamma(\alpha_{I_1}+\alpha_{N_1})\int_0^T (\pi_u^{mf,H})'\sigma_u^H dB_u^{\mathbb{Q}^H}}, \tag{5.17}$$

where k is a $\mathcal{F}_0^m$ measurable quantity. Together, this shows that

$$\int_0^T \left(\nu_u^H - (\sigma_t^H)' \left(\gamma\pi_u^{m,H} + \gamma(\alpha_{I_1} + \alpha_{N_1})\pi_u^{mf,H}\right)\right)' dB_u^{\mathbb{Q}^H} = k + \frac{1}{2}\int_0^T \left|\nu_u^H\right|^2 du, \tag{5.18}$$

(up to a potentially different $\mathcal{F}_0^m$ quantity k). If $\pi^{vv,H}$ denotes the strategy that replicates $(1/2)\int_0^T |\nu_u^H|^2 du$ then by uniqueness of replicating strategies we conclude

Proposition 5.6. *The market price of risk ν takes the form*

$$\nu^H = (\sigma^H)' \left(\gamma\pi^{m,H} + \gamma(\alpha_I + \alpha_N)\pi^{mf,H} + \pi^{vv,H}\right), \tag{5.19}$$

The price dynamics admits the three factor decomposition

$$dS_t^H = \beta_t^{m,H} d\mathcal{W}_t^{m,H} + \beta_t^{mf,H} d\mathcal{W}_t^{mf,H} + \beta_t^{vv,H} d\mathcal{W}_t^{vv,H} + dR_t^H, \tag{5.20}$$

where R^H is orthogonal to $\mathcal{W}^{j,H}, j \in \{m, mf, vv\}$ and is both a $(\mathbb{P}, \mathbb{F}^m)$, $(\mathbb{Q}, \mathbb{F}^m)$ martingale. $\beta^{m,B}$, $\beta^{mf,H}$ and $\beta^{vv,H}$ are explicitly given in (E.1) *of Appendix E. If $(\sigma^H)'(\pi^{m,H} + (\alpha_{I_1} + \alpha_{N_1})\pi^{mf,H})$ is non-stochastic then $\pi^{vv,H}$ is null and the price variation admits a two factor structure.*

Remark 5.7. The representation in (5.19), while well-defined, is implicit in that ν^H is written in terms of $\pi^{vv,H}$ which itself is determined from ν^H. However, an explicit representation is obtained through the identity

$$\mathcal{E}\left(-\int_0^\cdot (\pi_u^{vv,H})'\sigma_u^H dB_u^m\right)_T = \frac{e^{-\frac{1}{2}\gamma^2\int_0^T \left|(\sigma_u^H)'\left(\pi_u^{m,H} + (\alpha_{I_1}+\alpha_{N_1})\pi_u^{mf,H}\right)\right|^2 du}}{\mathbb{E}\left[e^{-\frac{1}{2}\gamma^2\int_0^T \left|(\sigma_u^H)'\left(\pi_u^{m,H} + (\alpha_{I_1}+\alpha_{N_1})\pi_u^{mf,H}\right)\right|^2 du}\Big|\mathcal{F}_0^m\right]},$$

which holds through uniqueness of solutions to a certain BSDE.

The three factor structure of prices is notable in light of the d-dimensional nature of the opportunity set (state variables). It amounts to a substantial reduction in the complexity of the sources of price changes, which is useful from an empirical perspective, in particular if the funds $\mathcal{W}^{j,H}, j \in \{m, mf, vv\}$ are traded. The specific factors emerging are also notable. All are tied to the evolution of the SDF of the uninformed agent which determines prices. The first two are natural given the decomposition of uninformed wealth in Corollary 5.3. The last one is more surprising, but stems from the fact that the volatility of uninformed gains from trade is stochastic and that fluctuations in this coefficient determine terminal uninformed wealth, hence marginal utility of terminal wealth. The portfolio $\pi^{vv,H}$ hedges this particular source of fluctuations in the SDF.[14]

[14] Adding noise traders with different signals G_{N_k} and risk aversions γ_{N_k} for $k = 1, ..., K$ does not affect the thee factor structure of prices. Corollary 5.3 continues to hold substituting $\sum_k \alpha_{N_k}$, γ_{N_k}, G_{N_k} for α_N, γ_N, G_N. Corollary 5.4 holds with dark pool portfolios of noise traders evaluated at G_{N_k}. Proposition 5.5 holds with α_{N_k} in line (N) and $\sum_k \alpha_{N_k}$ in line (U). As $F_{2T}^{H,G_{N_k}}$ does not affect uninformed gains from trade, it fails to affect the structures of stock and risk prices.

It is also of interest to note that the activity in the dark pool does not influence the evolution of prices, but that the mutual fund for public information trades does. The reason for this difference is that participation in the dark pool is endogenously comprised of the informed and the noise trader. In contrast all agents, including the uninformed, invest in the mutual fund for public information.

From (5.20), as R^H therein is both a $(\mathbb{P}, \mathbb{F}^m)$ and $(\mathbb{Q}^H, \mathbb{F}^m)$ martingale, it follows that asset risk premia $\mu^H := \sigma^H \nu^H$ also have a three factor structure, determined by the risk premia of the market $\mathcal{W}^{m,H}$, and the hedging $\mathcal{W}^{mf,H}, \mathcal{W}^{vv,H}$ portfolios,

$$\mu^H = \beta^{m,H}\mu^{m,H} + \beta^{mf,H}\mu^{mf,H} + \beta^{vv,H}\mu^{vv,H},$$

where $\mu^{j,H}, j \in \{m, mf, vv\}$ are the respective risk premia. In fact, the same factor structure characterizes the risk premium of any portfolio based on public information. Consistent with the static analysis in Dybvig and Ross (1985), this shows that the market factor alone is insufficient to assess the performance of a fund manager trading on public information. The perceived alpha generated by such fund management is in fact just equal to the compensation for an exposure to the two missing risk factors $\mathcal{W}^{mf,H}, \mathcal{W}^{vv,H}$. These additional factors naturally emerge in a dynamic NREE and cannot be ignored for performance evaluation. As in Hansen and Richard (1987), Proposition 5.6 also illustrates the importance of understanding the information structure in equilibrium. Indeed, (5.20) is a conditional factor model where factor loadings are adapted to the public filtration disseminated in equilibrium. Finally, Proposition 5.6 shows that factors and loadings evolve stochastically over time, and permits the derivation of their dynamics.

5.7. **On market completeness.** Lemma {1.2} proves the maps $h \to e^{-\gamma\Theta(X_T,h)}$ and $h \to \Psi(X_T)e^{-\gamma\Theta(X_T,h)}$ are *analytic* from $\mathbb{R}^d$ to $L^1(\mathbb{R})$ and $L^1(\mathbb{R}^d)$ respectively (see (Kramkov and Pulido, 2019, Ch.3) for definition of an analytic map from $\mathbb{R}^d$ to a Banach space X). Thus, Assumption 4.2, and the fact that H has a density relative to Lebesgue measure, imply that $(\mathbb{Q}^H, \mathbb{F}^m)$-local Martingales can be written as stochastic integrals relative to S^H (see (Kramkov and Pulido, 2019, Theorem 3.1) and Proposition {4.6}), where $\mathbb{Q}^H$ is from (5.7). Thus, the $(S^H, \mathbb{F}^m)$ market is complete: any $(\mathbb{Q}^H, \mathbb{F}^m)$-local Martingale can be replicated using the traded assets with prices S^H. Market completeness implies that any zero net supply claim with $\mathbb{F}^m$-adapted payoff has a unique price. This result generalizes Anderson and Raimondo (2008); Hugonnier, Malamud, and Trubowitz (2012) to the case of initial information signals

5.8. **Informed versus noise trader information.** The invertibility of $H(x,y)$ in y has a natural economic interpretation: the insider is in an advantageous position, not only in that she receives an informative signal at time 0, but also that given the public signal H and her own signal G_I, she can deduce the noisy signal G_N.

For the affine signal H of (5.1), one can recover the informed trader's signal from the noise trader's and public signals. This implies $\mathbb{F}^I = \mathbb{F}^N := \mathbb{F}^m \vee \mathfrak{s}(G_N)$. Despite the informational equivalence, the insider is still in an advantageous position, as the noise trader does not use his signal in a meaningful sense. Furthermore, in equilibrium $\omega_I \hat{\pi}^I_t = \Pi - \omega_U \hat{\pi}^U_t - \omega_N \hat{\pi}^{N,G_N}_t$ for $t \leq T$, and hence the insider's strategy is $\mathbb{F}^N$-predictable. Thus, if $\mathbb{F}^N \subset \mathbb{F}^I$ she would not be using all of her information.

6. Examples

We now provide explicit equilibrium expressions with various specifications of the underlying factor process X. It is useful to keep in mind that the function Θ from (4.1) is independent of the structure of X. To compute the PCE price, we plug in for Θ using $M, \tilde{M}, \zeta$ from (4.4) and use both (4.2) and the Markov property for X to conclude $S^H_t = S(t, X_t, H)$ for

$$S(t,x,h) = \frac{\mathbb{E}\left[\Psi(X_T)e^{-\gamma\left(\Pi'\Psi(X_T)+\frac{1}{2}X_T'MX_T - X_T'(\tilde{M}h-\zeta)\right)}\big| X_t = x\right]}{\mathbb{E}\left[e^{-\gamma\left(\Pi'\Psi(X_T)+\frac{1}{2}X_T'MX_T - X_T'(\tilde{M}h-\zeta)\right)}\big| X_t = x\right]}. \tag{6.1}$$

Using (2.1) and Ito's formula, the volatility is $\sigma^H_t = \sigma(t, X_t, H)$ where

$$\sigma(t,x,h)^{ij} = \left(a'(x)\nabla_x S^i(t,x,h)\right)^j; \qquad i,j = 1,...,d, \tag{6.2}$$

and ∇_x is the gradient with respect to x. For the market price of risk, recall that $\nu^H = \check{\nu}^H + \mu^H$. First, (5.5) and Ito imply $\check{\nu}^H_t = -a(X_t)'\nabla_x \chi^{\check{\nu}}(t, X_t, H)$ for

$$\chi^{\check{\nu}}(t,x,h) = \log\left(\mathbb{E}\left[e^{-\gamma\left(\Pi'\Psi(X_T)+\frac{1}{2}X_T'MX_T - X_T'(\tilde{M}h-\zeta)\right)}\big| X_t = x\right]\right). \tag{6.3}$$

Next, Lemma {3.1} and (7.11) yield $\mu^H_t = a(X_t)'\nabla_x \chi^\mu(t, X_t, H)$ where

$$\chi^\mu(t,x,h) = \log\left(\mathbb{E}\left[e^{-\left(\frac{1}{2}X_T'M^\mu X_T - X_T'(\tilde{M}^\mu h - \zeta^\mu)\right)}\big| X_t = x\right]\right), \tag{6.4}$$

for certain matrices M^μ, $\tilde{M}^\mu$ and vector ζ^μ. For the trading strategies, in view of Proposition 5.5 and since $\pi^{m,H} = \Pi$, we need only compute $\pi^{mf,H}$ and $\pi^{dp,g,H}$. To do so, fix a realization h for H. Using (5.14), (7.2) (c.f. Lemma {3.1}) and (7.11) it follows that $\pi^{mf,h}$ replicates $F_{1T}^h = F_1(X_T, h)$ where

$$F_1(x,h) = \frac{1}{2}x'\left(C_I^{-1} - M^\mu\right)x + x'\left(\left(\tilde{M}^\mu - C_I^{-1}\right)h - \zeta^\mu\right). \tag{6.5}$$

By Ito and Feynman-Kac $\pi_t^{mf,h} = (\sigma(t,X_t,h)')^{-1}a(X_t)'\nabla_x\chi^{mf}(t,X_t,h)$ where

$$\chi^{mf}(t,x,h) = \frac{\mathbb{E}\left[F_1(X_T,h)e^{-\gamma\left(\Pi'\Psi(X_T)+\frac{1}{2}X_T'MX_T - X_T'(\tilde{M}h-\zeta)\right)}\big|X_t = x\right]}{\mathbb{E}\left[e^{-\gamma\left(\Pi'\Psi(X_T)+\frac{1}{2}X_T'MX_TX_T'(\tilde{M}h-\zeta)\right)}\big|X_t = x\right]}. \tag{6.6}$$

Next, fix a realization g of either G_I or G_N. From (5.15), $\pi^{dp,g,h}$ replicates the payoff $F_{2T}^{h,g} = F_2(X_T,h,g) = x'C_I^{-1}(g-h)$. In a similar manner to $\pi^{mf,h}$, $\pi_t^{dp,g,h} = (\sigma(t,X,h)')^{-1}a(X_t)'\nabla_x\chi^{dp}(t,X_t,g,h)$ where $\chi^{dp}(t,x,g,h)$ is defined by the right hand side of (6.6), except with $F_2(X_T,h,g)$ replacing $F_1(X_T,h)$ therein. With σ^H, ν^H, $\pi^{m,H}$, $\pi^{mf,H}$ obtained, one can now identify $\pi^{vv,H}$ in Proposition 5.6.

All the expectations above either admit explicit solutions (e.g. Gaussian models), are integrals with respect to known density (e.g. square root model), or computable via linear PDEs or Monte Carlo simulation. Lastly, using (5.9) the above formulas for S,σ,ν are valid for the FCE, provided that $M,\tilde{M},\zeta$ are taken from (5.10).

6.1. **Gaussian Factor Process.** Consider when X is a d-dimensional OU process, and the payoff Ψ is linear-quadratic

$$dX_t = \kappa(\overline{X} - X_t)dt + \sigma_X dB_t;\quad \Psi^i(x) = \sum_{j=1}^d B^{ij}x^j + \frac{1}{2}\sum_{j,k=1}^d C(i)^{jk}x^jx^k;\ i = 1,...,d.$$

The matrices $C(i), i = 1,..,d$ are symmetric and, to enforce Assumption 2.1, $C^\Pi := \sum_{i=1}^d \Pi^i C(i)$ is non-negative definite.

We obtain explicit formulas for S^H,σ^H,ν^H as well as $\pi^{mf,H},\pi^{vv,H},\pi^{dp,g,H}$ by exploiting three facts. First, given $X_t = x$, $X_T \sim N(m(t,x),\Sigma(t))$ for

$$m(t,x) = \overline{X} + e^{-(T-t)\kappa}(x-\overline{X});\ \Sigma(t) = e^{-T\kappa}\left(\int_t^T e^{u\kappa}\sigma_X\sigma_X'e^{u\kappa'}du\right)e^{-T\kappa'}. \tag{6.7}$$

Second, if $Y \sim N(m,\Sigma)$ then for $D \in \mathbb{S}_+^d$ and $E \in \mathbb{R}^d$

$$\log\left(\mathbb{E}\left[e^{-\frac{1}{2}Y'DY+E'Y}\right]\right) = -\frac{1}{2}\log\left(|\mathcal{M}|\right) + \frac{1}{2}E'\mathcal{M}\Sigma E + E'\mathcal{M}m - \frac{1}{2}m'D\mathcal{M}m,$$

for the matrix $\mathcal{M} = (1 + \Sigma D)^{-1}$. Lastly, for $B, C \in \mathbb{R}$ and $i, j, k \in \{1, ..., d\}$

$$\frac{\mathbb{E}\left[(BY^i + CY^jY^k)e^{-\frac{1}{2}Y'DY + E'Y}\right]}{\mathbb{E}\left[e^{-\frac{1}{2}Y'DY+E'Y}\right]} = BF^i + CF^jF^k + CG^{jk}, \tag{6.8}$$

where $F = \mathcal{M}(m + \Sigma E)$ and $G = \mathcal{M}\Sigma$. We start with S from (6.1). Here, $D = \gamma(M + C^\Pi)$, $E = \gamma(\tilde{M}h - \zeta - B'\Pi)$, so that for $i = 1, ..., d$

$$S^i(t, x, h) = (BF(t, x, h))^i + \frac{1}{2}F(t, x, h)'C(i)F(t, x, h) + \frac{1}{2}\mathrm{Tr}\left(C(i)G(t)\right),$$

where $F(t, x, h) = \mathcal{M}(t)\left(m(t, x) + \mathcal{V}(t, h)\right)$ and $G(t) = \mathcal{M}(t)\Sigma(t)$ for the matrix $\mathcal{M}(t) = \left(1 + \gamma\Sigma(t)\left(M + C^\Pi\right)\right)^{-1}$ and vector $\mathcal{V}(t, h) = \gamma\Sigma(t)\left(\tilde{M}h - \zeta - B'\Pi\right)$. Using (6.2), the volatility matrix[15] is, for $i, j = 1, ..., d$

$$\sigma^{ij}(t, x, h) = \left(B\mathcal{M}(t)e^{-(T-t)\kappa}\sigma_X\right)^{ij} + \left(\sigma_X' e^{-(T-t)\kappa'}\mathcal{M}(t)'C(i)F(t, x, h)\right)^j. \tag{6.9}$$

Therefore, if the quadratic terms are present in Ψ then volatility is stochastic, linear in both x and h. This implies that the volvol for $i, j, p = 1, ..., d$

$$\partial_{x_p}\sigma^{ij}(t, x, h) = \left(\sigma_X' e^{-(T-t)\kappa'}\mathcal{M}(t)'C(i)\mathcal{M}(t)e^{-(T-t)\kappa}\right)^{jp},$$

is time varying, but independent of both x and h. As for ν^h, it is clear from (6.3), (6.4), (6.7) and (6.8) that $\check{\nu}^h, \mu^h$ and hence ν^h is (time-varying) affine in both x and h, with explicitly identifiable coefficients.

As for the trading strategies, from (6.5), (6.6), (6.8), and (6.9) we conclude that if the quadratic terms are present in Ψ, then $\pi^{mf,h}$ is the inverse of a matrix affine in x, h multiplied by a vector, also time-varying affine in x, h. Similar analysis for $\pi^{dp,g,h}$ shows $\pi^{dp,g,h}$ is also of this form (further discussion of the dark pool is given below). In light of Proposition 5.6 we also see $\pi^{vv,h}$ is of this form as well.[16]

6.1.1. *Linear payoff.* Consider when $C(i) = 0, i = 1, ..., d$. This is the continuous time, multi-dimensional analog of Grossman (1976); Hellwig (1980). The price and volatility functions are

$$S(t, x, h) = B\mathcal{M}(t)\left(m(t, x) + \mathcal{V}(t, h)\right); \quad \sigma(t, x, h) = \sigma(t) = B\mathcal{M}(t)e^{-(T-t)\kappa}\sigma_X.$$

[15]Assumption 4.2 holds provided for Lebesgue almost every $(t, x) \in [0, T] \times \mathbb{R}^d$ the matrix $R(t, x)$ is of full rank, where $R^{ij}(t, x) = B^{ji} + 2\left(C(j)F(t, x, 0)\right)^i$ for $i, j = 1, ..., d$.

[16]The uninformed portfolio is a linear combination of the market portfolio and the portfolio replicating F^h_{1T}. The uninformed does not need the dark pool to achieve his optimal allocation.

Thus, the price is linear in h, and h is recoverable from S provided B is of full rank (which also enforces Assumption 4.2). The volatility is non-degenerate and time varying, but deterministic and independent of h. Lastly, as easy calculations show $M \leq (1/\gamma)C_I^{-1}$, it follows that PCE volatility is greater than FCE volatility.

When $B = \sigma_X = 1$ and κ is diagonal, $\psi^i(x) = x^i$ and $X^1, ..., X^d$ are independent univariate OU processes. Here, $\Sigma(t)$ and $e^{-(T-t)\kappa}$ are diagonal, but unless the matrix M from (4.4) is diagonal (essentially if C_I, C_N are diagonal) then $\sigma(t)$ is not diagonal, and equilibrium prices are correlated even if the factor processes are not.

6.1.2. *Quadratic payoff.* We lastly highlight two phenomena in the quadratic case. For notational ease, assume $d = 1$ and $B = 0, C(1) = 1$. First, the initial price is

$$S(0, x, h) = \frac{1}{2}\left(\mathcal{M}(0)\left(m(0, x) + \mathcal{V}(0, h)\right)\right)^2 + \frac{1}{2}\mathcal{M}(0)\Sigma(0).$$

As $\mathcal{V}(0, h)$ is linear in h, $S_0^H = S(0, X_0, H)$ does not reveal H. However, (6.9) implies $\sigma(t, x, h)$ is linear in h, and hence as one may observe $d\langle S^H, X\rangle_t = \sigma_X \sigma(t, X_t, H)dt$, H is revealed after any positive time t. As H is "instantly observable", the equilibrium is nonetheless a REE in the sense of Definition 2.3.

Second, come back to the dark pool where the linear payoff $(g - h)X_T/C_I$ is synthesized. As the terminal asset payoff is $\Psi(X_T) = (1/2)X_T^2$, the dark pool does not involve static, or even deterministic trading. Indeed, (6.8) implies on $\{H = h, X_t = x\}$

$$\pi_t^{dp,g,h} = \frac{\sigma_X e^{-(T-t)\kappa}\mathcal{M}(t)(g - h)}{C_I \sigma(t, x, h)} = \frac{g - h}{C_I \mathcal{M}(t)\left(m(t, x) + \mathcal{V}(t, h)\right)}.$$

As such, the dark pool, comprised of $\pi^{dp,G_I,H}(t, X_\cdot, H)$ and $\pi^{dp,G_N,H}(t, X_\cdot, H)$ involves continuous, stochastic re-balancing with unbounded positions.

6.2. **Square Root Factor Process.** The square root model ("1/2 model") is generated by a mean-reverting square root process (MRSR) for X. The MRSR model is popular in the fixed income Cox, Ingersoll, and Ross (1985), and the stochastic volatility Heston (1993) literatures. The one-dimensional process X follows the stochastic differential equation (SDE) $dX_t = \kappa(\overline{X} - X_t)dt + \sigma_X \sqrt{X_t} dB_t$, and we consider the linear payoff $\Psi(x) = x$.

It is well known (c.f. Cox, Ingersoll, and Ross (1985)) that given $X_t = x$, $X_T = c(t)Y$ where $c(t) = (\sigma_X^2/4\kappa)(1 - e^{-\kappa(T-t)})$ and Y is a non-central chi-square random variable with $4\kappa\overline{X}/\sigma_X^2$ degrees of freedom and non-centrality parameter $(x/c(t))e^{-\kappa(T-t)}$.

Writing $f_Y(y,t,x)$ as the pdf of Y (which depends on t,x)

$$S(t,x,h) = \frac{\int_0^\infty c(t) y e^{-\frac{1}{2}\gamma M c(t)^2 y^2 + \gamma c(t) y \left(\tilde{M}h - \zeta - \Pi\right)} f_Y(y,t,x)dy}{\int_0^\infty e^{-\frac{1}{2}\gamma M c(t)^2 y^2 + \gamma c(t) y \left(\tilde{M}h - \zeta - \Pi\right)} f_Y(y,t,x)dy}.$$

The quadratic terms in the exponents imply closed form solutions for $S(t,x,h)$ are not available. However, numerical computation is easy.

6.2.1. *Numerical Implementation.* Fix $T = \Pi = 1$. To isolate the effects of information asymmetry, we equate the risk aversion (at 3) and weights (1/3) of all agents. We set $\tau_N = 1$ and $\mu_N = 0$ so that the noise trader, while thinking he will receive $G_I = X_T + Y_I$, in fact receives the noisier signal $G_N = G_I + Y_N$. We set the dispersion variances to $C_I = .5$ and $C_N = 1$. For the CIR process, we use $\kappa = \bar{X} = \sigma_X = 1$ and $X_0 = \bar{X}$, though our results are not sensitive to these parameter choices.

We first plot PCE and FCE equilibrium prices, volatilities and volvols as functions of the state variable X for the times $t = 0$, 0.50 and 0.90. In the plots, X varies between the 3^{rd} and 97^{th} quantile for the time-0 distribution of X_T. For each quantity we use a common y-axis to highlight temporal movements. For the FCE, we take $G_I = \overline{X}$, the mean of G_I (as $X_0 = \overline{X}$). For the PCE, we take two values for the noise component Y_N: a low value (-2 standard deviation), and a high value ($+2$ standard deviation). We do not plot PCE quantities for $Y_N = 0$ as they essentially coincide with their respective FCE quantities (this is expected, if $Y_N = 0$ then $H = G_I = G_N$).

Figure 1 plots the equilibrium price. Consistent with the REE statement in Theorem 4.3 (which applies as $\Psi(x) = x$), we see the price is monotonic in Y_N, with monotonicity in x holding as well. Furthermore, we see the price function steepens as the residual horizon decreases, approaching the identity map at $t = 1$.

Figure 2 plots equilibrium volatility. Here, volatility is also increasing in signal and state, however, the latter increase is not linear, but concave. To more clearly identify effects of the signal, Figure 3 plots the relative changes in the PCE volatility (over the FCE volatility). Here, volatility increase is most substantial around the long run mean $\overline{X}$ of X with generally larger increases as time progresses.

Figure 4 plots the relative changes (over the FCE) in the equilibrium volvol. We plot the changes because the absolute levels do not clearly communicate the signal effect. While generally the volvol is decreasing in the state, we see that unlike with the volatility and price, the relationship between the signal size and volvol function is not monotonic. Rather, a higher signal value will increase the volvol for low values of the state, but will decrease the volvol for higher values. Hence, volatility is more (less) sensitive to news for low (high) values of the state in the PCE.

We next plot the population-weighted optimal trading strategy for the uninformed agent in Figure 5. The dashed line is the (constant) FCE population weighted optimal strategy. Now, unlike with the price, volatility and volvol, the uninformed agent's optimal strategy in the case of $Y_N = 0$ does not coincide with the FCE optimal trading strategy. This difference is explained by Proposition 5.5. Indeed, in the PCE the uninformed has an additional (over the FCE) demand for the portfolio $\pi^{mf,h}$ which replicates the payoff F^h_{1T} from (5.14). Even when $h = 0$, the quantities $X_T' C_I^{-1} X_T$ and $\log\left(\mathbb{P}\left[H \in dh \middle| X_T\right]\right)$ (c.f. (7.11) below noting that $\mathbb{P}\left[H \in dh \middle| X_T\right] = \ell(T, X_T, h)$) contain a quadratic component in X which does not vanish. As such, there is a "baseline" demand for the public information fund, which holds even if the public and private signal realizations coincide. Interestingly, the uninformed agent's position size decreases with the market signal and increases with the state. It also varies with time and can be both smaller and larger than the FCE strategy. This illustrates the dynamic nature of the uninformed trading strategy.

Lastly, to highlight the dynamic aspect of equilibrium volatility in our continuous time framework, we consider sample paths for the FCE and PCE volatility. To illuminate informational effects, we also consider the no information equilibrium (NIE), where there is no private signal. Arguments similar to those in Section 3 show

$$S_t^{NIE} = S^{NIE}(t, X_t) : \qquad S^{NIE}(t, x) = \frac{\mathbb{E}\left[\Psi(X_T) e^{-\gamma \Pi' \Psi(X_T)} \middle| X_t = x\right]}{\mathbb{E}\left[e^{-\gamma \Pi' \Psi(X_T)} \middle| X_t = x\right]}.$$

In Figure 6, the left plots give two representative volatility sample paths. To highlight the differences, for each sample, the right plot shows the percentage difference versus the NIE volatility. For example, for FCE, the percentage difference is $100(\sigma_t^{FCE} - \sigma_t^{NIE})/\sigma_t^{NIE}$, with similar formulas for the PCE with high and low Y_N. We see that the PCE equilibrium volatility may be both above and below the FCE volatility, depending on the signal size, and there can be a significant (versus the NIE) increase or decrease in the volatility depending on the signal size. Furthermore, the NIE volatility may cross the FCE and PCE volatilities, so a uniform ordering is not possible. Finally, volatilities converge to the fundamental volatility as $t \to T$. Information becomes irrelevant in the limit as the asset price approaches the payoff function.

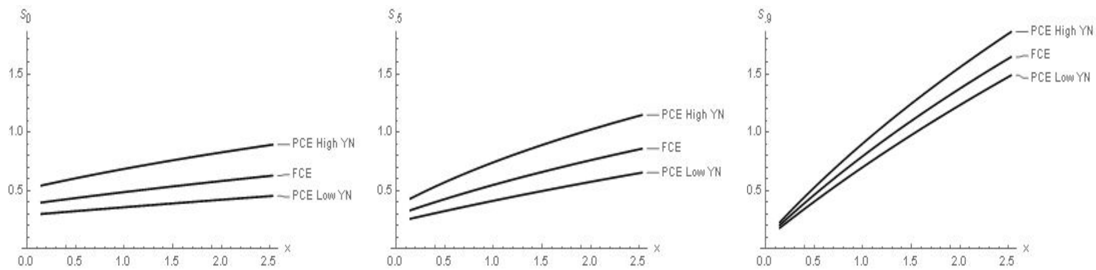

FIGURE 1. Equilibrium prices as a function of the state and signal for (left-right) $t = 0$, 0.5 and 0.9 in the 1/2 model.

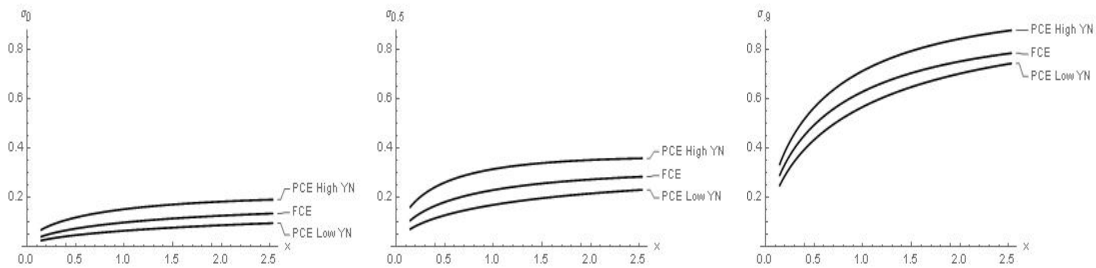

FIGURE 2. Equilibrium volatility as a function of the state and signal for (left-right) $t = 0$, 0.5 and 0.9 in the 1/2 model.

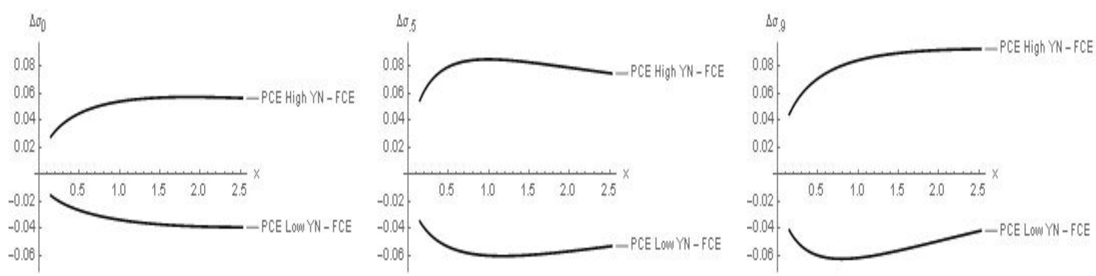

FIGURE 3. Equilibrium volatility changes relative to the FCE as a function of the state and signal for (left-right) $t = 0$, 0.5 and 0.9 in the 1/2 model.

7. MARKOVIAN NOISE

We now examine an economy with general distributions for the noises Y_I, Y_N. The purpose is to show the bi-linearity of the market signal H is almost entirely due to the Gaussian nature of the insider's noise component $Y_I = G_I - X_T$. In fact, the distribution of the noise trader's signal G_N plays a minimal role. This important fact enables us to consider other distributions for G_N. In particular, one could set G_N to be an independent copy of G_I and, up to verifying certain integrability conditions, the

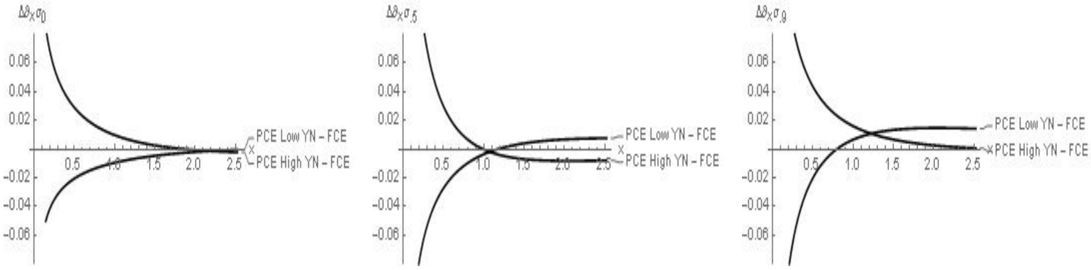

FIGURE 4. Equilibrium volvol changes relative to the FCE as a function of the state and signal for (left-right) $t = 0$, 0.5 and 0.9 in the 1/2 model.

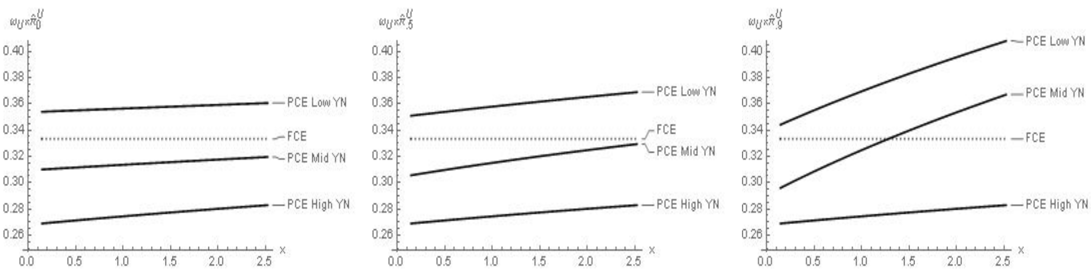

FIGURE 5. Uninformed strategy as a function of the state and signal for (left-right) $t = 0$, 0.5 and 0.9 in the 1/2 model.

equilibrium market signal would be the same. Throughout this section, Assumption 2.1 is in force. Regarding the insider (G_I) and noise trader (G_N) signals, we assume

Assumption 7.1.

(1) $G_I = X_T + Y_T^I$, where Y^I is a $\mathbb{R}^d$ valued Markov process starting at 0 and independent of B. Y_T^I has a smooth and bounded pdf $\hat{p}_I$.
(2) $G_N = \tau_N G_I + Y_T^N$, where Y^N is a $\mathbb{R}^d$ valued Markov process starting at y_0^N and independent of both B and Y^I. Y_T^N has a smooth and bounded pdf $\hat{p}_N$.

Clearly, to obtain $Y_T^I = Y_I$ and $Y_T^N = Y_N$ from (2.5), (2.7) respectively, we can take $Y_\cdot^I = \sqrt{1/T}\sqrt{C_I}W_\cdot^I$ [17], and $Y_\cdot^N = (1/T)\mu_n \cdot + \sqrt{1/T}\sqrt{C_N}W_\cdot^N$ where W^I, W^N are independent d-dimensional Brownian motions also independent of B. Next, we consider functions H satisfying a full range and joint invertibility condition.

Assumption 7.2. $\mathbb{F}^m = \mathbb{F}^B \vee \mathfrak{s}(H), H = H(G_I, G_N)$, where $H \in C^1(\mathbb{R}^{2d}; \mathbb{R}^d)$ and

[17] $\sqrt{C_N}$ is the unique positive definite symmetric square root of C_N.

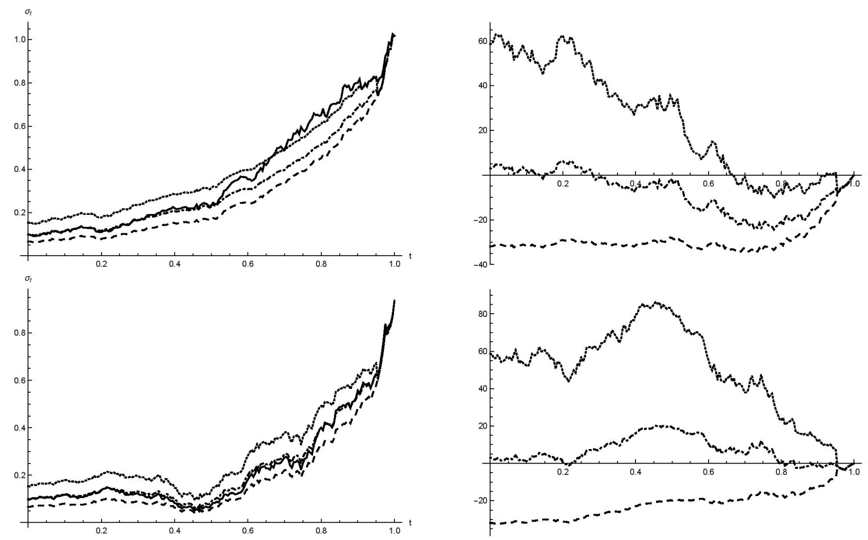

FIGURE 6. Sample paths for equilibrium volatility (left) and percentage difference relative to the NIE (right) in the 1/2 model. Thin dashes correspond to the PCE with high Y_N; medium dashes to the PCE with low Y_N; dot-dashes to the FCE, and the NIE volatility is a solid line.

(1) For each $x, y \in \mathbb{R}^d$ the ranges of $H(x, \cdot)$ and $H(\cdot, y)$ equal the range of H:

$$\mathcal{R}_H := \left\{H(\tilde{x}, \tilde{y}) \mid \tilde{x}, \tilde{y} \in \mathbb{R}^d\right\} = \left\{H(x, \tilde{y}) \mid \tilde{y} \in \mathbb{R}^d\right\} = \left\{H(\tilde{x}, y) \mid \tilde{x} \in \mathbb{R}^d\right\}.$$

(2) For each $x \in \mathbb{R}^d$ fixed, the map $y \to H(x, y)$ is invertible with inverse $G(x, h)$. Writing $J^G(x, h)$ as the Jacobian of G with respect to h evaluated at (x, h), the determinant $|J^G|$ is strictly positive and bounded function on $\mathbb{R}^d \times \mathcal{R}_H$.

(3) For each $y \in \mathbb{R}^d$ fixed, the map $x \to H(x, y)$ is invertible with inverse $\check{G}(y, h)$. Writing $J^{\check{G}}(y, h)$ as the Jacobian of $\check{G}$ with respect to h evaluated at (y, h), $|J^{\check{G}}|$ is strictly positive and bounded function on $\mathbb{R}^d \times \mathcal{R}_H$.

We again see Assumption 7.2 holds for H from (5.1). Now, under our assumptions, the factor process is time-homogenous strong Markov with transition density

$$p(\tau, x, y) := \frac{1}{dy}\mathbb{P}\left[X_\tau \in dy \middle| X_0 = x\right]; \quad \tau > 0, x, y \in E. \tag{7.1}$$

We now precisely define ℓ from (4.5), setting for $(t, x, h) \in [0, T] \times E \times \mathcal{R}_H$

$$\ell(t, x, h) := \int_{E\times\mathbb{R}^d} p(T - t, x, y)\hat{p}_I(g - y)\hat{p}_N(G(g, h) - \tau_N g)|J^G|(g, h)dydg. \tag{7.2}$$

Lemma {3.1} shows ℓ is indeed the conditional density of H given $X_t = x$. As ℓ is strictly positive, the Jacod equivalence condition $\ell(t, X_t, \cdot) = \mathbb{P}\left[H \in \cdot \middle| \mathcal{F}_t^B\right] \sim \mathbb{P}\left[H \in \cdot\right] = \ell(0, X_t, \cdot)$, holds almost surely on $t \leq T$, with density

$$(7.3) \quad p_t^h = \frac{\ell(t, X_t, h)}{\ell(0, X_0, h)} = \mathcal{E}\left(\int_0^{\cdot} (\mu_u^h)' dB_u\right)_t; \quad \mu_t^h := a(X_t)' \nabla_x \left(\log\left(\ell(t, X_t, h)\right)\right).$$

Also, Lemma {3.1} shows $B_{\cdot}^m = B_{\cdot} - \int_0^{\cdot} \mu_u^H du$ is a $(\mathbb{P}, \mathbb{F}^m)$-Brownian motion. The martingale preserving measure is defined through (c.f. (5.2) and associated footnote)

$$(7.4) \qquad \frac{d\widetilde{\mathbb{P}}^H}{d\mathbb{P}} = \frac{1}{p_T^H} = \mathcal{E}\left(-\int_0^{\cdot} (\mu_t^H)' dB_t^m\right)_T.$$

Now, let $\check{\nu} \in \mathcal{P}(\mathbb{F}^m)$, and assume $\check{\mathbb{Q}} = \mathbb{Q}^{\check{\nu}}$ defined through

$$(7.5) \qquad \frac{d\check{\mathbb{Q}}}{d\widetilde{\mathbb{P}}^H} = \mathcal{E}\left(-\int_0^{\cdot} \check{\nu}_t' dB_t\right)_T =: \check{Z}_T,$$

is a probability measure, where, as in (5.4) it is convenient to define martingale measures through densities with respect to $\widetilde{\mathbb{P}}^H$. In terms of the beliefs measure

$$\frac{d\check{\mathbb{Q}}}{d\mathbb{P}} = \frac{d\check{\mathbb{Q}}}{d\widetilde{\mathbb{P}}^H} \times \frac{d\widetilde{\mathbb{P}}^H}{d\mathbb{P}} = \frac{\check{Z}_T}{p_T^H} = \mathcal{E}\left(-\int_0^{\cdot} (\mu_t^H + \check{\nu}_t)' dB_t^m\right)_T$$

and under Assumption C.1 in Appendix C, the candidate PCE price is well defined with S from (7.6) below. The next result provides a necessary and sufficient condition for $\{\mathbb{F}^m, B^m, \nu, \sigma, S\}$ to be a PCE.

Theorem 7.3. *Let Assumptions 7.1, 7.2 hold. Let $\check{\nu}$ satisfy Assumption C.1 in Appendix C. Set $\nu := \check{\nu} + \mu^H$, and with $\check{\mathbb{Q}}$ from (7.5), define*

$$(7.6) \quad S_t := \mathbb{E}^{\check{\mathbb{Q}}}\left[\Psi(X_T) \middle| \mathcal{F}_t^m\right] = \mathbb{E}^{\check{\mathbb{Q}}}\left[\Psi(X_T) \middle| \mathcal{F}_0^m\right] + \int_0^t \sigma_u \left(dB_u^m + \nu_u du\right); \quad t \leq T.$$ [18]

Then, $\{\mathbb{F}^m, B^m, \nu, \sigma, S\}$ is a $\mathbb{F}^m$-PCE <u>if and only if</u> the SDF $\check{Z}_T$ satisfies the following market clearing condition:

$$(7.7) \quad \begin{aligned} \Pi'(\Psi(X_T) - S_0) &= \frac{1}{\gamma}\left(\widetilde{\mathbb{E}}^H\left[\check{Z}_T \log(\check{Z}_T) \middle| \mathcal{F}_0^m\right] - \log(\check{Z}_T)\right) \\ &\quad - \alpha_U\left(\widetilde{\mathbb{E}}^H\left[\check{Z}_T \log(\ell(T, X_T, H)) \middle| \mathcal{F}_0^m\right] - \log(\ell(T, X_T, H))\right) \\ &\quad - \alpha_I\left(\widetilde{\mathbb{E}}^H\left[\check{Z}_T \log(\hat{p}_I(g - X_T)) \middle| \mathcal{F}_0^m\right]\Big|_{g=G_I} - \log(\hat{p}_I(G_I - X_T))\right) \\ &\quad - \alpha_N\left(\widetilde{\mathbb{E}}^H\left[\check{Z}_T \log(\hat{p}_I(g - X_T)) \middle| \mathcal{F}_0^m\right]\Big|_{g=G_N} - \log(\hat{p}_I(G_N - X_T))\right). \end{aligned}$$

[18] σ in (7.6) is obtained through predictable representation (c.f. (Protter, 2005, Chapter IV)).

Theorem 7.3 identifies a two-step procedure for establishing a $\mathbb{F}^m$-PCE. The first step is to verify Assumption C.1, in particular part (2) on the non-degeneracy of the volatility matrix σ in (7.6), which implies both the uninformed and insider markets are complete. The second step is then to establish the terminal market clearing condition in (7.7), as completeness ensures market clearing at earlier times.

In fact, condition (7.7) is intuitive. The left hand side is the cumulative aggregate gain from trade at the terminal date. The right hand side is the sum of the agents' gains from trade. In essence (7.7) requires equality between the aggregate payoff on securities and the sum of agents' optimal terminal wealth levels. For each individual, optimal terminal gains is inverse marginal utility evaluated at the SDF $\check{Z}$ adjusted by the relevant distributions $\ell(T, X_T, H), \hat{p}_I\left(G_I - X_T\right)$ or $\hat{p}_I\left(G_N - X_T\right)$. The condition is a restriction on the SDF. It also imposes a measurability, i.e. informational, restriction on the beliefs-related part of the residual demand, i.e., the last two terms on the right hand side. These terms can only depend on G_I, G_N through H.

7.1. **Gaussian Insider Dispersion.** For $C \in \mathbb{S}^d_{++}$ define the density function for a d-dimensional Gaussian random variable with mean 0 and covariance matrix C

$$\hat{p}_C(y) := \frac{1}{(2\pi)^{d/2}\sqrt{|C|}} e^{-\frac{1}{2}y'C^{-1}y}; \quad y \in \mathbb{R}^d. \tag{7.8}$$

Suppose now that $\hat{p}_I = \hat{p}_{C_I}$ with C_I as in (2.5). The last two lines in (7.7) simplify to

$$\begin{aligned} &-\frac{1}{2}\left(\alpha_I+\alpha_N\right)\left(X_T'C_I^{-1}X_T - \widetilde{\mathbb{E}}^H\left[\check{Z}_T X_T' C_I^{-1}X_T \middle| \mathcal{F}_0^m\right]\right)\\ &\quad + \left(\alpha_I+\alpha_N\right)\left(\frac{\alpha_I G_I + \alpha_N G_N}{\alpha_I+\alpha_N}\right)' C_I^{-1}\left(X_T - \widetilde{\mathbb{E}}^H\left[\check{Z}_T X_T \middle| \mathcal{F}_0^m\right]\right). \end{aligned} \tag{7.9}$$

By normality of the noise in the informed signal, $\log\left(\hat{p}_I(\cdot)\right)$ is affine in its argument. It follows that the informed demand is affine in G_I and so is the noise trader demand with respect to G_N. The residual demand therefore aggregates individual signals into the composite signal $H = (\alpha_I G_I + \alpha_N G_N)/(\alpha_I + \alpha_N)$ and in the last line above we can pull H in and out of the conditional expectation, obtaining $-(\alpha_I+\alpha_N)\left(H'C_I^{-1}X_T - \widetilde{\mathbb{E}}^H\left[\check{Z}_T H' C_I^{-1} X_T \middle| \mathcal{F}_0^m\right]\right)$. Combining this with the other terms in (7.7), and recalling that $S_0 = \widetilde{\mathbb{E}}^H\left[\check{Z}_T \Psi(X_T)\middle|\mathcal{F}_0^m\right]$, we can re-write the market clearing condition is satisfied for $\check{Z}_T = k_0^H e^{-\gamma\Theta(X_T,H)}$ where k_0^H is a $\mathcal{F}_0^m$ measurable quantity and

$$\Theta(x,h) := \Pi'\Psi(x) + \frac{1}{2}(\alpha_I+\alpha_N)\left(x'C_I^{-1}x - 2h'C_I^{-1}x\right) - \alpha_U \log(\ell(T,x,h)). \tag{7.10}$$

$\check{\nu} = \check{\nu}^H$ is then found via predictable representation (c.f. (Protter, 2005, Chapter IV)), and under the relevant technical assumptions, the $\mathbb{F}^m$-PCE will follow.

Remarkably, this result holds for general conditional densities $\ell(T, x, h)$, and hence general distributions for the noise trader's signal G_N. We conclude that a PCE with affine signal $H = (\alpha_I G_I + \alpha_N G_N)/(\alpha_I + \alpha_N)$ exists for arbitrary distributions of G_N.

Even though the residual demand is affine in H, it is important to note that the conditional distribution of H given $X_T = x$, i.e., the function $\ell(T, x, h)$, is not Gaussian. As a result the uninformed demand and equilibrium price are not linear in H.

7.2. **Gaussian Noise Trader Dispersion.** Assume now the noise trader signal is Gaussian, taking the form in (2.7). In this case ℓ simplifies to (recall (4.3) and (7.8))

$$\ell(T, x, h) = \hat{p}_{\tilde{C}}\left(h - \frac{(\alpha_I + \alpha_N \tau_N)x + \alpha_N \mu_N}{\alpha_I + \alpha_N}\right), \tag{7.11}$$

Ignoring terms independent of x in Θ which do not affect pricing, effectively Θ from (7.10) reduces to Θ from (4.1) with $M, \tilde{M}, \zeta$ from (4.4). Then, under the relevant technical conditions, an $\mathbb{F}^m$-PCE follows. In fact, if either $\Psi(x) = x$, or $d = 1$ and Ψ is strictly monotone, the $\mathbb{F}^m$-PCE is a REE because the (vector) market signal H is recoverable from time 0 asset prices. The proof of this statement follows from (B.5) in Appendix B, specializing Θ from (4.1) to the case of $M, \tilde{M}, \zeta$ from (4.4).

7.3. **The Clearing Condition** (7.7) **and Breon-Drish (2015).** We conclude this section connecting the clearing condition (7.7) to the "exponential family" assumption in Breon-Drish (2015). Rather than making an identification such as $G_I = X_T + Y_I$, it is assumed in (Breon-Drish, 2015, Assumption 3) (using our notation and assuming X_T has a pdf) G_I, X_T are such that

$$\frac{d}{dx}\mathbb{P}\left[X_T \in dx \middle| G_I = g\right] = e^{kxg - f(kg)} m(x), \tag{7.12}$$

where $k > 0$, and where f and $m \geq 0$ are certain functions. Under this assumption, equilibrium was established in a static setting, with informed demand affine in the private signal. Now, come back to (7.7) and recall that $\hat{p}_I(g - x)$ is the conditional density of G_I given $X_T = x$. If (7.12) holds then Bayes rule implies the conditional density of G_I given $X_T = x$ also takes the form on the right side of (7.12), with possibly different f and m. Therefore, the analog of (7.9) gives the same composite signal H as in the Gaussian case.

We thus conjecture our results extend to general specifications $G_I = \phi(X_T, Y_I)$ and distributions for Y_I. We also highlight how our work and Breon-Drish (2015) extend the Gaussian setting. Breon-Drish (2015) considers a static model, while we consider a dynamic model. Using representation theorems under filtration enlargements, we decompose the dynamic problem into two stages. First we identify a terminal clearing condition using optimal terminal wealths, then we use representation results to obtain dynamic policies.

Lastly, we remark that for additive noise $G_I = X_T + Y_I$, if X_T, Y_I have pdfs then (7.12) implies Y_I is Gaussian. However, for general $G_I = \phi(X_T, Y_I)$ it is possible to identify other distributions for Y_I which enforce (7.12). Indeed, one can take $G_I = -Y_I/X_T$ where Y_I is Gamma distributed.

Appendix A. Model Assumptions and Admissible Strategies

Uncertainty. Consider the diffusion X and region E from (2.1). We assume

Assumption A.1. For some $\gamma \in (0,1]$

(1) $E = \bigcup_n E_n$ where for each n, E_n is open, connected, and bounded with $C^{2,\gamma}$ boundary[19]. $\bar{E}_n \subset E_{n+1}$ and if $d \geq 2$ then $E_{n+1} \setminus E_n$ is simply connected.

(2) $b \in C^{1,\gamma}(E; \mathbb{R}^d)$ and $a = \sqrt{A}$ for $A \in C^{2,\gamma}(E; \mathbb{S}^d_{++})$. Additionally, there exists a weak solution to (2.1).

Part (1) above assumes E may be filled up by "balls", and holds, for example, E is convex, open. The regularity conditions in (2) hold if b and A are C^2 and C^3 on E.

Admissible strategies for the uninformed. The set of admissible strategies for the uninformed coincides with the set of *permissible* strategies in Owen and Žitković (2009), and is appropriate for CARA preferences. For $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}^m_T$, define the density processes $Z^{\mathbb{Q}}_\cdot := d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}^m_\cdot}$ and $\mathring{Z}^{\mathbb{Q}}_\cdot := Z^{\mathbb{Q}}_\cdot / Z^{\mathbb{Q}}_0$ as well as the initial relative entropy $H_0\left(\mathbb{Q}\middle|\mathbb{P}\right) := \mathbb{E}\left[\mathring{Z}^{\mathbb{Q}}_T \log\left(\mathring{Z}^{\mathbb{Q}}_T\right) \middle| \mathcal{F}^m_0\right]$. Then, set

$$\begin{aligned} \mathcal{M}^m &:= \{\mathbb{Q} \sim \mathbb{P} \text{ on } \mathcal{F}^m_T \mid S \text{ is a } (\mathbb{Q}, \mathbb{F}^m) \text{ local martingale}\}; \\ \tilde{\mathcal{M}}^m &:= \left\{\mathbb{Q} \in \mathcal{M}^m \mid H_0\left(\mathbb{Q}\middle|\mathbb{P}\right) < \infty\right\}, \end{aligned} \tag{A.1}$$

[19] See (Pinsky, 1995, Chapter 3.2) for a precise definition of $C^{k,\gamma}$ boundaries and functions.

so that $\mathcal{M}^m$ is the set of equivalent local martingales measures (ELMM)[20] and $\tilde{\mathcal{M}}^m$ is the subset of ELMM with finite relative entropy. Now, recall the dynamics for S in (2.3), and that a strategy $\pi \in \mathcal{P}(\mathbb{F}^m)$ is S-integrable under $(\mathbb{P}, \mathbb{F}^m)$ if

$$\text{(A.2)} \qquad \int_0^T |\pi_u' \sigma_u \nu_u| du < \infty; \qquad \int_0^T \pi_u' \sigma_u \sigma_u' \pi_u du < \infty^{21}.$$

The admissible strategy set $\mathcal{A}^m$ consists of S-integrable $\pi \in \mathcal{P}(\mathbb{F}^m)$ with wealth process $\mathcal{W}^\pi_\cdot = \int_0^\cdot \pi_u' dS_u$ such that $\mathbb{E}^{\mathbb{Q}}\left[\mathcal{W}_T^\pi \middle| \mathcal{F}_0^m\right] \leq 0$ for all $\mathbb{Q} \in \tilde{\mathcal{M}}^m$. While $\mathcal{A}^m$ is similar to the "$L^2[G]$" set of Duffie (1986), neither set is contained within the other, and as shown in Owen and Žitković (2009), $\mathcal{A}^m$ is appropriate for CARA preferences.

Admissible strategies for the informed. To define the class of admissible strategies for the informed, construct $\mathcal{M}^I, \tilde{\mathcal{M}}^I$ as in (A.1) (with $\mathbb{F}^I$ replacing $\mathbb{F}^m$ therein, as well as in the density process and initial relative entropy definitions). Then set $\mathcal{A}^I$ as the class of $\pi \in \mathcal{P}(\mathbb{F}^I)$ such that (A.2) holds (with $\nu = \nu + \mu^{G_I}$ therein, see (2.6)) and $\mathbb{E}^{\mathbb{Q}}\left[\mathcal{W}_T^\pi \middle| \mathcal{F}_0^I\right] \leq 0$ for all $\mathbb{Q} \in \tilde{\mathcal{M}}^I$.

Admissible strategies for the noise trader. First, note that $\mathcal{M}^m$ from (A.1) is the same for the noise trader. Let $\tilde{\mathcal{M}}^{N,g}$ be the subset of $\mathcal{M}^m$ with finite relative entropy with respect to $\mathbb{P}^g$. The set of admissible strategies for the noise trader $\mathcal{A}^{N,g}$ is the class of $\pi \in \mathcal{P}(\mathbb{F}^m)$ such that (A.2) holds (with $\nu = \nu + \mu^g$ therein) and $\mathbb{E}^{\mathbb{Q}}\left[\mathcal{W}_T^\pi \middle| \mathcal{F}_0^m\right] \leq 0$ for all $\mathbb{Q} \in \tilde{\mathcal{M}}^{N,g}$.

Appendix B. Proof of Theorem 3.1

We now prove Theorem 3.1, with Assumptions 2.1, A.1 in force throughout.

Proof of Theorem 3.1. In light of (3.1) and the subsequent discussion, it suffices to show that $\mathbb{Q}^{G_I}$ and S^{G_I} are well defined, characterize the dynamics for S^{G_I}, prove $\mathbb{Q}^{G_I}$ has finite relative entropy with respect to $\mathbb{P}$, and that the equilibrium is a REE in the case of linear payoffs (monotone in dimension one).

We start with $\mathbb{Q}^{G_I}$, which is well defined, as non-negativity ensures $\mathbb{E}\left[e^{-\gamma \Pi' \Psi(X_T)} \middle| \mathcal{F}_0^m\right]$ has meaning. Next, recall Θ from (4.1), evaluated using $M, \tilde{M}, \zeta$ from (5.10). Part

[20]A process S is a local martingale if there exists a non-decreasing sequence of stopping times $\{\tau_n : n \in \mathbb{N}\}$ such that $\tau_n \to \infty$ and the stopped process $S_{\cdot \wedge \tau_n}$ is a martingale for each $n \in \mathbb{N}$.

[21](A.2) also ensures $\mathcal{W}^\pi$ is a $(\mathbb{P}, \mathbb{F}^m)$ semi-martingale.

(1) of Lemma {1.1} shows $\mathbb{E}\left[e^{-\gamma\Theta(X_T,g)}\right] < \infty$ for all $g \in \mathbb{R}^d$, and as such there is $\check{\nu}^g \in \mathcal{P}(\mathbb{F}^B)$ such that

$$\check{Z}^g_T := \frac{e^{-\gamma\Theta(X_T,g)}}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,g)}\right]} = \mathcal{E}\left(-\int_0^\cdot (\check{\nu}^g_u)' dB_u\right)_T. \tag{B.1}$$

Recall C_I from (2.5), p from (7.1), and (7.8). We first precisely define u from (4.5) setting, for $(t,x,g) \in [0,T]\times E\times\mathbb{R}^d$

$$u(t,x,g) := \mathbb{E}\left[\hat{p}_{C_I}(g - X_T)\big| X_t = x\right] = \int_E \hat{p}_{C_I}(g-y)p(T-t,x,y)dy. \tag{B.2}$$

Lemma {2.1} proves $\mathbb{P}\left[G_I \in \cdot \big| \mathcal{F}^B_t\right] = u(t,X_t,\cdot)$ and hence $\mathbb{P}\left[G_I \in \cdot\big|\mathcal{F}^B_t\right] \sim \mathbb{P}\left[G_I \in \cdot\right]$ almost surely $t\le T$ with density $\tilde{p}^g_t = u(t,X_t,g)/u(0,X_0,g)$. Furthermore, it follows that for each fixed g, $u(\cdot, g)$ is strictly positive and solves

$$u_t + Lu = 0 \text{ on } (0,T)\times E; \qquad u(T,\cdot) = \hat{p}_I(g-\cdot) \text{ on } E, \tag{B.3}$$

where $L := (1/2)\sum_{i,j=1}^d A^{ij}D^2_{ij} + \sum_{i=1}^d b^i D_i$. As $u(T,x,g) = \hat{p}_{C_I}(g-x)$, calculation shows $\tilde{p}^g_T e^{-\gamma\Pi'\Psi(X_T)}u(0,X_0,g) = \hat{p}_{C_I}(g)e^{-\gamma\Theta(X_T,g)}$, which implies $\mathbb{E}\left[\tilde{p}^g_T e^{-\gamma\Pi'\Psi(X_T)}\right] < \infty$ and $\check{Z}^g_T = \tilde{p}^g_T e^{-\gamma\Pi'\Psi(X_T)}/\mathbb{E}\left[\tilde{p}^g_T e^{-\gamma\Pi'\Psi(X_T)}\right]$. Using this, we see

$$\frac{\mathbb{E}\left[e^{-\gamma\Pi'\Psi(X_T)}\big|\mathcal{F}^m_0\right]}{\left(\mathbb{E}\left[\tilde{p}^g_T e^{-\gamma\Pi'\Psi(X_T)}\right]\right)\big|_{g=G_I}} = \mathbb{E}\left[\frac{\check{Z}^{G_I}_T}{\tilde{p}^{G_I}_T}\big| F^m_0\right] = \int_{\mathbb{R}^d}\mathbb{E}\left[\check{Z}^g_T\right]u(0,X_0,g)dg = 1.$$

Therefore, $\mathbb{Q}^{G_I}$ from (3.1) satisfies

$$\frac{d\mathbb{Q}^{G_I}}{d\mathbb{P}} = \frac{e^{-\gamma\Pi'\Psi(X_T)}}{\mathbb{E}\left[e^{-\gamma\Pi'\Psi(X_T)}\big|\mathcal{F}^m_0\right]} = \frac{\check{Z}^{G_I}_T}{\tilde{p}^{G_I}_T}. \tag{B.4}$$

Lemma {2.1} and Corollary {4.7} imply $1/\tilde{p}^{G_I}_T = \mathcal{E}\left(-\int_0^\cdot(\tilde{\mu}^{G_I}_t)'dB^m_t\right)_T$ and $\check{Z}^{G_I}_T = \mathcal{E}\left(-\int_0^\cdot(\check{\nu}^{G_I}_u)'dB_u\right)_T$. Therefore, with $\nu^{G_I} = \tilde{\mu}^{G_I} + \check{\nu}^{G_I}$ and using $dB^m_t = dB_t - \tilde{\mu}^{G_I}_t dt$ we see that $d\mathbb{Q}^{G_I}/d\mathbb{P} = \mathcal{E}\left(-\int_0^\cdot(\nu^{G_I}_u)'dB^m_u\right)_T$ and $B^m_\cdot + \int_0^\cdot \nu^{G_I}_u du$ is a $\mathbb{Q}^{G_I}$ Brownian motion. Turning to S^{G_I}, part (d) of Assumption 2.1 implies $\mathbb{E}^{\mathbb{Q}^{G_I}}\left[|\Psi(X_T)|\right] \le K_1(1 + \mathbb{E}^{\mathbb{Q}^{G_I}}\left[|X_T|^2\right])$. Also

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{G_I}}\left[|X_T|^2\right] &= \mathbb{E}\left[|X_T|^2\frac{e^{-\gamma\Pi'\Psi(X_T)}}{\mathbb{E}\left[e^{-\gamma\Pi'\Psi(X_T)}\big|\mathcal{F}^m_0\right]}\right] = \mathbb{E}\left[|X_T|^2\frac{e^{-\gamma\Theta(X_T,G_I)}}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,g)}\right]\big|_{g=G_I}}\frac{1}{\tilde{p}^{G_I}_T}\right]\\ &= \int_{\mathbb{R}^d}\frac{\mathbb{E}\left[|X_T|^2 e^{-\gamma\Theta(X_T,g)}\right]}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,g)}\right]}u(0,X_0,g)dg \le \hat{C}\left(\mathbb{E}\left[e^{\varepsilon_0|X_T|}\right] + \mathbb{E}\left[|G_I|^2\right]\right).\end{aligned}$$

The last inequality uses Lemma {1.3}. As $G_I = X_T + Y_I$, finiteness of the last expression follows from Assumption 2.1, part (c). With this, $S^{G_I}_\cdot := \mathbb{E}^{\mathbb{Q}^{G_I}}\left[\Psi(X_T)\big|\mathcal{F}^m_\cdot\right]$ is well defined. As Lemma {2.1} shows predictable representation holds for $(\widetilde{\mathbb{P}}^{G_I}, \mathbb{F}^m)$ local martingales with respect to B, we know it holds for $(\mathbb{Q}^{G_I}, \mathbb{F}^m)$ martingales with respect to $B^m_\cdot + \int_0^\cdot \nu_u^{G_I} du$ (c.f. (Amendinger, 2000, Theorem 4.6)). Therefore, there is an $\mathbb{F}^m$ predictable process σ^{G_I} such that $\Psi(X_T) = \mathbb{E}^{\mathbb{Q}^{G_I}}\left[\Psi(X_T)\middle|\mathcal{F}_0^m\right] + \int_0^T \sigma_t^{G_I}\left(dB_t^m + \nu_t^{G_I} dt\right)$. This gives the dynamics for S. As for the relative entropy, using (B.4)

$$\mathbb{E}\left[\frac{d\mathbb{Q}^{G_I}}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}^{G_I}}{d\mathbb{P}}\right)\right] = \int_{\mathbb{R}^d}\mathbb{E}\left[\check{Z}_T^g \log\left(\frac{\check{Z}_T^g}{\tilde{p}_T^g}\right)\right] u(0, X_0, g)dg;$$
$$\leq \int_{\mathbb{R}^d}\left(\mathbb{E}\left[\check{Z}_T^g\left(\log(\check{Z}_T^g) + (\log(\hat{p}_{C_I}(g - X_T)))^-\right)\right] + (\log(u(0, X_0, g)))^+\right) u(0, X_0, g)dg.$$

At this point, proving finiteness involves essentially the same calculations as in the PCE, and the interest of avoiding redundancies, we defer the PCE case: see {(3.10)}, {(3.8)} and the proof of Theorem 4.3 in Appendix D.

It remains to prove the equilibrium is a REE when either $\Psi(x) = x$, or in the one dimensional case when Ψ is strictly monotone. This clearly follows if one can recover G_I from $S_0^{G_I}$. First, consider $\Psi(x) = x$. By definition of S^{G_I},

$$\begin{aligned} S_0^{G_I} &= \frac{\mathbb{E}\left[X_T e^{-\gamma\Pi'\Psi(X_T)}\middle|\mathcal{F}_0^m\right]}{\mathbb{E}\left[e^{-\gamma\Pi'\Psi(X_T)}\middle|\mathcal{F}_0^m\right]} = \left(\frac{\mathbb{E}\left[X_T e^{-\gamma\Theta(X_T,g)}\right]}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,g)}\right]}\right)\Bigg|_{g=G_I}; \\ &= \nabla_\tau\phi(\tau)\big|_{\tau=C_I^{-1}G_I-\gamma\Pi}; \qquad \phi(\tau) := \log\left(\mathbb{E}\left[e^{\tau' X_T - \frac{1}{2}X_T' C_I^{-1} X_T}\right]\right). \end{aligned} \tag{B.5}$$

Above, the first equality follows from (B.4); the second from (B.1), (B.4) and the analog of (5.3) in the PCE case; and the third as $\Psi(x) = x$. Now, assume for $g_1 \neq g_2$ we have $v = S_0^{g_1} = S_0^{g_2}$. Then, with $\tau_i = (1/T)C_I^{-1}g_i - \gamma\Pi, i = 1, 2$ we see that τ_1, τ_2 are minimizers of the strictly convex function $\tau \to \phi(\tau) - \tau' v$. This contradiction proves the result. Next, suppose $d = 1$ and Ψ is strictly monotone. On $\{G_I = g\}$

$$\partial_g\left(S_0^g\right) = \frac{1}{C_I}\mathrm{Cov}^{\hat{\mathbb{P}}^g}\left(X_T, \Psi(X_T)\right); \qquad \frac{d\hat{\mathbb{P}}^g}{d\mathbb{P}} = \frac{e^{-\gamma\Pi'\Psi(X_T) - \frac{1}{2}X_T'C_I^{-1}X_T + X_T'C_I^{-1}g}}{\mathbb{E}\left[e^{-\gamma\Pi'\Psi(X_T) - \frac{1}{2}X_T'C_I^{-1}X_T + X_T'C_I^{-1}g}\right]}.$$

The strict monotonicity of $g \to S_0^g$ follows. Indeed, consider any random variable Ξ under any measure $\hat{\mathbb{P}}$. Since for all y, z, $(\Psi(y) - \Psi(z))(y - z) > 0$, by taking Ξ and $\tilde{\Xi}$ to be independent copies of Ξ; using the pointwise inequality at $y = \Xi$ and $z = \tilde{\Xi}$; and taking expectations under $\hat{\mathbb{P}}$ we see $\mathrm{Cov}^{\tilde{\mathbb{P}}}(\Xi, \Psi(\Xi)) > 0$. □

Appendix C. On Theorem 7.3

The proof of Theorem 7.3 is given in Appendix {3}. Presently we develop the necessary notation to state Assumption C.1. Throughout, Assumptions A.1, 7.1, 7.2 are in force. Recall ℓ, u in (4.5) (and (7.2), (B.2)) and note that $u(0, X_0, \cdot)$ is the pdf of G_I and $\ell(0, X_0, \cdot)$ is the pdf of H. Next, recall that $G_N = \tau_N G_I + Y_T^N$ and that Y_T^N is independent of G_I with pdf $\hat{p}_N$. Therefore, by convolution, G_N has pdf

$$u_N(g) = \int_{\mathbb{R}^d} u(0, X_0, z)\hat{p}_N(g - \tau_N z)dz. \tag{C.1}$$

Next, Lemma {3.1} proves $\mathbb{P}\left[H \in \cdot \middle| \mathcal{F}_t^B\right] \sim \mathbb{P}\left[H \in \cdot\right]$ almost surely on $t \leq T$, and identifies the density p^h and drift function μ^h as in (7.3). The martingale preserving measure $\widetilde{\mathbb{P}}^H$ is defined in (7.4), and for ease of notation, we will write $\widetilde{\mathbb{P}}$ for $\widetilde{\mathbb{P}}^H$ and $\widetilde{\mathbb{E}}$ for $\mathbb{E}^{\widetilde{\mathbb{P}}^H}$ respectively. By construction, B is a $(\widetilde{\mathbb{P}}, \mathbb{F}^m)$ Brownian motion. For $\check{\nu}$, we recall the measure $\check{\mathbb{Q}}$ and density $\check{Z}_T$ from (7.5), and extend $\check{Z}$ to earlier times

$$\check{Z}_\cdot := \mathcal{E}\left(-\int_0^\cdot \check{\nu}_t' dB_t\right)_\cdot = \frac{d\check{\mathbb{Q}}}{d\mathbb{P}}\Big|_{\mathcal{F}_\cdot^m}. \tag{C.2}$$

As Lemma {3.1} implies the PRP for B, provided $\check{Z}$ is a Martingale, the PRP holds for $B_\cdot + \int_0^\cdot \check{\nu}_t dt$ and $(\check{\mathbb{Q}}, \mathbb{F}^m)$ local martingales. With everything set up, we assume

Assumption C.1. $\check{\nu} \in \mathcal{P}(\mathbb{F}^m)$ is such that $\int_0^T \check{\nu}_u' \check{\nu}_u du < \infty$, and

(1) $(\check{\mathbb{Q}} \in \tilde{\mathcal{M}}^m)$ $\check{Z}$ from (C.2) is a $(\widetilde{\mathbb{P}}, \mathbb{F}^m)$ martingale with

$$\widetilde{\mathbb{E}}\left[\check{Z}_T \log\left(\check{Z}_T\right)\right] < \infty. \tag{C.3}$$

(2) (Non-Degeneracy) $\mathbb{E}^{\check{\mathbb{Q}}}\left[|\Psi(X_T)|\right] < \infty$. For the resultant $\mathbb{F}^m$ predictable, matrix valued process σ yielding

$$\mathbb{E}^{\check{\mathbb{Q}}}\left[\Psi(X_T)\middle|\mathcal{F}_t^m\right] = \mathbb{E}^{\check{\mathbb{Q}}}\left[\Psi(X_T)\middle|\mathcal{F}_0^m\right] + \int_0^t \sigma_u\left(dB_u + \check{\nu}_u du\right); \quad t \leq T,$$

σ is $\mathsf{Leb}_{[0,T]} \times \mathbb{P}$ almost surely of full rank.

(3) (Integrability) The following integrability conditions hold:

$$\begin{aligned}
&(a) \quad \widetilde{\mathbb{E}}\left[\check{Z}_T\left(\log\left(\ell(T, X_T, H)\right)\right)^-\right] < \infty;\\
&(b) \quad \widetilde{\mathbb{E}}\left[\check{Z}_T\left(\log\left(\hat{p}_I(g - X_T)\right)\right)^-\right] < \infty; \qquad \forall\, g \in \mathbb{R}^d,\\
&(c) \quad \int_{\mathbb{R}^d} \widetilde{\mathbb{E}}\left[\check{Z}_T\left(\log\left(\hat{p}_I(g - X_T)\right)\right)^-\right](u(0, X_0, g) + u_N(g))dg < \infty,\\
&(d) \quad \int_{\mathbb{R}^d} \widetilde{\mathbb{E}}\left[\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\left(\log\left(\hat{p}_I(\tilde{g} - X_T)\right)\right)^- \middle| \mathcal{F}_0^m\right]\right)\Big|_{\tilde{g} = G(g,H)}\right] u(0, X_0, g)dg < \infty.
\end{aligned} \tag{C.4}$$

Remark C.2. The integrability assumptions in (3) ensure the optimal wealth processes, for the respective agents, are martingales under the insider pricing measure $\check{\mathbb{Q}}^I$ on $\mathcal{F}^I_T$. The first is for the uninformed trader; the second and third are for the insider; and the third and fourth are for the noise trader.

Appendix D. Proof of Theorem 4.3

Throughout, Assumptions 2.1 and 4.2 are assumed to hold. The communicated signal is H from (5.1) and the insider and noise trader signals are given in (2.5) and (2.7). In light of Sections 7.1, 7.2, Theorem 4.3 will follow provided Assumption C.1 holds. Below, $\check{K}$ is a bounding constant, which might change from line to line.

Proof of Theorem 4.3. Fix $h \in \mathbb{R}^d$. By part (1) of Lemma {1.1}, $\mathbb{E}\left[e^{-\gamma\Theta(X_T,h)}\right] < \infty$. Thus, there is $\check{\nu}^h \in \mathcal{P}(\mathbb{F}^B)$ such that $e^{-\gamma\Theta(X_T,h)}/\mathbb{E}\left[e^{-\gamma\Theta(X_T,h)}\right] = \mathcal{E}\left(-\int_0^\cdot (\check{\nu}^h_t)' dB_t\right)_T$. Using Proposition {4.6} and part (2) of Lemma {4.8} we conclude

$$\frac{e^{-\gamma\Theta(X_T,H)}}{\widetilde{\mathbb{E}}\left[e^{-\gamma\Theta(X_T,H)}\middle|\mathcal{F}^m_0\right]} = \mathcal{E}\left(-\int_0^\cdot (\check{\nu}^H_t)' dB_t\right)_T. \tag{D.1}$$

Thus, $\check{Z}$ is a $(\widetilde{\mathbb{P}}, \mathbb{F}^m)$-local martingale with $\widetilde{\mathbb{E}}\left[\check{Z}_T\middle|\mathcal{F}^m_0\right] = 1$ almost surely; hence $(\widetilde{\mathbb{P}}, \mathbb{F}^m)$-martingale. We now verify Assumption C.1. For item (1), using Lemma {4.8}, (D.1), and $H \sim \ell(0, X_0, \cdot)$

$$\widetilde{\mathbb{E}}\left[\check{Z}_T \log\left(\check{Z}_T\right)\right] = \int_{\mathbb{R}^d} \ell(0, X_0, h)\left(\frac{\mathbb{E}\left[-\gamma\Theta(X_T,h)e^{-\gamma\Theta(X_T,h)}\right]}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,h)}\right]} - \log\left(\mathbb{E}\left[e^{-\gamma\Theta(X_T,h)}\right]\right)\right) dh.$$

Using first part (1) of Lemma {1.1} and Jensen's inequality, and second part (2) of Lemma {1.1 and {(1.2)} we conclude

$$\begin{aligned}\widetilde{\mathbb{E}}\left[\check{Z}_T \log\left(\check{Z}_T\right)\right] &\le \check{K}\int_{\mathbb{R}^d} \ell(0, X_0, h)\left((1 + h'h) + \mathbb{E}\left[|\Theta(X_T, h)|\right]\right) dh;\\ &\le \check{K}\int_{\mathbb{R}^d} \ell(0, X_0, h)\left(1 + \mathbb{E}\left[e^{\varepsilon_0|X_T|}\right] + 2h'h\right) dh.\end{aligned}$$

The formula for H in (5.1), Assumption 2.1 and {(1.2)} imply

$$\int_{\mathbb{R}^d} \ell(0, X_0, h) h'h dh = \mathbb{E}\left[H'H\right] \le \check{K}\mathbb{E}\left[|X_T|^2 + |Y_I|^2 + |Y_N|^2\right] < \infty. \tag{D.2}$$

The finiteness of the relative entropy follows. Turning to (2), that σ is of full rank follows from Assumption 4.2, Lemma {1.2}, and (Kramkov and Pulido, 2019, Theorem

3.1). As for the integrability statement, we deduce from Lemma {4.8}, $H \sim \ell(0, X_0, \cdot)$, Assumption 2.1, Lemma {1.3}, and (D.2) that

$$\begin{aligned} \mathbb{E}^{\check{\mathbb{Q}}}\left[|\Psi(X_T)|\right] &= \int_{\mathbb{R}^d} \ell(0, X_0, h) \frac{\mathbb{E}\left[|\Psi(X_T)| e^{-\gamma\Theta(X_T,h)}\right]}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,h)}\right]} dh \\ &\leq \check{K}\left(1 + \mathbb{E}\left[e^{\varepsilon_0|X_T|}\right] + \int_{\mathbb{R}^d} \ell(0, X_0, h) h'h dh\right) < \infty. \end{aligned}$$

We next prove (3), recalling that $\hat{p}_C$ from (7.8). (4.3) and (7.11) imply for $x \in E$ and $g, h \in \mathbb{R}^d$ that

$$(\log(\hat{p}_{C_I}(g - x)))^- \leq \check{K}\left(1 + x'x + g'g\right); \quad (\log(\ell(T, x, h)))^- \leq \check{K}\left(1 + x'x + h'h\right).$$

Thus, from (D.1), Lemmas {1.1}, {4.8} and $\widetilde{\mathbb{E}}\left[\check{Z}_T \middle| \mathcal{F}_0^m\right] = \widetilde{\mathbb{E}}\left[\check{Z}_T \middle| H\right] = 1$

$$\widetilde{\mathbb{E}}\left[\check{Z}_T\left(\log(\ell(T, X_T, H))\right)^-\right] \leq \check{K}\left(1 + \widetilde{\mathbb{E}}\left[H'H\right] + \int_{\mathbb{R}^d} \ell(0, X_0, h) \frac{\mathbb{E}\left[X_T' X_T e^{-\gamma\Theta(X_T,h)}\right]}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,h)}\right]} dh\right).$$

As $\widetilde{\mathbb{P}} = \mathbb{P}$ on $\mathfrak{s}(H)$, from Lemma {1.3} and (D.2) we deduce (C.4)(a). Continuing, similar calculations as above give

$$\widetilde{\mathbb{E}}\left[\check{Z}_T\left(\log(\hat{p}_{C_I}(g - X_T))\right)^-\right] \leq \check{K}\left(1 + g'g + \mathbb{E}\left[e^{\varepsilon_0|X_T|}\right] + \widetilde{\mathbb{E}}\left[H'H\right]\right).$$

From here, (D.2) implies (C.4)(b). (C.4)(c) (for $u(0, X_0, g)$) holds as $\int_{\mathbb{R}^d} g'gu(0, X_0, g)dg = \mathbb{E}\left[G_I' G_I\right] = \mathbb{E}\left[|X_T + Y_I|^2\right] < \infty$ in view of Assumption 2.1. Similarly, (C.4)(c) (for $u_N(g)$) holds as $G_N = \tau_N G_I + Y_N$ implies $\mathbb{E}\left[G_N' G_N\right] < \infty$. Lastly, to establish (C.4)(d) and finish the proof, first note for H from (5.1) we have $G(g, h) = (1/\alpha_N)((\alpha_I + \alpha_N)h - \alpha_I g)$. Thus, using calculations similar to those above

$$\begin{aligned} &\int_{\mathbb{R}^d} u(0, X_0, g)\widetilde{\mathbb{E}}\left[\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\left(\log(\hat{p}_{C_I}(\tilde{g} - X_T))\right)^- \middle| \mathcal{F}_0^m\right]\right)\Big|_{\tilde{g}=G(g,H)}\right] dg \\ &\quad \leq \check{K}\int_{\mathbb{R}^d} u(0, X_0, g)\widetilde{\mathbb{E}}\left[\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\left(1 + X_T'X_T + \tilde{g}'\tilde{g}\right) \middle| \mathcal{F}_0^m\right]\right)\Big|_{\tilde{g}=G(g,H)}\right] dg \\ &\quad = \check{K}\int_{\mathbb{R}^d} u(0, X_0, g)\left(1 + \widetilde{\mathbb{E}}\left[G(g, H)'G(g, H)\right] + \widetilde{\mathbb{E}}\left[\check{Z}_T X_T' X_T\right]\right) dg \\ &\quad \leq \check{K}\left(1 + \mathbb{E}\left[H'H\right] + \mathbb{E}\left[G_I' G_I\right] + \mathbb{E}\left[e^{\varepsilon_0|X_T|}\right]\right) < \infty. \end{aligned}$$

□

Appendix E. Proofs from Section 5

Proof of Proposition 5.6. To ease notation we remove the superscript "H" from all quantities throughout. First, we formally identify π^{vv} as the replicating strategy (which exists by completeness) for the terminal payoff $(1/2)\int_0^T |\nu_t|^2 dt$. Next, we prove (5.19), where k below is an $\mathcal{F}_0^m$ measurable random variable which might change from line to line. From the first order conditions for optimality of the uninformed trader, along with the fund decomposition in Proposition 5.5 we deduce (c.f. (5.17))

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(-\int_0^{\cdot} \nu_t' dB_t^m\right)_T = ke^{-\gamma_U \mathcal{W}_T^{\hat{\pi}^U}} = ke^{-\gamma\mathcal{W}_T^m - (\alpha_I+\alpha_N)\gamma\mathcal{W}_T^{mf}},$$

where we have set $\mathcal{W}_\cdot^j := \int_0^\cdot (\pi_t^j)'\sigma_t dB_t^{\mathbb{Q}}$ for $j \in \{m, mf, vv\}$, and where $B_\cdot^{\mathbb{Q}} = B_\cdot^m - \int_0^\cdot \nu_t dt$ is a $\mathbb{Q}$ Brownian motion with $dS_t = \sigma_t dB_t^{\mathbb{Q}}$. Taking logarithms, and using the definition of π^{vv} we deduce (using differential notation, and recalling (5.18)) that up to a $\mathcal{F}_0^m$ measurable quantity $\nu_t'(\sigma_t)^{-1}dS_t = \gamma(\pi_t^m)'dS_t + \gamma(\alpha_I+\alpha_N)(\pi_t^{mf})'dS_t + (\pi_t^{vv})'dS_t$. By martingale representation uniqueness, (5.19) follows. We now identify the β's from (5.20) by showing for $i = 1, \dots, d$ that $\beta^j, j \in \{m, mf, vv\}$ solve the linear system

$$\text{(E.1)}\qquad \begin{pmatrix} (\pi^m)'\Sigma\pi^m & (\pi^{mf})'\Sigma\pi^m & (\pi^{vv})'\Sigma\pi^m \\ (\pi^m)'\Sigma\pi^{mf} & (\pi^{mf})'\Sigma\pi^{mf} & (\pi^{vv})'\Sigma\pi^{mf} \\ (\pi^m)'\Sigma\pi^{vv} & (\pi^{mf})'\Sigma\pi^{vv} & (\pi^{vv})'\Sigma\pi^{vv} \end{pmatrix} \begin{pmatrix} (\beta^m)^i \\ (\beta^{mf})^i \\ (\beta^{vv})^i \end{pmatrix} = \begin{pmatrix} (\Sigma\pi^m)^i \\ (\Sigma\pi^{mf})^i \\ (\Sigma\pi^{vv})^i \end{pmatrix},$$

where $\Sigma = \sigma\sigma'$. First, it is easy to show for (t,ω) fixed, that (E.1) always admits a solution, which is unique provided $\left\{\pi^m, \pi^{mf}, \pi^{vv}\right\}$ are linearly independent. Furthermore, for any solution to (5.19), if one defines R by (5.20), then R is orthogonal to $\mathcal{W}^j, j \in \{m, mf, vv\}$. Lastly, write (E.1) as $\mathbf{A}\beta^i = \mathbf{C}^i$ for $i = 1, ..., d$. Calculation shows the dt terms in the Itô process decomposition for R^i are

$$\begin{aligned}(\sigma_t\nu_t)^i &- (\pi_t^m)'\sigma_t\nu_t(\beta_t^m)^i - (\pi_t^{mf})'\sigma_t\nu_t(\beta_t^{mf})^i - (\pi_t^{vv})'\sigma_t\nu_t(\beta_t^{vv})^i \\ &= (\gamma, \gamma(\alpha_I+\alpha_N), 1)\left(\mathbf{C}^i - \mathbf{A}\beta^i\right) = 0.\end{aligned}$$

The $(\mathbb{P}, \mathbb{F}^m)$ and $(\mathbb{Q}, \mathbb{F}^m)$ martingale properties readily follow, giving the result.

□

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