PRESENT BIAS

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ABSTRACT. Present bias is the inclination to prefer a smaller present reward to a larger later reward, but reversing this preference when both rewards are equally delayed. Such behavior violates stationarity of temporal choices, and hence exponential discounting. This paper provides a weakening of the stationarity axiom that can accommodate present-biased choice reversals. We call this new behavioral postulate *Weak Present Bias* and characterize the general class of utility functions that is consistent with it. We show that present-biased preferences can be represented as those of a decision maker who makes her choices according to conservative present-equivalents, in the face of uncertainty about future tastes.

Exponential discounting is extensively used in economics to study the trade-offs between alternatives that are obtained at different points in time. Under exponential discounting, the relative preference for early versus later rewards depends only on the temporal distance between the rewards (a property called Stationarity). However, experiments have shown that smaller immediate rewards are often preferred to later larger rewards, but this preference is reversed when the rewards are equally delayed. As an example, consider the following two choices:

Example 1.

A. \$100 today *vs* B. \$110 in a week

C. \$100 in 4 weeks *vs* D. \$110 in 5 weeks

Subjects in choice experiments often choose A over B, and D over C. This behavior extends to the domain of primary rewards, as shown by the following

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choice pattern exhibited by thirsty subjects in an experiment by McClure et al. (2007):

$$\text{A. Juice now} \quad vs \quad \text{B. Twice the amount of juice in 5 minutes}$$
$$\text{C. Juice in 20 minutes} \quad vs \quad \text{D. Twice the amount of juice in 25 minutes}$$

60% of all decision makers chose A over B, but only 30% chose C over D: the prospect of getting twice the amount of juice at the cost of waiting five more minutes became relatively less attractive when the earlier alternative was in the present. This specific pattern of choice reversal can be attributed to a bias we might have towards alternatives in the present, and hence is aptly called present bias or immediacy effect. This is a well documented time preference anomaly (Thaler 1981; Loewenstein and Prelec 1992; Frederick et al. 2002), replicated in the domains of both primary and monetary rewards.[1]

In Table 2 in Appendix I, we summarize the utility representations on the domain of temporal rewards (x, t) that can accommodate present-bias. Each model in the table assumes present bias, time-monotonicity *and additional conditions on temporal behavior idiosyncratic to the model*. For the purpose of illustration, let us compare two popular present-biased functional forms: Quasi-Hyperbolic (or β-δ) Discounting and Hyperbolic Discounting, in terms of additional behavioral implications they contain. Quasi-Hyperbolic Discounting additionally implies Quasi-stationarity: that violations of exponential discounting happen *only* when the present period ($t = 0$) is involved and the decision maker (DM)'s relative impatience between any two consecutive periods t and $t+1$ is constant for all $t > 0$. On the other hand, Hyperbolic Discounting implies that the relative impatience between two consecutive periods t and $t + 1$ decreases with t.

We provide a new axiom that accommodates present-biased behavior and is implied by all the models in Table 2 despite their idiosyncracies. We call this the Weak Present Bias axiom. The following example illustrates the behavioral pattern captured by Weak Present Bias.

[1] If preferences are stable across decision-times and the decision-makers are unable to ward against the behavior of their future selves, the same phenomenon creates dynamic inconsistency in temporal choices.

Example 2. Suppose that a DM in Example 1 instead chooses (B) \$110 in a week, over, (A) \$100 today. What can we infer about her choice between (D) \$110 in 5 weeks, versus, (C) \$100 in 4 weeks, if we allow for present-biased preferences?

The choice of B over A (B$\succsim$A in our notation) implies that the DM judges B more favorably than A, i.e, the size-of-the-prize factor (\$110>\$100) dominates a possible immediacy-factor (\$100 is available immediately and the DM might be present-biased) and the early factor (\$100 is available 1 week earlier). Equally delaying both alternatives preserves the direction of this net comparison: the early factor and the size-of-the-prize factor are present as before, but, the already inferior \$100 prize now also loses its potential immediacy-factor. Hence, B$\succsim$A must imply D$\succsim$C even if we allow for present bias[2]. We use this pattern to define a Weak Present Bias axiom, which weakens stationarity by allowing for present-biased reversals but rules out choice reversals inconsistent with present bias.

We characterize the general class of present-biased utility functions that satisfy basic tenets of rationality and our Weak Present Bias axiom. Our behavioral postulates are already assumed in the models from Table 2 and hence, our representation nests all of those models.

In the temporal rewards domain, we show that if a decision maker satisfies Weak Present Bias and some basic postulates of rationality, then, her preferences can be represented by the following utility function (henceforth called the *minimum representation*)

$$V(x,t) = \min{}_{u\in\mathcal{U}} u^{-1}(\delta^t u(x))$$

where $\delta \in (0,1)$ is the discount factor, and $\mathcal{U}$ is a set of continuous and increasing utility functions. The minimum representation can be interpreted as if the DM has not one, but a set $\mathcal{U}$ of potential future tastes or utilities. Each potential future taste suggests a different *present equivalent*[3] for the alternative (x,t). The DM resolves this multiplicity by considering the *minimal present equivalent* as her final utility.

[2]All the utility representations in Table 2 agree on this implication over choices.

[3]*Present equivalent* of an alternative (x,t) is the immediate prize that the DM would consider equivalent to (x,t). For a felicity function u defined on the space of all possible prizes x, and a discount factor of δ, the discounted utility from (x,t) is $\delta^t u(x)$. Hence the corresponding present equivalent is $u^{-1}(\delta^t u(x))$. The present is treated "favorably" in the minimum function: $u^{-1}(\delta^0 u(x)) = u^{-1}(u(x)) = x$ for all $u \in \mathcal{U}$, and hence, $\min{}_{u\in\mathcal{U}}(u^{-1}(\delta^0 u(x))) = x$.

We also extend Weak Present Bias and our representation result to the domain of consumption streams. We show that the preferences of any present-biased individual could be represented by

$$F(x_0, x_1, .., x_T) = \sum_{j=0}^{T} V(\min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^j u(x_j)))$$

as long as the preferences also satisfy coordinate independence. The individual behaves as if she evaluates conservative period-wise present equivalents through a continuous strictly increasing function V, and then additively aggregates across periods.

One could interpret conservative present-equivalents in two ways. A decision-maker, unsure about her future tastes, could be considering the minimum present-equivalent out of caution. Such a decision-maker might stand by her rationale of caution, when later confronted with the analysis of her behavior.[4] Alternatively, the minimum representation could model a decision-maker, who chooses under the whim of an evolutionarily-developed impulsive decision-system[5] that systematically underestimates uncertain future wants. When confronted with her behavior, such a decision-maker might regret not employing higher cognitive effort to make a more judicious choice.[6]

The idea that the uncertainty of future tastes shapes intertemporal choice, goes back to the nineteenth century. Eugen von Böhm-Bawerk, writing in 1889[7], connected the uncertainty of future wants to the human tendency of undervaluing future wants.

> "We systematically undervalue our future wants and also the means that serve to satisfy them...It may be that we possess inadequate

[4]The discussion on how the decision maker might perceive her choices, borrows from Gilboa (2010).

[5]Such an impulsive choice might be created through two decision-making systems suggested in the neuroeconomics literature (see Rangel et al. (2008) for details): The Pavlovian system that activates in evolutionarily "hard-wired" situations, and, the habitual control system that promotes favorable past actions. Sozou (1998) and Dasgupta and Maskin (2005) show that in situations where future consumption is uncertain, choice reversals à la present bias might be evolutionarily advantageous.

[6]Decision makers can deploy cognitive resources to control a prepotent response. They can override the Pavlovian and habitual control system by the the Goal-directed system, which assigns values to actions by careful computation of action-outcome associations and rewards (see Fehr and Rangel (2011)).

[7]See Bohm-Bawerk (1890) archives for reference.

> power to imagine and to abstract, or that we are not willing to put forth the necessary effort, but in any event we limn a more or less incomplete picture of our future wants and especially of the remotely distant ones." (1889, pp. 268-269)

Our representation result formalizes these ideas.

Finally, the minimum representation can also accommodate the stake-dependence of present bias, a choice pattern that most temporal models fail to accommodate. A present-biased decision-maker might expend high cognitive effort to fight off her present bias at large stakes. Her large stake choices would thus satisfy stationarity, whereas she would appear to be present-biased in her choices over smaller stakes (see Halevy 2015 for supporting evidence). We show how our representation can accommodate such choice patterns in Section 5.[8]

The paper is arranged as follows: Section 1 defines our Weak Present Bias axiom and provides the representation theorem for the temporal rewards domain. Section 2 presents the representation result for the domain of consumption streams. Sections 3 and 4 comment on the uniqueness and properties of our model. In Section 4, we show that the minimum representation accounts for present biased behavior without necessarily assuming Separability[9] properties (unlike β-δ or Hyperbolic Discounting, that impose Separability), and in this aspect this paper is connected to the papers of Benhabib et al. (2010) and Noor (2011). One choice pattern that most temporal models fail to accommodate is the stake dependence of present bias: we show how our representation can accommodate such preferences in Section 5. Section 6 extends the result of Section 1 to a domain with potentially non-separable time-risk behavior.[10] We provide an intuition of the proofs in Section 7. Section 8 surveys the literature closely related to this paper. The proofs of the main theorems appear in Appendix III: Proofs.

1. Main Result in The Temporal Rewards Domain

The decision maker under consideration has preferences $\succsim$ defined on pairs of timed alternatives $(x, t) \in \mathbb{X} \times \mathbb{T}$ where the first component x could be a

[8]The non-separability of prize and time within any period's utility plays an integral part here.
[9]That rewards and time are evaluated separately.
[10]In Appendix II: Extensions we show how a benevolent social planner can use insights from non-separable time-risk behavior to improve the welfare of present-biased individuals.

desirable reward (monetary or non-monetary) and the second component t is the time at which the reward is received. Let the time-domain be $\mathbb{T} = \{0, 1, 2, ...\infty\}$ or $T = [0, \infty)$ and the space of prizes be $\mathbb{X} = [0, M]$ for $M > 0$. We impose the following conditions on behavior.

A0: $\succsim$ is complete and transitive.

Completeness and transitivity are standard assumptions in the literature, though one can argue that they are more normative than descriptive in nature. Read (2001), Rubinstein (2003) and Ok and Masatlioglu (2007) discuss instances of present-biased intransitive preferences, and these would fall outside our domain of consideration due to (A0).

A1: CONTINUITY: $\succsim$ is continuous, i.e, the strict upper and lower contour sets of each timed alternative is open w.r.t the product topology.

Continuity is a technical assumption that is generally used to derive the continuity of the utility function over the relevant domain. When, $\mathbb{T} = \mathbb{R}_+$, the standard β-δ model does not satisfy continuity at $t = 0$.[11]

A2: DISCOUNTING: For $t,\ s \in \mathbb{T}$, if $t > s$ then $(x, s) \succ (x, t)$ for $x > 0$ and $(x, s) \sim (x, t)$ for $x = 0$. For all $y > x > 0$, there exists $t \in \mathbb{T}$ such that, $(x, 0) \succsim (y, t)$.

The Discounting axiom has two components. The first part says that the decision maker always prefers any non-zero reward at an earlier date, and the zero reward is equally valuable irrespective of when it is obtained. The second part states that any temporal reward (x, t) converges in desirability to the zero reward, as it is sufficiently delayed.[12]

A3: MONOTONICITY: For all $t \in \mathbb{T}$, $(x, t) \succ (y, t)$ if $x > y$.

The Monotonicity axiom requires that at any point in time, larger rewards are strictly preferred to smaller ones. Finally, in light of Example 2, we formally define our Weak Present Bias axiom below.

[11]Pan et al. (2015) axiomatize a model of Two Stage Exponential (TSE) discounting which incorporates the idea of β-δ discounting while maintaining continuity.

[12]In Appendix I, Claim 1 shows that the second part of A2 is indepndent of the other axioms.

A4: **WEAK PRESENT BIAS:** If $(y,t) \succsim (x,0)$ then, $(y,t+t_1) \succsim (x,t_1)$ for all $x,y \in X$ and $t,t_1 \in \mathbb{T}$.

The Stationarity axiom, used in characterizing Exponential Discounting, is stated below for comparison.

Stationarity: $(y,t_1) \succsim (x,t_2)$ if and only if, $(y,t+t_1) \succsim (x,t+t_2)$ for all $x,y \in X$ and $t,t_1,t_2 \in \mathbb{T}$.

Weak present bias, as defined in the fourth axiom is an intuitive weakening of Stationarity in light of present bias or immediacy effect. It allows for choice reversals if and only if they are consistent with present-bias. Other than the separable discounting models mentioned in Appendix I, this Weak Present Bias axiom is also satisfied by the non-separable models of present bias proposed by Benhabib et al. (2010)[13] and Noor (2011). Now we present our main representation result.

Theorem 1. *The following two statements are equivalent:*

i) The relation $\succsim$ defined on $\mathbb{X} \times \mathbb{T}$ satisfies axioms A0-A4.

ii) For any $\delta \in (0,1)$, there exists a set $\mathcal{U}_\delta$ of monotonically increasing continuous functions such that

$$F(x,t) = \min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^t u(x)) \tag{1}$$

represents the binary relation $\succsim$. The set $\mathcal{U}_\delta$ has the following properties: $u(0)=0$ and $u(M)=1$ for all $u \in \mathcal{U}_\delta$. $F(x,t)$ is continuous.

Proof. We prove the theorem in two parts: first for continuous time, and then for discrete time. See Appendix III: Proofs for details. The property of $u(0)=0$ comes from the first part of A2, and $u(M)=1$ is a normalization. □

For any timed alternative (x,t), $u^{-1}(\delta^t u(x))$ in (1) computes its "present equivalent", the amount in the present which the individual would deem equivalent to (x,t) if u were her utility function. For all present prizes, the present equivalents are trivially equal to the prize itself ($u^{-1}(\delta^0 u(x)) = x \ \forall u$), irrespective of the utility function under consideration, and thus there is no scope or need for prudence. For timed alternatives in the future, as long as $\mathcal{U}$ is not a singleton, the DM chooses the most conservative present equivalent due to the minimum functional,

[13] Benhabib et al. (2010) introduce the stake (y) dependent discount factor

$$\Delta(y,t) = \begin{cases} 1 & t=0 \\ (1-(1-\theta)rt)^{(1-\theta)} - \frac{b}{y} & t>0 \end{cases}$$

thus exhibiting prudence. This is the primary intuition of how this functional form treats the present differently from the future and thus incorporates present bias into it. The following example shows an easy application of the theorem to represent present-biased choices.

Example 3. Consider $\mathcal{U} = \{u_1, u_2\}$, where,

$$\begin{aligned} u_1(x) &= x^a \\ u_2(x) &= 1 - \exp(-bx) \end{aligned}$$

with $a = .99$, $b = .00021$, $\delta = .91$. The minimum representation with respect to this $\mathcal{U}$ would assign the following utilities to the timed alternatives in Example 1.

$$\begin{aligned} V(100, 0) &= \min(100, 100) = 100 \\ V(110, 1) &= \min(100.056, 99.995) = 99.995 \\ V(100, 4) &= \min(68.317, 68.48) = 68.317 \\ V(110, 5) &= \min(68.320, 68.344) = 68.320 \end{aligned}$$

Hence,

$$\begin{aligned} V(100, 0) &> V(110, 1) \\ V(100, 4) &< V(110, 5) \end{aligned}$$

Thus the minimum function with a simple $\mathcal{U}$ can be used to accommodate present biased choice reversals.

Here is how the behavioral axioms contribute to the final representation in Theorem 1: A0 and A1 jointly guarantee the existence of a continuous representation $F(x, t)$. A2 ensures that $F(x, t)$ is strictly decreasing in t for $x > 0$ and $F(0, t)$ is independent of t. A3 further ensures that F is increasing in x. A0-A3 together with Stationarity would imply exponential discounting: $F(x, t) = \delta^t u(x)$.

Now let us apply Theorem 1 to a popular model of present bias, the β-δ model (Phelps and Pollak 1968; Laibson 1997).

Example 4. The β-δ model (defined over discrete time periods) evaluates each alternative (x, t) as

$$U(x, t) = (\beta + (1 - \beta).1_{t=0})\delta^t u(x)$$

where u, δ, β have standard interpretation and $\beta < 1$. The indicator function $1_{t=0}$ takes a value of 1 only if $t = 0$ and is 0 otherwise, thus assigning a special role to the present. Given that the β-δ model satisfies axioms A0-4 included in Theorem 1, any such β-δ representation must have an alternative minimum representation, according to Theorem 1.

Below, we consider the simplest possible β-δ representation with linear felicity function $u(x) = x$, $\mathbb{T} = \{0, 1, 2, ..\}$ and provide the corresponding minimum representation.

Proposition 1. *The β-δ discounted utility representation with $u(x) = x$ has an alternative minimum representation* $\min_{y\in\mathbb{R}_+} u_y^{-1}(\delta^t u_y(x))$ *where $u_y : \mathbb{R} \to \mathbb{R}_+$ for all $y \in \mathbb{R}_+$ and*

$$u_y(x) \quad = \begin{cases} \dfrac{x}{\beta} & \textit{for } x \leq \beta\delta y \\ \delta y + (x - \beta\delta y)\dfrac{1-\delta}{1-\beta\delta} & \textit{for } \beta\delta y < x \leq y \\ x & \textit{for } x > y \end{cases}$$

Proof. Each function $u_y()$ is double-kinked, piece-wise linear, passes through the origin, and is parameterized by the point y where the second kink occurs. For any $y \in \mathbb{R}_+$, $x \leq u_y(x) \leq \dfrac{x}{\beta}$ for all $x \in \mathbb{R}_+$. As u_y is an increasing function, it must be that $x \geq u_y^{-1}(x) \geq \beta x$. Since, $x \leq u_y(x)$, we get $\delta^t u_y(x) \geq \delta^t x$, which implies,

$$u_y^{-1}(\delta^t u_y(x)) \geq u_y^{-1}(\delta^t x) \geq \beta\delta^t x$$

Finally, for $x = y$, $\delta^t u_y(x) = \delta^t x < \delta x$ and, hence, $u_y(\delta^t u_y(x)) = \beta\delta^t x$.

Therefore, $V(x,t) = \min_{y\in\mathbb{R}_+} u_y^{-1}(\delta^t u_y(x)) = (\beta + (1-\beta).1_{t=0})\delta^t x$, which finishes our proof.

We provide a minor extension of the main representation result along the lines of β-δ preferences in Appendix II: Extensions. □

2. Extension to consumption streams

In this section, we extend the representation derived in Section 1 to deterministic consumption streams. Supose, the DM's preferences $\succsim$ are defined over

$[0, M]^{T+1}$, the set of all consumption streams of finite length[14] with $T \geq 2$.[15] Let, $\mathbb{T} = \{0, 1, 2, .., T\}$. We impose the following conditions on behavior.

D0: $\succsim$ is complete and transitive.

D1: CONTINUITY: $\succsim$ is continuous, that is the strict upper and lower contour sets of each consumption stream are open w.r.t the product topology.

Given a consumption stream $\boldsymbol{x} \in [0, M]^{T+1}$, $a \in [0, M]$, and $i \in \mathbb{T}$, let us define $(\boldsymbol{x}_{-i}a)$ as the new consumption stream obtained by replacing the i-th component of stream $\boldsymbol{x}$ by a. Let us define $\boldsymbol{x}_{-i,j}a_i b_j$ as the new consumption stream obtained by replacing the i-th and j-th components of a stream $\boldsymbol{x}$ by a and b respectively. We are writing consumption-vectors in bold-face to distinguish it from one-period consumption. Let $\mathbf{0}$ be the consumption stream of all zero consumptions. We would use $\mathbf{0}_{-s}y$ as notation for the stream $(0, .. \underbrace{y}_{\text{in period } s}, .., 0)$.

D2: DISCOUNTING: If $0 \leq s < t \leq T$, then $\mathbf{0}_{-s}x \succsim \mathbf{0}_{-t}x$ for all $x \geq 0$, with the relation being strict if and only if $x > 0$.

D2 is weak enough to not be at odds with preferences for increasing sequences (Loewenstein and Prelec (1993)).[16] For example in a $T = 2$ world, it allows for

$$(10, 11, 0) \succsim (11, 10, 0)$$

which is not possible under standard exponentially discounted monotonic utility with impatience.

[14]Our representation results are for finite streams, and not for infinite streams. One reason behind this is that the utility for infinite streams is undefined for many of the prominent present biased discounting functions, that the current paper intends to generalize over. For example, if someone wanted to calculate the additive discounted utility $\sum_{t=1}^{\infty} D(t)u(c_t)$ of a constant infinite stream $(c, c, c, ...)$ using a simple hyperbolic discounting function $D(t) = \frac{1}{1+rt}$ with $r = 1$, then one would get a diverging series $u(c) \lim_{T\to\infty} \sum_{t=1}^{T} \frac{1}{1+t}$.

[15]$T \geq 2$ is necessary for preferences to reveal present bias.

[16]The modal subject in the Loewenstein and Prelec (1993) study preferred a free dinner at a French restaurant over that at a Greek restaurant, and preferred a free dinner at the French restaurant earlier rather than later, but then preferred "Dinner at the French restaurant in 2 months and dinner at the Greek restaurant in 1 month" over "Dinner at the Greek restaurant in 2 months and dinner at the French restaurant in 1 month".

Axiom A2 had two parts. The first part was about time-monotonicity, and the second part was about consequences of long delays. D2 only inherits the first part. Claims 2 and 3 in Appendix I show that for finite consumption streams, the behavioral content of a potential second part of D2 is already implied by the other axioms.

D3: MONOTONICITY: For any $(x_0, x_1, ..x_T)$, $(y_0, y_1, ..y_T) \in [0, M]^{T+1}$, $(x_0, x_1, ..x_T) \succsim (y_0, y_1, ..y_T)$ if $x_t \geq y_t$ for all $0 \leq t \leq T$. The preference is strict if at least one of the inequalities $x_t \geq y_t$ is strict.

D4: WEAK PRESENT BIAS: If $\mathbf{0}_{-t}y \succsim \mathbf{0}_{-0}x$ then, $\mathbf{0}_{-(t+t_1)}y \succsim \mathbf{0}_{-t_1}x$ for all $x, y \in \mathbb{X}$ and $t, t_1 \in \mathbb{T}$.

We state the Stationarity axiom in the context of consumption streams, to compare it with the weaker WPB axiom.
Stationarity: $(x_0, x_1.., x_T) \succsim (x_0, y_1.., y_T)$ if and only if, $(x_1.., x_T, x_0) \succsim (y_1.., y_T, x_0)$ for all $x_i, y_i \in X$ and $i \in \mathbb{T}$.

D5: COORDINATE INDEPENDENCE: For all $i \in \mathbb{T}$, $\boldsymbol{x}, \boldsymbol{y} \in [0, M]^{T+1}$ and $a, b \in [0, M]$, one has $(\boldsymbol{x}_{-i}, a) \succsim (\boldsymbol{y}_{-i}, a) \iff (\boldsymbol{x}_{-i}, b) \succsim (\boldsymbol{y}_{-i}, b)$.

Thus, if two alternatives have a common i^{th} coordinate, then the preference between those two alternatives are independent of the value taken by the common coordinate. One could use induction to show that preferences between alternatives are independent to the common values taken by any finite set of common coordinates. Coordinate independence and similar axioms have been previously used in the literature to derive representations that are utility-additive across states or time periods. Such properties are waived or relaxed in non-additive representations, for example, while modeling non-expected utility, habit-formation, or ambiguity aversion.

Theorem 2. *The following two statements are equivalent:*

i) The relation $\succsim$ on $[0, M]^{T+1}$ satisfies properties D0-D5.

ii) For any $\delta \in (0, 1)$, there exists a set $\mathcal{U}_\delta$ of monotonically increasing continuous functions and a continuous strictly increasing function V such that

$$F(x_0, x_1, .., x_T) = \sum_{j=0}^{T} V(\min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^j u(x_j)))$$

represents the binary relation $\succsim$. The set $\mathcal{U}_\delta$ has the following properties: $u(0) = 0$ and $u(M) = 1$ for all $u \in \mathcal{U}_\delta$. $F(.)$ is continuous. V is unique upto affine transformations.

Proof. We provide an intuition for the proof in Section 7. See Appendix III: Proofs for the full proof. □

In our representation, for each period $j \in \mathbb{T}$, conservative period-wise present equivalents are first evaluated through a continuous strictly increasing function V, and then additively aggregated across periods.

The utility is additively separable across periods. But, the utility within a period, $V(\min_{u\in\mathcal{U}_\delta} u^{-1}(\delta^t u(x_t)))$, is not necessarily separable in time and reward. Compare this to the representation $G = \sum_t \delta_t u(x_t)$, where the utility is additively separable across periods, and separable in time and reward within each period. G rules out preferences for increasing sequences, and the within-period separability plays a part in it. For example, for $M > m$, $\boldsymbol{x} \in \mathbb{R}^{T+1}$,$s < t$,

$$G(\boldsymbol{x}_{-s,t}m_sM_t) - G(\boldsymbol{x}_{-s,t}M_sm_t) \quad = \quad (u(m) - u(M))(\delta_s - \delta_t)$$

is negative if we assume monotonicity in rewards and time ($\delta_s > \delta_t$ whenever s pre-dates t). The minimum representation, on the other hand, can accommodate preferences for increasing sequences à la Loewenstein and Prelec (1993).[17]

One might wonder if the use of zero-vector $\mathbf{0}$ as the baseline consumption in D4 comes at a cost of generality. As defined previously, $\boldsymbol{w}_{-i,j}a_ib_j$ is the new consumption stream obtained by replacing the i-th and j-th components of a stream $\boldsymbol{w}$ by a and b respectively. D4 and D5 together imply D4′, a stronger version of D4.

D4′: For any $\boldsymbol{z}, \boldsymbol{w} \in [0, M]^{T+1}$, $x, y \in [0, M]$ and $t, t_1 \in \mathbb{T}$, if $\boldsymbol{w}_{-0,t}0_0y_t \succsim \boldsymbol{w}_{-0,t}x_00_t$ then, $\boldsymbol{z}_{-t_1,(t+t_1)}0_{t_1}y_{t+t_1} \succsim \boldsymbol{z}_{-t_1,(t+t_1)}x_{t_1}0_{t+t_1}$.

D4′ does not use the zero-vector at all. But, a period-consumption of 0 is still useful in D4′ while revealing intertemporal tradeoffs. The attractiveness of any

[17]For example, take $F(x_0, x_1, x_2) = \sum_t u^{-1}(\delta^t u(x_t))$ where, $\delta = .89$ and

$$\begin{aligned} u(x) &= x \text{ if x} \leq 10 \\ &= 10 + (x - 10)\frac{1.1}{.89} \text{ if x} > 10 \end{aligned}$$

$F(10, 11, 0) = 20 > 19.9 = F(11, 10, 0)$.

non-zero consumption changes with delay (see D2). Hence, while comparing the tradeoff between later-larger y and earlier-smaller x in D4′, we use 0 in the paired stream for the paired-periods. That way, we can study the tradeoff from equal delaying of y and x without worrying about any delay-induced tradeoffs in the paired-periods.

3. Uniqueness

The uniqueness results about the sets $\mathcal{U}_\delta$, are formulated for the main representation theorem (Theorem 1) in the temporal rewards domain, but they apply equally to the other representation theorems (for example to Theorem 2). We start with a basic question about the representation in Theorem 1: Could we have come up with an alternative representation result for the same preferences with a non-exponential discounting function being used to calculate the present equivalents? For example, starting with A0-A4 could we have ended up with a representation of the form:

$$V'(x,t) = \min_{u \in \mathcal{U}} u^{-1}(\Delta(t)u(x)) \tag{2}$$

where $\Delta(t) = \frac{1}{1+rt}$? It turns out that, if we start with any $\Delta(t)$ that is not exponential, i.e, $\dfrac{\Delta(t+t_1)}{\Delta(t)} \neq \Delta(t_1)$ for some t, t_1 , there would either exist some binary relation which satisfies all the axioms in this paper, but cannot be represented by the function in (2), or, the representation in (2) with a permissible set of utilities $\mathcal{U}$ would represent preferences which do not satisfy at least one of the axioms in this paper.

To get an intuition for this, note that for any separable utility representation $\Delta(t)u(x)$, the relation $(y,t) \sim (x,0)$ implies $(y,t+t_1) \succ (x,t_1)$ or $(y,t+t_1) \prec (x,t_1)$ depending on $\frac{\Delta(t+t_1)}{\Delta(t)} \gtrless \frac{\Delta(t_1)}{\Delta(0)}$. When $\frac{\Delta(t+t_1)}{\Delta(t)} < \frac{\Delta(t_1)}{\Delta(0)}$, any minimum representation with a singleton $\mathcal{U}$ would simultaneously imply $(y,t) \sim (x,0)$ and $(y,t+t_1) \prec (x,t_1)$, and thus violate WPB. Thus, minimum representation is not consistent with A0-A4 when $\frac{\Delta(t+t_1)}{\Delta(t)} < \frac{\Delta(t_1)}{\Delta(0)}$ for some t, t_1.

When $\frac{\Delta(t+t_1)}{\Delta(t)} > \frac{\Delta(t_1)}{\Delta(0)}$, for all u, we have that $u^{-1}(\Delta(t)u(x)) \geq y$ implies

$$\begin{aligned} & u^{-1}(\Delta(t+t_1)u(x)) > u^{-1}(\Delta(t_1)u(y)) \\ \implies & \min_{u \in \mathcal{U}} u^{-1}(\Delta(t+t_1)u(x)) > \min_{u \in \mathcal{U}} u^{-1}(\Delta(t_1)u(y)) \end{aligned}$$

Thus, any minimum representation necessarily has the property that $(y,t) \sim (x,0)$ implies $(y,t+t_1) \succ (x,t_1)$. Hence, for no $\mathcal{U}$ could we represent a preference order with both $(y,t) \sim (x,0)$ and $(y,t+t_1) \sim (x,t_1)$. Thus, when $\frac{\Delta(t+t_1)}{\Delta(t)} > \frac{\Delta(t_1)}{\Delta(0)}$, the minimum representation loses its flexibility to accommodate stationarity locally, and hence cannot represent the whole class of preferences captured by A0-A4. Stating this result more formally,

Proposition 2. *Given the axioms A0-4, the representation result in (2) is unique in the discount function $\Delta(t) = \delta^t$ used to calculate the present equivalents.*

Proof. See Appendix III: Proofs. $\square$

One of the limitations of our representation is that it does not identify a unique discount factor δ. Fishburn and Rubinstein (1982) impose A0-A3 along with Stationarity on preferences to derive a exponential discounting representation. In their representation, given those conditions on preferences, and given any $\delta \in (0,1)$, there exists a continuous increasing function f such that (x,t) is weakly preferred to (y,s) if and only if $\delta^t u(x) \geq \delta^s u(y)$. They also have the following result: if (u,δ) is a representation for a preference $\succsim$ then so is (v,β) where $\beta \in (0,1)$ and $v = u^{\frac{\log\beta}{\log\delta}}$. A similar result holds for our representations.

Proposition 3. *For $\delta, \alpha \in (0,1)$, if $(\delta, \mathcal{U}_\delta)$ is a representation of $\succsim$, then so is $(\alpha, \mathcal{F}_\alpha)$,where the set $\mathcal{F}_\alpha$ is constructed by the functions $v = u^{\frac{\log\alpha}{\log\delta}}$ for $u \in \mathcal{U}_\delta$.*

The proof of this proposition is elementary, and is hence omitted. In Theorem 3 we provide a representation over a richer domain where the discount factor $\delta \in (0,1)$ is unique.

For the next result, define

$$\mathcal{F} = \{u : [0,M] \to \mathbb{R}_+ : u(0) = 0,\ u \text{ is strictly increasing and continuous}\}$$

We would be considering the topology of compact convergence on the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let $co(A)$ and $\bar{A}$ define the convex hull and closure of a set of functions A with respect to the defined topology, and $\bar{co}(A)$ define the convex closure of the set A.

Proposition 4. *If $\mathcal{U},\mathcal{U}' \subset \mathcal{F}$ are such that $\bar{co}(\mathcal{U}) = \bar{co}(\mathcal{U}')$, and the minimum representation exists for both $\mathcal{U}$, $\mathcal{U}'$, then, $\min_{u\in\mathcal{U}} u^{-1}(\delta^t u(x)) = \min_{u\in\mathcal{U}'} u^{-1}(\delta^t u(x))$.*[18]

Proof. See Appendix III: Proofs. □

Proposition 5. *i) If there exists a concave function $f \in \mathcal{U}$, and if $\mathcal{U}_1$ is the subset of convex functions in $\mathcal{U}$, then $\min_{u\in\mathcal{U}}(u^{-1}(\delta^t u(x))) = \min_{u\in\mathcal{U}\setminus\mathcal{U}_1}(u^{-1}(\delta^t u(x)))$.*

ii) If $u_1, u_2 \in \mathcal{U}$ and u_1 is concave relative to u_2, then, $\min_{u\in\mathcal{U}} u^{-1}(\delta^t u(x)) = \min_{u\in\mathcal{U}\setminus\{u_2\}} u^{-1}(\delta^t u(x))$.

Proof. See Appendix III: Proofs. □

4. Properties of the representation

Suppose two DMs have preferences $\succsim_1$ and $\succsim_2$ that satisfy the axioms in Section 1, and, have minimum representations w.r.t sets $\mathcal{U}_{\delta,1}$ and $\mathcal{U}_{\delta,2}$ respectively, for some $\delta \in (0,1)$. When can we say that one of them has a higher/ lower proclivity for immediate rewards? As immediacy effect in our representation is generated through prudence in evaluating uncertainty over future tastes, increased uncertainty of future tastes (in the set inclusion sense $\mathcal{U}_{\delta,2} \subset \mathcal{U}_{\delta,1}$) should imply higher proclivity for the present. Below we show that set-inclusion of tastes is sufficient, but not necessary for us to be able to compare the proclivity of immediate gratification across DMs.

Consider the following partial relation defined on the set of all preferences $\succsim$ over $\mathbb{X} \times \mathbb{T}$ that satisfy A0-4.

Definition 1. We say that $\succsim_2$ has a lower proclivity for immediate gratification than $\succsim_1$, if DM2 chooses the delayed reward over an immediate reward whenever DM1 does the same, i.e, $\forall x, y \in \mathbb{X}$ and $t \in \mathbb{T}$

$$(x,t) \succsim_1 (y,0) \implies (x,t) \succsim_2 (y,0)$$

To see why this is an incomplete order consider the following example: if $\succsim_1$

[18] This Uniqueness result is weaker than the following class of uniqueness result that has been derived for multi-utility representations in other contexts (for example in Cerreia-Vioglio et al. (2015)): For a preference relation $\succsim$ that under certain postulates gives a multi-utility representation, there exists a minimal set of utilities $\mathcal{U}^*_{\delta,\succsim}$, such that if $\mathcal{U}_\delta$ constitutes a multi-utility representation of $\succsim$ then $\mathcal{U}^*_{\delta,\succsim} \subseteq \bar{co}(\mathcal{U}_\delta)$, and the set $\mathcal{U}^*_{\delta,\succsim}$ is unique up to its closed convex hull.

and $\succsim_2$ are both β-δ preferences, they can only be compared according to the definition above if, (1) $\delta_1 = \delta_2$ but $\beta_1 \leq \beta_2$, or, (2) the $\beta_1 = \beta_2$ and $\delta_1 \leq \delta_2$.

Proposition 6. *Let $\succsim_1$ and $\succsim_2$ be two binary relations which allow for minimum representation w.r.t sets $\mathcal{U}_{\delta,1}$ and $\mathcal{U}_{\delta,2}$ respectively, for some $\delta \in (0,1)$. The following two statements are equivalent:*

i) $\succsim_2$ has a lower proclivity for immediate gratification than $\succsim_1$.

ii) Both $\mathcal{U}_{\delta,1}$ and $\mathcal{U}_{\delta,1} \cup \mathcal{U}_{\delta,2}$ provide minimum representations of $\succsim_1$.

Proof. See Appendix III: Proofs. Note both sets $\mathcal{U}_{\delta,1}$ and $\mathcal{U}_{\delta,2}$ have been considered under the same δ for normalization. □

This result can be used to extend the idea of naivete and sophistication (O'Donoghue and Rabin (1999)) about future preferences for present-biased individuals in a dynamic decision making context. For example, a naive present biased individual might falsely believe that her future "period τ preferences $\succsim_\tau$" are $\min_{u\in\hat{\mathcal{U}}} u^{-1}(\delta^t u(x))$, whereas, they actually are $\min_{u\in\mathcal{U}} u^{-1}(\delta^t u(x))$, where $\hat{\mathcal{U}} \subset \mathcal{U}$.

In Appendix 8, we define Weak Future Bias (can be motivated along the lines of Example 2), and provide a corresponding representation.

A note about Separability in $\mathbb{X} \times \mathbb{T}$:

Formally, the property of Separability (that rewards and time are evaluated separately) could be defined as:

Separability: For all $x_1, x_2, y_1, y_2 \in \mathbb{X}$, and $s_1, s_2, t_1, t_2 \in \mathbb{T}$, if $(x_1, s_1) \sim (x_2, s_2)$, $(y_1, s_1) \sim (y_2, s_2)$ and $(x_1, t_1) \sim (x_2, t_2)$ then $(y_1, t_1) \sim (y_2, t_2)$.

It is easy to show that A0-A3 together with Stationarity imply Separability of preferences.[19] On the other hand, our characterizing axioms A0-A4 do not imply separability. One useful way to confirm this is to check that Noor (2011)'s Magnitude Effect Discounting (MED) model, defined as

$$V(x,t) \quad = \quad (\delta(x))^t u(x) \ \text{ where } \delta() \text{ is increasing in } x$$

(with the standard continuity properties) satisfies all axioms A0-A4 (and hence has an equivalent minimum representation), and is non-separable. Thus this paper provides a representation for WPB without necessarily assuming Separability

[19] Firstly, $(x_1, s_1) \sim (x_2, s_2)$ and $(x_1, t_1) \sim (x_2, t_2)$ together imply that $s_1 - s_2 = t_1 - t_2$,which when taken together with Stationarity and $(y_1, s_1) \sim (y_2, s_2)$, imply $(y_1, t_1) \sim (y_2, t_2)$.

properties (unlike β-δ discounting or Hyperbolic Discounting), and in this aspect this paper is closely related to the papers of Benhabib et al. (2010) and Noor (2011).[20]

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5. Stake Dependent Present Bias

Consider a decision maker who makes the following 2 pairs of choices.

Example 5.

$$
\begin{aligned}
\$100 \text{ today} &\succ \$110 \text{ in a week} \\
\$110 \text{ in 5 weeks} &\succ \$100 \text{ in 4 weeks} \\
\$11 \text{ in a week} &\sim \$10 \text{ today} \\
\$11 \text{ in 5 weeks} &\sim \$10 \text{ in 4 weeks}
\end{aligned}
$$

Both pairs of choices are consistent with Weak Present Bias, but there is a classical choice reversal (or a local violation of Stationarity) only in the first pair.[21] Such a choice pattern is at odds with the models in Appendix I[22], but not necessarily at odds with economic intuition or empirical evidence. Suppose that a DM's present bias is psychologically driven by the fear of unexpected events that might cause a reward to never be delivered to her future self. In such a scenario, the higher the stakes involved (first pair of choices), the higher could be the manifestation of this fear, and the more present-biased the DM might appear in her actions. The opposite phenomenon, when a DM appears strictly present-biased for smaller stakes but appears stationary at larger stakes (for e.g, choice

[20]Imposing the Hexagon/ Separability condition alongside my WPB axiom and other regularity conditions (A0 to A4), leads to preferences also being represented by $V(x,t) = D(t)u(x)$ where for all $k \in \mathbb{T}$, $\arg\min_t \frac{D(t+k)}{D(t)} = 0$, u is continuous strictly increasing, and $D(t)$ is strictly decreasing. The condition on $D()$ is similar to that of Delay Independent Diminishing Imaptience from Chakraborty et al. (2020).

[21]This behavior closely parallels stake-dependence in discounting behavior: in studies that vary the outcome sizes, subjects appear to exhibit greater patience toward larger rewards. For example, Thaler (1981) finds that respondents were on average indifferent between \$15 now and \$60 in a year, \$250 now and \$350 in a year, and \$3000 now and \$4000 in a year, suggesting a (yearly) discount factor of 0.25, 0.71 and 0.75 respectively. Noor (2009) and Noor (2011) explain how curvature of felicity function in an exponentially discounted model is not enough to account for this anomaly.

[22]For example, if one tries to fit a β-δ model to this data, the second pair of choices immediately suggest $\beta = 1$, which in turn is inconsistent with the first pair of choices.

reversal in only the second pair of choices above) could happen, if the subject gets better at fighting off their temporal biases by spending higher cognitive resources at higher stakes. Stake dependence of present bias is an understudied facet of temporal behavior. The closest precedent for an experimental study appears in Halevy (2015) where the author finds evidence of such stake dependence. The minimum function mentioned in Example 3 can account for such choices.

6. An extension to risky prospects

In this section, we extend the representation derived in Section 1 to binary risk. This extension serves the following three goals. First, it shows that the representation in Section 1 has a natural extension to simple binary lotteries, with zero being one of the lottery outcomes. Second, through the extended representation we are able to accommodate experimental evidence that is inconsistent with "risk-time separable" models of behavior. Finally, through this extension, we will be able to identify a unique discount factor δ for the DM.

We start by presenting the experimental evidence that violates time-risk separability. Each alternative offered to the subjects in the experiments was a triplet (x, p, t) where p is the probability with which prize x was attained at time t, the subjects got 0 with the complementary probability. x was offered in Dutch Guilder, t was measured in in weeks.

	Prospect A	Prospect B	% chosing A	% chosing B	N
1	(100,1,0)	(110,1,4)	82%	18%	60
2	(100,1,26)	(110,1,30)	37%	63%	60
3	(100,.5,0)	(110,.5,4)	39%	61%	100
4	(100,.5,26)	(110,.5,30)	33%	67%	100

Table 1. From Keren and Roelofsma (1995).

The data can be interpreted in the following way: People have an affinity for both certainty and immediacy. The loss in either certainty or immediacy has a similar disproportionate effect on behavior (compare rows 2-3 with row 1). Further, this disproportional effect for a loss of immediacy is prominent when all choices are certain. There is little evidence of present-biased reversals over risky prospects (compare rows 1-2, with rows 3-4), which is the key intuition behind non-separability of time and risk.

We will consider preferences over triplets $(x, p, t) \in \mathbb{X} \times \mathbb{P} \times \mathbb{T}$, which describe the prospect of receiving a reward $x \in \mathbb{X}$ at time $t \in \mathbb{T}$ with a probability$p \in [0, 1]$. $\mathbb{X} = [0, M]$ is a positive reward interval, $\mathbb{P} = [0, 1]$ is the unit interval of probability, and $\mathbb{T} = [0, \infty)$ is the time interval. We impose the following conditions on behavior, which closely mirror axioms A0-A4.

B0: $\succsim$ is complete and transitive.

B1: CONTINUITY: $\succsim$ is continuous, that is the strict upper and lower contour sets of each risky timed alternative are open w.r.t the product topology.

B2: DISCOUNTING: For $t, s \in \mathbb{T}$, if $t > s$ then $(x, p, s) \succ (x, p, t)$ for $x, p > 0$ and $(x, p, s) \sim (x, p, t)$ for $x = 0$ or $p = 0$. For $y > x > 0$, there exists $T \in \mathbb{T}$ such that, $(x, q, 0) \succsim (y, 1, T)$.

B3: PRIZE AND RISK MONOTONICITY: For all $t \in \mathbb{T}$, $(x, p, t) \succsim (y, q, t)$ if $x \geq y$ and $p \geq q$. The preference is strict if at least one of the two following inequalities is strict.

B4: WEAK PRESENT BIAS: If $(y, 1, t) \succsim (x, 1, 0)$ then, $(y, 1, t + t_1) \succsim (x, 1, t_1)$ for all $x, y \in \mathbb{X}$, $\alpha \in [0, 1]$ and $t, t_1 \in \mathbb{T}$.

B5: PROBABILITY-TIME TRADEOFF: For all $x, y \in \mathbb{X}$, $p, q, \theta \in (0, 1]$, and $t, s, D \in \mathbb{T}$, $(x, p\theta, t) \succsim (x, p, t + D) \implies (y, q\theta, s) \succsim (y, q, s + D)$.

The B5 axiom (pioneered by Baucells and Heukamp 2012) says that passage of time and introduction of risk have similar effects on behavior, and there is a consistent way in which time and risk can be traded off across the domain of behavior. This axiom additionally implies calibration properties that we will utilize to pin down a unique discount factor δ for any DM. True to the spirit of the experimental findings discussed above, the Weak Present Bias axiom (B4) is only imposed for prizes with certainty.

Additionally, (B4) when combined with (B5) captures a decision maker's joint bias towards certainty as well as the present, i.e, it embeds Weak Present Bias *as well as Weak Certainty Bias*[23] in itself. This underlines the insight that once risk and time can be traded-off, Weak Present Bias and Weak Certainty Bias are behaviorally equivalent. Such relations between time and risk preferences

[23]Weak Certainty Bias can be defined on $\mathbb{X} \times \mathbb{P}$ in the following fashion: If $(y, p) \succsim (x, 1)$ then, $(y, p\alpha) \succsim (x, \alpha)$ for all $x, y \in \mathbb{X}$ and $\alpha \in [0, 1]$.

have been discussed previously in Halevy (2008), Baucells and Heukamp (2012), Saito (2009), Fudenberg and Levine (2011), Epper and Fehr-Duda (2012) and Chakraborty et al. (2020). In Section 7, we will discuss how the Weak Certainty Bias postulate connects the current work to previous literature on risk preferences. We are now ready for our extended representation result.

Theorem 3. *The following two statements are equivalent:*

i) The relation $\succsim$ *on* $\mathbb{X} \times \mathbb{P} \times \mathbb{T}$ *satisfies properties B0-B5.*

ii) There exists a unique $\delta \in (0,1)$ *and a set* $\mathcal{U}$ *of monotonically increasing continuous functions such that* $F(x,p,t) = \min_{u \in \mathcal{U}}(u^{-1}(p\delta^t u(x)))$ *represents the relation* $\succsim$*. For all the functions* $u \in \mathcal{U}$, $u(M) = 1$ *and* $u(0) = 0$*. Moreover,* $F(x,p,t)$ *is continuous.*

Proof. See Appendix III: Proofs. □

The next example shows a potential application of this representation in light of Keren and Roelofsma (1995)'s experimental results.

Example 6. Consider the set of functions $\mathcal{U}$ and parameters considered in Example 3. When applied to the representation derived in Theorem 3, they predict the choice pattern obtained in the original Keren and Roelofsma (1995) experiment. In Appendix II, we discuss a timing game, where non-separable time-risk relations can be used towards welfare improving policy design.

7. An outline of the proofs

This section outlines the proofs of Theorems 1, 2, and 3 chronologically and places the methodology used in the proofs against the context of previous literature.

For Theorem 1, we will focus on the case of $\mathbb{T} \in [0, \infty)$, as it is less technical than the discrete-time case, but conveys the main idea behind the proof nonetheless. For any timed alternative (z, τ), there exists an immediate reward $x \in \mathbb{X}$ such that $(z, \tau) \sim (x, 0)$. This follows from monotonicity, continuity, connectedness of the prize-domain and this guarantees that any (timed) alternative has a well defined present equivalent with respect to $\succsim$. Given the present equivalents with respect to $\succsim$ are well defined, one possible utility representation $V : \mathbb{X} \times \mathbb{T} \to \mathbb{R}_+$ is the function that assigns to every alternative (z, τ), the present equivalent according to the relation $(z, \tau) \sim (x, 0)$. The crux of the remaining proof lies in showing that under Weak Present Bias, we can construct a set of utilities $\mathcal{U}_\delta$ such that the previously defined V function can be rewritten as

$$V(z,\tau) = x \quad = \quad \min_{u\in\mathcal{U}_\delta} u^{-1}(\delta^\tau u(z))$$

The proof is constructive. For any point $x^* \in (0,M)$, we construct an indexed utility-function $u_{x^*}(.)$ in the following steps.
i) Assign $u_{x^*}(0) = 0$, $u_{x^*}(x^*) = 1$.
ii) At any $x \in (x^*, M]$, calibrate the utility according to the temporal relation with x^*. For $x > x^*$, there exists $t > 0$ such that $(x,t) \sim (x^*,0)$. Define, $u_{x^*}(x) = \delta^{-t}$ (for the $\delta \in (0,1)$ under consideration) and re-label x as x_t. Note that x^* would be x_0.
iii) Defining the utility at some $y \in (0,x^*)$ is more involved: one needs to consider x^* and all the $x_t \in (x^*, M]$ points and their relation with y. We define

$$u_{x^*}(y) = \delta^{\max\{\tau:\ (x_t,t+\tau)\sim(y,0) \text{ for some t and } x_t\}}$$

The proof then proceeds to show that the maximum is well defined in step (iii), and the constructed $u_{x^*}()$ is strictly increasing, continuous, and has the following crucial property: If $(z,t) \sim (x,0)$ then, $\delta^t u_{x^*}(z) \geq u_{x^*}(x)$ and subsequently, $u_{x^*}^{-1}(\delta^t u_{x^*}(z)) \geq x$, with the weak inequality replaced by equality if $x = x^*$. The asymmetric construction of $u_{x^*}()$ on the left and right of x^*, especially, the part where points to the left are allocated utility conservatively, is necessary for this to hold.
Next we define the set of functions as $\mathcal{U}_\delta = \{u_{x^*}(.) :\ \ x^* \in (0,M)\}$. It readily follows from the aforementioned property of constructed utility functions that $\min_{u\in\mathcal{U}_\delta} u^{-1}(\delta^t u_x(z)) = x$ whenever $(z,t) \sim (x,0)$. The proof for the same theorem in the discrete-time case gets more technical as steps (ii) and (iii) above cannot be performed anymore, and they have to be replaced by careful interval-wise construction.

The proof of Theorem 2 has two steps. The first step utilizes completeness, transitivity, continuity and coordinate independence to derive an additively separable representation, $\sum_{t\in\mathbb{T}} W_t(c_t) = \sum_{t\in\mathbb{T}} W_0(W_0^{-1}(W_t(c_t)))$ over streams, which can be interpreted as aggregating W_0(present equivalent of consumption in period t) over all the periods t. The second step shows that the present equivalent $W_0^{-1}(W_t(c_t))$ can indeed be written as $\min_{u\in\mathcal{U}_\delta} u^{-1}(\delta^t u(c_t))$, and this requires two substeps. First, one needs to show that the axioms on consumption streams imply behavioral properties similar to A0-A4 (from Theorem 1) on the restricted domain of consumption streams that can be meaningfully interpreted as temporal rewards (i.e, streams that deliver a non-zero reward in no more than one period). Second, one needs to show that a result similar to 1 holds over this restricted domain even when time-periods are discrete and finite. Combining all the steps together lets us write the representation on streams as $\sum_{t\in\mathbb{T}} W_0(\min_{u\in\mathcal{U}_\delta} u^{-1}(\delta^t u(c_t)))$.

Theorem 3 restricts its domain to $\mathbb{T}=\mathbb{R}_+$, unlike Theorem 1. Thus, the proof of this theorem only requires part of the machinery developed in Theorem 1. The postulates on behavior connect time and risk in the following way: Given the Probability-Time Tradeoff axiom, the $\mathbb{X}\times\mathbb{P}\times\mathbb{T}$ domain is isomorphic to either of the reduced domains of $\mathbb{X}\times\mathbb{P}$ or $\mathbb{X}\times\mathbb{T}$. For example, there exists unique $\delta\in(0,1)$ such that $(x,p,t)\sim(x,p\delta^t,0)$ and $(x,p,t)\sim(x,1,t+\log_\delta p)$ for all $(x,p,t)\in\mathbb{X}\times\mathbb{P}\times\mathbb{T}$. The axioms on $\mathbb{X}\times\mathbb{P}\times\mathbb{T}$ domain imply completeness, transitivity, continuity, risk monotonicity (Discounting respectively), Weak Certainty Bias (Weak Present Bias respectively) for a preference defined on the reduced domain of $\mathbb{X}\times\mathbb{P}$ ($\mathbb{X}\times\mathbb{T}$ respectively for $\mathbb{T}=\mathbb{R}_+$). Proving Theorem 3, is equivalent to proving that there is a minimum representation on $\mathbb{X}\times\mathbb{P}$ or $\mathbb{X}\times\mathbb{T}$ of the forms $\min_{u\in\mathcal{U}}(u^{-1}(pu(x)))$ or $\min_{u\in\mathcal{U}}(u^{-1}(\delta^t u(x)))$ respectively. In the Appendix, we show how the reduction from the richer domain to $\mathbb{X}\times\mathbb{P}$ or $\mathbb{X}\times\mathbb{T}$ works, and then prove that a relation on $\mathbb{X}\times\mathbb{P}$ satisfies completeness, transitivity, continuity, risk monotonicity and Weak Certainty Bias *if and only if* the relation on $\mathbb{X}\times\mathbb{P}$ can be represented by the functional form of $\min_{u\in\mathcal{U}}(u^{-1}(pu(x)))$. The proof is similar to that of Theorem 1.

This result on the reduced $\mathbb{X}\times\mathbb{P}$ domain brings us to a very interesting connection that the present work has with Cerreia-Vioglio et al. (2015). In an influential paper, Cerreia-Vioglio et al. (2015) consider preferences over lotteries ($\mathcal{L}$) defined over a real interval of outcomes. To account for violations of the Independence Axiom[24] based on a DM's bias towards certainty or sure prizes[25], they relax it in favor of Negative Certainty Independence (NCI) axiom defined below. This NCI axiom when reduced to the domain of binary lotteries on $\mathbb{X}\times\mathbb{P}$, conveys similar behavior as the Weak Certainty Bias axiom we have discussed above.

NCI: (Dillenberger 2010) For $p,q\in\mathcal{L}$, $x\in[w,b]$, and $\lambda\in(0,1)$,

[24]For $p,q,r\in\mathcal{L}$, and $\lambda\in(0,1)$, $p\trianglerighteq q$ if and only if $\lambda p+(1-\lambda)r\trianglerighteq\lambda q+(1-\lambda)r$.
[25]We denote the lottery that gives the outcome $x\in[w,b]$ for sure as $L_x\in\mathcal{L}$.

$$q \trianglerighteq L_x \implies \lambda p + (1-\lambda)q \trianglerighteq \lambda p + (1-\lambda)L_x$$

Cerreia-Vioglio et al. (2015) show that if $\trianglerighteq$ satisfies NCI and basic rationality postulates, then there exists a set of continuous and strictly increasing functions $\mathcal{W}$, such that the relation $\trianglerighteq$ can be represented by a continuous function $V(p) = \inf_{u \in \mathcal{W}} c(p,u)$, where $c(p,u)$ is the certainty equivalent of the lottery p with respect to $u \in \mathcal{W}$. Our representation over atemporal binary lotteries $\mathbb{X} \times \mathbb{P}$ is a minimum representation that closely parallels the infimum representation obtained by Cerreia-Vioglio et al. (2015). We show that their infimum representation can be replaced with a minimum representation under the axioms in our domain. Our proof is essentially constructive, in part because we cannot use results from convex analysis in our domain, breaking norm from papers that deal with preferences over lotteries.

8. Related Literature

This paper is closely linked to the literature that explores the conditions under which a "rational" person may have present-biased preferences. Sozou (1998), Dasgupta and Maskin (2005) and Halevy (2008) explain particular environments that can give rise to present-biased choices through uncertainty-averse behavior. While telling an uncertainty story *sufficient* to explain present bias, these models explicitly assume the particular structure of risk or uncertainty with a relevant risk attitude, and these assumptions are central to establishing behavior consistent with present bias in these models. We do not explicitly assume any uncertainty framework or uncertainty attitude, but obtain a subjective state space representation that *is necessary and sufficient* for present bias. Our representation also looks similar to the max-min expected utility representation of Gilboa and Schmeidler (1989) used in the uncertainty or ambiguity aversion literature, though there is no explicit connection. There are other variants of the minimum or infimum functional in previous literature, for example, Cerreia-Vioglio (2009) and Maccheroni (2002), used in different contexts.

There is also a sizable literature on the behavioral characterizations of temporal preferences, that this project adds to. Olea and Strzalecki (2014), Hayashi (2003) and Pan et al. (2015) characterize the behavioral conditions necessary and sufficient for β-δ discounting, Loewenstein and Prelec (1992) characterize Hyperbolic discounting, and, Koopmans (1972), Fishburn and Rubinstein (1982) do the same for exponential discounting. Gul and Pesendorfer (2001) study a two-period model where individuals have preferences over sets of alternatives that represent second-period choices. Their axioms provide a representation that identifies the decision maker's commitment ranking, temptation ranking and cost of self-control. Kreps (1979) shows that a preference for flexible menus could be interpreted as if the decision-maker has uncertainty over future tastes and is trying to maximize the expected utility of menu-choice over future tastes. Our paper is also related to Noor et al. (2017), Gabaix and Laibson (2017) and Galperti and Strulovici (2017) that provide explanations for why people exhibit preference reversals.

Wakai (2008) re-interprets Gilboa and Schmeidler (1989)'s multiple prior utility in a temporal context, with states becoming time-periods. The axiomatic representation is a maxmin over the set of discount-weights assigned to present and future utilities. Though the maxmin feature in Wakai (2008) might resemble the minimum representation, it is unrelated to present-bias, and instead related to the decision-maker's attitude towards utility variations between adjacent periods.

Conclusion

This paper provides a behavioral definition of (Weak) Present Bias and characterizes a general class of utility functions consistent with such behavior in both the temporal rewards domain, as well as in the consumption streams domain. Whereas present bias is typically modelled using a discounted utility model with a fixed utility u, this paper shows that it can also be modelled more generally using an uncertain-tastes model. Having a more general representation for present bias, also helps us accommodate empirical anomalies (for example, stake dependent present biased behavior) that previous models could not account for. We hope that this paper generates further interest in theoretical and applied work directed towards forming a better understanding of intertemporal preferences.

Appendix I

MODELS OF PRESENT BIAS

Consider the general separable discounted utility function defined over temporal rewards (x, t),

$$V(x, t) = \Delta(t)u(x)$$

Here, $\Delta(t)$ is the discount factor, and $u(x)$ is the felicity function. Below, we give a brief summary of the literature on different discounting models which accommodate present bias, in terms of the discount functions they propose.

	Model	Author(s)	$\Delta(t)$
1	Quasi-hyperbolic discounting	Phelps and Pollak (1968)	$(\beta + (1-\beta)_{t=0})(1+g)^{-t}, \beta < 1, g > 0$
2	Proportional discounting	Herrnstein (1981)	$(1+gt)^{-1}, g > 0$
3	Power discounting	Harvey (1986)	$(1+t)^{-\alpha}, \alpha > 0$
4	Hyperbolic discounting	Loewenstein and Prelec (1992)	$(1+gt)^{-\alpha/\gamma}, \alpha > 0, g > 0$
5	Constant sensitivity	Ebert and Prelec (2007)	$\exp[-(at)^b], a > 0, 0 < b < 1$

Table 2. Models of temporal behavior

Claim 1. Consider the preference $\succsim$ over $[0,1] \times \{0, 1, ..\infty\}$ represented by

$$U(x,t) \quad = \quad \begin{cases} \frac{\delta^t + 1}{2} x & \text{for } t \geq 1 \\ x & \text{for } t = 0 \end{cases}$$

for $\delta < 1$. $\succsim$ satisfies all of A0-A4 but the second part of A2.

Proof. The composite discount $\frac{\delta^t+1}{2} < 1$ is decreasing in t. $\succsim$ is complete, transitive, continuous, and, monotonic in x and t.

$\succsim$ also satisfies WPB. Take $y > x > 0$ and $t > 0$ with $(y, t) \succsim (x, 0)$.[26] □

[26]When $x = 0$ WPB holds trivially.

$$
\begin{aligned}
(y,t) &\succsim (x,0)\\
\iff \frac{\delta^t+1}{2}y &\geq x\\
\iff \frac{y}{x} \geq \frac{2}{\delta^t+1} &\geq \frac{\delta^{t_1}+1}{\delta^{t+t_1}+1}\\
\iff \frac{\delta^{t+t_1}+1}{2}y &\geq \frac{\delta^{t_1}+1}{2}x\\
(y,t+t_1) \iff &\succsim (x,t_1)
\end{aligned}
$$

The third step follows from the following algebra

$$
\begin{aligned}
\frac{2}{\delta^t+1} &\geq \frac{\delta^{t_1}+1}{\delta^{t+t_1}+1}\\
\iff 2\delta^{t+t_1}+2 &\geq \delta^{t+t_1}+\delta^t+\delta^{t_1}+1\\
\iff \delta^{t+t_1}+1 &\geq \delta^t+\delta^{t_1}\\
\iff (1-\delta^t)(1-\delta^{t_1}) &\geq 0
\end{aligned}
$$

But, $\lim_{t\to\infty} U(x,t) = \frac{x}{2}$ and hence $\succsim$ satisfies all A0-A4 but the second part of the Discounting (A2).

Discussion of D2 axiom:

Our D2 axiom is different from the A2 axiom, because it does not include the following property D2-2 about asymptotic discounting:

D2-2: For $y^0 > x > 0$, and for any $(y^1, y^2, y^3, ..y^m, ..)$ and $(n^1, n^2, .., n^m, ..)$, where $\mathbf{0}_{-0}y^i \sim \mathbf{0}_{-n_i}y^{i-1}$ $\forall i \in \{1, 2, .., m, ..\}$, $0 < n^i \leq T$, there exists $t \in \mathbb{N}$ such that, $y^m \leq x$ whenever $\sum_1^m n^i \geq t$.

In D2-2, we define an iterative process, where y^i is the present equivalent of the consumption y^{i-1}, that is available $n^i > 0$ periods in the future. Due to the added restriction that the DM can only consider time delays of upto T periods, we have approximated arbitrarily long delays by a sequence of delays, none greater than T. D2-2 states that any y_0 keeps falling arbitrarily in present-equivalent value, as one increases the *total discounting* $\sum_1^m n^i$ it is subjected to. D2-2 would have been the natural counterpart of the second part of A2. Claim 3 shows that D2-2 is already implied by the other axioms in a finite-period setting and is hence redundant.

Claim 2. Suppose $\succsim$ defined on $[0, M]^{T+1}$ satisfies D0-D5. Then for any $y^0 > x > 0$, and the sequence $(y^1, y^2, y^3, ..y^m, ..)$ formed using $n^i = 1$, there exists $t \in \mathbb{N}$ such that, $y^m \leq x$ whenever $\sum_1^m n^i \geq t$.

Proof: By Discounting and Monotonicity, (y^i) must be monotonically decreasing: $y^i < y^{i-1}$.
Suppose the claim we are after is not true. Then, the entire sequence $(y^1, y^2, y^3, ..y^m, ..)$ formed with $n^i = 1$ is bounded below by x. Let

$$l = \inf\{y^i : i \in \{0, 1, 2..\}\}$$

l must also be the limit of the monotonically decreasing sequence (y^i). As x is a lower bound of the points in the sequence, $l \geq x > 0$. That l is strictly greater than zero, would drive the rest of the argument. Let the present equivalent of $\mathbf{0}_{-1}l$ be $l' < l$.[27]
Take any point $z \in (l', l)$. By Monotonicity, $\mathbf{0}_{-0}z \succ \mathbf{0}_{-0}l' \sim \mathbf{0}_{-1}l$. Then, by continuity of preferences there must exist $\epsilon > 0$, such that $\mathbf{0}_{-0}z \succ \mathbf{0}_{-1}w$ for all $w \in (l - \epsilon, l + \epsilon)$. Because l is the limit of the sequence (y^i), there must exist y^k, an element of the sequence (y^i), in $(l - \epsilon, l + \epsilon)$. Hence, $\mathbf{0}_{-0}z \succ \mathbf{0}_{-1}y^k$. As $\mathbf{0}_{-0}y^{k+1} \sim \mathbf{0}_{-1}y^k$, hence $y^{k+1} < z < l$, which is a contradiction to l being a lower bound of (y^i).

Claim 3. If $\succsim$ satisfies D0-D5, then it also satisfies D2-2.
Proof: Take any any $y^0 > x > 0$. From Claim 2, we know that when considering the sequence with delays of $n^i = 1$, there exists a t (that depends on x, y_0) such that t^{th} element onwards, the sequence falls below x. For any fixed $y^i > 0$, higher the delay, smaller must be the present equivalent (by Discounting and Monotonicity). Thus, even when considering arbitrary delays n_i of length $1 \leq n_i \leq T$, t^{th} and later elements of the new sequence would be smaller than x . Thus the total delay length of tT would suffice: $y^m \leq x$ whenever $\sum_1^m n^i \geq tT$.

Appendix II: Extensions

Beta-Delta Extension

In Example 4, the set values taken by the set of functions is bounded above and below at each non-zero point x of the domain by $[\frac{x}{\beta}, x]$, and this brings us to

[27] The present equivalent of $\mathbf{0}_{-1}l$ and generally any $x \in [0, M]^T$ always exists under D0-D5.

our next result. Our next theorem characterizes the behavioral axiom A5 similar to Quasi-stationarity that is necessary and sufficient for the functions in $\mathcal{U}_\delta$ to be similarly bounded.

We start by introducing two more axioms.

A5: EVENTUAL STATIONARITY: For any $x > z > 0 \in \mathbb{X}$, there exists $t_1 \in \mathbb{T}$, such that for all $t \geq 0$, $(z,t) \succ (x, t+t_1)$ and $(z,0) \succ (x_t, t_1+t)$ for all x_t such that $(x,0) \sim (x_t,t)$.

A6: NON-TRIVIALITY: For any $x \in \mathbb{X}$, and $t \in \mathbb{T}$, there exists $z \in \mathbb{X}$, such that $(z,t) \succ (x,0)$.

A5 is the more crucial axiom. The first part, that for any $x > z > 0 \in X$, there exists a sufficient delay $t_1 \in \mathbb{T}$, such that $(z,0) \succ (x,t_1)$ is already implied by Discounting (A2). The behavioral content that has been added is the existence of delay t_1 for which we additionally have $(z,t) \succ (x,t+t_1)$ for all $t \geq 0$: This intuitively means once the later larger prize is "sufficiently" delayed, the relative rates at which the attractiveness of the earlier and later rewards fall with further delay (increasing values of t) become consistent with stationarity (note the similarity to Quasi-stationarity). The last and third part of the axiom, $(z,0) \succ (x_t, t_1+t)$ for any x_t such that $(x,0) \sim (x_t,t)$, also has the same interpretation. The A5 property provides a crucial separation between two popular classes of present-biased discounting functions: β-δ discounting and Hyperbolic discounting , as only the former satisfies it, but the latter does not. We show this more formally in Proposition 7 in Appendix III: Proofs.

The last axiom basically means that the space of prizes is rich enough to have exceedingly better outcomes, and hence, is imposed *only when* $\mathbb{X} = \mathbb{R}_+$ and not for $\mathbb{X}$ =$[0, M]$. (Compare Theorem 4 and Corollary 1)

Theorem 4. *Let* $\mathbb{T} = \{0,1,2,...\infty\}$ *and* $\mathbb{X} = \mathbb{R}_+$. *The following two statements are equivalent:*

i) The relation $\succsim$ *satisfies properties A0-A6.*

ii) There exists a set $\mathcal{U}_\delta$ *of monotonically increasing continuous functions such that*

$$F(x,t) = \min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^t u(x)) \tag{3}$$

represents the binary relation $\succsim$. The set $\mathcal{U}_\delta$ has the following properties:$u(0) = 0$ for all $u \in \mathcal{U}$, $\sup_u u(x)$ is bounded above, $\inf_u u(x) > 0$ $\forall x > 0$, $\inf_u \dfrac{u(z)}{u(x)}$ is unbounded in z for all $x > 0$. $F(x,t)$ is continuous.

We omit the proof of this result. This theorem implies that any "minimum-representation" of hyperbolic discounting must require a set of functions which would take unbounded set values at some point of the domain. The immediate conclusion one can draw from here is that one cannot generate any variant of Hyperbolic discounting (coupled with any felicity function) with a minimum representation over a finite set $\mathcal{U}$ of utilities. This theorem also has a straightforward corollary, where we consider the prize domain $\mathbb{X}$ =$[0, M]$ and drop A6.

Corollary 1. *Let $\mathbb{T} = \{0, 1, 2, ...\infty\}$ and $\mathbb{X} = [0, M]$. The following two statements are equivalent:*

i) The relation $\succsim$ satisfies properties A0-A5.

ii) There exists a set $\mathcal{U}_\delta$ of monotonically increasing continuous functions such that

$$F(x,t) = \min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^t u(x))$$

represents the binary relation $\succsim$. The set $\mathcal{U}_\delta$ has the following properties:$u(0) = 0$, $u(1) = 1$ for all $u \in \mathcal{U}_\delta$, $\inf_u u(x) > 0$ $\forall x$. $F(x,t)$ is continuous.

Application to a Timing Game

In this section we are going to study dynamic decision-making games for a present-biased DM whose preferences are consistent with the time-risk relations outlined in Keren and Roelofsma (1995). Present-biased preferences, when extended to a dynamic context[28], create time inconsistent preferences, which in turn results in inefficient decision making and loss in long-term welfare. The goal of this section is to convince the reader about the importance of axiomatization of risk-time relations, by showing that risk-time relations have important welfare implications for such a present-biased individual.

Consider the following game adopted from O'Donoghue and Rabin (1999). Suppose a DM gets a coupon to watch a free movie, over the next four Saturdays.

[28]We are imposing Time Invariance of preferences following Halevy (2015). We will make precise assumptions about sophitication/ naivete as we go.

He has to redeem the coupon an hour before the movie starts. her free ticket is issued subject to availability of tickets, and if there are no available tickets, the coupon is wasted. Hence there is some risk while redeeming the coupon. The movies on offer are of increasing quality- the theater is showing a mediocre movie this week, a good movie next week, a great movie in two weeks and Forrest Gump in three weeks. Our DM perceives the quality of these movies as 30, 40, 60 and 90 on a scale of $0 - 100$. In our problem, the DM can make a decision maximum 4 times, at $\tau = 1, 2, 3, 4$ (measured in weeks). The DM's utility at calendar time τ from watching a movie of quality x with probability p at calendar time $t + \tau$(in weeks) is given by:

$$U^\tau(x, p, \tau + t) \quad = \quad \begin{cases} p^{100}\alpha^t x & \text{for } p^{100}\alpha^t \geq \alpha^{\frac{1}{2}} \\ \left(\dfrac{\alpha}{\beta}\right)^{\frac{1}{2}} p\beta^t x & \text{for } p^{100}\alpha^t < \alpha^{\frac{1}{2}} \end{cases}$$

Where, $\beta = .99$, $\alpha = (.99)^{100} \approx .36$. This utility function (which is inspired by Pan et al. (2015)'s Two Stage Exponential discounting model) has the following interpretation: The DM has a long run weekly discount factor of .99 that sets in after a delay of half a week for $p = 1$. Before reaching the cut-off, the DM is extremely impatient, with a smaller discount factor of $\alpha = \beta^{100} \approx .36$, and hence is biased towards the present and very short-run outcomes. Similarly, the DM also proportionally undervalues probabilities close to 1. *The utility function(s) U^τ define a preference that satisfies all the axioms in Section 6, and hence have a minimum representation.* The DM is time-inconsistent, as her preferences between future options differ between any two decision periods τ_1 and τ_2 for $\tau_1, \tau_2 \in \{1, 2, 3, 4\}$. Let us assume that the DM is aware of her future preferences, that is she is sophisticated, a notion pioneered by Pollak (1968). We are going to use the following notion of equilibrium for this game.

Definition 2. (O'Donoghue and Rabin (1999)) A Perception Perfect Strategy for *sophisticates* is a strategy $s^s = (s_1^s, s_2^s, s_3^s, s_4^s)$, such that such that for all $t < 4$, $s_t^s = Y$ if and only if $U^t(t) \geq U^t(\tau')$ where $\tau' = \min_{\tau > t}\{s_\tau^s = Y\}$.

In any period, sophisticates correctly calculate when their future selves would redeem the coupon if they wait now. They then decide on redeeming the coupon if and only if doing it now is preferred to letting their future selves do it. We consider the following two cases:

Case 1: Suppose, there is not much demand for movie tickets in that city, and the DM knows that he can always book a ticket through her coupon and $p = 1$ for all alternatives under consideration.

In this case, the unique Perception Perfect Strategy is $s^s = (Y, Y, Y, Y)$. The knowledge that the future selves are going to be present biased creates an unwinding effect: The period 2 sophisticate would choose to use the coupon towards the good movie as he knows that the period 3 sophisticate would end up using the coupon for the great movie instead of going for Forrest Gump due to present bias. The period 1 sophisticate in turns correctly understands that waiting now would only result in watching the good movie and hence decides to see the mediocre movie right now instead.

Case 2: Suppose, due to persistent demand for movie tickets in that city, and the DM knows that redeeming a coupon results in a movie ticket in only 99% of cases.

The unique Perception Perfect Strategy is $s^s = (N, N, N, Y)$. The unwinding from the previous case does not happen here due to the risk involved in redeeming the coupon. Once the present is risky (equivalent to having a front end delay due to Probability Time Tradeoff), the bias previously assigned to the present vanishes, stopping the unraveling. The DM waits until the final period to cash in her coupon when the expected returns are the highest to the long run self.

		t				s^s_τ
		1	2	3	4	
τ	4				90	Y
	3			60	54.2	Y
	2		40	36.1	53.6	Y
	1	30	24	35.8	53	Y

		t				s^s_τ
		1	2	3	4	
τ	4				54.2	Y
	3			36.1	53.6	N
	2		24	35.8	53	N
	1	18	24	35.8	52.57	N

TABLE 3. The Left table is for Case 1 ($p = 1$), the right table is for Case 2 ($p = .9$). The entries in the table provide $U^\tau(x, p, t)$. The sophisticated DM compares the quantities in red row-wise for each τ when making a decision.

It would be instructive to compare the two cases in terms of welfare implications. Since present-biased preferences are often used to model self-control problems rooted in the pursuit of immediate gratification, we would compare welfare from the long run perspective. This outcome in Case 1 is consistent with the following general result in O'Donoghue and Rabin (1999): When benefits are immediate, the sophisticates "preprorate", i.e, they do it earlier than it might be optimal. For example, considering the long term self's interests, given a long term weekly discount factor of .99 for movie quality, the equilibrium outcome of watching the mediocre movie (quality of 30) in the first week, instead of Forrest Gump (quality of 90) definitely results in sub-optimal welfare in Case 1. For example, considering the choices from a $\tau = 0$ self gives $U^0(30, 1, 1) = 18$, and $U^0(90, 1, 4) = 53$. On the other hand, the introduction of a small amount of risk in Case 2, stops the unraveling in terms of "preprorating" (preponing consumption), thus helping the DM attain the most efficient outcome in equilibrium, thus reversing the O'Donoghue and Rabin (1999) result. In fact, not only is the highest level of available welfare achieved in Case 2 after the introduction of risk, *the equilibrium welfare improves from Case 1 to Case 2 in the absolute sense, even though apriori Case 2 seems to be worse than Case 1 for the DM*!

$$U^0(30, 1, 1) = 18 \quad < \quad U^0(90, .99, 4) = 52$$

Similarly, the introduction of risk in this set-up would also help a naive DM make an efficient choice in equilibrium.

Appendix III: Proofs

Theorem 1: *Let* $\mathbb{T} = \{0, 1, 2, ...\infty\}$ *or* $\mathbb{T} =[0, \infty)$ *and* $\mathbb{X} = [0, M]$ *for* $M > 0$. *The following two statements are equivalent:*

i) The relation $\succsim$ *defined on* $\mathbb{X} \times \mathbb{T}$ *satisfies properties A0-A4.*

ii) For any $\delta \in (0, 1)$ *there exists a set* $\mathcal{U}_\delta$ *of monotinically increasing continuous functions such that*

$$F(x, t) = \min_{u \in \mathcal{U}_\delta}(u^{-1}(\delta^t u(x)))$$

represents the binary relation $\succsim$. *Moreover,* $u(0) = 0$ *and* $u(M) = 1$ *for all* $u \in \mathcal{U}_\delta$. $F(x, t)$ *is continuous.*

Proof: We start by showing (ii) implies (i). To show Weak Present Bias, we follow the following steps

$$\begin{array}{rcr}
 & (y, t) \succsim (x, 0) & \\
\implies & \min_{u \in \mathcal{U}_\delta}(u^{-1}(\delta^t u(y))) \geq \min_{u \in \mathcal{U}}(u^{-1}(u(x))) & \\
\implies & \min_{u \in \mathcal{U}_\delta}(u^{-1}(\delta^t u(y))) \geq x & \\
\implies & u^{-1}(\delta^t u(y)) \geq x & \forall u \in \mathcal{U}_\delta
\end{array}$$

$$\begin{array}{rcr}
\implies & \delta^t u(y) \geq u(x) & \forall u \in \mathcal{U}_\delta \\
\implies & \delta^{t+t_1} u(y) \geq \delta^{t_1} u(x) & \forall u \in \mathcal{U}_\delta \\
\implies & u^{-1}(\delta^{t+t_1} u(y)) \geq u^{-1}(\delta^{t_1} u(x)) & \forall u \in \mathcal{U}_\delta \\
\implies & \min_{u \in \mathcal{U}_\delta}(u^{-1}(\delta^{t+t_1} u(y))) \geq \min_{u \in \mathcal{U}_\delta}(u^{-1}(\delta^{t_1} u(x))) & \\
\implies & (y, t + t_1) \succsim (x, t_1) &
\end{array}$$

To show Monotonicity and Discounting, let us show $(x, t) \succ (y, s)$, when, either $x > y$ and $t = s$, or, $x = y$ and $t < s$. As all the functions $u \in \mathcal{U}_\delta$ are strictly increasing, and $\delta \in (0, 1)$,

$$
\begin{aligned}
\delta^t u(x) &> \delta^s u(y) \ \ \forall u \in \mathcal{U}_\delta \\
\iff u^{-1}(\delta^t u(x)) &> u^{-1}(\delta^s u(y)) \ \ \forall u \in \mathcal{U}_\delta \\
\iff \min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^t u(x)) &> \min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^s u(y)) \\
\iff (x,t) &\succ (y,s)
\end{aligned}
$$

For proving the second statement under Discounting, start with any $u_1 \in \mathcal{U}_\delta$. For $z > x > 0$, and $\delta \in (0,1)$ there must exist t big enough such that

$$
\begin{aligned}
u_1(x) &> \delta^t u_1(z) \\
\iff u_1^{-1}(u_1(x)) &> u_1^{-1}(\delta^t u_1(z)) \\
\iff x &> \min_{u \in \mathcal{U}_\delta} u^{-1}((\delta^t u(z))
\end{aligned}
$$

Hence, there exists t big enough such that $(x,0) \succ (z,t)$.
That $\succsim$ satisfies continuity follows directly from the definition of continuity on the utility function.
Now, we will prove the other direction of the representation theorem. We will first deal with the case of $\mathbb{T}$ =$[0,\infty)$. A similar proof technique would be used in the proof of Theorem 3.

Proof for the case when $\mathbb{T}$ =$[0,\infty)$.

Proof. For every $x^* \in (0,M)$, we are going to provide an increasing utility function u_{x^*} on $[0,M]$ which would have $\delta^\tau u_{x^*}(x) \geq u_{x^*}(y)$ if $(x,\tau) \succsim (y,0)$. Additionally it would also have $\delta^t u_{x^*}(x_t) = u_{x^*}(x^*)$ for all $(x^*,0) \sim (x_t,t)$.
Fix $u_{x^*}(x^*) = 1$, $u_{x^*}(0) = 0$.
For any $x \in (x^*, M]$, by Discounting there exists a delay T large enough, such that $(x^*,0) \succ (x,T)$. Hence, it must be true that $(x,0) \succ (x^*,0) \succ (x,T)$. By Continuity there must exist $t(x) \in \mathbb{T}$ such that, $(x,t(x)) \sim (x^*,0)$. Define the utility at x as

$$
u_{x^*}(x) = \delta^{-t(x)} \tag{4}
$$

It would be helpful to introduce the following additional notation to move seamlessly between prizes and time in terms of indifference of time-prize pairs w.r.t $(x^*, 0)$. Let $(x^*, 0) \sim (M, t_{max})$. For $t \in [0, t_{max}]$, define x_t as the value in $[x^*, M]$ such that $(x_t, t) \sim (x^*, 0)$. Using continuity, we can say that all points in the interval $[x^*, M]$ can be written as x_t for some $t \geq 0$. This notation essentially implements the inverse of the $t(x)$ function defined in the previous paragraph.
Now, for $x \in (0, x^*)$, define

$$u_{x^*}(x) = \inf\{\delta^\tau : \text{ There exists } t \text{ such that } (x_t, t+\tau) \sim (x, 0)\} \tag{5}$$

Firstly, we will show that the infimum in (5) can be replaced by minimum. Let the infimum be obtained at a value $I = \delta^{\tau^*}$. Consider a sequence of delays $\{\tau_n\}$ that converge above to τ^*, and $(x_{t_n}, t_n + \tau_n) \sim (x, 0)$. Note that $t_n \in [0, t_{max}]$ where $(x^*, 0) \sim (M, t_{max})$. Hence, $\{t_n\}$ must lie in this compact interval, and must have a convergent subsequence $\{t_{n_k}\}$. If t^* is the corresponding limit of $\{t_{n_k}\}$, we know that $t^* \in [0, t_{max}]$. Similarly, x_t can be considered a continuous function in t (this also follows from the continuity of $\succsim$). Therefore, $x_{t_{n_k}} \to x_{t^*}$ when $t_{n_k} \to t^*$. Thus, we have $(x_{t_{n_k}}, t_{n_k} + \tau_{n_k}) \sim (x, 0)$ for all elements of $\{n_k\}$. As, $n_k \to \infty$, $x_{t_{n_k}} \to x_{t^*}$, $t_{n_k} + \tau_{n_k} \to t^* + \tau^*$. Then, using the continuity of $\succsim$, $(x_{t^*}, t^* + \tau^*) \sim (x, 0)$. Hence, the infimum can be replaced by a minimum.
Now we will show that the utility defined in (4) and (5) has the following properties : 1) It is increasing. 2) $\delta^t u_{x^*}(x_t) = u_{x^*}(x^*)$ for all $(x^*, 0) \sim (x_t, t)$. 3) $(x, \tau) \succsim (y, 0)$ implies $\delta^\tau u_{x^*}(x) \geq u_{x^*}(y)$, 4) u is continuous. The first two properties are true by definition of u. We will show the third and fourth in some detail.
Consider $(x, \tau) \succsim (y, 0)$. In the case of interest, $\tau > 0$ and hence, $x > y$.
Now let $x > y \geq x^*$. Let, $u(y) = \delta^{-t_1}$, which means, $(y, t_1) \sim (x^*, 0)$. Given $(x, \tau) \succsim (y, 0)$, we must have

$$(x, \tau + t_1) \succsim (y, t_1) \sim (x^*, 0)$$

Hence, if $(x, t_2) \sim (x^*, 0)$, then,

$$\begin{aligned} t_2 &\geq \tau + t_1 \\ \iff u_{x^*}(x) = \delta^{-t_2} &\geq \delta^{-(\tau+t_1)} \\ \iff \delta^\tau u_{x^*}(x) &\geq \delta^{-t_1} = u_{x^*}(y) \end{aligned}$$

If, $x \geq x^* \geq y$, the proof follows from the way the utility has been defined.
Let $y < x \leq x^*$. Let, $u_{x^*}(x) = \delta^{t_1}$, which means, $(x_t, t + t_1) \sim (x, 0)$ for some $x_t \in [x^*, M]$. Given $(x, \tau) \succsim (y, 0)$, we must have

$$(x_t, t + t_1 + \tau) \succsim (x, \tau) \succsim (y, 0)$$

Hence, $u_{x^*}(y) \leq \delta^{\tau + t_1} = \delta^\tau u_{x^*}(x)$.
Now we turn to proving the continuity of u_{x^*}. The continuity at x^* from the right, or on $(x^*, M]$ is easy to see.
Next, for any $r = \delta^s \in (0, 1)$, define

$$f(r) = \sup\{y : \ (x_t, t + s) \sim (y, 0)\} = \hat{y} \tag{6}$$

The supremum can be replaced by a maximum, and the proof is similar to the one before. Suppose there is a sequence of $\{y_n\}$ that converges up to a value $\hat{y}$, and, $(x_{t_n}, t_n + s) \sim (y_n, 0)$. Note that t_n lies in a compact interval $[0, t_{max}]$, and hence has a convergent subsequence t_{n_k} that converges to a point in that interval $\hat{t} \in [0, t_{max}]$. Now, x_t is continuous in t (in the usual sense), and hence, x_{t_n} also converges to $x_{\hat{t}}$. Further, $y_{n_k} \to \hat{y}$ as $n_k \to \infty$. Therefore, using, $(x_{t_{n_k}}, t_{n_k} + s) \sim (y_{n_k}, 0)$, as, $n_k \to \infty$, it must be that $(x_{\hat{t}}, \hat{t} + s) \sim (\hat{y}, 0)$. Hence, the supremum in (6) must have been attained from $x_{\hat{t}}$, and hence the supremum can be replaced by a maximum. Further given this is a maximum, we can say that $\hat{y} \in (0, x^*)$. The f function is well defined, strictly increasing and is the inverse function of u_{x^*} from $(0, 1)$ to $(0, x^*)$, in the sense that, $u(f(r)) = r$. This function can be used to show the continuity of u at the point x^*.
Finally, the function u can be easily normalized to have $u_{x^*}(M) = 1$. (By dividing the function from before by $u_{x^*}(M)$.)
Now, consider $\mathcal{U}_\delta = \{u_{x^*}() : \ x^* \in (0, M]\}$. By construction of the functions, it must be that

$$\begin{aligned}
(x, t) \succsim (y, 0) \quad &\iff \quad \delta^t u(x) \geq u(y) \ \forall u \in \mathcal{U}_\delta \\
(x, t) \sim (y, 0) \quad &\iff \quad \delta^t u(x) \geq u(y) \ \forall u \in \mathcal{U}_\delta \\
& \qquad\quad \text{and } \delta^t u_y(x) = u_y(y) \text{ for some } u_y \in \mathcal{U}_\delta
\end{aligned}$$

Given $\succsim$ is complete, transitive and satisfies continuity, there exists a continuous function $\bar{F} : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ such that $\bar{F}(a) \geq \bar{F}(b)$ if and only if $a \succsim b$ for

$a, b \in \mathbb{X} \times \mathbb{T}$. (Following Theorem 1, Fishburn and Rubinstein (1982)).
We define $G : \mathbb{X} \to \mathbb{R}$ as $G(x) = \bar{F}(x, 0)$. The function G would be strictly monotonic and continuous. Also define $F : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ as $F(x,t) = G^{-1}(\bar{F}(x,t))$. As any alternative has a unique present equivalent, F is well defined, is a monotonic continuous transformation of $\bar{F}$ (hence represents $\succsim$) and $F(x, 0) = x$ for all $x \in \mathbb{X}$. By definition the function F assigns to every alternative its present equivalent as the corresponding utility. Therefore, the present equivalent utility representation is continuous.
We will show that the function W defined below also assigns to every alternative (z, τ) an utility exactly equal to its present equivalent:

$$W(x,t) \quad = \quad \min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^t u(x)) = F(x,t)$$

Consider any $(z, \tau) \sim (y_1, 0)$. By definition of $\mathcal{U}_\delta$ and by construction of its constituent functions, it must be that for all $u \in \mathcal{U}_\delta$, $\delta^\tau u(z) \geq u(y_1)$ and there exists a function u_{y_1} such that $\delta^\tau u_{y_1}(z) = u(y_1)$. This is equivalent to the following statement: For all $u \in \mathcal{U}_\delta$, $u^{-1}(\delta^\tau u(z)) \geq y_1$ and there exists a function u_{y_1} such that $u_{y_1}^{-1}(\delta^\tau u_{y_1}(z)) = y_1$.
Therefore, $W(z, \tau) = \min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^\tau u(z))$ is a continuous utility representation of the relation $\succsim$.

□

Proof for the case of $\mathbb{T}$ =$\{0, 1, 2, ...\}$.
This proof would be more technical and we will break down the proof of this case into the following Lemmas.

Lemma 1. *Under Axioms A0-A4, for a fixed x_0, and any x_t and t such that $(x_t, t) \sim (x_0, 0)$, there exists a continuous strictly increasing function u such that $\delta^t u(x_t) = u(x_0)$ and $\delta^t u(z_1) \geq u(z_0)$ for all $(z_1, t) \succsim (z_0, 0)$. Further, $u(0) = 0$, $u(M) = 1$.*

Proof. SUBSTEP 1.1: Choose $x_0^* = x_0$. To build the function, we start by finding points x_t^* on $[x_0^*, M]$ which would be indifferent to x^* at different time delays (t), i.e, $(x_0^*, 0) \sim (x_t^*, t)$. By the Discounting axiom, we know that there exists a smallest integer $n \geq 1$ such that $(x_0^*, 0) \succsim (M, n)$. We would use this definition of n throughout this proof. If $(x_0, 0) \succ (M, n)$, choose $x_n^* = M$.
We define x_{-1}^* in the following way

$$x^*_{-1} = \min\{x \in \mathbb{X} : (x, 0) \succsim (x^*_j, j+1),\ j = 0, 1, 2, ...n\}$$

The idea is to look at the present equivalents of $(x^*_j, j+1)$ and take the maximum of those present equivalents to define x^*_{-1}, this is similar to assigning conservative utilities to points below x^* as we did in the continuous exercise. The alternative way to express the same is to look at the intersection of the weak upper counter sets of $(x^*_j, j+1)$ on $\mathbb{X} \times \{0\}$, and then take the minimal value from that set.
Next we will use this to define x^*_{-2}, then use x^*_{-1} and x^*_{-2} to define x^*_{-3}. In general, for $i \in \{-1, -2, -3...\}$ define x^*_i recursively as the minimum of the set

$$\{x \in \mathbb{X} : (x, 0) \succsim (x^*_j, j-i), j = i+1, i+2, ...n\}$$

The definition uses the same idea as before. We consider the intersection of the weak upper counter sets of $(x^*_j, j-i)$ on $\mathbb{X} \times \{0\}$ and take its minimum. The set is non-empty (x^*_0 belongs to it, for example), closed and the minimum exists due to the continuity, monotonicity and discounting properties.
The interative process always has $x^*_i > 0$ at each step, as, $(x, t) \succ (x, t+1) \succ (0, t+1) \sim (0, 0)$ for $x \in (0, M]$ and $t \in \mathbb{T}$.
Substep 1.2: Next we show that for every x^*_i with $i \leq -1$, there exists $j \in \{0, 1, ..n\}$ such that $(x^*_j, j-i) \succsim (x^*_i, 0)$. The proof is by induction. For $i = -1$, it is immediate from the definition of x^*_{-1}. Suppose, it holds for all $i \geq -m$. Consider x^*_{-m-1}. By construction, there must exist $k \in \{-m, -m+1, ..n\}$ such that $(x^*_{-m-1}, 0) \sim (x^*_k, k+m+1)$. If $k \in \{0, 1, ..n\}$ we are done already. If not, by the induction hypothesis, there exists $j \in \{0, 1, ..n\}$ such that $(x^*_j, j-k) \succsim (x^*_k, 0)$, which gives, $(x^*_j, j+m+1) \succsim (x^*_k, k+m+1)$, and hence, $(x^*_j, j+m+1) \succsim (x^*_{-m-1}, 0)$, completing the proof.
Substep 1.3: We will now show that the sequence $\{...x^*_{-2}, x^*_{-1}, x^*_0, x^*_1, x^*_2, ..\}$ converges below to 0. Suppose not (we are going for a proof by contradiction), that is there exists $z > 0$ such that $x_i \geq z$ for all $i \in \mathbb{Z}$. As, $M > z > 0$, there must exist t_1 big enough such that $(z, 0) \succ (M, t_1)$. Consider the element $x^*_{-t_1}$ from the sequence in consideration. Using the result from the previous paragraph, it must be true that there exists $j \in \{0, 1, .., n\}$, such that $(x^*_j, j+t_1) \succsim (x^*_{-t_1}, 0)$. Now, as $M \geq x^*_j$, we must have, $(M, t_1) \succsim (x^*_j, j+t_1) \succsim (x^*_{-t_1}, 0) \succ (z, 0)$, which provides a contradiction.
Substep 1.4: Now, we calibrate the utility at points inside the intervals we have

constructed. Consider any $y_0 \in (x_0^*, x_1^*)$.
We are going to find a $y_1, y_2, ..y_{n-1}$ recursively, as before.
Finding y_1: In the next paragraph we define how to get the first element of the sequence y_1.
For each point $y \in (x_1^*, x_2^*]$, take reflections of length 1, i.e, find x_y such that $(y,1) \sim (x_y, 0)$. Note that, $(x_1^*, 0) \succ (y,1) \succ (x_0^*, 0)$, where the first relation follows from $(y,1) \succsim (x_1^*, 0)$ resulting in a contradiction[29], and the second follows from $y > x_1^*$. Hence, $x_y \in (x_0^*, x_1^*)$. Let, x_{x_2} be the reflection for the point x_2^*. For any $y \in (x_1^*, x_2^*]$, let $f(y) = x_0^* + (x_y - x_0^*)\dfrac{(x_1^* - x_0^*)}{(x_{x_2} - x_0^*)}$. Now, for $y_0 \in (x_0^*, x_1^*)$, define y_1 as $f^{-1}(y_0)$.
We can check that this method satisfies the 2 following conditions:

1) Consider two such sequences (y^1) and (y^2), one starting from y_0^1, and another from y_0^2, with $y_0^1 > y_0^2$ and both points in the interval (x_0^*, x_1^*). We will have $y_1^1 > y_1^2$.

2) All points in intervals (x_1^*, x_2^*) are included by some y_1 from the sequence. This follows from monotonicity and discounting.

Now, the recursive step:
For each point $y \in (x_i^*, x_{i+1}^*]$ with $i > 0$, take reflections of length $j \in \{i, i-1.., 1\}$ conditional on those reflections being in the corresponding $(x_{i-j}^*, x_{i+1-j}^*]$.[30] For any y, at least one of these reflections must exist, and in particular the one with length i always exists, as $(x_1^*, 0) \succsim (x_{i+1}^*, i) \succsim (y, i)$[31] and $(y, i) \succ (x_i^*, i) \sim (x_0^*, 0)$. Now, for each such reflection that lies in the proper interval, find the corresponding sequence of $\{y_0, y_1, ..y_{i-1}\}$ that reflection belongs to, and denote the smallest y_0 from that collection of sequences as $x_y \in [x_0^*, x_1^*]$. Note that if $x_{x_{i+1}}$ is the i-th reflection of x_{i+1}^*, then, $x_{x_{i+1}} \leq x_1^*$.[32] Define the $1:1$ strictly increasing function f from $(x_i^*, x_{i+1}^*]$ to $(x_0^*, x_1^*]$ in the following way: For any $y \in (x_i^*, x_{i+1}^*]$, $f(y) = x_0^* + (x_y - x_0^*)\dfrac{(x_1^* - x_0^*)}{(x_{x_{i+1}} - x_0^*)}$. Now, define y_i as $f^{-1}(y_0)$. The conditions (1) and (2) mentioned above are still satisfied for the extended sequence. Note that in every step of this iterative process, for every sequence that we construct, we

[29] It implies $(y, 2) \succsim (x_1^*, 1) \sim (x_0^*, 0) \sim (x_2^*, 2)$

[30] There are only two possible cases, either the reflections lie in the interval, or, they are less than every point in that interval. This follows from $(x_{i+1-j}^*, 0) \succsim (x_{i+1}^*, j) \succ (y, j)$

[31] The first relation follows from the fact that $(x_{i+1}^*, i) \succsim (x_1^*, 0)$ would imply $(x_{i+1}^*, i+1) \succsim (x_1^*, 1) \sim (x_0^*, 0)$ which would give a contradiction.

[32] As, $(x_1^*, 0) \succsim (x_{i+1}^*, i)$ by definition of x_{i+1}^*.

have $(y_i, 0) \succsim (y_j, j-i)$ for $j > i$.
Substep 1.5: Having found $y_1, y_2, ..y_{n-1}$, for $i \leq -1$, define y_i recursively in the following way. Start by finding y_i' as the minimum of the set

$$\{y \in \mathbb{X} : (y, 0) \succsim (y_j, j-i), j = i+1, i+2, ...n-1\}$$

Define x_i' as the minimum of the set

$$\{y \in \mathbb{X} : (y, 0) \succsim (x_j^*, j-i), j = i+1, i+2, ...n-1\}$$

Define $x_{i+1}^{'}$ as the minimum of the set

$$\{y \in \mathbb{X} : (y, 0) \succsim (x_j^*, j-i-1), j = i+2, i+3, ...n\}$$

Note that $x_{i+1}^{'} \leq x_{i+1}^*$, with the equality being guaranteed for all $i < -1$.
Finally, for $i < -1$, define

$$y_i = x_{i+1}^* - (x_{i+1}^* - y_i')\frac{(x_{i+1}^* - x_i^*)}{(x_{i+1}^* - x_i')} \tag{7}$$

and for $i = -1$, define

$$y_i \quad = \quad x_{i+1}^* - (x_{i+1}^{'} - y_i')\frac{(x_{i+1}^* - x_i^*)}{(x_{i+1}^{'} - x_i')}$$

Given $y_0^1 > y_0^2$ determines the order of $y_t^1 > y_t^2$, for $t \in \{1, 2, ..n-2\}$, our inductive procedure make sure this holds true for all $t \leq -1$ too.
One can check for covering properties of the sequences by induction. Suppose all points in the intervals (x_i^*, x_{i+1}^*) are covered by y_i for some sequence, for $i \geq j$ for some integer j. We are going to show that all points in (x_{j-1}^*, x_j^*) are also covered by y_{j-1} for some sequence. Take any point $y \in (x_{j-1}^*, x_j^*)$, and consider its corresponding y' as defined in Equation 7. Consider the reflections from point y' of sizes $1, ..n-j+1$, i.e, the points at those temporal distances which are indifferent to it, conditional on being in the corresponding intervals. By the induction hypothesis, each of those reflection end points must be coming from some $y_0 \in (x_0^*, x_1^*)$. Take the sequence with smallest y_0, and that sequence would result in having $y \in (x_{j-1}^*, x_j^*)$ as its next element.
Substep 1.6: Now, define u on $\mathbb{X}$ as follows: Set $u(x_n^*) = 1$. For the sequence

..., $x^*_{-2}, x^*_{-1}, x^*_0, x^*_1, ..$, let $u(x^*_i) = \delta^{i-n}$ for all positive and negative integers i. Define $u(0) = 0$, and the function is continuous at zero, as the sequence (x^*_i) converges to zero.

Next, let us define u on (x^*_{n-1}, x^*_n) as any continuous and increasing function with $\inf_{(x^*_{n-1}, x^*_n)} u(x) = \delta = u(x^*_{n-1})$ and $\sup_{(x^*_{n-1}, x^*_n)} u(x) = 1 = u(M)$. We can extend each dual sequence with some as $u(y_i) = \delta^{i-n} u(y_0)$. This finishes the construction of a u that satisfies the conditions mentioned in the Lemma. □

Lemma 2. *Under Axioms A0-A4, there exists a continuous present equivalent utility function $F : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ that represents $\succsim$. Moreover, for $\delta \in (0,1)$, $F(z,\tau) = \min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^\tau u(z))$ for some set $\mathcal{U}_\delta$ of strictly monotonic, continuous functions, $u(0) = 0$ and $u(M) = 1$ for all $u \in \mathcal{U}_\delta$.*

Proof. This part follows from Lemma 1, and the proof of the continuous case.

□

Proposition 2: *Given the axioms A0-4, the representation form in (2) is unique in the discounting function $\Delta(t) = \delta^t$ inside the present equivalent function.*

Proof. We start with the case where $\Delta(t)$ is such that $\dfrac{\Delta(t+t_1)}{\Delta(t)} < \Delta(t_1)$ for some t, t_1. Consider any singleton $\mathcal{U} = \{u\}$.

$$
\begin{aligned}
(y,t) &\sim (x,0) \\
\implies u^{-1}(\Delta(t)u(y)) &= x \\
\implies \Delta(t)u(y) &= u(x) \\
\implies \Delta(t+t_1)u(y) &= \frac{\Delta(t+t_1)}{\Delta(t)}u(x) < \Delta(t_1)u(x) \\
\implies u^{-1}(\Delta(t+t_1)u(y)) &< u^{-1}(\Delta(t_1)u(x)) \\
\implies (x,t_1) &\succ (y,t+t_1)
\end{aligned}
$$

Hence, the relation implied by the representation contradicts Weak Present Bias.

Now assume the opposite, let there exists some $t, t_1 > 0$ such that $\dfrac{\Delta(t+t_1)}{\Delta(t)} > \Delta(t_1)$. Now suppose we started with a relation $\succsim$ which has $(y,t) \sim (x,0)$ as well as $(y, t+t_1) \sim (x, t_1)$ for all t, t_1 and some x, y. Such preferences are admissible under our axioms, and the easiest example would be the preferences under exponential discounting. We will show below that such preferences cannot be represented by the functional form we started with for any set of functions $\mathcal{U}$.

$$
\begin{aligned}
(y,t) &\sim (x,0) \\
\implies \min_{u\in\mathcal{U}}(u^{-1}(\Delta(t)u(y))) &= \min_{u\in\mathcal{U}}(u^{-1}(u(x))) = x \\
\implies \Delta(t)u(y) &\geq u(x) \;\; \forall u \in \mathcal{U} \\
\implies \Delta(t+t_1)u(y) &\geq \frac{\Delta(t+t_1)}{\Delta(t)}u(x) > \Delta(t_1)u(x) \;\; \forall u \in \mathcal{U} \\
\implies u^{-1}(\Delta(t+t_1)u(y)) &> u^{-1}(\Delta(t_1)u(x)) \;\; \forall u \in \mathcal{U} \\
\implies \min_{u\in\mathcal{U}}(u^{-1}(\Delta(t+t_1)u(y))) &> \min_{u\in\mathcal{U}}(u^{-1}(\Delta(t_1)u(x))) \\
\implies (y,t+t_1) &\succ (x,t_1)
\end{aligned}
$$

This completes our proof.

□

Theorem 2: *The following two statements are equivalent:*

i) The relation $\succsim$ on $[0,M]^{T+1}$ satisfies properties D0-D5.

ii) For any $\delta \in (0,1)$, there exists a set $\mathcal{U}_\delta$ of monotonically increasing continuous functions and a continuous strictly increasing function V such that

$$F(x_0,x_1,..,x_T) = \sum_{j=0}^{T} V(\min_{u\in\mathcal{U}_\delta} u^{-1}(\delta^j u(x_j)))$$

represents the binary relation $\succsim$. The set $\mathcal{U}_\delta$ has the following properties: $u(0)=0$ and $u(M)=1$ for all $u \in \mathcal{U}_\delta$. $F(.)$ is continuous. V is unique upto affine transformations.

Proof. We will prove this theorem in two steps. In the first step we would derive a period-wise additive representation, following Wakker (2013). In the second and third steps, we would borrow some of the proof techniques used in Theorem 1 while adapting it to a finite period setting.

Step 1: In this step, we will construct a set of continuous monotonic functions $V_j(\cdot)$ for $j \in \mathbb{T}$, such that $\sum_{j\in\mathbb{T}} V_j$ represents $\succsim$.
We start by assigning $V_j(0) = 0 \;\forall j \in \mathbb{T}$ and $V_0(1) = 1$. (WLOG, assuming

$M \geq 1$) We would be using the grid $[0,1]$ on the zero-th period as a utility measuring stick. In the following, we would be using comparisons of streams along only two or three dimensions, without caring about the rest of the stream, and this is allowed due to the Coordinate Independence (CI) axiom. Hence while working with comparisons along i^{th} and j^{th} dimensions, we would be denoting the stream $(w_1, w_2, ..w_i, .., w_j, .w_T)$ by (w_i, w_j) in short, to save on notation. Similarly, we would use (w_i, w_j, w_k) when only three dimensions need to be considered for the argument to be made.

SUBSTEP 1.1: Defining an initial grid

We are going to construct a grid of $(T+1)$ coordinates, with the t^{th} coordinate being constructed for period $t-1$ consumption. Let $w_j^0 = 0$ (equating w_j^0 to the 0-point of the j^{th} coordinate) for all $j \in \mathbb{T}$, and $w_1^1 = 1$. For every $j \neq 1$, by induction on natural numbers z we recursively define points w_j^z such that

$$(w_1^0, w_j^z) \quad \sim \quad (w_1^1, w_j^{z-1})$$

On each axis j, this should either define an infinite sequence, or there should exist a natural number $z-1$, such that no corresponding w_j^z can be found. Now, we would use $[w_2^0, w_2^1]$[33] as a scale to recursively define a similar sequence $\{w_1^z\}$:

$$(w_1^z, w_2^0) \quad \sim \quad (w_1^{z-1}, w_2^1)$$

SUBSTEP 1.2: Calibration of utility on the grid

For any $i, j \in \mathbb{T} - \{1\}$, and for any natural number z_i, z_j, by construction,

$$(w_1^0, w_i^{z_i+1}, w_j^{z_j}) \quad \sim (w_1^1, w_i^{z_i}, w_j^{z_j}) \quad \sim (w_1^0, w_i^{z_i}, w_j^{z_j+1})$$

For $i = 1, j \neq 2$,

$$(w_1^{z_1+1}, w_2^0, w_j^{z_j}) \quad \sim (w_1^{z_1}, w_2^1, w_j^{z_j}) \quad \sim (w_1^{z_1}, w_2^0, w_j^{z_j+1})$$

The first step follows from construction, and the last step follows from the preceding $i, j \in \mathbb{T} - \{1\}$ case under CI.

For $i = 1, j = 2$,

$$(w_1^{z_1}, w_2^{z_2+1}, w_k^{z_k}) \quad \sim (w_1^{z_1}, w_2^{z_2}, w_k^{z_k+1}) \quad \sim (w_1^{z_1+1}, w_2^{z_2}, w_k^{z_k})$$

[33] Note that $w_2^1 \leq w_1^1 \leq M$.

The first step follows from the $i, j \in \mathbb{T} - \{1\}$ case, the second from the $i = 1, j \neq 2$ case, using CI in both steps.

At this point, we can define functions $V_j(\cdot)$ on the grid of points from the sequences defined on the coordinates, with $V_j(w_j^{z_j}) = z_j$. Further, the subject should be indifferent between any two points v and w on the grid, if $\sum_j V_j(v_j) = \sum_j V_j(w_j)$. This can be showed by inductively using the indifference properties described above, repeatedly. Further, $\sum_j V_j(v_j) \geq \sum_j V_j(w_j)$ if and only if $v \succsim w$.

Substep 1.3: Doubling the density of the grid

Showing this requires the intermediate result that if $w_1^1 > v_1^1 > v_1^0 = w_1^0$, then $w_1^m > v_1^m$, where v_{dim}^i is constructed just like w_{dim}^i was. The proof is by induction[34]. Suppose the result holds for all $m \leq n-1$. This would imply that along $j = 2$, $w_2^1 > v_2^1$. Fixing all $j \neq 1, 2$ coordinate values to w_j^0, we have $(w_1^n, w_2^0) \sim (w_1^{n-1}, w_2^1) \succ (v_1^{n-1}, v_2^1) \sim (v_1^n, v_2^0)$. Thus, $w_1^n > v_1^n$.

To show that there is a way to exactly double the density of the grid, we will start by showing that any new utility scale smaller or larger than exactly half of the previous scale, can be modified in the right direction: In particular, we will show the case that for any candidate $\{v_1^k\}$, with $w_1^1 > v_1^2 > v_1^1 > v_1^0 = w_1^0$, there exists a larger scale $\{y_1^k\}$ such that $w_1^1 > y_1^2 > v_1^2$ and $y_0^0 = w_0^0$.

Suppressing all $j \neq 1, 2$ coordinate values (which are assumed to be w_j^0), we see that $(w_1^1, w_2^0) \succ (v_1^2, v_2^0) \sim (v_1^1, v_2^1)$. Therefore, by continuity, there must exist points $s_1 > v_1^1$ and $s_2 > v_2^1$, such that $(v_1^1, v_2^1) \prec (s_1, v_2^1) \prec (s_1, s_2) \prec (w_1^0, w_2^1)$.

Now, if $(w_1^0, s_2) \succsim (s_1, w_2^0)$ then, set $y_1^1 = s_1$ and define y_1^2 thence. As, $y_1^1 = s_1 > v_1^1$ we would have $s_2 > y_2^1 > v_2^1$

Otherwise, set $y_2^1 = s_2$ and find y_1^1 thence.

Such a choice ensures that $(v_1^2, v_2^0) \sim (v_1^1, v_2^1) \prec (y_1^1, y_2^1) \sim (y_1^2, y_2^0) \prec (s_1, s_2) \prec (w_1^1, w_2^0)$ and hence, $w_1^1 > y_1^2 > v_1^2$.

Now, appealing to the connectedness of the space, we can conclude that there must exist an utility scale $[z_1^0, z_1^1]$, that is exactly half of $[w_1^0, w_1^1]$, that is $z_1^0 = w_1^0$, $z_1^2 = w_1^1$ and $x_1^1 = w_1^{\frac{1}{2}}$. When we extend the utility scale z to all coordinates, we get a grid which is double the density of the previous w grid (follows by induction).

Substep 1.4: Constructing a dense grid by an iterative repetition of Substep 1.3

[34] If w_1^m exists, then so does v_1^m.

The substep above can be infinitely repeated to get a mesh of points that is dense in the underlying space, and the minimum distance between two points in the mesh is zero in the limit (details in Chapter III, Wakker (2013)). It can also be shown that there are points on the grid, arbitrarily close to the two extremes of the $[0, M]$ axis.

SUBSTEP 1.5: Defining the functions V_j on $[0, M]$.

For $j \in \mathbb{T}$ and $x_j \in [0, M]$, let,

$$V_j(x_j) \quad = \quad \sup\{V_j(w_j^{b_j}) : \text{where } w_j^{b_j} \text{is a point on the dense grid and } x_j \succsim w_j^{b_j}\}$$

We had already normalized $V_j(0) = 0$. The proof that the function is well defined, continuous and always takes finite values, is simple and uses a similar argument as used in Theorem 1. The joint cardinality of the functions V_j follows from the fact that the value of the functions are unique subject to the $(n+1)$ arbitrary choices made while initializing the V_js.

Showing $(x_0, x_1, x_2...x_T) \succ (y_0, y_1, y_2, ..y_T)$ implies $\sum_{j\in\mathbb{T}} V_j(x_j) > \sum_{j\in\mathbb{T}} V_j(y_j)$, requires replacing every $x_j \in (0, M)$ not on the dense grid with a point $\tilde{x}_j$ from the grid infinitesimally smaller than $x_j \in (0, M)$, and every point y_j not on the dense grid with a point $\tilde{y}_j$ from the grid infinitesimally larger than y_j[35] such that $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2...\tilde{x}_T) \succ (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, ..\tilde{y}_T)$, and this is possible by continuity of preferences. Hence

$$\sum_{j\in\mathbb{T}} V_j(x_j) > \sum_{j\in\mathbb{T}} V_j(\tilde{x}_j) > \sum_{j\in\mathbb{T}} V_j(\tilde{y}_j) > \sum_{j\in\mathbb{T}} V_j(y_j) \tag{8}$$

where the first and last inequality follow from the monotonicity of the V_js and the second inequality follows from the points being compared being on the grid. A similar trick shows that $(x_0, x_1, x_2...x_T) \succsim (y_0, y_1, y_2, ..y_T)$ implies $\sum_{j\in\mathbb{T}} V_j(x_j) \geq \sum_{j\in\mathbb{T}} V_j(y_j)$.

This brings us to

STEP 2: Preferences are represented by

$$\sum_{j\in\mathbb{T}} V_j(x_j) = \sum_{j\in\mathbb{T}} V_0(V_0^{-1}(V_j(x_j))$$

[35] The lowest point 0 is already on the grid on each axis, and hence it does not need to be changed in the $\tilde{x}$-vector. If the maximal point M on the j^{th} dimension is not on the grid for, then first it is replaced by any arbitrary point y'_j on the grid in the $\tilde{y}$-vector. Then, $\sum V_j(\tilde{x}_j) > \sum_{j\in\mathbb{T}} V_j(\tilde{y}_j)$ is first written for the perturbed $\tilde{y}$-vector, and as the actual $\sum_{j\in\mathbb{T}} V_j(\tilde{y}_j)$ is the supremum of all possible summations with perturbed $\tilde{y}$-vector, $\sum_{j\in\mathbb{T}} V_j(\tilde{x}_j) \geq \sum_{j\in\mathbb{T}} V_j(\tilde{y}_j)$ holds even for the actual $\tilde{y}$-vector with the M's intact.

where the alternative form implies *evaluating the present equivalent of every periodwise-consumption by* V_0*, then aggregating additively.* At this point, we show that the axioms on consumption streams imply behavioral properties similar to A0-A4 (from Theorem 1) on the restricted domain of consumption streams that can be meaningfully interpreted as temporal rewards (i.e, streams that deliver a non-zero reward in no more than one period). We say that $(x, s) \succsim (y, t)$ iff

$$(0, .. \underbrace{x}_{\text{in period } s}, .., 0) \succsim (0, .. \underbrace{y}_{\text{in period } t}, .., 0)$$

By making appropriate changes at different points in the proof (as mentioned later), one can establish that present equivalents here too could be represented as $V_0^{-1}(V_j(x_j)) = \min_{u \in \mathcal{U}_\delta} u^{-1}(\delta^j u(x_j))$, and hence,

$$\sum_{j \in \mathbb{T}} V_j(x_j) = \sum_{j \in \mathbb{T}} V_0(\min_{u \in \mathcal{U}} u^{-1}(\delta^j u(x_j)))$$

STEP 3: Below, we metion how our proof of Theorem 1 needs to be adapted to the current finite period setting. Lemma 1 and 2 can be proved similarly, but the substeps 1.1 to 1.5 in Lemma 1 need to changed to substeps 3.1 to 3.5.

SUBSTEP 3.1: Choose $x_0^* = x_0$. We start by finding points x_t^* in $[x_0^*, M]$ which would be indifferent to x^* at different time delays $t \in \mathbb{T}$, i.e, $(x_0^*, 0) \sim (x_t^*, t)$. If $(M, T) \succ (x_0, 0)$, then while defining points $x_{T+1}, x_{T+2}..$, we would have to use

$$x_{T+i} = \max\{x \in \mathbb{X} : (x_{T+i-t}, 0) \succsim (x, t) \ \forall t = \{1, 2, ..T\}\}$$

In the following we would also consider D2-2 under the discounting axiom, as it already follows from D0-D4 (See Appendix I). By the Discounting axiom, we know that there exists a smallest integer $n \geq 1$ at which this iterative process terminates, as either $x_n = M$ or alternatively the subsequent x_{n+i}s are all larger than M. If $x_n > M > x_{n-1}$ redefine $x_n = M$.
As before, we define x_{-1}^* in the following way

$$x_{-1}^* = \min\{x \in \mathbb{X} : (x, 0) \succsim (x_j^*, j + 1), \ j = 0, 1, 2, ...T - 1\}$$

Expressed differently, the idea is to look at the present equivalents of all comparable $(x_j^*, j + 1)$ and take the maximum of those present equivalents to define x_{-1}^*. The alternative way to express the same is to look at the intersection of the weak upper counter sets of $(x_j^*, j + 1)$ on $\mathbb{X} \times \{0\}$, and then take the minimal value from that set.

Next we will use this to define x^*_{-2}. In general, for $i \in \{-1, -2, -3...\}$ define x^*_i recursively as the minimum of the set

$$\{x \in \mathbb{X} : (x, 0) \succsim (x^*_j, j - i), j = i + 1, i + 2, ..., i + T\}$$

The set is non-empty (M belongs to it, for example), closed and the minimum exists due to the continuity, monotonicity and discounting properties.
The iterative process always has $x^*_i > 0$ at each step, using continuity and $(x, t) \succ (0, 0)$ for $x \in (0, M]$ and $t \in \mathbb{T}$.
Substep 3.2: Substep 1.2 is not required here.
Substep 3.3: We will now show that the sequence $\{...x^*_{-2}, x^*_{-1}, x^*_0, x^*_1, x^*_2, ..\}$ converges below to 0. Suppose not (we are going for a proof by contradiction), that is there exists $z > 0$ such that $x^*_i \geq z$ for all $i \in \mathbb{Z}$. Given $x^*_i > z > 0$ for $i \geq 0$, by the Discounting axiom, for each $i\geq 0$, there must exist $t_1(i)$ big enough such that when the corresponding x^*_i is recursively discounted for a total of $t_1(i)$ occasions (see the Discounting axiom), it becomes an inferior choice than consuming z in the present. This would lead to a contradiction with the iterative construction of the sequence (especially elements below $x^*_{\min_{i\in 1,2,,n}\{i-t(i)\}}$) never reaching below z.
Substep 3.4: Now, we calibrate the utility at points inside the intervals we have constructed. Consider any $y_0 \in (x^*_0, x^*_1)$.
We are going to find a $y_1, y_2, ..y_{n-1}$ recursively, as before.
Finding y_1: In the next paragraph we define how to get the first element of the sequence y_1.
For each point $y \in (x^*_1, x^*_2]$, take reflections of length 1, i.e, find x_y such that $(y, 1) \sim (x_y, 0)$. Note that, $(x^*_1, 0) \succsim (y, 1) \succ (x^*_0, 0)$ where the first relation holds as $(y, 1) \succ (x^*_1, 0)$ would result in a contradiction[36], and the second follows from $y > x^*_1$. Hence, $x_y \in (x^*_0, x^*_1)$. Let, x_{x_2} be the reflection for the point x^*_2. For any $y \in (x^*_1, x^*_2]$, let $f(y) = x^*_0 + (x_y - x^*_0)\dfrac{(x^*_1 - x^*_0)}{(x_{x_2} - x^*_0)}$. Now, for $y_0 \in (x^*_0, x^*_1)$, define y_1 as $f^{-1}(y_0)$.
We can check that this method satisfies the 2 following conditions:

1) Consider two such sequences (y^1) and (y^2), one starting from y^1_0, and another from y^2_0, with $y^1_0 > y^2_0$ and both points in the interval (x^*_0, x^*_1). We will have $y^1_1 > y^2_1$.

[36] It implies $(y, 2) \succ (x^*_1, 1) \sim (x^*_0, 0) \sim (x^*_2, 2)$

2) All points in intervals (x_1^*, x_2^*) are included by some y_1 from the sequence. This follows from construction.

Now, the recursive step:

For each point $y \in (x_i^*, x_{i+1}^*]$ with $i > 1$, take reflections of length $j \in \{\min\{T, i\}, (\min\{T, i\} - 1), .., 1\}$ conditional on those reflections being in the corresponding $(x_{i-j}^*, x_{i+1-j}^*]$[37].

For any y, at least one of such reflections must exist in the interval, in particular the one with the same length which created x_i^*: Let x_i^* have been a reflection of x_k^*, i.e, $(x_i^*, i-k) \sim (x_k^*, 0)$ with $i - k \leq \min\{T, i\}$.

Now, for each such reflection of $y \in (x_i^*, x_{i+1}^*]$ that lies in the proper interval, find the corresponding subsequence $\{y_{i-\min\{T,i\}}, .y_k, ..y_{i-1}\}$ that the reflection belongs to. Now consider the y_k-component of those subsequences and denote the smallest y_k-component from that collection of sequences as $x_y \in [x_k^*, x_{k+1}^*]$. Let $x_{x_{i+1}}$ be the corresponding point for $y = x_{i+1}^*$. Define the $1:1$ strictly increasing function f from $(x_i^*, x_{i+1}^*]$ to $(x_k^*, x_{k+1}^*]$ in the following way: For any $y \in (x_i^*, x_{i+1}^*]$, $f(y) = x_k^* + (x_y - x_k^*)\dfrac{(x_{k+1}^* - x_k^*)}{(x_{x_{i+1}} - x_k^*)}$. Now, define the next member y_i of any sequence $\{y_1, .y_k, ..y_{i-1}\}$ as $f^{-1}(y_k)$. The conditions (1) and (2) mentioned above are still satisfied for the extended sequence. Note that in every step of this iterative process, for every sequence that we construct, we have $(y_i, 0) \succsim (y_j, j - i)$ for $j > i$.

Substep 3.5: Having found $y_1, y_2, ..y_{n-1}$, for $i \leq -1$ define y_i recursively in the following way. Start by finding y_i' as the minimum of the set

$$\{y \in \mathbb{X} : (y, 0) \succsim (y_j, j - i), j = i + 1, i + 2, ... \max\{n - 1, i + T\}\}$$

Define x_i' as the minimum of the set

$$\{y \in \mathbb{X} : (y, 0) \succsim (x_j^*, j - i), j = i + 1, i + 2, ... \max\{n - 1, i + T\}\}$$

Define x_{i+1}'' as the minimum of the set

$$\{y \in \mathbb{X} : (y, 0) \succsim (x_j^*, j - i - 1), j = i + 2, i + 3, ... \max\{n, i + 1 + T\}\}$$

[37]There are only two possible cases, either the reflections lie in the interval, or, they are less than every point in that interval. This follows from $(x_{i+1-j}^*, 0) \succsim (x_{i+1}^*, j) \succsim (y, j)$.

Note that $x''_{i+1} \leq x^*_{i+1}$, with the equality being guaranteed for all $i < -1$.
Finally, for $i < -1$, define

$$y_i = x^*_{i+1} - (x^*_{i+1} - y'_i)\frac{(x^*_{i+1} - x^*_i)}{(x^*_{i+1} - x'_i)} \tag{9}$$

and for $i = -1$[38], define

$$y_{-1} \quad = \quad x^*_0 - (x'_0 - y'_{-1})\frac{(x^*_0 - x^*_{-1})}{(x''_0 - x'_{-1})}$$

Given $y^1_0 > y^2_0$ determines the order of $y^1_t > y^2_t$, for $t \in \{1, 2, ..n-1\}$, our inductive procedure makes sure that this holds true for all $t \leq -1$ too.
One can check for covering properties of the sequences by induction, just as before.
This completes the description of how the proof of Theorem 1 needs to be adapted in Step 3 of the proof of Theorem 2.

$\square$

Proposition 4: *If $\mathcal{U}, \mathcal{U}' \subset \mathcal{F}$ are such that $\bar{co}(\mathcal{U}) = \bar{co}(\mathcal{U}')$, and the functional form in (1) allows for a continuous minimum representation for both of those sets, then,* $\min_{u \in \mathcal{U}} u^{-1}(\delta^t u(x)) = \min_{u \in \mathcal{U}'} u^{-1}(\delta^t u(x))$.

Proof. We will prove this in 2 steps.
First we will show that for any set A, $\min_{u \in A} u^{-1}(\delta^t u(x)) = \min_{u \in \bar{A}} u^{-1}(\delta^t u(x))$, where $\bar{A}$ is the closure of the set A.
It is easy to see the direction that $\min_{u \in A} u^{-1}(\delta^t u(x)) \geq \min_{u \in \bar{A}} u^{-1}(\delta^t u(x))$.
We will prove the other direction by contradiction. Suppose, $\min_{u \in A} u^{-1}(\delta^t u(x)) > \min_{u \in \bar{A}} u^{-1}(\delta^t u(x))$. This would imply that there exists $v \in \bar{A} \backslash A$ and some $\epsilon > 0$, such that $v^{-1}(\delta^t v(x)) + \epsilon < u^{-1}(\delta^t u(x))$ for all $u \in A$. By definition of the topology of compact convergence and given that v belongs to the set of limit points of A, there must exist a sequence of functions $\{v_n\} \subset A$ which converges to v in the topology of compact convergence, i.e, for any compact set $K \subset \mathbb{R}_+$, $\lim_{n \to \infty} \sup_{x \in K} |v_n(x) - v(x)| = 0$. It can be shown that under this condition,

[38] $i = -1$ is a special case, as the upper bound of y_{-1} is x^*_0. An adjustment would be required in case in case x_n was readjusted to be equal to M, and $n \leq T$, in which case x^*_0 is larger than the the present equivalent x''_0 of (x^*_n, n).

$v_n^{-1}(\delta^t v_n(x))$ would also converge to $v^{-1}(\delta^t v(x))$ where $v_n \in \mathcal{U}$.[39] This constitutes a violation of $v^{-1}(\delta^t v(x)) + \epsilon < u^{-1}(\delta^t u(x))$ for all $u \in A$. Hence, it must be $\min_{u\in A} u^{-1}(\delta^t u(x)) = \min_{u\in \bar{A}} u^{-1}(\delta^t u(x))$.

As a second part of this proof, we will show that for any set A, $\min_{u\in A}(u^{-1}(\delta^t u(x))) = \min_{u\in co(A)}(u^{-1}(\delta^t u(x)))$.

It is easy to see that $\min_{u\in A}(u^{-1}(\delta^t u(x))) \geq \min_{u\in co(A)} u^{-1}(\delta^t u(x))$, as $A \subset co(A)$. We will again use proof by contradiction to show the opposite direction. We assume that there exists a $\bar{u} \in \bar{co}(A)$ and $(x,t) \in \mathbb{X} \times \mathbb{T}$, such that $\bar{u} = \sum_{i=1}^n \lambda_i u_i$, $\sum_{i=1}^n \lambda_i = 1$ and $\bar{u}^{-1}(\delta^s \bar{u}(y)) < \min_i u_i^{-1}(\delta^s u_i(y))$. This would imply that $u_i(\bar{u}^{-1}(\delta^s \bar{u}(y))) < \delta^s u_i(y)$ for all i.

Now,

$$\begin{aligned}
\delta^s \bar{u}(y) &= \delta^s \sum_i \lambda_i u_i(y) \\
&= \sum_i \lambda_i \delta^s u_i(y) \\
&> \sum_i \lambda_i u_i(\bar{u}^{-1}(\delta^s \bar{u}(y))) \\
&= \bar{u}(\bar{u}^{-1}(\delta^s \bar{u}(y))) \\
&= \delta^s \bar{u}(y)
\end{aligned}$$

This gives us a contradiction. Note that the equality right after the inequality comes from the definition of $\bar{u}$.

Hence, we have, $\min_{u\in A} u^{-1}(\delta^t u(x)) = \min_{u\in co(A)} u^{-1}(\delta^t u(x))$.

□

Proposition 5: *i) If there exists a concave function $f \in \mathcal{U}$, and if $\mathcal{U}_1$ is the subset of convex functions in $\mathcal{U}$, then* $\min_{u\in\mathcal{U}} u^{-1}(\delta^t u(x)) = \min_{u\in\mathcal{U}\setminus\mathcal{U}_1} u^{-1}(\delta^t u(x))$.

ii) If $u_1, u_2 \in \mathcal{U}$ and u_1 is concave relative to u_2, then, $\min_{u\in\mathcal{U}} u^{-1}(\delta^t u(x)) = \min_{u\in\mathcal{U}\setminus\{u_2\}} u^{-1}(\delta^t u(x))$.

[39] As, $v_n \to v$ in the topology of compact convergence, $v_n \to v$ point wise, hence, $\delta^t v_n(x) \to \delta^t v(x)$. Now, as $v_n^{-1} \to v^{-1}$ compact convergence (proof later in the appendix), $v_n^{-1}(\delta^t v_n(x)) \to v^{-1}(\delta^t v(x))$.

Proof. If a function u is convex,

$$\begin{aligned} u^{-1}(\delta^t u(x)) &= u^{-1}(\delta^t u(x) + (1-\delta^t)u(0)) \\ &\geq u^{-1}(u(\delta^t x + (1-\delta^t)0)) \\ &= \delta^t x \end{aligned}$$

Similarly for concave f , we would have, $f^{-1}(\delta^t f(x)) \leq \delta^t x$ which completes the proof of part (i). Note that this result is expected given concave functions give rise to more conservative present equivalents.
For part (ii), note that

$$\begin{aligned} u_1^{-1}(\delta^t u_1(x)) &= u_1^{-1}(\delta^t u_1(u_2^{-1}(u_2(x)))) \\ &\leq u_1^{-1}(u_1(u_2^{-1}(\delta^t u_2(x)))) \\ &= u_2^{-1}(\delta^t u_2(x)) \end{aligned}$$

Where the inequality arises from the fact that u_1 is concave relative to u_2.

□

Proposition 6: *Let $\succsim_1$ and $\succsim_2$ be two binary relations which allow for minimum representation w.r.t sets $\mathcal{U}_{\delta,1}$ and $\mathcal{U}_{\delta,2}$ respectively. The following two statements are equivalent:*
i) $\succsim_1$ allows a higher premium to the present than $\succsim_2$.
ii) Both $\mathcal{U}_{\delta,1}$ and $\mathcal{U}_{\delta,1} \cup \mathcal{U}_{\delta,2}$ provide minimum representations for $\succsim_1$.

Proof. The direction from (i) to (ii): Consider any $(x,t) \in \mathbb{X} \times \mathbb{T}$ such that $(x,t) \sim_1 (y,0)$. Using (i), we must have, $(x,t) \succsim_2 (y,0)$.
Hence,

$$\begin{aligned} \min_{u \in \mathcal{U}_{\delta,2}} u^{-1}(\delta^t u(x)) &\geq y \\ \implies \min_{u \in \mathcal{U}_{\delta,1} \cup \mathcal{U}_{\delta,2}} u^{-1}(\delta^t u(x)) &= y \end{aligned}$$

Hence,

$$\min_{u \in \mathcal{U}_{\delta,1} \cup \mathcal{U}_{\delta,2}} u^{-1}(\delta^t u(x)) = \min_{u \in \mathcal{U}_{\delta,1}} u^{-1}(\delta^t u(x)) \tag{10}$$

To go in the opposite direction, let us assume, $(x,t) \succsim_1 (y,0)$.
Given, (10), it must be that

$$\begin{aligned}
\min_{u\in\mathcal{U}_{\delta,1}\cup\mathcal{U}_{\delta,2}} u^{-1}(\delta^t u(x)) &= \min_{u\in\mathcal{U}_{\delta,1}} u^{-1}(\delta^t u(x)) \geq y \\
\implies u^{-1}(\delta^t u(x)) &\geq y \quad \forall u \in \mathcal{U}_{\delta,1}\cup\mathcal{U}_{\delta,2} \\
\implies u^{-1}(\delta^t u(x)) &\geq y \quad \forall u \in \mathcal{U}_{\delta,2} \\
\implies \min_{u\in\mathcal{U}_{\delta,2}} u^{-1}(\delta^t u(x)) &\geq y
\end{aligned}$$

Hence, $(x,t) \succsim_2 (y,0)$, which completes the proof.

□

Proposition 7. *Eventual stationarity is satisfied by β-δ discounting, but not hyperbolic discounting.*

Proof. Now for any $x > z > 0 \in X$, choose $t_1 > \log_{\frac{1}{\delta}}\left(\frac{u(x)}{u(z)}\right)$.

$$\begin{aligned}
t_1 &> \log_{\frac{1}{\delta}}\left(\frac{u(x)}{u(z)}\right) \\
\iff \left(\frac{1}{\delta}\right)^{t_1} &> \frac{u(x)}{u(z)} \\
\implies u(z) &> \delta^{t_1}u(x) > \beta\delta^{t_1}u(x) \\
\implies \beta\delta^t u(z) &> \beta\delta^{t+t_1}u(x) \\
(z,t) &\succ (x,t+t_1)
\end{aligned}$$

Also, $(x,0) \sim (x_t,t)$ implies, $u(x) = \beta\delta^t u(x_t)$, which implies,

$$\begin{aligned}
u(z) &> \delta^{t_1}u(x) = \beta\delta^{t+t_1}u(x_t) \\
(z,0) &\succ (x_t,t+t_1)
\end{aligned}$$

This shows that $\beta-\delta$ does indeed satisfy A5.
Now consider the simple variant of Hyperbolic discounting model when $\alpha = \gamma = 1$. Fix any felicity function u and $x > z > 0 \in X$. We will show that there does not exist t_1,such that $(z,t) \succ (x,t+t_1)$ for all $t \geq 0$.

$$\begin{aligned} (z,t) &\succ (x,t+t_1) \text{ for all } t \geq 0 \\ \iff \frac{u(z)}{1+t} &> \frac{u(x)}{1+t+t_1} \text{ for all } t \geq 0 \\ \iff \frac{1+t+t_1}{1+t} &> \frac{u(x)}{u(z)} \text{ for all } t \geq 0 \\ \iff 1+\frac{t_1}{1+t} &> \frac{u(x)}{u(z)} \text{ for all } t \geq 0 \end{aligned}$$

Note that the last statement is not possible, as for fixed t_1 the LHS$\downarrow 1$ as $t \uparrow \infty$, whereas, the RHS is always a fixed number, that is strictly greater than one. Hence, hyperbolic discounting does not satisfy A5. □

Theorem 3: The following two statements are equivalent:

i) The relation $\succsim$ satisfies properties B0-B5.

ii) There exists a continuous function $F : \mathbb{X} \times \mathbb{P} \times \mathbb{T} \to \mathbb{R}$ such that $(x,p,t) \succsim (y,q,s)$ if and only if $F(x,p,t) \geq F(y,q,s)$. The function F is continuous, increasing in x, p and decreasing in $t \in \mathbb{T}$. There exists a unique $\delta \in (0,1)$ and a set $\mathcal{U}$ of monotinically increasing continuous functions such that $F(x,p,t) = \min_{u \in \mathcal{U}} u^{-1}(p\delta^t u(x))$ and $u(0) = 0$ for all $u \in \mathcal{U}$.

Proof. Showing that (ii) implies (i) is omitted.
We will prove the direction (i) to (ii) in the following steps.

Step 1: Recall the Probability Time Tradeoff axiom: for all $x, y \in \mathbb{X}$, $p, q \in (0,1]$,and $t, s \in \mathbb{T}$, $(x, p\theta, t) \succsim (x, p, t+\Delta) \implies (y, q\theta, s) \succsim (y, q, s+\Delta)$.
This axiom has calibration properties that we will use. Given Monotinicity, $(x,1,0) \succ (x,1,1) \succ (x,0,0)$ for any $x \in \mathbb{X}$. By continuity, there must exist a probability $\delta \in (0,1)$ such that $(x,\delta,0) \sim (x,1,1)$. Note that Probability-Time Tradeoff Axiom helps us write $(x,\delta,\tau+1) \sim (x,1,\tau)$ for all $x \in \mathbb{X}$ and $\tau \in \mathbb{T}$. This δ is our candidate single period discount factor, and we will now show that $(x,q,t) \sim (x,q\delta^t,0)$ for arbitrary $t > 0$.
For integer t's this follows by induction.
For $t = \frac{1}{b}$ where b is a positive integer, there must exists $\delta_1 \in \mathbb{P}$ such that $(x,1,\dfrac{1}{b}) \sim (x,\delta_1,0)$. Now applying Probability Time Tradeoff (PTT) repeatedly b times, $(x,1,1) \sim (x,\delta_1^b,0)$. Therefore, $\delta_1 = \delta^{\frac{1}{b}}$ and $(x,\delta^{\frac{1}{b}},0) \sim (x,1,\dfrac{1}{b})$.
The case for rational and then real t follows.

The axioms on $\mathbb{X} \times \mathbb{P} \times \mathbb{T}$ domain imply completeness, transitivity, continuity, risk monotonicity, Weak Certainty Bias for a preference defined on the reduced domain of $\mathbb{X} \times \mathbb{P}$. Proving Theorem 3, is equivalent to proving that there is a minimum representation on $\mathbb{X} \times \mathbb{P}$ of the form $\min_{u \in \mathcal{U}}(u^{-1}(pu(x)))$. Henceforth, we are going to concentrate on finding a representation of the reduced domain of $\mathbb{X} \times [0,1]$.

Step 2: The rest of the proof will have a similar flavor to the ones the reader has already encountered. For every $x^* \in \mathbb{X}$, we are going to provide an increasing utility function u on $[0, M]$ which would respect all the relations of the form $(x,p) \succsim (y,1)$, i.e, have $pu(x) \geq u(y)$ and also have $pu(y) = u(x^*)$ for all $(x^*,1) \sim (y,p)$. We skip the details as they are very similar to the continuous time case from Theorem 1.

Finally, we prove a result that we have used in our Uniqueness propositions.

□

Proposition 8. *Let f_n be a set of bijective, increasing, continuous functions. Let $f_n \to f$ "locally uniformly"/ compactly (equivalent notions in $\mathbb{R}^n$.), where f is bijective, increasing, continuous. Then, $f_n^{-1} \to f^{-1}$ compactly.*

Proof. Consider the composite function $g_n = f_n o f^{-1}$. Note that g_n is also bijective, increasing, continuous. As f_n converges locally uniformly to f, g_n converges locally uniformly to the identity function $g(x)$.
To see this, note that for any $\epsilon_1 > 0$

$$\begin{aligned}\sup_{x\in[c,d]} |g_n(x) - g(x)| &= \sup_{x\in[c,d]} |f_n(f^{-1}(x)) - f(f^{-1}(x))| \\ &= \sup_{y\in[f^{-1}(c),f^{-1}(d)]} |f_n(y) - f(y)| \\ &\leq \epsilon_1\end{aligned}$$

for $n \geq N_0$ for some N_0.
Choose an interval $[a,b]$. Now, there would exist n_1, n_2 such that $g_n(a-1) \leq a$ and $g_n(b+1) > b$ for $n \geq n_1$ and $n \geq n_2$ respectively. Similarly, there exists n_3 such that $\sup_{x\in[a-1,b+1]} |g_n(x) - g(x)| < \epsilon$ for $n \geq n_3$.
Finally, for $N \geq max\{n_1, n_2, n_3\}$

$$\begin{aligned}\sup_{x\in[a,b]}|g_n^{-1}(x)-g(x)| &\leq \sup_{x\in[g_n(a-1),g_n(b+1)]}|g_n^{-1}(x)-x| \\ &= \sup_{t\in[a-1,b+1]}|g_n^{-1}(g_n(t))-g(t)| \\ &= \sup_{t\in[a-1,b+1]}|t-g(t)| \\ &< \epsilon\end{aligned}$$

Therefore, $g_n^{-1} = fof_n^{-1}$ converges locally uniformly to the identity function. Therefore, f_n^{-1} converges locally uniformly to f^{-1}. □

Appendix IV: Result on Future Bias

There is is an alternative relaxation of Stationarity that is complementary to WPB: when combined with Weak Future Bias, it yields the Stationarity Axiom.

A4*: **WEAK FUTURE BIAS:** If $(x,0)\succsim(y,t)$ then, $(x,t_1)\succsim(y,t+t_1)$ for all $x,y\in X$ and $t,t_1\in\mathbb{T}$.

Theorem 5. *Let $\mathbb{T}=[0,\infty)$ and $\mathbb{X}=[0,M]$. The following two statements are equivalent:*

i) The relation $\succsim$ satisfies properties A0-A3 and A4.*

ii) There exists a set $\mathcal{U}_\delta$ of monotonically increasing continuous functions such that

$$F(x,t)=\max_{u\in\mathcal{U}}u^{-1}(\delta^t u(x))$$

represents the binary relation $\succsim$. The set $\mathcal{U}_\delta$ has the following properties: $u(0)=0$ and $u(M)=1$ for all $u\in\mathcal{U}_\delta$. $F(x,t)$ is continuous.

Weak Future Bias is characterized by a weakly optimistic attitude towards uncertainty about tastes in the future. The proof is similar to that of Theorem 1, and is omitted.

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