Pairwise stable matching in large economies*

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Abstract

We formulate a stability notion for two-sided pairwise matching problems with individually insignificant agents in distributional form. Matchings are formulated as joint distributions over the characteristics of the populations to be matched. Spaces of characteristics can be high-dimensional and need not be compact. Stable matchings exist with and without transfers, and stable matchings correspond precisely to limits of stable matchings for finite-agent models. We can embed existing continuum matching models and stability notions with transferable utility as special cases of our model and stability notion. In contrast to finite-agent matching models, stable matchings exist under a general class of externalities.

JEL-Classification: C62, C71, C78, D47.

Keywords: Stable matching; economies in distributional form; large markets.

1 Introduction

This paper provides a novel stability notion for pairwise matchings in two-sided matching markets modeled via the joint statistical distribution of the characteristics of the agents involved. Stable matchings exist in full generality with and without transfers between agents and even in the presence of externalities. The stable matchings in our model exactly capture the limit behavior of stable matchings in large finite matching markets in terms of the joint distribution of characteristics. In matching models with transfers, our stability notion is equivalent to existing stability notions specific to settings with transfers. We also show that the model can be reformulated in terms of individual agents.

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In the traditional theory of stable matchings, agents can be split into two sides. A matching, in the simplest setting, specifies which agent on one side is matched to which agent on the other side, if at all. The matching is stable if no two agents on opposing sides would rather be matched to each other than to their current matches, and no agent would rather be alone. This is, in essence, the marriage model of Gale and Shapley (1962). Extensions allow for transfers between the matched agents as in Shapley and Shubik (1971) and Demange and Gale (1985). Stable matching models provide the natural frictionless benchmark for the analysis of markets in which interacting participants can neatly be divided into two groups, characteristics of participants matter, and all interaction is between matched agents. One can match workers to firms, students to schools, and medical residents to hospitals. None of these markets can be separated from the rest of the economy, but integrating matching theory with other economic models poses serious technical problems due to matching theory being fundamentally discrete. Wine, milk, and hours worked are divisible; people are not. From the perspective of matching theory, interactions with the rest of the economy amount to externalities within the matching market. But the combination of indivisibilities and externalities in matching problems is in general incompatible with the existence of stable matchings. We deal with the problem the way most economists deal with the molecular indivisibilities found in wine and milk: Scale the analysis so that even people look divisible. Matching markets are taken to be so large that individuals are negligible relative to market size. The economy is large; people are not.

Our approach is guided by the following desiderata:

Compatibility There is a rich existing literature on stable matching in large markets under transferable utility. We do not want to provide an alternative to this literature; we want to generalize it.

Generality We want to provide a unified framework for the study of stable matchings. As such we allow for non-transferable utility, transferable utility, and everything in between. Characteristics need not lie in a finite, one-dimensional, or compact set.

Approximability A model in which agents are negligible relative to market size is at best an approximation to a model in which finitely many agents have little influence. Stable matchings in the large market model should be interpretable as the limit of stable matchings of familiar finite-agent matching models.

Embeddability Embedding a matching model in a wider economic model reduces everything outside the matching to externalities, so stable matchings should exist even in the presence of externalities.

Existence Stable matchings should always exist. Otherwise, our model models nothing.

We show that these desiderata can be satisfied by choosing the right model and stability notion. The most popular approach to modeling large economies is the individualistic approach introduced by Aumann (1964) in general equilibrium theory and by Schmeidler (1973) in non-cooperative game theory. In this framework, the economy is represented by a nonatomic probability space of agents and a function that maps agents to their characteristics. We show below in Example 1 that naively adopting this approach to matching theory gives us a model in which stable matchings need not exist. Such an individualistic model is also hard to relate to finite-agent models. If the number of agents changes, the dimension of the model changes and there is no common space for comparisons.

These problems can be overcome by adopting the distributional approach to modeling large economies. The distributional approach was introduced by Hart, Hildenbrand, and Kohlberg (1974) and Hildenbrand (1975) in general equilibrium theory and by Mas-Colell (1984) in non-cooperative game theory. Closely related on a technical level is the distributional formulation of Bayesian games by Milgrom and Weber (1985). The distributional approach to modeling large economies disposes of the agents completely and only uses the distribution of their characteristics as the data of the model. Since one can compare distributions of characteristics independently of the set of agents having the characteristics, this allows us to relate the limit model and its stable matchings to finite-agent matching models and their stable matchings. Distributional matching models are not new; the distributional approach has already proven useful in matching with transferable utility and even in unifying the econometric treatment of stable matchings; see Chiappori and Salanié (2016). But the distributional approach has not yet proven useful for the analysis of matching markets without transfers. The problem is that it is unclear what a stable matching should be in a distributional setting without transfers. In models with transfers, stability has been defined in terms of the utility level an agent of a specific type receives. Implicitly, it is assumed that agents of the same type end up equally well off in any stable matching. Such a "law of one price" can only be guaranteed with transfers. We show that, nevertheless, stable matchings can be defined in a natural way so that stable matchings always exist (Theorem 1) and exist even with externalities (Theorem 5), that stable matchings in the large economy correspond exactly to limits of stable matchings in finite matching models that approximate the large economy under the topology of weak convergence of measures (Theorem 2), and that our stability notion agrees with the widely used existing stability notion for distributional models with transfers mentioned above when transfers are available (Theorem 3 and Theorem 4). Although we formulate everything in terms of distributions, we show that one can reinterpret the distributional model in individualistic terms (Theorem 7) by a purification argument.

Since many of the main ideas are more transparent in a simplified continuum version of the marriage model of Gale and Shapley (1962), we provide an overview in that setting in the next section. Throughout the paper, we rely heavily on concepts from the topology of metrizable spaces and weak convergence of measures. A mathematical appendix at the beginning of the supplemental online appendix contains the essentials. Proofs that are short and instructive are in the main text, the rest in Section 9. There is a supplemental online appendix containing the mathematical appendix and more specialized material.

Related Literature

Much of the literature on stable matching in large economies focuses on the asymptotic behavior of large finite matching models without ever using a limit model. We refer to

Kojima (2017) for an overview of this approach; here, we focus on limit matching models for which existence results are available. Owing to our novel approach to stability, our paper is the first to prove the existence of stable matchings in a large economy framework that allows for (but does not require) non-transferable utility with general spaces of characteristics and both sides representing individually insignificant agents. Other large economy models of stable matching fall into three categories:

1. Finite-type models. Baïou and Balinski (2002), Echenique, Lee, and Shum (2010) and Echenique, Lee, Shum, and Yenmez (2013) prove, among other things, the existence of stable matchings in a distributional marriage model with finitely many types. Azevedo and Hatfield (2015) prove the existence of stable matchings and nonemptiness of the core for a many-to-many matching model with finitely many contracts and types. In comparison to finite-agent models, their model allows for complementarities. Galichon, Kominers, and Weber (2019) prove the existence of stable matchings in a general two-sided finite-type model that allows for both non-transferable utility and (imperfectly) transferable utility and can be used for the econometric estimation of matching models. With finitely many types, stable matchings can simply be defined in terms of some blocking pair having positive mass in the distribution, something not possible when no type has positive mass.

2. One-sided continuum models. Azevedo and Leshno (2016) have an individualistic model in which a continuum of students is matched to a finite number of colleges and use it to prove that there is generically a unique stable matching under a richness condition on preferences and relate stable matchings to simple conditions for markets to clear. Che, Kim, and Kojima (2019) study a similar setting in a distributional framework and show, using fixed-point methods from nonlinear functional analysis, that stable matchings exist even with complementarities. Fuentes and Tohmé (2018) show that the existence result of Che, Kim, and Kojima (2019) for finitely many colleges implies existence for countably many colleges. When colleges are large, one can directly apply the usual notion of stability, so the conceptual problems of agents on both sides being insignificant play no role. Indeed, each blocking possibility can be realized by one large college and a mass of students. When agents on both sides are individually insignificant, this is not possible.

3. Two-sided continuum models with transfers. In the case of (perfectly) transferable utility and finitely many agents, stable matchings can be identified with solutions to the dual of a linear programming problem. There exists an infinite-dimensional version of this linear programming problem in which one optimizes over spaces of measures, the optimal transport problem of Kantorovich.[1] Using a duality result for the Kantorovich optimal transport problem, Gretsky, Ostroy, and Zame (1992, 1999) develop a distributional approach to stable matching under transferable utility for general compact metric type spaces. Chiappori, McCann, and Nesheim (2010) provide further results and the generalization to separable, completely metrizable type spaces.

[1] An introduction to optimal transport theory geared towards economists and econometricians is given by Galichon (2016). For advanced material on the mathematics of optimal transport, see Villani (2003) and Villani (2009).

An individualistic model of cooperative games that subsumes stable matching with imperfectly transferable utility is given by Kaneko and Wooders (1986). An existence result under the assumption that characteristics lie in a compact metric space is given by Kaneko and Wooders (1996). Feasible payoffs in the model may only be approximately realizable, though, so the model is best interpreted as a model of approximately stable matchings (or cores, in the more general setting).

Closest to our model is the distributional model of stable matching with imperfectly transferable utility and compact metric type spaces by Nöldeke and Samuelson (2018).[2] Nöldeke and Samuelson develop a general nonlinear duality theory and apply it to contract theory and matching theory. Their matching model is less general than ours, and their methods cannot be applied to models without transfers, but the additional structure of their model allows them to obtain results on the lattice structure of stable matchings.

With transferable and imperfectly transferable utility, one traditionally defines stable matchings in terms of the payoffs a type gets in a matching and this is how all papers mentioned above define stability. This requires that payoffs in a matching depend on types only, there is equal treatment of types and this is assumed in the papers just mentioned. Without transfers, this is generally not possible, and stability has to be defined differently. However, we can prove using our stability notion that equal treatment and a stronger form of equal treatment is a consequence of stability and transfers; see Section 5. In this paper, the "law of one price" is a result, not an assumption. This allows us to show that our stability notion is equivalent to the ones used by Gretsky, Ostroy, and Zame (1992, 1999), Chiappori, McCann, and Nesheim (2010), and Nöldeke and Samuelson (2018).[3]

Another literature deserves mention here: The literature on ex-ante investments in competitive matching markets. Many match-relevant investments are made before agents join matching markets. People go to university before they know which firm they are going to work for eventually. Under imperfect competition, there will be a hold-up problem and resulting inefficiencies. To isolate whether other inefficiencies are possible, one needs a competitive matching model in which the classic hold-up problem cannot occur, and perfect competition rarely obtains with finitely many agents. A number of papers have studied whether other inefficiencies remain under perfectly competitive matching by using continuum models; the following list is not complete. Cole, Mailath, and Postlewaite (2001) and Iyigun and Walsh (2007) study investments under transferable utility in the optimal transport framework but focus on one-dimensional characteristics. Dizdar (2018) uses the general optimal transport framework to study investments under transferable utility. Peters and Siow (2002) study investment in the non-transferable utility context in which characteristics are one-dimensional and all agents rank agents on the other side the same way. Matchings are assumed to be assortative, but no explicit stability argument is given. We see in Example 2 that our model and stability notion provide appropriate foundations. Nöldeke and Samuelson (2015) study investment under imperfectly transferable utility. The present paper provides a unified competitive matching framework for this literature in which all these models can be embedded and analyzed with the same solution concept.

[2] We became aware of each others' work only in July 2016 at the Game Theory Society World Congress.

[3] In the optimal transport context, we only nest those models in which the surplus function is continuous.

2 Overview

In this section, we provide an overview of the central ideas and results of this paper in a setting without transfers, both sides of the market having the same size, and individual rationality constraints not binding. This allows us to focus on the central arguments without getting lost in details. The only major part of the paper we do not discuss here are the results on equal treatment of types that rely crucially on the availability of transfers.

The data of the classical, finite marriage model consist of a group of agents divided into two subgroups, women and men.[4] Women are assumed to have preferences over men and the option to stay single and men are assumed to have preferences over women and the option to stay single. In the simplest case, the one we focus on in this section, there are exactly as many women as men, every woman prefers every man to being single, and every man prefers every woman to being single. In that case, a matching simply pairs women and men in a one-to-one way and the matching is stable if there is no woman and no man who prefer each other strictly to their respective partners in the matching. All these definitions make sense even when the set of agents is infinite and one might try to obtain a continuum version of the marriage model by simply assuming there to be a continuum of agents on both sides and naively applying the usual stability notion. However, unlike in the case of finitely many agents, stable matchings need not exist as the following example shows.[5]

Example 1. Let the set of women be $A_W = [0,1]$ and the set of men be $A_M = [0,1/2] \times \{1,2\}$. All women have the same preferences and all men have the same preferences. Every woman is indifferent between $(x,1)$ and $(x,2)$, but prefers $(x,1)$ to $(y,1)$ if $x > y$. Men prefer x to y if $x > y$. Everyone prefers being matched to someone to not being matched at all. A matching in this context is simply a bijection $f : A_W \to A_M$. Since stable matchings will not exist in this example, we refrain from putting even more restrictions on a matching such as preserving measure. The matching f is stable if there is no blocking pair, that is, there are no $a \in A_W$ and $b \in A_M$ such that a prefers b to $f(a)$ and b prefers a to $f^{-1}(b)$.

Now let f be any matching; we show it is not stable. Pick any $x \in [0,1/2]$. Without loss of generality, assume that $f^{-1}(x,1) < f^{-1}(x,2)$. Let $a \in A_W$ satisfy $f^{-1}(x,1) < a < f^{-1}(x,2)$ and let $(y,m) = f(a)$. We must either have $y > x$ or $y < x$. If $y > x$, then $f^{-1}(x,2)$ and (y,m) can block the matching. If $y < x$, then a and $(x,1)$ can block the matching.

If all women rank all men the same way and all men rank all women the same way, as is the case in the example above, we would expect in analogy with finite-agent matching models a stable matching to be positive assortative; higher-ranked women are matched to higher-ranked men. But this is not possible here since it would require woman x to be matched with both $(x,1)$ and $(x,2)$, which is impossible if we literally take x, $(x,1)$, and $(x,2)$ to be indivisible agents.

[4]We focus on heterosexual marriage markets for ease of exposition. Clearly, this is not the only application of interest.

[5]The absence of transfers in the example is inessential. Example 3 in the online appendix redoes the analysis under transferable utility.

But one does not have to literally think of a continuum of indivisible agents. Instead of indivisible agents, we work with perfectly divisible types of agents; we formulate our matching model in terms of the distributions of the characteristics of agents. In this, we follow Gretsky, Ostroy, and Zame (1992, 1999). Each agent has, implicitly, a type that specifies both their preferences over types of agents on the other side and characteristics agents on the other side might care about. The matching model is then given in terms of distributions over types. Formally, there are two sets of *types* W and M. For each $w \in W$ there is strict preference $\succ_w$ over M and for each $m \in M$ there is a strict preference $\succ_m$ over W. These preferences are assumed to be acyclic;[6] this is for example guaranteed if they are irreflexive and transitive. We assume in this section that everyone finds everyone on the other side acceptable, so we need not include the option of staying single in the preferences. We assume there are Polish (that is, separable and completely metrizable) topologies on W and M, respectively, such that the sets

$$\{(w, m, m') \in W \times M \times M \mid m \succ_w m'\}$$

and

$$\{(m, w, w') \in M \times W \times W \mid w \succ_m w'\}$$

are open in the corresponding product topologies, essentially a continuity assumption. The first condition, for example, says that if a woman of type w prefers a man of type m to a man of type m' then a woman with a type sufficiently similar to w prefers a man of type sufficiently similar to m to a man of type sufficiently similar to m'. The assumption guarantees that the topologies on types provide the economically correct notion of similarity. Finally, we close the model by specifying two Borel probability measures ν_W and ν_M on W and M, respectively, as the *population distributions*. That ν_W and ν_M are probability measures means that we have the same number of women as men in our model, a restriction we will drop for our general model.

For now, we define matchings so that everyone is matched, and a matching simply specifies the joint distribution over matched types. Formally, a *matching* μ is a Borel probability measure on $W \times M$ with W-marginal ν_W and M-marginal ν_M. The marginal conditions correspond to accounting identities. For example, that μ has W-marginal ν_W means that for every Borel set $B \subseteq W$, $\nu_W(B) = \mu(B \times M)$, so that the number of couples in which the types of the woman lie in B and the type of the man lies in M is equal to the number of women whose type lies in B. Given the underlying assumption that all women are matched in a matching, this must clearly be the case.

Having disposed of agents, we have also disposed of blocking pairs and therefore of the usual stability notion. This is not a problem if there are only finitely many types, so that all blocking possibilities occur, if at all, with positive mass. With transfers, other definitions of stability are available but those do not translate to the non-transferable utility setting in an obvious way. But the central point of this paper is that stability can be defined in a natural way and that our stability notion actually agrees with the stability notion used for distributional matching models with transfers; see Section 5. A

[6] Recall that the relation $\succ$ on the set S is *acyclic* if there is no finite sequence $\langle s_1, s_2, \ldots, s_n \rangle$ with values in S such that $s_1 \succ s_2 \succ \cdots \succ s_n \succ s_1$.

matching μ is nothing but a probability distribution over the pairs of types of matched agents. We define a matching to be stable if the matching with finitely many types obtained by independently sampling finitely many pairs of types from the distribution μ is stable with probability one. For this, it suffices to look at samples consisting of two sampled pairs. Denoting the probability distribution on $(W \times M) \times (W \times M)$ of two independent samples from $W \times M$ according to μ by $\mu \otimes \mu$ (the product measure, see the Mathematical Appendix), the matching μ is *stable* if the set of all pairs of couple types $((w, m), (w', m')) \in (W \times M) \times (W \times M)$ such that

$$m' \succ_w m \text{ and } w \succ_{m'} w'$$

or

$$m \succ_{w'} m' \text{ and } w' \succ_m w$$

has $\mu \otimes \mu$-probability zero. It does not really matter which couple type is sampled first and which couple type is sampled second. As a consequence, it suffices to check only one of the two conditions.

With this notion of stability, stable matchings always exist under minimal regularity assumptions; Theorem 1. Our proof reduces the existence problem to the already solved existence problem for finite-agent matching problems via finite approximations and a compactness argument.

We illustrate our stability notion for a simple test case for which there is broad agreement on what a stable matching should look like. We take a model in which all women rank all men the same way and all men rank all women the same way. It represents the Example 1 in a distributional way and does admit a stable matching. Such a model has already been used in Peters and Siow (2002) and, more recently, Diamond and Agarwal (2017). However, existing papers plainly assume stable matchings are positive assortative in analogy with finite-agent matching models without providing a stability argument for the continuum model. Our stability notion delivers just that.

Example 2. We let $W = M = [0, 1]$ and $\nu_W = \nu_M$ be the uniform distribution.[7] Moreover, everyone prefers to be matched to someone with a higher number and being matched to not being matched at all. Given the structure of the problem, we would expect positive assortative matching to be the only stable matching. If F is the two-dimensional cumulative distribution function of the matching μ, this amounts to $F(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$. If μ fails to be positive assortative, there will be some x such that, with obvious notation, $\mu(w < x, m > x) > 0$ and $\mu(w > x, m < x) > 0$, showing μ to be instable. Since stable matchings always exist, the positive assortative matching is the unique stable matching.

Our stability notion allows us to relate stable matchings in the continuum model to stable matchings in finite-agent matching models. Let μ be a stable matching for our general continuum model. We get with probability one a finite-type stable matching from

[7] The restriction to uniform distributions of types is less restrictive than it might seem. As long as the distributions of types on the real line have no atoms, the same argument works by interpreting x and y not as types but as quantiles of types. The relevant material on quantile functions and copulas may be found in Galichon (2016, Appendix C).

n independent draws from μ given by the sample distribution. We can interpret the resulting matching as a stable matching between n agents on either side. Moreover, it holds with probability one that for large n the resulting finite type stable matching is close to μ in the topology of weak convergence (see the Mathematical Appendix.) This way, we can interpret μ as the limit of stable matchings with finitely many agents. Indeed, the stable matchings in our continuum model are exactly the limits of finite-agent matching models from the distributional point of view; Theorem 2. Therefore, every other stability notion that captures the limiting distributional behavior of stable matchings of large finite-agent models must be equivalent to ours.

It is useful to interpret the stable matchings of the continuum model not just as approximations of stable matchings for large finite matching models but as genuine matchings of infinitely many agents. The distributional model is easy to work with, but a model with agents is easier to interpret. We will often explain concepts in terms of agents even though our model formally has none. One can extend the model to a model in which actual indivisible individuals are matched. Example 1 above shows that just interpreting types as individuals will not do. However, it is possible to find probability spaces $(A_W, \mathcal{A}_W, \tau_W)$ and $(A_M, \mathcal{A}_M, \tau_M)$ and measurable type functions $t_W : A_W \to W$ and $t_M : A_M \to M$ satisfying $\nu_W = \tau_W \circ t_W^{-1}$ and $\nu_M = \tau_M \circ t_M^{-1}$, respectively, such that for any matching μ there exists a measurable bijection $\phi : A_W \to A_M$ with a measurable inverse that preserves measure and induces the distributional matching; the function $w \mapsto \big(t_W(w), t_M(\phi(w))\big)$ has distribution μ under τ_W. The general version of this result is Theorem 7. In words, ϕ matches the actual agents in A_W with the agents in A_M in a one-to-one fashion and induces the distributional matching μ. We can, therefore, think of our distributional matchings as the empirical distributions of deterministic matchings of individuals. Looking at our stability notion through the individualistic lens, a matching is stable if the probability that a randomly chosen woman and a randomly chosen man can form a blocking pair is zero. Blocking pairs may exist, but they will have a hard time finding each other.

So far, nothing in the continuum model has allowed us to go beyond what is possible in finite-agent models. An area where this is possible is matching with externalities. We focus on wide-spread externalities that only depend on the matching itself and therefore show up at the distributional level. Implicitly, we will assume that the population is so large that no pair of agents has an influence on the distribution and, therefore, the externality: Individuals take externalities as given. Let $\mathcal{M}$ be the space of matchings, endowed with the topology of weak convergence (see the Mathematical Appendix.) Now, for each $w \in W$, the preference relation $\succ_w$ is defined on $M \times \mathcal{M}$ and for each $m \in M$, the preference relation $\succ_m$ is defined on $W \times \mathcal{M}$. Even though we assume that no agent can influence the matching, welfare analysis requires that every agent has preferences also over the matching they are part of. We strengthen the assumption that preferences are acyclic to preferences being asymmetric and negatively transitive. The corresponding continuity assumption for the extended model is that the sets

$$\{(w, m, m', \mu, \mu') \in W \times M \times M \times \mathcal{M} \times \mathcal{M} \mid (m, \mu) \succ_w (m', \mu')\}$$

and

$$\{(m, w, w', \mu, \mu') \in M \times W \times W \times \mathcal{M} \times \mathcal{M} \mid (w, \mu) \succ_m (w', \mu')\}$$

are open in the corresponding product topologies. Everything else is kept as before. We now call the matching μ *stable* if the set of all pairs of couple types $((w, m), (w', m')) \in (W \times M) \times (W \times M)$ such that

$$(m', \mu) \succ_w (m, \mu) \text{ and } (w, \mu) \succ_{m'} (w', \mu)$$

or

$$(m, \mu) \succ_{w'} (m', \mu) \text{ and } (w', \mu) \succ_m (w, \mu)$$

has $\mu \otimes \mu$-probability zero. In this setting, stable matchings still exist; Theorem 5. The indivisibilities that might prevent existence with finitely many agents play no role at our distributional level. The crucial step in the existence proof is proving existence first for the case in which W and M are finite. We prove this case by a topological fixed-point argument. Jagadeesan (2017) has previously shown,[8] in a somewhat different context that involves further complications, that one can formulate a suitable topological analog to the order-theoretic method used by Fleiner (2003) and Hatfield and Milgrom (2005) to obtain the existence of stable matchings for discrete distributional matching-models with non-trivial indifferences. Since the space of matchings and thus externalities is connected, nontrivial indifferences are unavoidable. The approach is robust to taking account of externalities in the preferences and this is how our proof works.

3 The Model and Stability

In this section, we introduce an extended matching model and prove the existence of stable matchings. The extended matching model allows for unbalanced markets, binding individual rationality constraints, and contract choices within matched pairs.

We start by defining the environment. As before, the model-relevant characteristics of agents on both sides of the market are given by nonempty sets of *types* W and M, respectively. To allow agents to stay single, we use extended spaces $W_\emptyset = W \cup \{\emptyset\}$ and $M_\emptyset = M \cup \{\emptyset\}$ with $\emptyset \notin W \cup M$ being a dummy type. An agent matched with an agent of type $\emptyset$ is really just an unmatched agent. We call a pair in $W_\emptyset \times M_\emptyset$ a *couple type*. Matched agents can enter contracts between them, and the contracts available to them may depend on their couple type. Formally, there is a set C of *contracts* and a *contract correspondence* $\mathbb{C} : W_\emptyset \times M_\emptyset \to 2^C$ specifying the set of contracts actually available to the matched agents. Depending on the context, contracts might stand for intra-household allocations, transfers, wages, or, of course, contracts. A triple (w, m, c) with (w, m) a couple type and $c \in C$ is a *couple-contract type*. Types also specify (strict) preferences. For each $w \in W$, there is a relation $\succ_w$ on $M_\emptyset \times C$, and for each $m \in M$, there is a relation $\succ_m$ on $W_\emptyset \times C$. That $(m, c) \succ_w (m', c')$ means that a woman of type w prefers to enter contract c with a man of type m to entering contract c' with a man of type m'. Similarly, $(w, c) \succ_m (w', c')$ means that a man of type m prefers to enter contract c with

[8] The paper Jagadeesan (2017) has been superseded by Jagadeesan and Vocke (2021).

a woman of type w to entering contract c' with a woman of type w'. Types play three roles: They specify the contracts available to a couple, the preferences of an agent, and the characteristics of an agent that agents on the other side of the market might care about.

We endow the sets W, M, and C with Polish (that is, separable and completely metrizable) topologies, providing us with a notion of closeness for types and contracts. The restriction to Polish spaces of types and contracts is a relatively harmless technical restriction; almost every space used in probabilistic modeling is Polish.[9]

We also endow $W_\emptyset$ and $M_\emptyset$ with the topologies that make $\emptyset$ an isolated point and such that the topologies on W and M are just the respective subspace topologies. For notational convenience, we let $\succ_\emptyset$ be the empty relation under which no elements are comparable, so that $\succ_w$ and $\succ_m$ are defined even when $w = \emptyset$ or $m = \emptyset$, respectively. Throughout the paper, we make the following three assumptions:

Acyclicity of Preferences: The relation $\succ_w$ on $M_\emptyset \times C$ is acyclic for each $w \in W$ and the relation $\succ_m$ on $W_\emptyset \times C$ is acyclic for each $m \in M$.

Continuity of Preferences: The set

$$\{(m, c, m', c', w) \mid (m, c) \succ_w (m', c')\}$$

is open in

$$M_\emptyset \times C \times M_\emptyset \times C \times W,$$

and the set

$$\{(w, c, w', c', m) \mid (w, c) \succ_m (w', c')\}$$

is open in

$$W_\emptyset \times C \times W_\emptyset \times C \times M.$$

Regularity of the Contract Correspondence: The correspondence $\mathbb{C} : W_\emptyset \times M_\emptyset \to 2^C$ is continuous with nonempty and compact values.

The assumption that preferences are acyclic is extremely weak but will suffice for most of our results. The continuity assumption on preferences ties the notion of closeness specified by the topologies on W, M, and C to how the agents themselves view the types.[10]

In practice, it is usually more convenient to work with utility functions than preferences. A sufficient condition for both assumptions on preferences to be satisfied is the existence of continuous functions $u_W : W \times M_\emptyset \times C \to \mathbb{R}$ and $u_M : W_\emptyset \times M \times C \to \mathbb{R}$ such that

$$(m, c) \succ_w (m', c') \text{ if and only if } u_W(w, m, c) > u_W(w, m', c')$$

[9]Appendix E of Dudley (2002) contains a discussion of why Polish spaces are nowadays used as the basis for probability theory and topological measure theory.

[10]Our continuity condition is equivalent to the functions mapping w to $\succ_w$ and m to $\succ_m$, respectively, being continuous in the Kannai topology introduced by Kannai (1970), which can be equivalently defined to be the weakest topology that makes these preference functions continuous. However, the Kannai topology may fail to be even Hausdorff without some form of local non-satiation, so we will not work directly with the Kannai topology.

and

$$(w, c) \succ_m (w', c') \text{ if and only if } u_M(w, m, c) > u_M(w', m, c').$$

If all preferences are asymmetric and negatively transitive (and therefore the asymmetric part of complete and transitive preference relations), and W, M, and C are Euclidean or, more generally, locally compact spaces, then this sufficient condition is also necessary by a theorem of Mas-Colell (1977). Preferences can then always be represented by such parametrized jointly continuous utility functions. We will use such utility representations frequently in examples and assume them to hold when discussing (possibly imperfectly) transferable utility.

To close the model, we specify nonzero, finite, Borel *population measures* ν_W and ν_M on W and M, respectively. In general, we denote the space of finite Borel measures on a Polish space X by $\mathcal{M}(X)$, so $\nu_W \in \mathcal{M}(W)$ and $\nu_M \in \mathcal{M}(M)$. We do allow for unbalanced markets in which $\nu_W(W) \neq \nu_M(M)$ and the population measures need not be probability measures. Our model is invariant to normalizing both measures jointly, so the absolute numbers $\nu_W(W)$ and $\nu_M(M)$ are economically meaningless. However, their relative size $\nu_W(W)/\nu_M(M)$ is meaningful and represents the number of women per man in the analyzed population. Finite-agent models correspond to the case in which ν_W and ν_M both have finite support, and all values are rational numbers. Indeed, by multiplying each of these rational numbers by the least common multiple of all denominators, one obtains an equivalent model in which types occur in positive integer quantities.

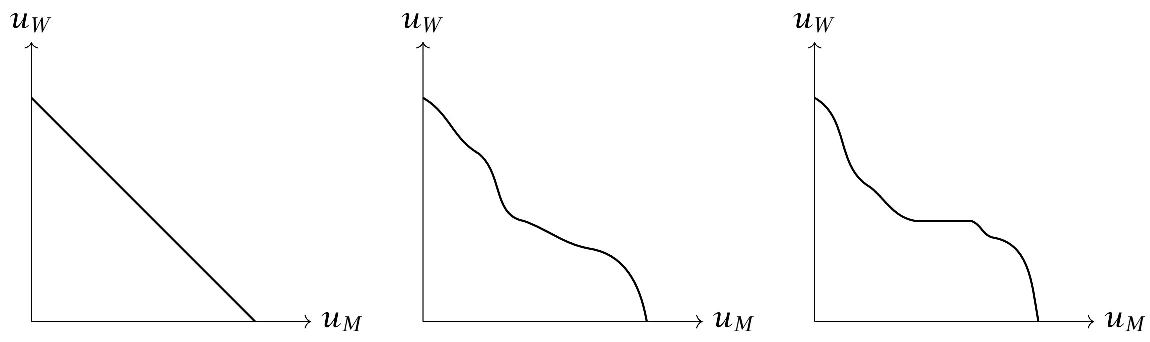
Before we go to the definitions of matchings and the stability notion, we briefly discuss how our model relates to existing matching models. Our model subsumes widely used models with non-transferable utility, transferable utility, and imperfectly transferable utility. In the classic marriage model of Gale and Shapley (1962), no transfers or contracts are allowed. In our framework, this corresponds to the degenerate case of a single contract $C = \{c\}$ and the contract correspondence $\mathbb{C}$ having the constant value $C = \{c\}$. Notationally, we will suppress C and $\mathbb{C}$ when working with the classic marriage model.

Assume for now that preferences are given by jointly continuous functions $u_W : W \times M_\emptyset \times C \to \mathbb{R}$ and $u_M : W_\emptyset \times M \times C \to \mathbb{R}$. We will require every couple type to make (weakly) efficient contract choices so that it is not possible to make both agents involved better off by choosing a different contract. If we only care about the utilities obtained, we might let the couple choose directly from the utility possibility frontier. Under reasonable sufficient conditions known from general equilibrium theory, the utility possibility frontier will be homeomorphic to the unit interval.[11]

For the discussion of matching with transfers, we focus on the case that $C = [0, 1]$ and that $\mathbb{C}$ is constant with value C. Note that under the formulation with $C = [0, 1]$, the utility levels are not given net of the outside options; we do allow for the utility possibility frontier to lie wholly below the utility levels the agents would get by staying unmatched. It is even possible for the utility possibility frontier to lie wholly above the utility levels

[11]Stated in terms of the set of feasible utility pairs, this set should be compact, comprehensive from below (up to the outside utility levels of the agents), and the outside utility levels of the agents should be dominated (in the weak Pareto sense) by every point in the utility possibility frontier. For a proof, see Arrow and Hahn (1971, Chapter 5, Section 2) who show, more generally, that the utility possibility frontier for n agents is homeomorphic to an $n-1$-dimensional simplex.

the agents would get by staying unmatched; a possibility we will explicitly exclude in Section 5. Examples for the possible shapes of utility possibility frontiers are given in the figures below.



If the utility possibility frontier is a line with constant slope -1 for all $w \in W$ and $m \in M$, as in the left figure, we have a model of *transferable utility*. Matching with transferable utility has a long tradition, starting with the work of Koopmans and Beckmann (1957) and Shapley and Shubik (1971). With transferable utility, the value $u_W(w, m, c) + u_M(w, m, c)$ does not depend on c and we can define a surplus function $S : W \times M \to \mathbb{R}$ by

$$S(w, m) = u_W(w, m, 1) + u_M(w, m, 1).$$

The surplus function, together with the utilities for staying single, carries all relevant information for the analysis of stability. As formulated here, transferable utility depends on specific choices of the functions u_W and u_M and is not invariant under transformations that preserve ordinal rankings. The exact ordinal implications of transferable utility have recently been characterized by Chiappori and Gugl (2020).[12] Matching markets with transferable utility have a very special structure. Maximizing aggregate surplus is a linear program and the solutions to the dual program can be interpreted as stable matchings or market equilibria. This duality holds even if types are not discrete, so transferable utility is the one case in which existence has been known for two-sided continuum models with general types for some time; see Gretsky, Ostroy, and Zame (1992, 1999) and Chiappori, McCann, and Nesheim (2010).

For our purposes equally well-behaved is the more general case of *imperfectly transferable utility*, as depicted in the middle figure. It is characterized by $u_W(w, m, \cdot)$ being an increasing[13] function and $u_M(w, m, \cdot)$ being a decreasing function for all $w \in W$ and $m \in M$. With imperfectly transferable utility, weak efficiency coincides with the more stringent usual definition of efficiency: It is impossible to make any agent in a couple better off without making the other worse off. Matching markets with imperfectly transferable utility have been studied in detail by Demange and Gale (1985). Existence proofs for such markets with finitely many agents may be found in Crawford and Knoer (1981) and Quinzii (1984). Parallel to our work, the existence of stable matchings under imperfectly transferable utility in a distributional model with continuum (but compact) type spaces was shown by Nöldeke and Samuelson (2018). Nöldeke and Samuelson use a different stability notion than we do, but their stability notion coincides with ours under

[12] For a gentle exposition, see Section 3.1 in Chiappori (2017).

[13] We take increasing functions to be strictly increasing and decreasing functions to be strictly decreasing.

the assumptions they make; see Theorem 3.

It is possible that weak efficiency and strict efficiency diverge, as in the figure on the right. Importantly, the set of strictly efficient points might not be closed then. To guarantee the existence of stable matchings in the full generality of our model, we need to be content with weak efficiency.

An alternative to the approach taken in this section would be to let $C = \mathbb{R}^2$ and u_W and u_M be given by $u_W(w, m, (r, r')) = r$ and $u_M(w, m, (r, r')) = r'$. In that case, $\mathbb{C}$ would just provide the feasible utility allocations. This is the approach used traditionally in cooperative game theory with non-transferable utility, starting with the work of Aumann and Peleg (1960). It is also the approach used by Galichon, Kominers, and Weber (2019) to provide a general framework for the empirical analysis of matching problems with finitely many types. For most purposes, the two approaches are equivalent. The assumption commonly made in the literature on cooperative games that the set of feasible utility allocations is comprehensive from below ensures that one can recover this set from the utility possibility frontier.

Having specified the environment, our next task is to define matchings and what it means for a matching to be stable. Given our distributional point of view, a matching simply specifies how likely it is to observe certain couple types that have entered certain contracts and is, therefore, a distribution of couple-contract types. A matching can still be defined via some marginal conditions, but we now have to take account of singles and make sure that couples only choose among contracts available to them. Let $G_{\mathbb{C}} \subseteq W_\emptyset \times M_\emptyset \times C$ be the graph of the contract correspondence $\mathbb{C}$. A *matching* is a Borel measure $\mu \in \mathcal{M}(W_\emptyset \times M_\emptyset \times C)$ such that

(i) $\nu_W(B) = \mu(B \times M_\emptyset \times C)$ for every Borel set $B \subseteq W$,

(ii) $\nu_M(B) = \mu(W_\emptyset \times B \times C)$ for every Borel set $B \subseteq M$,

(iii) $\mu\Big(W_\emptyset \times M_\emptyset \times C \Big\backslash G_{\mathbb{C}}\Big) = 0$, and

(iv) $\mu\Big(\{(w, m, c) \mid w = m = \emptyset\}\Big) = 0$.

Condition (iv) is simply a convenient normalization. We want to emphasize that nothing in the definition of a matching is inherently "random." We think of a matching (after normalizing the measure) as the empirical distribution of couple-contract types induced by a deterministic matching of agents. Section 7 supplies a formal foundation for this point of view.

Finally, we define stability, the idea at the heart of this paper. Even if a matching μ need not be a probability measure anymore, the product measure $\mu \otimes \mu$ is well-defined and we call the matching μ *stable* if $\mu \otimes \mu(I) = 0$ for a certain *instability set* $I \subseteq W_\emptyset \times M_\emptyset \times C \times W_\emptyset \times M_\emptyset \times C$ containing all pairs of couple-contract types witnessing to instability. A pair of couple-contract types is witness to the instability of a matching if there exists a blocking pair as before, if the contract choice is inefficient, or if an individual rationality constraint is violated. There are two conditions specifying that blocking possibilities exist between pairs (w, m, c) and (w', m', c') of couple-contract types:

$$(m', c'') \succ_w (m, c) \text{ and } (w, c'') \succ_{m'} (w', c') \text{ for some } c'' \in \mathbb{C}(w, m'),$$
$$(m, c'') \succ_{w'} (m', c') \text{ and } (w', c'') \succ_m (w, c) \text{ for some } c'' \in \mathbb{C}(w', m).$$

Both couple-contract types could have an inefficient contract choice, which gives us two more conditions.

$$(m, c'') \succ_w (m, c) \text{ and } (w, c'') \succ_m (w, c) \text{ for some } c'' \in \mathbb{C}(w, m),$$
$$(m', c'') \succ_{w'} (m', c') \text{ and } (w', c'') \succ_{m'} (w', c') \text{ for some } c'' \in \mathbb{C}(w', m').$$

Lastly, each four of the (individual) types in the two couple-contract types could have their individual rationality constraint violated, giving us four more conditions:

$$(\varnothing, c'') \succ_w (m, c) \text{ for some } c'' \in \mathbb{C}(w, \varnothing),$$
$$(\varnothing, c'') \succ_{w'} (m', c') \text{ for some } c'' \in \mathbb{C}(w', \varnothing),$$
$$(\varnothing, c'') \succ_m (w, c) \text{ for some } c'' \in \mathbb{C}(m, \varnothing),$$
$$(\varnothing, c'') \succ_{m'} (w', c') \text{ for some } c'' \in \mathbb{C}(m', \varnothing).$$

A violation of individual rationality constraints could also come from an inefficient contract choice of an unmatched agent, so these constraints serve double duty.

It follows from the continuity of preferences and the regularity of the contract correspondence (namely, its lower hemicontinuity) that each of these eight conditions specifies an open subset of $W_\varnothing \times M_\varnothing \times C \ \times \ W_\varnothing \times M_\varnothing \times C$. We let I be the union of these sets. Clearly, I is open.

An equivalent way to define stability would be to say that the matching μ is stable if there are no couple contract types (w, m, c) and (w', m', c') and neighborhoods V and V' of (w, m, c) and (w', m', c'), respectively, satisfying $\mu(V) > 0$, $\mu(V') > 0$, and $V \times V' \subseteq I$. The equivalence is a straightforward consequence of the topology of $(W_\varnothing \times M_\varnothing \times C) \ \times \ (W_\varnothing \times M_\varnothing \times C)$ having a countable basis of open rectangles and I being open.

Stable matchings exist. To prove this, we approximate a given matching problem by finite matching problems for which Lemma 3 will guarantee the existence of stable matchings. A compactness argument allows us then to extract a stable matching for the limit matching problem. To make this argument work, we use the topology of weak convergence of measures.[14] Recall that the sequence $\langle \mu_n \rangle$ of measures in $\mathcal{M}(X)$ with X Polish converges to the measure $\mu \in \mathcal{M}(X)$ under the topology of weak convergence of measures if

$$\lim_{n \to \infty} \int g \, \mathrm{d}\mu_n = \int g \, \mathrm{d}\mu$$

for every bounded continuous function $g : X \to \mathbb{R}$. Whenever we make topological arguments for spaces of measures, it will be understood that we are using the topology of weak convergence of measures.

Our compactness argument can be split into two distinct parts. We first show that our sequence of finite matching problems must have a subsequence converging to some limit

[14]The most important facts related to weak convergence can be found in the Mathematical Appendix.

measure in Lemma 1 and then show using Lemma 2 that the limit measure will indeed be a matching.

Lemma 1. *Let $\langle \nu_W^n, \nu_M^n, \mu_n \rangle$ be a sequence in $\mathcal{M}(W) \times \mathcal{M}(M) \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C)$ such that $\langle \nu_W^n \rangle$ converges to $\nu_W \in \mathcal{M}(W)$, $\langle \nu_M^n \rangle$ converges to $\nu_M \in \mathcal{M}(M)$, and μ_n is a matching for population distributions ν_W^n and ν_M^n for each natural number n. Then a subsequence of $\langle \mu_n \rangle$ converges.*

The proof of Lemma 1 is a straightforward application of Prohorov's characterization of relative compactness in the topology of weak convergence of measures. Intuitively, we need to make sure that no mass "escapes to infinity." No mass can escape to infinity unless mass of the sequence of population measures escapes to infinity, which is not possible when population measures converge. The argument does not require W, M, or C to be compact.

Lemma 2. *The following set is closed:*

$$\Big\{(\nu_W, \nu_M, \mu) \in \mathcal{M}(W) \times \mathcal{M}(M) \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \mid \mu \text{ is a matching for the population measures } \nu_W \text{ and } \nu_M\Big\}.$$

Most of the proof of Lemma 2 consists in showing that conditions (i) and (ii) in the definition of a matching are preserved under taking limits. The topology of weak convergence of measures does not guarantee convergence of the measure of every Borel set, so we show that it suffices that convergence holds for an appropriately chosen class of well-behaved Borel sets for which we actually get convergence.

To get the compactness argument off the ground, we prove that stable matchings exist for discrete matching problems with finitely many agents by approximating the matching problem with one in which there are only finitely many contracts.

Lemma 3. *A stable matching exists whenever ν_W and ν_M have finite supports and take on only rational values.*

We are now ready for the proof of our main existence theorem.

Theorem 1. *There is at least one stable matching.*

Proof. Let $\langle \nu_W^n, \nu_M^n \rangle$ be a sequence of pairs of measures on W and M, respectively, such that $\langle \nu_W^n \rangle$ converges to ν_W, $\langle \nu_M^n \rangle$ converges to ν_M and ν_W^n and ν_M^n have finite support and only rational values for all n. This is possible since measures with finite supports are dense in the space of all measures and, clearly, every measure with finite support is the limit of a sequence of measures with the same finite support and rational values.

For each n, we can choose a stable matching μ_n for the finite matching problem given by population distributions ν_W^n and ν_M^n by Lemma 3. By passing to a subsequence and using Lemma 1, we can assume without loss of generality that $\langle \mu_n \rangle$ converges to some measure μ, which is again a matching for the population measures ν_W and ν_M by Lemma 2. The continuity assumption on preferences and the lower hemicontinuity of $\mathbb{C}$ guarantee that I is open. Therefore,

$$\mu \otimes \mu(I) \leq \liminf_n \mu_n \otimes \mu_n(I) = 0$$

by the Portmanteau theorem (see the Mathematical Appendix) and the fact that taking products preserves weak convergence. □

Importantly, our existence result needs no further compactness assumptions. Indeed, Lemma 1 and Lemma 2 show that all the needed compactness is automatic in the space of matchings. Nöldeke and Samuelson (2018) include value functions in their solution concept and therefore need the space of value functions to be compact in an appropriate topology. They do this by using their duality theory to show that the value functions form a uniformly compact set of continuous functions independently of the population distributions. Their argument depends crucially on type spaces being compact, which is not needed here.

In the usual setting with finitely many agents, a stable matching continues to be stable if we remove matched couples from the population. Indeed, this can only reduce the blocking possibilities of other agents. The same holds true in our model, and we note the following lemma for later reference.[15]

Lemma 4. *Let $\mu, \lambda \in \mathcal{M}(W_\emptyset \times M_\emptyset \times C)$, with μ being a stable matching and $\lambda(B) \leq \mu(B)$ for every Borel set $B \subseteq W_\emptyset \times M_\emptyset \times C$. Then $\lambda \otimes \lambda(I) = 0$.*

4 Relation to Large Finite Matching Markets

We show in this section that stable matchings as defined in this paper are exactly the limits of stable matchings of finite-agent matching problems that approximate our limit matching model.

If a sequence of stable matchings for matching problems with finitely many agents converges, it converges to a stable matching for the limiting population measures. If it does not converge, a subsequence will. Indeed, we showed this much when proving Theorem 1. Hence, *at least one* stable matching for the limiting population measures captures the limiting behavior of a sequence of stable matchings for large, finite populations. Next, we show that *all* stable matchings do so; they are all the limits of sequences of stable matchings for large, finite populations.

We first have to define what a matching problem is. We hold the type spaces, the set of contracts, the contract correspondence, and the preferences fixed. So we only vary the population measures and define a *matching problem* to be a pair (ν_W, ν_M) of population measures. The matching problem (ν_W, ν_M) is *finite* if both ν_W and ν_M take on only rational values and have a finite support.[16] As already mentioned above, by multiplying both ν_W and ν_M with a multiple of all denominators occurring in nonzero values, one obtains an equivalent matching problem in which finitely many types occur in positive integer quantities. Each of these quantities can be interpreted as the number of agents of this type. Similarly, a matching μ is *finite* if it has a finite support and only rational values. Note that a stable matching for a finite matching problem need not be finite.

[15] See Villani (2009, Theorem 4.6) for a similar result in the optimal transport context.

[16] It can be shown that a finite measure that takes on only rational values can take on only finitely many values; see for example Tsakas (2014, Proposition 1). In the present context, such a measure must have a finite support. Our definition of a finite matching problem contains, therefore, a redundancy.

Theorem 2. *Let μ be a matching for the matching problem (ν_W, ν_M). Then μ is stable if and only if there are sequences $\langle \nu_W^n \rangle$, $\langle \nu_M^n \rangle$, and $\langle \mu_n \rangle$ such that*

(i) the matching problem (ν_W^n, ν_M^n) is finite for each n and μ_n is a finite stable matching for it,

(ii) the sequence $\langle \nu_W^n \rangle$ converges to ν_W, the sequence $\langle \nu_M^n \rangle$ converges to ν_M, and $\langle \mu_n \rangle$ converges to μ.

Proof. That (i) and (ii) imply that μ is stable was implicitly already shown in Lemma 2 and the proof of Theorem 1. For the converse, let μ be a stable matching. Normalize it to a probability measure $\bar{\mu} \in \mathcal{M}(W_\emptyset \times M_\emptyset \times C)$ by letting $\bar{\mu}(B) = \mu(B)/\mu(W_\emptyset \times M_\emptyset \times C)$ for every Borel set $B \subseteq W_\emptyset \times M_\emptyset \times C$. For each sequence $\omega = \langle \omega_n \rangle \in (W_\emptyset \times M_\emptyset \times C)^\infty$, we can form the sequence $\langle \bar{\mu}_n^\omega \rangle$ of sample distributions given by

$$\bar{\mu}_n^\omega(B) = n^{-1} \,\#\{m \leq n \mid \omega_m \in B\}$$

for every Borel set $B \subseteq W_\emptyset \times M_\emptyset \times C$ and each natural number n. For each ω and each natural number n, we have

$$\bar{\mu}_n^\omega \otimes \bar{\mu}_n^\omega(I) = n^{-2}\#\{(l,m) \mid (\omega_l, \omega_m) \in I \text{ and } l, m \leq n\}.$$

Since μ is stable, we have $\bar{\mu} \otimes \bar{\mu}(I) = 0$ and $\bar{\mu} \otimes \bar{\mu}$ is the marginal distribution of $\otimes_n \bar{\mu}$ on any pair of factors $W_\emptyset \times M_\emptyset$. Since there are only countably many pairs of such factors, it follows that for $\otimes_n \bar{\mu}$-almost all ω and each natural number n, $\bar{\mu}_n^\omega \otimes \bar{\mu}_n^\omega(I) = 0$. Moreover, by Varadarajan's version of the Glivenko–Cantelli theorem, Varadarajan (1958), the sequence $\langle \bar{\mu}_n^\omega \rangle$ converges to $\bar{\mu}$ for $\otimes_n \bar{\mu}$-almost all ω. So we can choose some sequence $\omega \in (W_\emptyset \times M_\emptyset \times C)^\infty$ such that $\bar{\mu}_n^\omega \otimes \bar{\mu}_n^\omega(I) = 0$ for each natural number n and such that $\langle \bar{\mu}_n^\omega \rangle$ converges to $\bar{\mu}$. Indeed, $\otimes_n \bar{\mu}$-almost every ω will do. Let $\langle q_n \rangle$ be a sequence of positive rational numbers converging to $\mu(W_\emptyset \times M_\emptyset \times C)$. For each natural number n, let $\mu_n = q_n \cdot \bar{\mu}_n^\omega$ and define ν_W^n and ν_M^n by

$$\nu_W^n(B) = \mu_n(B \times M_\emptyset \times C)$$

for every Borel set $B \subseteq W$ and

$$\nu_M^n(B) = \mu_n(W_\emptyset \times B \times C)$$

for every Borel set $B \subseteq M$, respectively. Clearly, $\langle \mu_n \rangle$ converges to μ and μ_n is a finite stable matching for the matching problem (ν_W^n, ν_M^n) for each natural number n. It remains to show that $\langle \nu_W^n \rangle$ converges to ν_W and $\langle \nu_M^n \rangle$ converges to ν_M. Let $B \subseteq W$ be a ν_W-continuity set. Then $B \times M_\emptyset \times C$ is a μ-continuity set and

$$\lim_n \nu_W^n(B) = \lim_n \mu_n(B \times M_\emptyset \times C) = \mu(B \times M_\emptyset \times C) = \nu_W(B).$$

It follows that $\langle \nu_W^n \rangle$ converges to ν_W. Similarly, $\langle \nu_M^n \rangle$ converges to ν_M. □

Let us take a look at what Theorem 2 does not say. The sequences of population measures $\langle \nu_W^n \rangle$ and $\langle \nu_M^n \rangle$ shown to exist do not just depend on the limiting population measures ν_W and ν_M, they depend on the matching μ itself. What is not true is that for any population measures ν_W and ν_M, we can find sequences of population measures $\langle \nu_W^n \rangle$ and $\langle \nu_M^n \rangle$ converging to ν_W and ν_M, respectively, such that for every stable matching μ for the population measures ν_W and ν_M, there exists a sequence $\langle \mu_n \rangle$ converging to μ such that μ_n is a stable matching for the population measures ν_W^n and ν_M^n for each natural number n. Formally, the correspondence that maps each matching problem to its set of stable matchings may not be lower hemicontinuous.

This failure of lower hemicontinuity is not just an artifact of our distributional model; the phenomenon is known to occur in finite matching theory. Indeed, Pittel (1989) has shown in a model with randomly drawn preferences, no transfers, and the same number of women and men, that the number of stable matchings grows incredibly fast with the number of agents, but Pittel's result is not robust to small changes of populations: Ashlagi, Kanoria, and Leshno (2017) have shown that when the number of women differs from the number of men by even one, the set of stable matchings collapses essentially to a unique stable matching as the number of agents grows. A difference of only a single person in population sizes must vanish in the limit, so a reasonable limit model cannot preserve the distinction.

Our model is different,[17] but we can use ideas inspired by Ashlagi, Kanoria, and Leshno (2017) to construct an example in which a sequence of population measures converges, but a stable matching for the limiting population measures is not the limit of any sequence of stable matchings for the given sequence of population measures. This is Example 4 in the Online Appendix. The solution correspondence fails to be lower hemicontinuous in a substantial way. Examples where lower hemicontinuity fail have been known for quite some time in the setting of transferable utility. Indeed, the "master-servant"- example of Edgeworth (1881) is such an example.

The failure of lower hemicontinuity of the solution correspondence shows that one cannot simply transport the structure theory of stable matchings, as first reported by Knuth (1976), from finite matching theory to the distributional model by limit arguments; the limit of a sequence of men-optimal matchings for the sequence of "unbalanced" populations in our Example 4 is far from the men-optimal matching for the limit matching problem. However, failure of lower hemicontinuity is not a robust phenomenon. This is indeed a general phenomenon; the solution correspondence is continuous for a topologically large (residual) set of matching problems by a theorem of Fort (1949). This allows us to recover some of the structure theory for the marriage model generically. We show in the Online Appendix that if type spaces are compact and a weak assumption on preferences holds, extremal matchings exist for a residual set of matching problems; Theorem G1. Moreover, a version of the Lone Wolf Theorem of McVitie and Wilson (1970), which says that the same agents are unmatched in every stable matching (a special case of the rural hospital theorem), holds for a residual set of matching problems; Theorem G2. However, our methods do not allow us to recover the full lattice structure even on a residual set; lattice operations need not be continuous.

The situation is better when we allow for transfers. Stable matchings have, under

[17] Indeed, we explain in our concluding remarks that such random finite matching models cannot be analyzed by our methods.

reasonable assumptions, the structure of a lattice in terms of payoff assignments. For transferable utility, this was shown by Gretsky, Ostroy, and Zame (1999). For imperfectly transferable utility, this was shown only recently by Nöldeke and Samuelson (2018). The case of imperfectly transferable utility is considerably harder; under transferable utility one can separate the matching between agents and the assignment of payoffs. In our general model, there need not even be functions that assign payoffs to types. Transfers help here, as we will see in the next section.

5 Equal Treatment

In large matching models with transferable or imperfectly transferable utility, stability is usually defined in terms of functions that assign to each type of agent in a matching the payoff they receive in the matching. In particular, agents with the same type must be equally well off. While such an equal treatment of types is not guaranteed in our general model, it holds under weak conditions in the presence of transfers for stable matchings. Under transfers, the law of one price holds. This allows us to show that our stability notion is equivalent to existing stability notions for models with transfers and provides a more basic foundation for these stability notions.

Since type spaces come endowed with a topology, we can also think of a stronger form of equal treatment in which similar types are similarly well off so that the function from types to their obtained utilities is continuous. We get such a strong form of equal treatment when type spaces are compact and population distributions have full support. This allows us to give a characterization of stable matchings that coincides with the stability notion used by Nöldeke and Samuelson (2018); see Theorem 3.[18] The weaker form of equal treatment holds without compactness assumptions, and we get a correspondingly weaker characterization of stable matchings that coincides with the notion usually used for transferable utility models based on optimal transport techniques.

We will make two assumptions that guarantee enough possibilities for transferring utility are available. The first assumption has already been discussed in Section 3.

Imperfectly Transferable Utility: We let $C = [0, 1]$, and for all $w \in W_\emptyset$ and $m \in M_\emptyset$, we let $\mathbb{C}(w, m) = [0, 1]$. Preferences are represented by jointly continuous functions $u_W : W \times M_\emptyset \times C \to \mathbb{R}$ and $u_M : W_\emptyset \times M \times C \to \mathbb{R}$ with u_W increasing in the last coordinate and u_M decreasing in the last coordinate.

Our notion of imperfectly transferable utility does not require that every agent can transfer all utility above their outside option to their partner; we need to assume this explicitly. The assumption holds automatically in other formulations of transferable utility, such as the one used by Nöldeke and Samuelson (2018). For notational ease, we make the outside payoff an agent can obtain explicit by functions $u_W^\emptyset : W \to \mathbb{R}$ and $u_M^\emptyset : M \to \mathbb{R}$ defined by $u_W^\emptyset(w) = u_W(w, \emptyset, 1)$ and $u_M^\emptyset(m) = u_M(\emptyset, m, 0)$, respectively.

[18] Indeed, this section grew out of our attempts to understand how the framework of Nöldeke and Samuelson (2018) relates to ours.

Bounds on Transfers do not bind: For all $w \in W$ and $m \in M$

$$u_W(w, m, 0) \leq u_W^{\emptyset}(w) \text{ and } u_M(w, m, 1) \leq u_M^{\emptyset}(m).$$

We are now ready to give our first characterization of stable matchings.

Theorem 3. *Assume imperfectly transferable utility, that bounds on transfers do not bind, that W and M are compact, and that ν_W and ν_M have full support. Then, a matching μ is stable if and only if there are continuous functions $V_W : W \to \mathbb{R}$ and $V_M : M \to \mathbb{R}$ such that*

$$V_W(w) = u_W(w, m, c)$$

for μ-almost all $(w, m, c) \in W \times M_{\emptyset} \times [0, 1]$ and

$$V_M(m) = u_M(w, m, c)$$

for μ-almost all $(w, m, c) \in W_{\emptyset} \times M \times [0, 1]$ and such that

(i) $V_W(w) \geq u_W^{\emptyset}(w)$ for all $w \in W$,

(ii) $V_M(m) \geq u_M^{\emptyset}(m)$ for all $m \in M$,

(iii) $u_W(w, m, c) \leq V_W(w)$ if $u_M(w, m, c) \geq V_M(m)$ for all $(w, m) \in W \times M$.

It follows from the duality theory of Nöldeke and Samuelson (2018) that any functions V_W and V_M satisfying (i)-(iii) must automatically be continuous.

Most of the effort in proving Theorem 3 consists of showing that if μ is stable then there exists a continuous function $V_W : W \to \mathbb{R}$ such that $V_W(w) = u_W(w, m, c)$ for μ-almost all $(w, m, c) \in W \times M_{\emptyset} \times [0, 1]$. We do this by taking a random sequence $\langle (w_n, m_n, c_n) \rangle$ obtained from independent draws from μ (normalized). With probability one, no two terms in the sequence combine to a pair in the instability set and the set of women types sampled is dense in W. This can be shown to imply that the set $\{(w_n, u_W(w_n, m_n, c_n)) \mid n \in \mathbb{N}\}$ is the graph of a uniformly continuous function on a dense subset of W. Extending this function by continuity to all of W gives us the desired function V_W. It should be noted that in a (purely) transferable utility setting, Gretsky, Ostroy, and Zame (1992) prove that when type spaces are compact and the surplus function continuous, then the solution of the dual optimal transport problem can be taken to have continuous values. Their method of proof is different from ours. They construct continuous value functions from measurable value functions by what they call a "shrink-wrap"-argument. Our approach delivers continuous value functions directly.

General Polish spaces of types can be approximated from within by compact subspaces, and this allows us to obtain a weak form of equal treatment and a corresponding characterization of stable matchings with possibly discontinuous functions V_W and V_M.

Theorem 4. *Assume imperfectly transferable utility and that bounds on transfers do not bind. Then, a matching μ is stable if and only if there are measurable functions $V_W : W \to \mathbb{R}$ and $V_M : M \to \mathbb{R}$ such that*

$$V_W(w) = u_W(w, m, c)$$

for μ-almost all $(w,m,c) \in W \times M_\emptyset \times [0,1]$,

$$V_M(m) = u_M(w,m,c)$$

for μ-almost all $(w,m,c) \in W_\emptyset \times M \times [0,1]$, and such that

(i) $V_W(w) \geq u_W^\emptyset(w)$,

(ii) $V_M(m) \geq u_M^\emptyset(m)$,

(iii) $u_W(w,m,c) \leq V_W(w)$ *if* $u_M(w,m,c) \geq V_M(m)$.

for $\nu_W \otimes \nu_M$-almost all $(w,m) \in W \times M$.

Note that the conditions (i)-(iii) need to hold only almost surely in Theorem 4. In Theorem 3, continuity of V_W and V_M and the full support conditions imply that the exceptional sets of measure zero must be empty, something one cannot conclude when these functions are only required to be measurable. The compactness assumptions in Theorem 3 cannot be dispensed with; Example 5 in the Online Appendix shows that continuous versions of V_W and V_M need not exist otherwise. For the existence of V_W, the compactness of W is not essential; see Proposition E1 in the Online Appendix.

In the optimal transport literature, stability is derived via Kantorovich duality from the dual of a linear optimization problem, and the full force of the theory requires some integral boundedness restrictions so that the value of the linear program stays finite; see Villani (2009, Theorem 5.10). This is not required for our existence result Theorem 1 or Theorem 4. Our approach covers, therefore, even transferable utility problems not covered by optimal transport techniques.

6 Externalities

Our model can be extended to allow for widespread externalities in which the preferences of agents depend on the matching itself. The externalities we consider are widespread externalities no single agent can influence. Besides classical externalities and peer effects, this allows for modeling market forms and institutions outside the matching market under consideration. With finitely many agents, indivisibilities of the population may preclude the existence of stable matchings; see Example 6 in the online appendix.

Apart from these problems with indivisibilities, externalities pose some conceptual problems for matching markets with finitely many agents. In that case, each agent has to have some notion of how their behavior impacts others and what kind of response it might cause. Starting with Sasaki and Toda (1996), a number of authors have analyzed such matching markets with finitely many agents and externalities using fairly sophisticated farsightedness ideas. Our approach sidesteps the main problems occurring with finitely many agents and allows for a much simpler treatment.

The idea that large aggregate externalities might be compatible with stability and finitely many agents was explored by Fisher and Hafalir (2016), but they still had to make special assumptions to deal with indivisibilities. Closer to our approach is the treatment of widespread externalities by Hammond, Kaneko, and Wooders (1989). The main difference is in how they topologize allocations or matchings. The topology used by them is not compact and their solution concept is best interpreted as an approximate solution concept. Most closely related to our approach is the treatment by Noguchi and Zame (2006) who study the existence of Walrasian equilibria under widespread externalities.

We first have to adapt our environment and assumptions to a setting with externalities. Preferences now include matchings and even more general measures over couple-contract types. That agents have preferences even over non-matchings is not needed for our existence result, Theorem 5. The space of matchings is closed and even compact. But to interpret stable matchings as limits of finite approximately stable matchings in Theorem 6, we need continuity on the larger space, so that preferences are well-defined in approximating matching problems.[19] We also assume that preferences are asymmetric and negatively transitive (and thus the asymmetric part of a complete and transitive weak preference relation). This will greatly simplify the existence proof.

Rationality of Preferences with Externalities: The relation $\succ_w$ on $M_\emptyset \times C \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C)$ is asymmetric and negatively transitive for each $w \in W$ and the relation $\succ_m$ on $W_\emptyset \times C \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C)$ is asymmetric and negatively transitive for each $m \in M$.

Continuity of Preferences with Externalities: The set

$$\left\{(m, c, \mu, m', c', \mu', w) \mid (m, c, \mu) \succ_w (m', c', \mu')\right\}$$

is open in

$$M_\emptyset \times C \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \times M_\emptyset \times C \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \times W,$$

and the set

$$\left\{(w, c, \mu, w', c', \mu', m) \mid (w, c, \mu) \succ_m (w', c', \mu')\right\}$$

is open in

$$W_\emptyset \times C \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \times W_\emptyset \times C \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \times M.$$

Regularity of the Contract Correspondence: The correspondence $\mathbb{C} : W_\emptyset \times M_\emptyset \to 2^C$ is continuous with nonempty and compact values.

The interpretation of these assumptions is similar to the intrepretation in the model without externalities, but the continuity of preferences in externalities is now a more significant restriction.

Agents now have preferences over matchings and even over distributions over couple-contract types that are not matchings. Fixing a matching μ and using the induced preferences, we have a standard matching problem that comes with an *instability set* $I(\mu)$. We now say that the matching μ is *stable* if $\mu \otimes \mu(I(\mu)) = 0$.

Theorem 5. *There is at least one stable matching in the extended model with externalities.*

[19] Since the space of actual matchings is closed, we can always extend preferences continuously to the ambient space.

Without externalities, stable matchings correspond exactly to the limits of stable matchings for finite matching problems by Theorem 2. Clearly, this need not be the case here. Indeed, there may be no stable matching at all in an approximating finite population of agents. But at least for compact spaces of types and contracts, and preferences that admit utility representations, each stable matching corresponds to the limit of "nearly stable" finite matchings for finite matching problems.[20] For each pair of jointly continuous utility representations $u_W : W \times M_\emptyset \times C \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \to \mathbb{R}$ and $u_M : W_\emptyset \times M \times C \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \to \mathbb{R}$, each $\epsilon > 0$, and each matching μ we define the *ϵ-instability set* $I_\epsilon(\mu)$ so that $I_\epsilon(\mu)$ replaces the strict preferences defining $I(\mu)$ by strict inequalities in terms of u_W and u_M that have to hold with a gap of at least ϵ. For example, we replace the condition

$$(m', c'', \mu) \succ_w (m, c, \mu) \text{ and } (w, c'', \mu) \succ_{m'} (w', c', \mu) \text{ for some } c'' \in \mathbb{C}(w, m')$$

in the definition of $I(\mu)$ by

$$u_W(w, m', c''\mu) > u_W(w, m, c, \mu) + \epsilon \text{ and } u_M(w, m', c'', \mu) > u_M(w', m', c', \mu) + \epsilon$$

for the definition of $I_\epsilon(\mu)$. The notion of a matching problem translates directly to the model with externalities; a *matching problem* is still a pair (ν_w, ν_M) of population measures as in Section 4.

Theorem 6. *Assume in the extended model with externalities that W, M, and C are compact and let $u_W : W \times M_\emptyset \times C \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \to \mathbb{R}$ and $u_M : W_\emptyset \times M \times C \times \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \to \mathbb{R}$ be (jointly) continuous utility representations. Let μ be a matching for the matching problem (ν_W, ν_M). Then μ is stable if and only if there are sequences $\langle \nu_W^n \rangle$, $\langle \nu_M^n \rangle$, and $\langle \mu_n \rangle$ such that*

(i) the matching problem (ν_W^n, ν_M^n) is finite for each n and μ_n is a finite matching for it,

(ii) the sequence $\langle \nu_W^n \rangle$ converges to ν_W, the sequence $\langle \nu_M^n \rangle$ converges to ν_M, and $\langle \mu_n \rangle$ converges to μ,

(iii) and for all $\epsilon > 0$, there exists a natural number N such that $\mu_n \otimes \mu_n(I_\epsilon(\mu_n)) = 0$ for each $n \geq N$.

7 Individualistic Representation

We talked about agents, but our model has none. In this section, we show that one can enrich the model so that matchings can be formulated at the level of individual agents. There are measure spaces of women and men, and each matching matches a unique woman to a unique man or lets her be by herself. This exercise has two purposes: First, it shows there is nothing random about a matching in our distributional model; the underlying matching of agents is deterministic. Second, it clarifies our notion of stability by taking it to the level of agents. Nevertheless, our distributional model is much easier

[20]The existence of the utility representations is needed for the given formulation, one could use a similar result without them by using the mathematical machinery of uniform spaces.

to handle for practical purposes. Example 1 shows that we cannot simply interpret types as agents.

In the individualistic representation of a matching, we require all couples and individuals to choose feasible contracts. Not even a measure zero set of exceptions is allowed. In the individualistic representation of a stable matching, we further require all couples and individuals to choose efficient feasible contracts. Again, not even a measure zero set of exceptions is allowed. To make this precise, we first have to define what it means for a contract to be efficient. This is straightforward, but requires some care in handling dummy types. So let $w \in W$ and $m \in M$. Then $c \in \mathbb{C}(w,m)$ is *efficient* if there is no $c' \in \mathbb{C}(w,m)$ such that both $(m,c') \succ_w (m,c)$ and $(w,c') \succ_m (w,c)$. We deal with unmatched agents next. The contract $c \in \mathbb{C}(w,\emptyset)$ is *efficient* if there is no $c' \in \mathbb{C}(w,\emptyset)$ such that $c' \succ_w c$, and the contract $c \in \mathbb{C}(\emptyset,m)$ is *efficient* if there is no $c' \in \mathbb{C}(\emptyset,m)$ such that $c' \succ_m c$. Finally, every $c \in \mathbb{C}(\emptyset,\emptyset)$ is taken to be *efficient*.

We are now able to state our individualistic representation theorem.

Theorem 7. *There exist measure spaces $(A_W, \mathcal{A}_W, \tau_W)$ and $(A_M, \mathcal{A}_M, \tau_M)$, and measurable type functions $t_W : A_W \to W_\emptyset$ and $t_M : A_M \to M_\emptyset$ such that $\nu_W(B) = \tau_W \circ t_W^{-1}(B)$ for every Borel set $B \subseteq W$, such that $\nu_M(B) = \tau_M \circ t_M^{-1}(B)$ for every Borel set $B \subseteq M$, and such that for every matching μ, there is a pair of measurable functions $\phi : A_W \to A_M$ and $\chi : A_W \to C$ such that*

(i) the measurable function ϕ is a bijection with a measurable inverse that preserves measure; $\tau_M(S) = \tau_W \circ \phi^{-1}(S)$ for every measurable set $S \subseteq A_M$,

(ii) for every Borel set $B \subseteq W_\emptyset \times M_\emptyset \times C \setminus \{(\emptyset,\emptyset)\} \times C$,

$$\mu(B) = \tau_W\Big(\Big\{a_W \in A_W \mid (t_W(a_W), t_M(\phi(a_W)), \chi(a_W)) \in B\Big\}\Big),$$

(iii) and for every $a_W \in A_W$,

$$\chi(a_W) \in \mathbb{C}\Big(t_W(a_W), t_M(\phi(a_W))\Big).$$

Moreover, if μ is stable, then ϕ and χ can be chosen to satisfy the following condition:

(iv) For every $a_W \in A_W$, the contract choice $\chi(a_W)$ is efficient.

Formally, proving Theorem 7 amounts to proving a so-called "purification"-theorem for measure-valued maps. Our theorem is related to but does not follow from existing results on the purification of measure-valued maps such as Podczeck (2009), Loeb and Sun (2009), Wang and Zhang (2012), and Greinecker and Podczeck (2015). The additional complication we face comes from requiring the matching to be represented by a measurable, measure-preserving isomorphism. This requires, intuitively, having many agents of every type.[21] A related, somewhat weaker, such representation is given in Gretsky, Ostroy,

[21] Saturation or, equivalently, super-nonatomicity as in the purification results mentioned above does not suffice. Every nonatomic Borel probability measure on a Polish space extends to a saturated (superatomless) probability measure on a larger σ-algebra by the main result of the appendix of Podczeck (2009). In particular, Example 1 is compatible with the spaces of agents being saturated probability spaces. In contrast, saturation is enough to derive individualistic Nash equilibria from distributional Nash equilibria in large games; see Carmona and Podczeck (2009) and Keisler and Sun (2009). Our additional assumption, homogeneity, has been interpreted as an anonymity property in Khan and Sun (1999).

and Zame (1992, Section 1.5.1), but the individualistic matchings obtained there need not be invertible and measurable in both directions.

We briefly sketch the proof of Theorem 7. The full proof is somewhat complex, but the basic idea is not. We explain this idea for the simple marriage model without transfers, both sides of the market having the same size, individual rationality constraints not binding, and, in addition, both W and M being finite. If both W and M are finite, we can take the unit interval $[0,1]$ endowed with the Borel sets and the uniform distribution λ for both $(A_W, \mathcal{A}_W, \tau_W)$ and $(A_M, \mathcal{A}_M, \tau_M)$. So let ν_W and ν_M be probability measures. It is easy to show that there are measurable functions $t_W : [0,1] \to W$ and $t_M : [0,1] \to M$ such that $\nu_W = \lambda \circ t_W^{-1}$ and $\nu_M = \lambda \circ t_M^{-1}$. Let $\mu \in \mathcal{P}(W \times M)$ be a matching for the matching problem (ν_W, ν_M). We utilize two properties that will in general only hold when W and M are finite:

1. If X and Y are finite sets, $f : [0,1] \to X$ a measurable function, and $\rho \in \mathcal{P}(X \times Y)$ a probability measure with X-marginal $\lambda \circ f^{-1}$, then there exists a measurable function $g : [0,1] \to Y$ such that the function $(f,g) : [0,1] \to X \times Y$ given by $(f,g)(\omega) = \big(f(\omega), g(\omega)\big)$ satisfies $\lambda \circ (f,g)^{-1} = \rho$.

2. If X is a finite set and $f : [0,1] \to X$ and $g : [0,1] \to X$ are measurable functions such that $\lambda \circ f^{-1} = \lambda \circ g^{-1}$, then there exists a measurable bijection $h : [0,1] \to [0,1]$ with a measurable inverse that preserves measure, so $\lambda = \lambda \circ h^{-1}$, such that $f(\omega) = g(h(\omega))$ for almost all $\omega \in [0,1]$.

The first property means there will be functions $f_W : [0,1] \to W$ and $f_M : [0,1] \to M$ such that $\lambda \circ (t_W, f_M)^{-1} = \mu = \lambda \circ (f_W, t_M)^{-1}$. Now, the second property means there will be a measure-preserving measurable bijection $\phi : [0,1] \to [0,1]$ with measurable inverse such that $(t_W(\omega), f_M(\omega)) = \Big(f_W(\phi(\omega)), t_M(\phi(\omega))\Big)$. The function ϕ can then be taken to be our individualistic matching. Now, the two properties we used hold for the unit interval with the Borel sets and the uniform distribution in general only when X and Y are finite or, for the first property, at most countable. But there do exist probability spaces such that both properties hold for arbitrary Polish spaces X and Y; such probability spaces differ a lot from most probability spaces one usually encounters. In particular, they cannot be Polish themselves. One example of such a probability space is given by the product measure on $\{H, T\}^{\mathbb{R}}$ corresponding to a continuum of independent fair coin-flips. The proof for the simple marriage model without transfers, both sides of the market having the same size, individual rationality constraints not binding, and general Polish types spaces can then directly be given as here. To prove Theorem 7 in full generality, one needs to take account of unmatched agents and patch up violations of (iii) and (iv) on sets of measure zero. For this, one uses appropriate measurable selections from the correspondence mapping types of couples to feasible contracts and efficient contracts, respectively. Finally, a close look at the proof shows that we can arbitrarily change the type functions t_W and t_M and the same spaces of agents will still work for the resulting matching problem.

In our representation for stable matchings, there might still be some "blocking pairs." What the representation does ensure is that two agents that could form a blocking pair have a hard time finding each other. It is tedious but straightforward to verify that the set of $a_W \in A_W$ and $a_M \in A_M$ that could form a blocking pair has $\tau_W \otimes \tau_M$-measure zero. Part of the tedium is that one has to define blocking pairs for pairs of agents and take account of the outside options ø. We sketch the argument for the simplest case, the case of the marriage model without contracts, non-binding individual rationality constraints, and both ν_W and ν_M being probability measures. Let μ be a stable matching. Then $\phi : A_W \to A_M$ is a measure-preserving measurable bijection with a measurable inverse such that the function $a_W \mapsto (t_W(a_W), t_M(\phi(a_W)))$ has τ_W-distribution μ. That ϕ preserves measure implies that the function $a_M \mapsto (t_W(\phi^{-1}(a_M)), t_M(a_M))$ has τ_M-distribution μ, too. It follows that the function $(a_W, a_M) \mapsto (t_W(a_w), t_M(\phi(a_W)), t_W(\phi^{-1}(a_M)), t_M(a_M))$ has $\tau_W \otimes \tau_M$-distribution $\mu \otimes \mu$. So for $\tau_W \otimes \tau_M$-almost all a_W and a_M,

$$(t_W(a_w), t_M(\phi(a_W)), t_W(\phi^{-1}(a_M)), t_M(a_M)) \notin I$$

since $\mu \otimes \mu(I) = 0$. And for such a_W and a_M, it is not the case that a_W prefers a_M to $\phi(a_W)$ and a_M prefers a_W to $\phi^{-1}(a_M)$.

8 Concluding Remarks

We have provided foundations for large two-sided matching markets from the distributional point of view. Our model represents exactly the distributional properties of large finite matching markets that are preserved under weak convergence. Even though individual agents are negligible, stability has a simple and natural interpretation in the limit model, and we nest existing models with transfers. Stable matchings exist and exist even in the presence of widespread externalities. We also provided an individualistic interpretation of our distributional model and used it to clarify the economic meaning of our stability notion.

It is time to take a look at what our stability notion does not deliver. There is a problem in treating our model as a limit model of econometric matching models. In econometric models of matching, the payoff usually includes an idiosyncratic additive component that is stochastically independent between pairs of agents that might be matched. This implies in our distributional framework that there is a jointly measurable function $\epsilon : W \times M \to \mathbb{R}$, representing the purely idiosyncratic part, such that for ν_W-almost all w and ν_W-almost all $w' \in W$, the random variables $\epsilon(w, \cdot)$ and $\epsilon(w', \cdot)$ are stochastically independent. By Sun (1998, Proposition 1.1), the random variable $\epsilon(w, \cdot)$ must be constant for ν_W-almost all w and therefore deterministic; a degenerate case. One approach to deal with the problem is to characterize the projection of stable matchings onto observable characteristics, not to worry whether unobservable characteristics converge or not, and only look at the limit of the observable part. In a special econometric version of the marriage model, Menzel (2015) did exactly that. The details of how the idiosyncratic part of preferences is modeled will generally matter. A technical hurdle in adapting our approach to this more general problem is that we make much use of the instability set being open. When only some characteristics are observable, we have to work with the projection of an open set, and such a projection need not be open. These problems are, of course, not particular to our approach, they haunt all the existing models nested by ours.

The source of the problem just mentioned is that the idiosyncratic component of payoffs depends on both sides. In the econometric transferable utility model of Choo and Siow (2006), this problem does not arise and our methods guarantee existence in a generalization of the model of Choo and Siow (2006) to imperfectly transferable utility.[22] There are nonempty finite sets I and J of categories of women and men, respectively. Every agent has an idiosyncratic additional payoff for each category on the other side and the option to remain unmatched. To reflect this in our distributional approach, we let $W = I \times \mathbb{R}^{\#J+1}$ and $M = J \times \mathbb{R}^{\#I+1}$, and we let ν_W and ν_M be strictly positive, finite Borel measures on W and M, respectively. We identify for each $i \in I$ the measure ν_i on $\mathbb{R}^{\#J+1}$ given by

$$\mu_i(B) = \nu_W(\{i\} \times B)/\nu_W(\{i\} \times \mathbb{R}^{\#J+1})$$

for each Borel set $B \subseteq \mathbb{R}^{\#J+1}$ with the distribution of the idiosyncratic payoff component. Implicitely, there is some law of large numbers in action that guarantees that the ex-post distribution of the idiosyncratic component equals the ex-ante distribution. We can define and interpret ν_j similarly for $j \in J$. We also let $F^{ij} : \mathbb{R} \to \mathbb{R}$ be a strictly decreasing continuous surjection for each $i \in I$ and $j \in J$. We can then define a continuous correspondence with nonempty and compact values $\mathbb{C} : W \times M \to \mathbb{R}^2$ by

$$\mathbb{C}(w, m) = \mathbb{C}((i, \alpha_1, \dots, \alpha_{\#J}, \alpha_0), (j, \beta_1, \dots, \beta_{\#I}), \beta_0) =$$

$$= \{(u, v) \in \mathbb{R}^2 \mid u = F^{ij}(v - \beta_i) + \alpha_j, u \geq \alpha_0, v \geq \beta_0\} \cup \{\alpha_0 - 1, \beta_0 - 1\}.$$

We extend $\mathbb{C}$ to all of $W_\emptyset \times M_\emptyset$ by letting $\mathbb{C}(\emptyset, \emptyset)$ be an arbitrary pair of numbers, $\mathbb{C}(w, \emptyset) = \{(\alpha_0, 0)\}$ and $\mathbb{C}(\emptyset, m) = \{(0, \beta_0)\}$. Payoffs will then depend only on contracts and are given by the coordinate projections.

For technical reasons related to the problem of having two-sided idiosyncratic payoff-components, our model cannot be used to study the asymptotic stochastic behavior of large finite marriage models in which every agent's preferences are independently and uniformly chosen from the set of strict rankings of agents on the other side, the approach of Pittel (1989) and many subsequent papers. For simplicity, we look at the classical marriage model with nonbinding individual rationality constraints and balanced populations. Define the women's *rank function* $R_W : W \times M \to [0, 1]$ by

$$R_W(w, m) = \nu_M\Big(\{m' \in M \mid m \succ_w m'\}\Big).$$

The measurability of the rank function follows from Fubini's theorem.[23] Now take a large finite matching problem in which preferences of every agent are independently selected from a uniform distribution over all strict rankings of agents on the other side. Take α and α' between 0 and 1 and choose m and m' at random from the population of men. For a large population, the probability that an individual woman ranks m above α is almost independent of the probability that she ranks m' above α'. So for a large population,

[22]We are grateful to Pierre-André Chiappori for suggesting this model.

[23]Let $G = \{(w, m, m') \mid m \succ m'\}$. By Fubini's theorem, $\nu_W \otimes \nu_M \otimes \nu_M(G) = \int\int\int 1_G \, d\nu_W \, d\nu_M \, d\nu_M$. In particular, the function $(w, m) \mapsto \int 1_G(w, m, \cdot) \, d\nu_M = R_W(w, m)$ is measurable.

the fraction of women who rank m above α and m' above α' is, with high probability, close to the product of the fraction of women who rank m above α times the fraction of women who rank m' above α'. This means that with high probability, $R_W(\cdot, m)$ and $R_W(\cdot, m')$ are close to independent for m and m' chosen at random. If we want to have a limit model that takes account of this asymptotic independence, we need to specify Polish spaces W and M, appropriate preferences, and population measures ν_W and ν_M such that for $\nu_M \otimes \nu_M$-almost all (m, m'), the random variables $R_W(\cdot, m)$ and $R_W(\cdot, m')$ are independent. Again, it follows from Sun (1998, Proposition 1.1) that for ν_M-almost all $m \in M$, the function $R(\cdot, m)$ is constant ν_W-almost surely. In particular, almost all women rank almost all men exactly the same way. This is as far from the model of Pittel (1989) as can be. Indeed, for large n, most agents will have ranking functions that are close to uniformly distributed.

We see the definition of stability as the main contribution of this paper. This definition allows us to work in a distributional model with a rich set of types even though all effective coalitions are finite and seemingly invisible at the level of analysis. There is no reason to think this approach is restricted to matching theory alone. In club theory as developed in Cole and Prescott (1997) and Ellickson, Grodal, Scotchmer, and Zame (1999), agents in individualistic continuum economies form clubs with finitely many members to organize their consumption and production decisions. A major restriction in this literature is that the external characteristics of agents (those relevant to other agents) and characteristics of clubs must belong to a finite set, a restriction that could be overcome by the distributional approach when unstable club internal decisions can be identified by random sampling as in our stability definition.

9 Omitted Proofs

9.1 Proofs omitted from Section 3

Proof of Lemma 1. It suffices to show that the sequence $\langle \mu_n \rangle$ is tight. Take some $\epsilon > 0$. Since the families $\{\nu_W^n\}$ and $\{\nu_M^n\}$ of population distributions come from converging sequences, they have compact closure and are therefore tight. So there are compact sets $K_W \subseteq W$ and $K_M \subseteq M$ such that both $\nu_W^n(W \setminus K_W) < \epsilon/2$ and $\nu_M^n(M \setminus K_M) < \epsilon/2$ holds for every natural number n. Let $K_W^\varnothing = K_W \cup \{\varnothing\}$ and $K_M^\varnothing = K_M \cup \{\varnothing\}$. Now

$$\mathbb{C}(K_W^\varnothing \times K_M^\varnothing) = \bigcup_{(w,m) \in K_W^\varnothing \times K_M^\varnothing} \mathbb{C}(w, m)$$

is compact as the forward image of a compact set under a compact-valued upper hemicontinuous correspondence; Aliprantis and Border (2006, 17.8). We show that

$$\mu_n\Big(W_\varnothing \times M_\varnothing \times C \ \setminus \ K_W^\varnothing \times K_M^\varnothing \times \mathbb{C}(K_W^\varnothing \times K_M^\varnothing)\Big) < \epsilon$$

for every natural number n. Indeed, in order for a couple-contract type to be in $W_\varnothing \times M_\varnothing \times C$ but not in $K_W^\varnothing \times K_M^\varnothing \times \mathbb{C}(K_W^\varnothing \times K_M^\varnothing)$ it has to either have a first term not in $K_W^\varnothing$ and therefore be in $W_\varnothing \setminus K_W^\varnothing \times M_\varnothing \times C$, or it has to have a second term not in $K_M^\varnothing$ and therefore be in $W_\emptyset \times (M_\emptyset \backslash K_M^\emptyset) \times C$, or it has to have both first two terms in $K_W^\emptyset \times K_M^\emptyset$ but the third term not in $\mathbb{C}(K_W^\emptyset \times K_M^\emptyset)$ and therefore be in $K_W^\emptyset \times K_M^\emptyset \times C \backslash \mathbb{C}(K_W^\emptyset \times K_M^\emptyset)$. Now $W_\emptyset \backslash K_W^\emptyset \times M_\emptyset \times C$ has μ_n-measure $\nu_W^n(W \backslash K_W) < \epsilon/2$ by condition (i) in the definition of a matching, $W_\emptyset \times (M_\emptyset \backslash K_M^\emptyset) \times C$ has μ_n-measure $\nu_M^n(M \backslash K_M) < \epsilon/2$ by condition (ii) in the definition of a matching, and, finally, $K_W^\emptyset \times K_M^\emptyset \times C \backslash \mathbb{C}(K_W^\emptyset \times K_M^\emptyset)$ has μ_n measure zero by condition (iii) in the definition of a matching. Now $\epsilon/2 + \epsilon/2 + 0 = \epsilon$, so we obtain the desired inequality and therefore the tightness of the sequence $\langle \mu_n \rangle$. □

Proof of Lemma 2. Since the topology of weak convergence of measures is metrizable, it suffices to prove that the limit (ν_W, ν_M, μ) of a convergent sequence $\langle \nu_W^n, \nu_M^n, \mu_n \rangle$ with values in the set must again lie in the set.

We first show that for every Borel set $B \subseteq W$, we have $\mu(B \times M_\emptyset \times C) = \nu_W(B)$, which is equivalent to showing $\mu \circ \pi_W^{-1}(B) = \nu_W(B)$, with $\pi_W : W \times M_\emptyset \times C \to W$ being the canonical projection. Since Borel measures on Polish spaces are tight, for each Borel set $B \subseteq W$,

$$\mu \circ \pi_W^{-1}(B) = \sup\left\{\mu \circ \pi_W^{-1}(K) \mid K \text{ compact and } K \subseteq B\right\}.$$

and

$$\nu_W(B) = \sup\{\nu_W(K) \mid K \text{ compact and } K \subseteq B\}.$$

It therefore suffices to prove the result for B compact. So let B be compact and $d : W \times W \to \mathbb{R}$ be a compatible metric. For each $\delta > 0$, let $B_\delta = \{x \in W \mid d(x, B) < \delta\}$. Note that $B_\delta \downarrow B$ as $\delta \downarrow 0$. For each $x \in B$ and $\epsilon > 0$, the boundary $\partial B_\epsilon(x)$ of the ball $B_\epsilon(x)$ is a subset of the sphere $S_\epsilon(x) = \{y \in W \mid d(x, y) = \epsilon\}$. This boundary can therefore only have positive $\mu \circ \pi_W^{-1}$-measure or ν_W-measure for countably many ϵ, since no finite measure space can allow for an uncountable disjoint family of measurable sets with positive measure.[24] For each $x \in B$, there is therefore some ϵ_x^1 such that $0 < \epsilon_x^1 < 1$ and

$$\mu \circ \pi_W^{-1}\left(\partial B_{\epsilon_x^1}(x)\right) = \nu_W\left(\partial B_{\epsilon_x^1}(x)\right) = 0.$$

The family $\{B_{\epsilon_x^1} \mid x \in B\}$ is an open cover of the compact set B and has, therefore, a finite subcover. Let B^1 be the union of this subcover. Then $B \subseteq B^1 \subseteq B_1$, B^1 is open and since the boundary of a finite union of sets is a subset of their boundaries, we have

$$\mu \circ \pi_W^{-1}\left(\partial B^1\right) = \nu_W\left(\partial B^1\right) = 0.$$

Given that B^n is defined, we can repeat the procedure to obtain B^{n+1} so that $B^{n+1} \subseteq B^n$ and $B^{n+1} \subseteq B_{1/n}$ and

$$\mu \circ \pi_W^{-1}(\partial B^n) = \nu_W(\partial B^n) = 0.$$

By the Portmanteau theorem,

$$\mu \circ \pi_W^{-1}(B^n) = \lim_m \mu_m \circ \pi_W^{-1}(B^n) = \lim_m \nu_W^m(B^n) = \nu_W(B^n)$$

[24]Indeed, if $\mathcal{F}$ is a disjoint family of measurable sets, the family of all $F \in \mathcal{F}$ whose measure exceeds $1/n$ must be finite for every natural number n. Since the countable union of finite sets is countable, the conclusion follows.

for all n. Now, since measures are downward-continuous,

$$\mu \circ \pi_W^{-1}(B) = \lim_n \mu \circ \pi_W^{-1}(B^n) = \lim_n \nu_W(B^n) = \nu_W(B).$$

Similarly, one can show that $\mu(W_\emptyset \times B \times A) = \nu_M(B)$ for every Borel set $B \subseteq M$.

Finally, since $G_{\mathbb{C}}$ is closed as the graph of a compact-valued and upper hemicontinuous correspondence, $W_\emptyset \times M_\emptyset \times C \setminus G_{\mathbb{C}}$ is open and therefore, by another use of the Portmanteau theorem,

$$\mu(W_\emptyset \times M_\emptyset \times C \setminus G_{\mathbb{C}}) \leq \liminf_n \mu_n(W_\emptyset \times M_\emptyset \times C \setminus G_{\mathbb{C}}) = 0.$$

□

Proof of Lemma 3. We can assume without loss of generality that $W = \operatorname{supp} \nu_W$ and $M = \operatorname{supp} \nu_M$. For each $w \in W_\emptyset$ and $m \in M_\emptyset$ let $\langle c_{wm}^n \rangle$ be a sequence in $\mathbb{C}(w,m)$ such that $\{c_{wm}^n \mid n \in \mathbb{N}\}$ is dense in $\mathbb{C}(w,m)$. Define $\mathbb{C}_n : W_\emptyset \times M_\emptyset \to 2^C$ by $\mathbb{C}_n(w,m) = \{c_{wm}^k \mid k \leq n\}$. A stable matching exists when we replace $\mathbb{C}$ by $\mathbb{C}_n$. To see this, represent the matching problem by an individualistic matching problem with actual agents. This is possible since ν_W and ν_M have finite support and take on only rational values. Extend all preferences to strict linear orders. This is possible since preferences are acyclic. With strict linear orders, weakly efficient and strictly efficient contract choices coincide. So one can apply the extended deferred acceptance algorithm with wages of Crawford and Knoer (1981) and Kelso and Crawford (1982) with the set of efficient contracts to obtain a stable matching for the extended preferences. Since extending preferences cannot reduce blocking possibilities, the matching continues to be stable under the original preferences. The induced distribution of couple-contract types gives us a distributional stable matching.

Note that every matching, stable or not, for the restricted correspondence $\mathbb{C}_n$ is also a, not necessarily stable, matching for the unrestricted correspondence $\mathbb{C}$. So we can find a sequence $\langle \mu_n \rangle$ of matchings such that μ_n is a stable matching for the restricted correspondence $\mathbb{C}_n$. By passing to a subsequence and using Lemma 1, we can assume without loss of generality that $\langle \mu_n \rangle$ converges to some measure μ, which is again a matching by Lemma 2.

It remains to prove that μ is stable. For each $n \in \mathbb{N}$, let I_n be the instability set for the matching problem with contracts involved in the instability conditions of the form "for some $c'' \in \mathbb{C}(w,m)$" taken from the restricted correspondence $\mathbb{C}_n$. Note that I_n itself can still contain contracts not available under $\mathbb{C}_n$. The continuity assumption on preferences guarantees that I_n is open for every natural number n, and, together with $\bigcup_n \mathbb{C}_n(w,m) = \{c_{wm}^n \mid n \in \mathbb{N}\}$ being dense in $\mathbb{C}(w,m)$, also that $I = \bigcup_n I_n$. If $k \leq n$, then $I_k \subseteq I_n$, so $\mu_n \otimes \mu_n(I_k) = 0$ for $k \leq n$. Therefore,

$$\mu \otimes \mu(I_k) \leq \liminf_n \mu_n \otimes \mu_n(I_k) = 0$$

by the Portmanteau theorem. Finally,

$$\mu \otimes \mu(I) = \mu \otimes \mu\left(\bigcup_k I_k\right) \leq \sum_k \mu \otimes \mu(I_k) = 0.$$

□

Proof of Lemma 4. Clearly, λ is absolutely continuous with respect to μ and has a Radon-Nikodym derivative g with values in $[0,1]$. Using Fubini's theorem,

$$\begin{aligned}
\lambda \otimes \lambda(I) &= \int 1_I \, \mathrm{d}\lambda \otimes \lambda \\
&= \int \int 1_I(x,y) \, \mathrm{d}\lambda(x)\mathrm{d}\lambda(y) \\
&= \int g(y) \int g(x) 1_I(x,y) \, \mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&= \int g(x)g(y) 1_I(x,y) \, \mathrm{d}\mu \otimes \mu(x,y) \\
&\leq \int 1_I \, \mathrm{d}\mu \otimes \mu \\
&= \mu \otimes \mu(I) \\
&= 0.
\end{aligned}$$

□

9.2 Proof of Theorem 3 and Theorem 4

We will prove Theorem 3 and Theorem 4 in two parts each. For Theorem 3, we first show, under the given assumptions, that there exist for each stable matching continuous functions $V_W : W \to \mathbb{R}$ and $V_M : M \to \mathbb{R}$ such that $V_W(w) = u_W(w,m,c)$ for μ-almost all $(w,m,c) \in W \times M_\varnothing \times [0,1]$ and $V_M(m) = u_M(w,m,c)$ for μ-almost all $(w,m,c) \in W_\varnothing \times M \times [0,1]$. This is done in Lemma 9. We then show that if there exist such continuous functions for a matching μ, stable or not, and if population distributions have full support, then stability is characterized by (i)-(iii) of Theorem 3. This is done in Lemma 11.

Similarly, we prove Theorem 4 by first showing that under the given assumptions there are for each stable matching measurable functions $V_W : W \to \mathbb{R}$ and $V_M : M \to \mathbb{R}$ such that $V_W(w) = u_W(w,m,c)$ for μ-almost all $(w,m,c) \in W \times M_\varnothing \times [0,1]$ and $V_M(m) = u_M(w,m,c)$ for μ-almost all $(w,m,c) \in W_\varnothing \times M \times [0,1]$. This is done in Lemma 10. We then show that if there exist such measurable functions for a matching μ, stable or not, then stability is characterized by (i)-(iii) of Theorem 4. This is done in Lemma 12.

By far, the most work is required in proving Lemma 9. Lemma 10 is mostly a corollary, obtained by approximating general type spaces by compact subspaces. We formulate the arguments in a one-sided way, proving the existence of an appropriate function V_W.

The proof of Lemma 9 will require some preliminary work. To simplify stability arguments, we want every woman to be able to provide every man with every utility level the man might get in some stable matching. In general, this will not be possible. But we can extend the model by adding additional contracts in a way that makes this possible while not changing the set of stable matchings. We do this by making sure that the additional contracts will violate the individual rationality constraints. This is done in Lemma 5. In the extended model, there are continuous compensation functions selecting the right contract to do the compensating; see Lemma 6. We then construct a continuous function that serves as a kind of "modulus of continuity" for the payoffs in a stable matching; see Lemma 7. If randomly selected pairs of points in a compact metrizable space and real numbers satisfy a given modulus of continuity, we can obtain a continuous function satisfying the modulus continuity almost surely by sampling countably many points. This will almost surely give us the graph of a continuous function on a dense subset which we can extend by continuity to the whole compact metrizable space. The underlying distribution of pairs will be supported on the graph of this continuous function; see Lemma 8. Applying this lemma to the matching context gives us then Lemma 9.

Lemma 5. *Let W and M be compact. There exist $\alpha > 0$ and continuous functions*

$$u_W^* : W \times M_\varnothing \times [0 - \alpha, 1 + \alpha] \to \mathbb{R}$$

and

$$u_M^* : W_\varnothing \times M \times [0 - \alpha, 1 + \alpha] \to \mathbb{R}$$

such that

(i) $u_W^*(w, m, c) = u_W(w, m, c)$ *and* $u_M^*(w, m, c) = u_M(w, m, c)$ *for all* $w \in W$, $m \in M$, *and* $c \in [0, 1]$,

(ii) $u_W(w, \varnothing, 1 + \alpha) = u_W^\varnothing(w)$ *for all* $w \in W$ *and* $u_M^*(\varnothing, m, 0 - \alpha) = u_M^\varnothing(m)$ *for all* $m \in M$,

(iii) u_W *is increasing in the third coordinate and* u_M *is decreasing in the third coordinate,*

(iv) There are numbers b, t *satisfying* $0 - \alpha < b < t < 1 + \alpha$ *such that*

$$u_M^*(w, m, t) < u_M^\varnothing(m) \text{ and } u_W^*(w, m, b) < u_W^\varnothing(w)$$

for all $w \in W$ *and* $m \in M$ *and such that*

$$u_M^*(w, m, 1 + \alpha) < u_M^*(w', m, t) \text{ and } u_M^*(w, m, 0 - \alpha) > u_M^*(w', m, b)$$

for all $w, w' \in W$ *and* $m \in M$.

Proof. Since outside options are isolated points in the type spaces, we only have to consider the extension for $w \in W$ and $m \in M$, for outside options we simply rescale contracts. Take an arbitrary $\alpha > 0$ and for all $w \in W$, $m \in M$, and $\delta \in [0, \alpha]$, we let $u_W^*(w, m, 1+\delta) = u_W(w, m, 1)+\delta$, $u_M^*(w, m, 1+\delta) = u_M(w, m, 1)-\delta$, $u_W^*(w, m, 0-\delta) = u_W(w, m, 0) - \delta$, and $u_M^*(w, m, 0 - \delta) = u_M(w, m, 0) + \delta$. Then it is clear that u_W^* and u_M^* are continuous and satisfy (i)-(iii). Moreover, a simple continuity and compactness argument shows that (iv) is satisfied for α large enough, since the continuous functions $u_W^\varnothing$ and $u_M^\varnothing$ are bounded on the compact spaces W and M, respectively. □

It follows from (i), (iii), and the assumption that bounds on transfers do not bind that all additional contracts violate the individual rationality constraint. As a consequence, the complement of the instability set in the extended model will be the same as the complement of the instability set of the original model.

Lemma 6. *In any extended model as shown to exist in Lemma 5, there exists a unique function* $\chi : W \times (W \times M \times [b,t]) \to [0-\alpha, 1+\alpha]$ *such that*

$$u_M^*(w,m,c) = u_M^*\Big(w', m, \chi(w', (w,m,c))\Big)$$

for all w, m, c, w'. *The function* χ *is continuous.*

We call the unique function $\chi : W \times (W \times M \times [b,t]) \to [0-\alpha, 1+\alpha]$ shown to exist in Lemma 6 the *compensation function.* For notational ease, we write $\chi_{w'}(w,m,c)$ for $\chi(w',(w,m,c))$.

Proof of Lemma 6. For each $w, w' \in W$, $m \in M$ and $c \in [b,t]$,

$$u_M^*(w', m, 0-\alpha) \geq u_M^*(w,m,c) \geq u_M^*(w', m, 1+\alpha)$$

by (iii) and (iv) of Lemma 5. By the intermediate value theorem, there exists $c' \in [0-\alpha, 1+\alpha]$ such that $u_M^*(w,m,c) = u_M^*(w',m,c')$. Since u_M^* is decreasing in its third argument, there can be at most one such c. Therefore, χ is well defined and unique. To see that χ is continuous, note that

$$\chi(w',(w,m,c)) = \operatorname*{arg\,min}_{c' \in [0-\alpha, 1+\alpha]} |u_M^*(w,m,c) - u_M^*(w',m,c')|$$

and apply the maximum theorem. □

Lemma 7. *Let* W *and* M *be compact. There exists a continuous function* $\omega : W \times W \to \mathbb{R}_+$ *such that* $\omega(w,w) = 0$ *for all* $w \in W$ *and such that*

$$|u_W(w,m,c) - u_W(w',m',c')| \leq \omega(w,w')$$

for all

$$((w,m,c),(w',m',c')) \in W \times M_\emptyset \times [0,1] \ \times\ W \times M_\emptyset \times [0,1]$$

that are not in the instability set I.

Proof. We work in the extended model as shown to exist by Lemma 5. Since this does not change the complement of the instability set, the results apply to the original model, too. By abuse of notation, we write u_W and u_M for the extensions u_W^* and u_M^*.

Let $((w,m,c),(w',m',c')) \notin I$. We derive a number of inequalities by simple stability and continuity arguments. If both $m \in M$ and $m' \in M$, then

$$\begin{aligned} u_W(w,m,c) &\geq u_W(w,m',\chi_w(w',m',c')), \\ u_W(w',m',c') &\geq u_W(w',m,\chi_{w'}(w,m,c)), \end{aligned}$$

which implies

$$\begin{aligned} |u_W(w,m,c) - u_W(w',m',c')| \leq & |u_W(w,m,c) - u_W(w',m,\chi_{w'}(w,m,c))| \\ & + |u_W(w',m',c') - u_W(w,m',\chi_w(w',m',c'))|. \end{aligned}$$

If $m = \varnothing = m'$, then

$$|u_W(w,m,c) - u_W(w',m',c')| = |u_W^{\varnothing}(w) - u_W^{\varnothing}(w')|.$$

If $m \in M$ and $m' = \varnothing$, then

$$\begin{aligned} u_W(w,m,c) - u_W(w',m',c') &\le u_W(w,m,c) - u_W(w',m,\chi_{w'}(w,m,c)), \\ u_W(w',m',c') - u_W(w,m,c) &= u_W^{\varnothing}(w') - u_W(w,m,c) \le u_W^{\varnothing}(w') - u_W^{\varnothing}(w). \end{aligned}$$

Similarly, if $m = \varnothing$ and $m' \in M$, then

$$\begin{aligned} u_W(w',m',c') - u_W(w,m,c) &\le u_W(w',m',c') - u_W(w,m',\chi_{w'}(w,m,c)), \\ u_W(w,m,c) - u_W(w',m',c') &= u_W^{\varnothing}(w) - u_W(w',m',c') \le u_W^{\varnothing}(w) - u_W^{\varnothing}(w'). \end{aligned}$$

Collecting inequalities, we obtain

$$\begin{aligned} |u_W(w,m,c) - u_W(w',m',c')| \le &|u_W(w,m,c) - u_W(w',m,\chi_{w'}(w,m,c))| \\ &+ |u_W(w',m',c') - u_W(w,m',\chi_w(w',m',c'))| \\ &+ |u_W^{\varnothing}(w) - u_W^{\varnothing}(w')|, \end{aligned}$$

with the first two terms only being effective if $m \in M$ or $m' \in M$, respectively. A fortiori, $|u_W(w,m,c) - u_W(w',m',c')|$ can be no larger than

$$\begin{aligned} &\max_{m \in M, c \in [b,t]} |u_W(w,m,c) - u_W(w',m,\chi_{w'}(w,m,c))| \\ &+ \max_{m' \in M, c' \in [b,t]} |u_W(w',m',c') - u_W(w,m',\chi_w(w',m',c'))| \\ &+ |u_W^{\varnothing}(w) - u_W^{\varnothing}(w')|. \end{aligned}$$

This last expression depends only on w and w' and we take it to be the value of $\omega(w,w')$. Clearly, $\omega(w,w') = 0$ if $w = w'$. The continuity of ω follows from the maximum theorem. □

Lemma 8. *Let K be a compact metrizable space and $\omega : K \times K \to \mathbb{R}$ a continuous function such that $\omega(x,x) = 0$ for all $x \in K$. If μ is a Borel measure on $K \times \mathbb{R}$ such that $|r - r'| \le \omega(x,x')$ for $\mu \otimes \mu$-almost all pairs $((x,r),(x',r'))$, then μ is supported on the graph of a unique continuous function from the support of the K-marginal of μ to $\mathbb{R}$.*

Proof. Without loss of generality, we can assume that μ is a probability measure. Consider the space $(K \times \mathbb{R})^\infty$ endowed with the product measure $\mu^\infty = \bigotimes_n \mu$ and let $\langle x_n, r_n \rangle \in (K \times \mathbb{R})^\infty$ be a random sequence.

The space K has a countable basis; pick an open set O in such a basis. If $\mu(O \times \mathbb{R}) = 0$, then μ^∞-almost surely $x_n \notin O$ for each natural number n. If $\mu(O \times \mathbb{R}) > 0$, then μ^∞-almost surely $x_n \in O$ for some natural number n. So μ^∞-almost surely, the closure of the random set $\{x_n \mid n \in \mathbb{N}\}$ equals the support of the K-marginal of μ.

Now μ^∞-almost surely, $|r_m - r_n| \le \omega(x_m, x_n)$ for $m, n \in \mathbb{N}$. Indeed, this holds, by assumption, for fixed m and n, and there are only countably many such pairs of natural numbers. In particular, $r_m = r_n$ whenever $x_m = x_n$ holds μ^∞-almost surely, so the random set $\{(x_n, r_n) \mid n \in \mathbb{N}\}$ is μ^∞-almost surely the graph of a function g^∞. Let d be any metric that metrizes K. We show that g^∞ is uniformly continuous with respect to d. Let $\epsilon > 0$. The set $\omega^{-1}([0,\epsilon))$ is an open neighborhood of the diagonal $D_K = \{(x,y) \in K \times K \mid x = y\}$. Define the metric d_1 on $K \times K$ by $d_1((x,y),(x',y')) = d(x,x') + d(y,y')$ and observe that $d_1((x,y), D_K) = d(x,y)$. Since D_K is compact and the function $(x,x) \mapsto d_1((x,x), \omega^{-1}([\epsilon,\infty)))$ continuous, the function must take on a minimal value $\delta > 0$. Then for $d(x_m, x_n) < \delta$, we get $\omega(x_m, x_n) < \epsilon$ and, since $|r_m - r_n| \leq \omega(x_m, x_n)$, also $|r_m - r_n| < \epsilon$. So g^∞ is uniformly continuous and extends, by Aliprantis and Border (2006, 3.11), to a unique continuous function g defined on the closure of $\{x_n \mid n \in \mathbb{N}\}$, which equals the support of the K-marginal of μ. Now for μ-almost all (x,r), we must have $|r - r_n| \leq \omega(x, x_n)$ for each natural number n. But this implies that (x,r) lies on the graph of g, since g is continuous.

Next, to see that g is unique, assume that g' is another continuous function from the support of the K-marginal of μ to $\mathbb{R}$ whose graph supports μ. Take another random sequence $\langle x'_n, r'_n \rangle \in (K \times \mathbb{R})^\infty$. Now μ^∞-almost surely, the closure of the set $\{x'_n \mid n \in \mathbb{N}\}$ equals the support of the K-marginal of μ as above. Since g and g' coincide μ^∞-almost surely, we have μ^∞-almost surely that $g(x'_n) = r'_n = g'(x'_n)$ for each natural number n. But two continuous functions that agree on a dense set must coincide, so $g' = g$.

□

Lemma 9. *Let W and M be compact and let μ be a stable matching. Then there exists a unique continuous function $V_W : \operatorname{supp} \nu_W \to \mathbb{R}$ such that*

$$V_W(w) = u_W(w,m,c)$$

for μ-almost all $(w,m,c) \in W \times M_\varnothing \times [0,1]$.

Proof. Let $\omega : W \times W \to \mathbb{R}_+$ be a function as guaranteed to exist by Lemma 7. Let μ^W be the trace of μ on $W \times M_\varnothing \times [0,1]$. That is, $\mu^W(B) = \mu(B \cap W \times M_\varnothing \times [0,1])$ for every Borel set $B \subseteq W_\varnothing \times M_\varnothing \times [0,1]$. Define $h : W \times M_\varnothing \times [0,1] \to W_\varnothing \times M_\varnothing \times [0,1] \times \mathbb{R}$ by

$$h(w,m,c) = (w,m,c,u_W(w,m,c)).$$

We show that the $W \times \mathbb{R}$-marginal of $\mu^W \circ h^{-1}$ satisfies the conditions of Lemma 8. To see this, let $\pi : W \times M_\varnothing \times [0,1] \times \mathbb{R} \to W \times \mathbb{R}$ be the canonical projection. The $W \times \mathbb{R}$-marginal of $\mu^W \circ h^{-1}$ is then simply $\mu^W \circ h^{-1} \circ \pi^{-1}$. Now

$$\begin{aligned}
&\mu^W \circ h^{-1} \circ \pi^{-1} \otimes \mu^W \circ h^{-1} \circ \pi^{-1}\Big(\{((w,r),(w',r')) \mid |r - r'| > \omega(w,w')\}\Big) \\
&= \mu^W \circ h^{-1} \otimes \mu^W \circ h^{-1}\Big(\{((w,m,c,r),(w',m',c',r')) \mid |r - r'| > \omega(w,w')\}\Big) \\
&\leq \mu^W \circ h^{-1} \otimes \mu^W \circ h^{-1}\Big(\{((w,m,c,r),(w',m',c',r')) \mid (w,m,c),(w',m',c') \in I\}\Big) \\
&= \mu^W \otimes \mu^W(I) \leq \mu \otimes \mu(I) = 0.
\end{aligned}$$

Let $V_W : \operatorname{supp} \nu_W \to \mathbb{R}$ be the unique function shown to exist by Lemma 8. We have

$$\begin{aligned}
0 &= \mu^W \circ h^{-1} \circ \pi^{-1}\Big(\{(w,r) \mid V_W(w) \neq r\}\Big) \\
&= \mu^W\Big(\{(w,m,c) \mid V_W(w) \neq u_W(w,m,c)\}\Big),
\end{aligned}$$

so V_W has the desired properties. Moreover, since any other function V'_W with the desired properties must satisfy the last two equations in place of V_W, uniqueness follows from the uniqueness part of Lemma 8. □

Lemma 10. *Let μ be a stable matching. Then there exists a measurable function $V_W : W \to \mathbb{R}$ such that*

$$V_W(w) = u_W(w, m, c)$$

for μ-almost all $(w, m, c) \in W \times M_\emptyset \times [0, 1]$.

Proof. Since ν_W and ν_M are tight, there exist increasing sequences $\langle K_W^n \rangle$ and $\langle K_M^n \rangle$ of compact subsets of W and M, respectively, such that $\nu_W(W) = \lim_n \nu_W(K_W^n)$ and $\nu_M(M) = \lim_n \nu_M(K_M^n)$. Let $K_{W\emptyset}^n = K_W^n \cup \{\emptyset\}$ and $K_{M\emptyset}^n = K_M^n \cup \{\emptyset\}$. For each natural number n, define μ_n by

$$\mu_n(B) = \mu(B \cap K_{W\emptyset}^n \times K_{M\emptyset}^n \times [0, 1])$$

for every Borel set $B \subseteq W_\emptyset \times M_\emptyset \times [0, 1]$. Then μ_n is a stable matching for appropriately chosen population measures supported on compact sets by Lemma 4. By Lemma 9, there exists for each natural number n a measurable function $V_n : W \to \mathbb{R}$ such that $V_n(w) = u_W(w, m, c)$ for μ_n-almost all $(w, m, c) \in K_W^n \times K_{M\emptyset}^n \times [0, 1]$. Let $V : W \to \mathbb{R} \cup \{\infty\}$ be given by $V(w) = \limsup_n V_n(w)$. Construct V_W from V by changing the value ∞ to some real number. We claim that V_W has the desired property. Consider the set

$$N = \{(w, m, c) \in W \times M_\emptyset \times [0, 1] \mid V_W(w) \neq u_W(w, m, c)\}.$$

It suffices to show that $\mu(N) = 0$. Suppose not. Since $\mu(N) = \lim_n \mu_n(N)$, there exists some natural number k such that $\mu_k(N) > 0$. Let $n \geq k$. We claim that $V_k(w) = V_n(w)$ for μ_k-almost all $(w, m, c) \in K_W^k \times K_{M\emptyset}^k \times [0, 1]$. Indeed, every set of μ_n-measure zero has μ_k-measure zero, so $V_n(w) = u_W(w, m, c) = V_k(w)$ for μ_k-almost all $(w, m, c) \in K_W^k \times K_{M\emptyset}^k \times [0, 1]$. It follows that $V_W(w) = \lim_n V_n(w) = V_k(w)$ for μ_k-almost all $(w, m, c) \in K_W^k \times K_{M\emptyset}^k \times [0, 1]$. Therefore $\mu_k(N) > 0$ is equivalent to

$$\mu_k\Big(\{(w, m, c) \in W \times M_\emptyset \times [0, 1] \mid V_k(w) \neq u_W(w, m, c)\}\Big) > 0,$$

which is impossible. □

Lemma 11. *Let μ be a matching. If ν_W and ν_M have full support and $V_W : W \to \mathbb{R}$ and $V_M : M \to \mathbb{R}$ are continuous functions such that*

$$V_W(w) = u_W(w, m, c)$$

for μ-almost all $(w, m, c) \in W \times M_\emptyset \times [0, 1]$ and

$$V_M(m) = u_M(w, m, c)$$

for μ-almost all $(w, m, c) \in W_\emptyset \times M \times [0, 1]$, then μ is a stable matching if and only if the following conditions are satisfied:

(i) $V_W(w) \geq u_W^\emptyset(w)$ for all $w \in W$,

(ii) $V_M(m) \geq u_M^{\o}(m)$ *for all* $m \in M$,

(iii) $u_W(w,m,c) \leq V_W(w)$ *whenever* $u_M(w,m,c) \geq V_M(m)$.

Proof. We first show that (i)-(iii) are satisfied if μ is stable. Let

$$N = \{(w,m,c) \in W \times M_{\o} \times [0,1] \mid u_W(w,m,c) < u_W^{\o}(w)\}.$$

Now, $N \times N \subseteq I$. Since $\mu \otimes \mu(I) = 0$, we get $\mu \otimes \mu(N \times N) = \mu(N)\mu(N) = 0$ and therefore $\mu(N) = 0$. Together with $V_W(w) = u_W(w,m,c)$ for μ-almost all $(w,m,c) \in W \times M_{\o} \times [0,1]$, this implies

$$\mu\Big(\{(w,m,c) \in W \times M_{\o} \times [0,1] \mid V_W(w) < u_W^{\o}(w)\}\Big) = 0.$$

Since μ is a matching, the open set $\{w \in W \mid V_W(w) < u_W^{\o}(w)\}$ has therefore ν_W-measure zero. But since ν_W has full support, every open set with ν_W-measure zero must be empty. This proves (i) and an analogous argument applies to (ii).

Next, we deal with (iii). Suppose that $u_M(w,m,c) \geq V_M(m)$, but $u_W(w,m,c) > V_W(w)$. We know from (i) and (ii) and the assumption that bounds on transfers do not bind that we can assume $c \neq 0$. So there is some c^* slightly smaller than c such that $V_M(m) < u_M(w,m,c^*)$ and $V_W(w) < u_W(w,m,c^*)$ by continuity. Also by continuity, there exists open neighborhoods O_w of w and O_m of m, such that $V_M(m') < u_M(w',m',c^*)$ and $V_W(w') < u_W(w',m',c^*)$ for all $w' \in O_w$ and $m' \in O_m$. Now $O_w \times M_{\o} \times [0,1] \ \times \ W_{\o} \times O_m \times [0,1]$ is a subset of

$$\Big\{((w,m,c),(w',m',c')) \in W \times M_{\o} \times [0,1] \ \times \ W_{\o} \times M \times [0,1] \mid \\ u_W(w,m',c^*) > V_W(w) \text{ and } u_M(w,m',c^*) > V_M(m')\Big\}$$

and the latter set coincides $\mu \otimes \mu$-almost surely with

$$\Big\{((w,m,c),(w',m',c')) \in W \times M_{\o} \times [0,1] \ \times \ W_{\o} \times M \times [0,1] \mid \\ u_W(w,m',c^*) > u_W(w,m,c) \text{ and } u_M(w,m',c^*) > u_M(w',m',c')\Big\},$$

a subset of the instability set I. It follows that

$$\mu \otimes \mu(O_w \times M_{\o} \times [0,1] \ \times \ W_{\o} \times O_m \times [0,1]) = 0.$$

Since this is the measure of a measurable rectangle and μ is a matching, this shows that

$$0 = \mu(O_w \times M_{\o} \times [0,1])\mu(W_{\o} \times O_m \times [0,1]) = \nu_W(O_w)\nu_M(O_m),$$

so $\nu_W(O) = 0$ or $\nu_M(O_m) = 0$. If $\nu_W(O_w) = 0$, then O_w is empty since ν_W has full support. If $\nu_M(O_m) = 0$, then O_m is empty since ν_M has full support. In either case, we obtain a contradiction.

For the other direction, assume that (i)-(iii) hold. Proving that $\mu \otimes \mu(I) = 0$ is somewhat tedious since I is defined by no less than eight conditions. Each of these conditions defines an open subset of $W_{\o} \times M_{\o} \times C \ \times \ W_{\o} \times M_{\o} \times C$ and I is the union of these eight open sets. It suffices, therefore, to show separately that each of these eight open sets has

$\mu \otimes \mu$-measure zero. We do one case here and leave the others to the industrious reader.[25] So let

$$I' = \Big\{((w,m,c),(w',m',c')) \in W_\emptyset \times M_\emptyset \times C \ \times \ W_\emptyset \times M_\emptyset \times C \mid (m',c'') \succ_w (m,c) \text{ and } (w,c'') \succ_{m'} (w',c') \text{ for some } c''\}.$$

We show that $\mu \otimes \mu(I') = 0$. We can rewrite I' as

$$\Big\{((w,m,c),(w',m',c')) \in W_\emptyset \times M_\emptyset \times C \ \times \ W_\emptyset \times M_\emptyset \times C \mid u_W(w,m',c'') > u_W(w,m,c) \text{ and } u_M(w,m',c'') > u_M(w',m',c') \text{ for some } c''\Big\},$$

which $\mu \otimes \mu$-almost surely coincides with

$$\Big\{((w,m,c),(w',m',c')) \in W_\emptyset \times M_\emptyset \times C \ \times \ W_\emptyset \times M_\emptyset \times C \mid u_W(w,m',c'') > V_W(w) \text{ and } u_M(w,m',c'') > V_M(m') \text{ for some } c''\Big\}.$$

This last set must be empty by (iii) and therefore have $\mu \otimes \mu$-measure zero. □

Lemma 12. *Let μ be a matching. If $V_W : W \to \mathbb{R}$ and $V_M : M \to \mathbb{R}$ are measurable functions such that*

$$V_W(w) = u_W(w,m,c)$$

for μ-almost all $(w,m,c) \in W \times M_\emptyset \times [0,1]$ and

$$V_M(m) = u_M(w,m,c)$$

for μ-almost all $(w,m,c) \in W_\emptyset \times M \times [0,1]$, then μ is a stable matching if and only if the following conditions are satisfied for $\nu_W \otimes \nu_M$-almost all $(w,m) \in W \times M$:

(i) $V_W(w) \geq u_W^\emptyset(w)$,

(ii) $V_M(m) \geq u_M^\emptyset(m)$,

(iii) $u_W(w,m,c) \leq V_W(w)$ if $u_M(w,m,c) \geq V_M(m)$.

Proof. Showing that (i)-(iii) hold almost surely if μ is stable, follows almost exactly as in the proof of Theorem 3. But whenever we showed that some set violating the condition is an open set of measure zero and therefore empty under the full support assumption, it now suffices that the set is measurable with measure zero.

Showing that μ is stable if conditions (i)-(iii) hold, works exactly as in the proof of Theorem 3, with the tiny modification that the set discussed at the end may not be empty but is already assumed to have measure zero. Neither the continuity of the value function nor the support being full played any other role in proving that direction. □

[25] The other condition involving blocking pairs is completely analogous to the one we verify here, showing that the four individual rationality conditions hold is straightforward, and the two conditions concerning efficient contract choices for both couples hold vacuously since there can be no inefficient contract choices under the assumption of imperfectly transferable utility.

9.3 Proof of Theorem 5

We prove Theorem 5 by first proving it for the special case that W, M, and C are all finite in Lemma 13. We then extend the result by approximation arguments to the case that ν_W and ν_M have finite support in Lemma 14, and then to the general case. The proof of Lemma 13 does most of the work. Similarly to Jagadeesan (2017), we give a topological version of the operators in Fleiner (2003) and Hatfield and Milgrom (2005) and obtain stable matchings from appropriate fixed points that can be shown to exist by Kakutani's fixed-point theorem.[26] The argument is robust to preferences getting continuously modified via externalities, and this allows us to include externalities. The assumption that preferences are asymmetric and negatively transitive allows us to construct a "choice correspondence" for the group of agents on one side that behaves much like the choice correspondence of an individual facing a decision problem and automatically takes care of issues such as the optimal rationing of agents of the same type.

Lemma 13. *A stable matching exists in the model with externalities if W, M, and C are finite.*

Proof. It will be convenient to slightly reformulate the existence problem and model; the resulting model is clearly equivalent in this special setting. We let Γ be the graph of $\mathbb{C}$. Let ν_W and ν_M be measures on W and M, respectively. Let $\bar{\kappa} \in \mathbb{R}^{W\times M\times C}$ be a vector such that $\bar{\kappa}(w,m,c) \geq \min\{\nu_W(w), \nu_M(m)\}$ for all $(w,m,c) \in \Gamma$ and $\bar{\kappa}(w,m,c) = 0$ for all $(w,m,c) \notin \Gamma$. Let

$$\mathcal{K} = \{\kappa \in \mathbb{R}^{W\times M\times C} \mid 0 \leq \kappa(w,m,c) \leq \bar{\kappa}(w,m,c)\}.$$

Also, let

$$\mathcal{M} = \Big\{\kappa \in \mathcal{K} \mid \sum_{m\in M, c\in C} \kappa(w,m,c) \leq \nu_W(w) \text{ and } \sum_{w\in W, c\in C} \kappa(w,m,c) \leq \nu_M(m) \text{ for all } w \in W, m \in M\Big\}.$$

$\mathcal{M}$ is the space of matchings and is a nonempty, convex, and compact subset of $\mathbb{R}^{W\times M\times C}$. Let $u_W : W \times M_\varnothing \times C \times \mathcal{M} \to \mathbb{R}$ and $u_M : W_\varnothing \times M \times C \times \mathcal{M} \to \mathbb{R}$ be continuous functions representing the preferences. Such functions exist by the main result of Mas-Colell (1977). For the purpose of the proof, we can assume without loss of generality that

$$\max_{c\in\mathbb{C}(w,\varnothing)} u_W(w,\varnothing,c,\kappa) = \max_{c\in\mathbb{C}(\varnothing,m)} u_M(\varnothing,m,c,\kappa) = 0$$

for all $w \in W$, $m \in M$, and $\kappa \in \mathcal{M}$. For the first case, just subtract the continuous function $(w,\kappa) \mapsto \max_{c\in\mathbb{C}(w,\varnothing)} u_W(w,\varnothing,c,\kappa)$ from u_W. The resulting function will in general not represent the same preferences over matchings, but for stability, only the induced preferences over $M_\varnothing \times C$ given each matching matter. Since $W \times M \times C \times \mathcal{M}$ is a compact subset of $W \times M \times C \times \mathbb{R}^{W\times M\times C}$ and u_W and u_M are, therefore, bounded, we can

[26] As Jagadeesan (2017) explains, other topological approaches such as the one used by Azevedo and Hatfield (2015) cannot be as easily adopted to allow for indifferences. Allowing for indifferences is necessary in the context of externalities.

assume, by the Tietze extension theorem, Aliprantis and Border (2006, 2.47), that u_W and u_M are defined, continuous, and bounded on all of $W \times M \times C \times \mathbb{R}^{W\times M\times C}$.

For each $(\kappa_W, \kappa_M) \in \mathcal{K} \times \mathcal{K}$, let

$$D_W(\kappa_W, \kappa_M) = \Big\{\tau \in \mathcal{K} \mid \tau \text{ maximizes } \sum_{w\in W, m\in M, c\in C} u_W(w, m, c, \kappa_W + \kappa_M - \bar{\kappa})\tau(w, m, c)$$
$$\text{under the constraints that } \sum_{m\in M, c\in C} \tau(w, m, c) \le \nu_W(w)$$
$$\text{and } \tau(w, m, c) \le \kappa_W(w, m, c) \text{ for all } w \in W, m \in M, c \in C\Big\}.$$

It is easy to see (use the maximum theorem) that $D_W : \mathcal{K} \times \mathcal{K} \to 2^{\mathcal{K}}$ is a correspondence with nonempty, convex, and compact values. Similarly, define $D_M : \mathcal{K} \times \mathcal{K} \to 2^{\mathcal{K}}$ by

$$D_M(\kappa_M, \kappa_W) = \Big\{\tau \in \mathcal{K} \mid \tau \text{ maximizes } \sum_{w\in W, m\in M, c\in C} u_M(w, m, c, \kappa_W + \kappa_M - \bar{\kappa})\tau(w, m, c)$$
$$\text{under the constraints that } \sum_{w\in W, c\in C} \tau(w, m, c) \le \nu_M(m)$$
$$\text{and } \tau(w, m, c) \le \kappa_M(w, m, c) \text{ for all } w \in W, m \in M, c \in C\Big\}.$$

Define $\phi_W : \mathcal{K} \times \mathcal{K} \to 2^{\mathcal{K}}$ by

$$\phi_W(\kappa_W, \kappa_M) = \bar{\kappa} - \kappa_W + D_W(\kappa_W, \kappa_M)$$

and $\phi_M : \mathcal{K} \times \mathcal{K} \to 2^{\mathcal{K}}$ by

$$\phi_M(\kappa_M, \kappa_W) = \bar{\kappa} - \kappa_M + D_M(\kappa_M, \kappa_W).$$

Finally, define $\phi : \mathcal{K} \times \mathcal{K} \to 2^{\mathcal{K}\times\mathcal{K}}$ by

$$\phi(\kappa_W, \kappa_M) = \phi_M(\kappa_M, \kappa_W) \times \phi_W(\kappa_W, \kappa_M).$$

By Kakutani's fixed-point theorem, ϕ has a fixed point (κ_W^*, κ_M^*). Let $\mu = \kappa_W^* + \kappa_M^* - \bar{\kappa}$. We show that μ is a stable matching. So let (κ_W^*, κ_M^*) satisfy the fixed-point conditions

$$\kappa_W^* \in \phi_M(\kappa_M^*, \kappa_W^*) = \bar{\kappa} - \kappa_M^* + D_M(\kappa_M^*, \kappa_W^*)$$

and

$$\kappa_M^* \in \phi_W(\kappa_W^*, \kappa_M^*) = \bar{\kappa} - \kappa_W^* + D_W(\kappa_W^*, \kappa_M^*).$$

Define $D_W^* : \mathcal{K} \to 2^{\mathcal{K}}$ by

$$D_W^*(\kappa_W) = \Big\{\tau \in \mathcal{K} \mid \tau \text{ maximizes } \sum_{w\in W, m\in M, c\in C} u_W(w, m, c, \mu)\tau(w, m, c)$$
$$\text{under the constraints that } \sum_{m\in M, c\in C} \tau(w, m, c) \le \nu_W(w)$$
$$\text{and } \tau(w, m, c) \le \kappa_W(w, m, c) \text{ for all } w \in W, m \in M, c \in C\Big\}.$$

and define $D_M^*: \mathcal{K} \to 2^{\mathcal{K}}$ by

$$D_M^*(\kappa_M) = \Big\{\tau \in \mathcal{K} \mid \tau \text{ maximizes} \sum_{w\in W, m\in M, c\in C} u_M(w,m,c,\mu)\tau(w,m,c)$$
$$\text{under the constraints that} \sum_{w\in W, c\in C} \tau(w,m,c) \le \nu_M(m)$$
$$\text{and } \tau(w,m,c) \le \kappa_M(w,m,c) \text{ for all } w\in W, m\in M, c\in C\Big\}.$$

D_W^* and D_M^* represent the "choice correspondences" for the preferences induced by our candidate stable matching. Note that this means the maximization problem is also solved for each $w \in W$ and for each $m \in M$, respectively, separately. Note also that these correspondences will only put weight on individually rational couple-contract types (for the preferences induced by μ). For example, if $\tau \in D_W^*(\kappa_W)$ and $\tau(w,m,c) > 0$, then $u_W(w,m,c,\mu) \ge 0 = \max_{c\in\mathbb{C}(w,\emptyset)} u_W(w,\emptyset,c,\mu)$.

Now, the fixed-point conditions imply

$$\mu = \kappa_W^* + \kappa_M^* - \bar{\kappa} \in D_W(\kappa_W^*, \kappa_M^*) \cap D_M(\kappa_M^*, \kappa_W^*) = D_W^*(\kappa_W^*) \cap D_M^*(\kappa_M^*).$$

Since

$$\mu = \kappa_W^* + \kappa_M^* - \bar{\kappa} \in D_W(\kappa_W^*, \kappa_M^*),$$

we have

$$\sum_{m\in M, c\in C} \mu(w,m,c) \le \nu_W(w)$$

for all $w \in W$ and, similarly,

$$\sum_{w\in W, c\in C} \mu(w,m,c) \le \nu_M(m)$$

for all $m \in M$. Also, μ is nonnegative and satisfies $\mu(w,m,c) = 0$ for $(w,m,c) \notin \Gamma$ for the same reason and is, therefore, a matching. It remains to show that it is a stable matching. That individual rationality constraints are satisfied follows from $\mu \in D_W^*(\kappa_W^*) \cap D_M^*(\kappa_M^*)$.

It remains to show that there are no blocking pairs. Assume for the sake of contradiction that there are (w,m,c) with $\mu(w,m,c) > 0$ and (w',m',c') with $\mu(w',m',c') > 0$ such that for some $c'' \in \mathbb{C}(w,m')$ both

$$(m', c'', \mu) \succ_w (m, c, \mu)$$

and

$$(w, c'', \mu) \succ_{m'} (w', c', \mu)$$

hold. Because $\mu(w,m,c) > 0$ and μ is a matching, we have $\mu(w,m',c'') < \nu_W(w)$. Similarly, $\mu(w,m',c'') < \nu_M(m')$. Together, we have

$$\mu(w,m',c'') < \min\{\nu_W(w), \nu_M(m')\} \le \bar{\kappa}(w,m',c'').$$

Since $(w,c'',\mu) \succ_{m'} (w',c',\mu)$, $\mu(w',m',c') > 0$, and $\mu \in D_M^*(\kappa_M^*)$, we must have

$\mu(w,m',c'') = \kappa^*_M(w,m',c'')$; the inferior option cannot be chosen by the correspondence D^*_M from κ^*_M if more of the better option is available. Similarly, we must have $\mu(w,m',c'') = \kappa^*_W(w,m',c'')$. Together with the definition of μ we get

$$\begin{aligned}\mu(w,m',c'') &= \kappa^*_W(w,m',c') + \kappa^*_M(w,m',c'') - \bar{\kappa}(w,m',c'') \\ &= 2\mu(w,m',c'') - \bar{\kappa}(w,m',c'') \\ &< \mu(w,m',c''),\end{aligned}$$

where the strict inequality follows from $\mu(w,m',c'') < \bar{\kappa}(w,m',c'')$. This contradiction proves that there are, indeed, no blocking pairs under μ, and μ is a stable matching.

□

Lemma 14. *A stable matching exists when ν_W and ν_M have finite support.*

Proof. We can assume without loss of generality that $W = \operatorname{supp} \nu_W$ and $M = \operatorname{supp} \nu_M$. For each $w \in W_\varnothing$ and $m \in M_\varnothing$ let $\langle c^n_{wm} \rangle$ be a sequence in $\mathbb{C}(w,m)$ such that $\{c^n_{wm} \mid n \in \mathbb{N}\}$ is dense in $\mathbb{C}(w,m)$. Define $\mathbb{C}_n : W_\varnothing \times M_\varnothing \to 2^C$ by $\mathbb{C}_n(w,m) = \{c^k_{wm} \mid k \leq n\}$. A stable matching exists when we replace $\mathbb{C}$ by $\mathbb{C}_n$ by Lemma 13.

Note that every matching, stable or not, for the restricted correspondence $\mathbb{C}_n$ is also a, not necessarily stable, matching for the unrestricted correspondence $\mathbb{C}$. So we can find a sequence $\langle \mu_n \rangle$ of matchings such that μ_n is a stable matching for the restricted correspondence $\mathbb{C}_n$. By passing to a subsequence and using Lemma 1, we can assume without loss of generality that $\langle \mu_n \rangle$ converges to some measure μ, which is again a matching by Lemma 2.

It remains to prove that μ is stable. For each pair $p = \big((w,m,c),(w',m',c')\big)$ of couple-contract types in $I(\mu)$, there exists some contract c^* in $\mathbb{C}(w^*,m^*)$, with $w^* = w, w', \varnothing$ and $m^* = m, m', \varnothing$ that ensures that one of the conditions witnessing to the instability of p holds. By the continuity condition on preferences, one can choose this contract to be of the form $c^k_{w^*m^*}$ for some k. We can choose by our strengthened continuity assumption an open neighborhood O_p of p and an open neighborhood U_p of μ such that every $p' \in O_p$ can be blocked by the contract $c^k_{w^*m^*}$ whenever preferences are induced by some $\mu' \in U_p$. For n large enough, $c^k_{w^*m^*} \in \mathbb{C}_n(w,m)$ and $\mu_n \in U_p$. It follows that $\mu(O_p) \leq \liminf_n \mu_n(O_p) = 0$ by the Portmanteau theorem.

Now suppose for the sake of contradiction that $\mu \otimes \mu\big(I(\mu)\big) > 0$. Since Borel measures are regular, there exists then a compact set $K \subseteq I(\mu)$ such that $\mu \otimes \mu(K) > 0$. Now the family $(O_p)_{p \in K}$ is an open cover of K and K is therefore covered by finitely many open sets of $\mu \otimes \mu$-measure zero, in contradiction to $\mu \otimes \mu(K) > 0$. □

Proof of Theorem 5. Let $\langle \nu^n_W, \nu^n_M \rangle$ be a sequence of pairs of measures on W and M, respectively, such that $\langle \nu^n_W \rangle$ converges to ν_W, $\langle \nu^n_M \rangle$ converges to ν_M and ν^n_W and ν^n_M have finite support n. This is possible since measures with finite supports are dense in the space of all measures.

For each n, we can choose a stable matching μ_n for the finite matching problem given by population distributions ν^n_W and ν^n_M by Lemma 14. By passing to a subsequence and using Lemma 1, we can assume without loss of generality that $\langle \mu_n \rangle$ converges to some measure μ, which is again a matching for the population measures ν_W and ν_M by Lemma 2. So far, everything works as in the proof of Theorem 1.

The matching μ is stable. Indeed, for each pair $p = ((w, m, c), (w', m', c'))$ of couple-contract types in $I(\mu)$ we can choose by our strengthened continuity assumption and the lower hemicontinuity of $\mathbb{C}$ an open neighborhood O_p of p and an open neighborhood U_p of μ such that $O_p \subseteq I(\mu')$ for $\mu' \in U_p$. Since $\mu_n \in U_p$ for n large enough, we have $O_p \subseteq I(\mu_n)$ for n large enough. Hence, $\mu \otimes \mu(O_p) \leq \liminf_n \mu_n \otimes \mu_n(O_p) = 0$ by the Portmanteau theorem. We can conclude as in the proof of Lemma 14 that $\mu \otimes \mu(I(\mu)) = 0$. □

9.4 Proof of Theorem 6

Proof of Theorem 6. We first show that (i)-(iii) implies that μ is stable. Now $I(\mu) = \bigcup_{\epsilon \downarrow 0} I_\epsilon(\mu)$, so $\mu \otimes \mu(I(\mu)) > 0$ would imply $\mu \otimes \mu(I_\epsilon(\mu)) > 0$ for some $\epsilon > 0$. But exactly as in the proof of Theorem 5, one can show $\mu \otimes \mu(I_\epsilon(\mu)) = 0$.

For the other direction, assume that $\mu(I(\mu)) = 0$. As in the proof of Theorem 2, we can show that there are sequences $\langle \nu_W^n \rangle$, $\langle \nu_M^n \rangle$, and $\langle \mu_n \rangle$ such that (i) and (ii) hold and such that $\mu_n \otimes \mu_n(I(\mu)) = 0$ for all n. We show that (iii) holds, too, for fixed $\epsilon > 0$.

For this, we show that the functions $\phi_W : \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \to C(W \times M_\emptyset \times C)$ and $\phi_M : \mathcal{M}(W_\emptyset \times M_\emptyset \times C) \to C(W_\emptyset \times M \times C)$ given by $\phi_W(\mu) = u_W(\cdots, \mu)$ and $\phi_M(\mu) = u_M(\cdots, \mu)$, respectively, are continuous when the range is endowed with the sup-norm. To see this, for example, for ϕ_W, note that the function

$$\mu \mapsto \sup_{w,m,c} |u_W(w, m, c, \mu) - u_W(w, m, c, \mu')|$$

is continuous for each μ' by the maximum theorem since u_W is continuous and W, M, and C are compact. It follows that there exists a neighborhood U of μ in $\mathcal{M}(W_\emptyset \times M_\emptyset \times C)$ such that

$$\sup_{w,m,c} |u_W(w, m, c, \mu) - u_W(w, m, c, \mu')| < \epsilon/2$$

and

$$\sup_{w,m,c} |u_M(w, m, c, \mu) - u_M(w, m, c, \mu')| < \epsilon/2$$

for all $\mu' \in U$. Note that $\mu_n \otimes \mu_n(I(\mu)) = 0$ is equivalent to $I(\mu) \cap \operatorname{supp} \mu_n \times \operatorname{supp} \mu_n = \emptyset$ for each n since μ_n is finite. Similarly, $\mu_n \otimes \mu_n(I_\epsilon(\mu_n)) = 0$ is equivalent to $I_\epsilon(\mu_n) \cap \operatorname{supp} \mu_n \times \operatorname{supp} \mu_n = \emptyset$. Let N be such that $\mu_n \in U$ for $n \geq N$. It is straightforward but slightly tedious to verify that $I_\epsilon(\mu_n) \cap \operatorname{supp} \mu_n \times \operatorname{supp} \mu_n = \emptyset$ for $n \geq N$. □

It should be noted that we only used the compactness to show that the functions ϕ_W and ϕ_M are continuous. For fixed utility representations, we could make this an explicit assumption and can prove an analogous result without compactness assumptions.

9.5 Proof of Theorem 7

We need some definitions for the proof of Theorem 7. A probability space (Ω, Σ, τ) is *saturated* if for every two Polish spaces X and Y, every Borel probability measure μ on $X \times Y$ and every measurable function $f : \Omega \to X$ with distribution equal to the X-marginal of μ, there exists a measurable function $g : \Omega \to Y$ such that the function $(f, g) : \Omega \to X \times Y$ given by $(f, g)(\omega) = (f(\omega), g(\omega))$ has distribution μ, that is, $\mu = \tau \circ (f, g)^{-1}$.

An *automorphism* of the probability space (Ω, Σ, τ) is a measurable bijection $h : \Omega \to \Omega$ with measurable inverse such that $\tau(A) = \tau(h(A))$ for all $A \in \Sigma$. A probability space (Ω, Σ, τ) is *homogeneous* if for every two measurable functions $f : \Omega \to X$ and $g : \Omega \to X$ with X Polish such that $\tau \circ f^{-1} = \tau \circ g^{-1}$, there exists an automorphism h such that $f(\omega) = g(h(\omega))$ for almost all ω.

An extensive discussion of these concepts can be found in Fajardo and Keisler (2002), where it is also shown that probability spaces that are both saturated and homogeneous exist.[27]

Proof of Theorem 7. We first ignore (iii) and (iv) and then patch up our solution so that even these conditions hold. Extend ν_W to all of $W_\emptyset$ by assigning mass $\nu_M(M)$ to the point $\emptyset \in W_\emptyset$, and extend ν_M to all of $M_\emptyset$ by assigning mass $\nu_W(W)$ to the point $\emptyset \in M_\emptyset$. The measures ν_W and ν_M thus extended satisfy $\nu_W(W_\emptyset) = \nu_W(W) + \nu_M(M) = \nu_M(M_\emptyset)$ and we take them without loss of generality to be probability measures. We take $(A_W, \mathcal{A}_W, \tau_W)$ and $(A_M, \mathcal{A}_M, \tau_M)$ to be the same saturated and homogeneous, but otherwise arbitrary, probability space (Ω, Σ, τ).

Let X be any Polish space and $g : \Omega \to X$ be any measurable function. By saturation, there exists $h : \Omega \to W_\emptyset$ such that $\tau \circ (g, h)^{-1} = \tau \circ g^{-1} \otimes \nu_W$. In particular, $\tau \circ h^{-1} = \nu_W$ and we can take t_W to be h. Similarly, we can find a function $t_M : \Omega \to M_\emptyset$ such that $\tau \circ t_M^{-1} = \nu_M$.

Now let μ be a matching and let $\mu_\emptyset$ be the measure on $W_\emptyset \times M_\emptyset \times C$ obtained from μ by letting $\mu_\emptyset(B) = \mu(B)$ for every Borel set $B \subseteq W_\emptyset \times M_\emptyset \times C \setminus \{\emptyset, \emptyset\} \times C$, but

$$\mu_\emptyset(W_\emptyset \times M_\emptyset \times C) = 1$$

and such that $\mu_\emptyset(G_{\mathbb{C}}) = 1$. So $\mu_\emptyset(W_\emptyset \times M_\emptyset \times C) = 1$, the $W_\emptyset$-marginal of $\mu_\emptyset$ is ν_W, and the $M_\emptyset$-marginal of $\mu_\emptyset$ is ν_M. By saturation, there exist measurable functions $f_W : \Omega \to M_\emptyset$, $\chi_W : \Omega \to C$, $f_M : \Omega \to W_\emptyset$, and $\chi_M : \Omega \to C$ such that

$$\tau \circ (t_W, f_W, \chi_W)^{-1} = \mu_\emptyset = \tau \circ (f_M, t_M, \chi_M)^{-1}.$$

By homogeneity, there exists an automorphism $\phi : \Omega \to \Omega$ such that

$$\big(t_W(\omega), f_W(\omega), \chi_W(\omega)\big) = \big(f_M(\phi(\omega)), t_M(\phi(\omega)), \chi_M(\phi(\omega))\big)$$

for τ-almost all $\omega \in \Omega$. In particular,

$$\tau \circ \big(t_W, t_M(\phi(\omega)), \chi_W\big)^{-1} = \mu_\emptyset.$$

There might still be some $\omega \in \Omega$ such that $\chi_W(\omega) \notin \mathbb{C}(t_W(\omega), t_M(\phi(\omega)))$. The corre-

[27]The notion of homogeneity used in Fajardo and Keisler (2002) is more permissive in that they require only automorphisms of sets of measure 1 that may be smaller than the whole probability space. But in their proof of their Theorem 3B.12, which shows that homogeneous and saturated probability spaces exist, they obtain the automorphisms as the realization of automorphisms of the underlying measure algebra using a result from Maharam (1958), which actually delivers automorphisms in our stronger sense.

spondence $\mathbb{C} : W_\emptyset \times M_\emptyset \to 2^C$ is upper hemicontinuous with nonempty and compact values and therefore also measurable with nonempty and closed values. By the Kuratowski–Ryll-Nardzewski measurable selection theorem, Aliprantis and Border (2006, 18.13), there exists a measurable function $s : W_\emptyset \times M_\emptyset \to C$ such that $s(w, m) \in \mathbb{C}(w, m)$ for all $w \in W_\emptyset$ and $w \in W_\emptyset$. Let

$$N = \Big\{\omega \in \Omega \;\Big|\; \chi_W(\omega) \notin \mathbb{C}(t_W(\omega), t_M(\phi(\omega)))\Big\}$$
$$= \Big\{\omega \in \Omega \;\Big|\; (t_W(\omega), t_M(\phi(\omega)), \chi_W(\omega)) \notin G_{\mathbb{C}}\Big\}.$$

Since $\mu_\emptyset(G_{\mathbb{C}}) = 1$, we have $\tau(N) = 0$. Define $\chi : \Omega \to C$ by

$$\chi(\omega) = \begin{cases} s\Big(t_W(\omega), t_M(\phi(\omega))\Big) & \text{if } \omega \in N, \\ \chi_W(\omega) & \text{otherwise.} \end{cases}$$

The functions ϕ and χ have the desired properties apart from, possibly, (iv).

Now assume that μ is in addition stable. Let $E : W_\emptyset \times M_\emptyset \to 2^C$ be the correspondence such that $E(w, m)$ consists of all efficient $c \in \mathbb{C}(w, m)$. Note that efficient contract choices are maximal elements under the weak Pareto ordering for couples and this ordering is acyclic with an open graph. It follows from a version of the maximum theorem that E is upper hemicontinuous with nonempty and compact values, Hildenbrand (1974, Theorem 3 on page 29).[28] In particular, E is a measurable correspondence with nonempty closed values. By the Kuratowski–Ryll-Nardzewski measurable selection theorem, there exists a measurable function $s' : W_\emptyset \times M_\emptyset \to C$ such that $s'(w, m) \in E(w, m)$ for all $w \in W_\emptyset$ and $m \in M_\emptyset$. Now let

$$N' = \Big\{\omega \in \Omega \;\Big|\; \chi_W(\omega) \notin E(t_W(\omega), t_M(\phi(\omega)))\Big\}.$$

We show that $\tau(N') = 0$. Since μ is stable, we have $\mu \otimes \mu(I) = 0$ and therefore also $\mu_\emptyset \otimes \mu_\emptyset(I) = 0$. Define $I' \subseteq I$ by

$$I' = \Big\{((w, m, c), (w', m', c')) \in W_\emptyset \times M_\emptyset \times C \;\times\; W_\emptyset \times M_\emptyset \times C \;\Big|$$
$$c \notin E(w, m) \text{ or } c' \notin E(w', m')\Big\}.$$

Note that

$$\mu_\emptyset\Big(\{(w, m, c) : c \notin E(w, m)\}\Big) \leq \mu_\emptyset \otimes \mu_\emptyset(I') \leq \mu_\emptyset \otimes \mu_\emptyset(I) = 0.$$

Since $\mu_\emptyset = \tau \circ \Big(t_W, t_M(\phi(\omega)), \chi_W\Big)^{-1}$, we must have $\tau(N') = 0$. In the present case, define $\chi : \Omega \to C$ by

$$\chi(\omega) = \begin{cases} s'\Big(t_W(\omega), t_M(\phi(\omega))\Big) & \text{if } \omega \in N', \\ \chi_W(\omega) & \text{otherwise.} \end{cases}$$

The functions ϕ and χ have the desired properties. □

[28]The cited result assumes that preferences are transitive and irreflexive, not merely acyclic. But the proof works without modification for acyclic preferences using the fact that maximal elements exist for acyclic relations on nonempty finite sets.

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