Econometrica Supplementary Material

SUPPLEMENT TO "REPUTATION AND SOVEREIGN DEFAULT"

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MANUEL AMADOR
Department of Economics, University of Minnesota, Federal Reserve Bank of Minneapolis, and NBER

CHRISTOPHER PHELAN
Department of Economics, University of Minnesota, Federal Reserve Bank of Minneapolis, and NBER

APPENDIX SA: EXISTENCE AND UNIQUENESS OF A q^o AND A CONTINUOUS q^c

HERE WE SHOW that given $\{F_\tau\}_{\tau=0}^{\infty}$, there exists a unique q^o and a unique continuous q^c that satisfy the integral equations described in the main body of the paper.

Instead of working with vector-valued operators, the idea of the proof is to substitute the equation for q^o into q^c. Then, to prove the existence, uniqueness, and continuity of q^c, we construct a contraction T mapping the space of bounded, continuous functions to itself and where q^c is a fixed point of this mapping.

First, define $T^o\{f\}(\tau)$ as

$$T^o\{f\}(\tau) = \int_0^\infty \Bigg[\bigg(\int_0^s (i+\lambda)e^{-(i+\lambda)\tilde{s}}\, d\tilde{s} + e^{-(i+\lambda)s} f(\tau+s) \bigg) \big(1 - F_\tau(\tau+s)\big) + \int_0^s \bigg(\int_0^{\tilde{s}} (i+\lambda)e^{-(i+\lambda)\Delta}\, \mathrm{d}\Delta \bigg) dF_\tau(\tau+\tilde{s}) \Bigg] \epsilon e^{-\epsilon s}\, ds. \tag{S1}$$

In words, $T^o\{f\}(\tau)$ is the value of a bond given an opportunistic government where upon a type switch, the owner receives an arbitrary payoff $f(\cdot) \in [0, 1]$.

Next, likewise define $T^c\{g\}(\tau)$ as

$$T^c\{g\}(\tau) = \frac{i+\lambda}{i+\lambda+\delta} + \int_0^\infty e^{-(i+\lambda+\delta)s} g(\tau+s)\delta\, ds. \tag{S2}$$

In words, $T^c\{g\}(\tau)$ is the value of a bond given a commitment type government where upon a type switch, the owner receives an arbitrary payoff $g(\cdot) \in [0, 1]$.

Finally, let $T\{f\}(\tau) \equiv T^c\{T^o\{f\}\}(\tau)$. Here, T is the value of a bond given a commitment type government where upon two type switches (from commitment to opportunistic and back again), the owner receives an arbitrary payoff $f(\cdot) \in [0, 1]$.

We now proceed to show that T^o and T^c are each well defined, and that T is a contraction on the space of bounded continuous functions. First, we can rewrite T^c and T^o as

$$T^c\{g\}(\tau) = \overline{q} + \delta H_0(-\tau) \int_\tau^\infty H_0(s) g(s)\, ds,$$

Manuel Amador: amador.manuel@gmail.com
Christopher Phelan: cphelan@umn.edu

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$$T^o\{f\}(s) = \epsilon \int_0^\infty \int_0^{\tilde{s}} \big(1 - F_s(s+\hat{s})\big) e^{-\epsilon \tilde{s}}\, dH_1(\hat{s})\, d\tilde{s} + \epsilon \int_0^\infty H_2(\tilde{s}) f(s+\tilde{s}) \big(1 - F_s(s+\tilde{s})\big)\, d\tilde{s},$$

where

$$\overline{q} = \frac{i+\lambda}{i+\lambda+\delta}, \qquad H_0(s) = e^{-(i+\lambda+\delta)s}, \qquad H_1(s) = \big(1 - e^{-(i+\lambda)s}\big), \qquad H_2(s) = e^{-(i+\lambda+\epsilon)s},$$

and where we used integration by parts to rewrite T^o.

Plugging the equation for T^o back into T^c we obtain that q^c is a fixed point of the operator, T, now written as

$$T\{f\}(\tau) = g_0(\tau) + \delta\epsilon H_0(-\tau) \int_\tau^\infty \int_0^\infty g_1(s, \tilde{s})\, d\tilde{s}\, ds,$$
$$\text{where } g_1(s,\tilde{s}) = H_0(s) H_2(\tilde{s}) \big(1 - F_s(s+\tilde{s})\big) f(s+\tilde{s})$$

and where

$$g_0(\tau) = \overline{q} + \delta\epsilon H_0(-\tau) \int_\tau^\infty \int_0^\infty \int_0^{\tilde{s}} H_0(s) e^{-\epsilon\tilde{s}} \big(1 - F_s(s+\hat{s})\big)\, dH_1(\hat{s})\, d\tilde{s}\, ds.$$

We now argue that for any bounded nonnegative continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the iterated integral, $\int_0^\infty \int_0^\infty g_1(s,\tilde{s})\, d\tilde{s}\, ds$, exists. We show this in three steps.

(a) Given that f is continuous, it follows that the function g_1 is measurable in $\mathbb{R}_+^2$, given our assumption that $F_s(s+\tilde{s})$ is measurable, together with H_0, H_2, and f continuous (g_1 is the product of measurable functions and, thus, is itself measurable).

(b) The integral $\int_0^\infty g_1(s,\tilde{s})\, d\tilde{s}$ exists given $s \in \mathbb{R}_+$. That f is nonnegative and bounded implies that there exists a $M > 0$ such that $0 \le f \le M$. In addition, that $F_s(s+\tilde{s}) \in [0,1]$ implies $0 \le g_1(s,\tilde{s}) \le H_0(s)H_2(\tilde{s})M \equiv \bar{g}(s,\tilde{s})$. Given $s \in \mathbb{R}_+$, the function $\bar{g}(s,\cdot)$ is integrable in $\mathbb{R}_+$, and it thus follows that $g_1(s,\cdot)$ is bounded by two integrable functions and, thus, is also integrable.

(c) From the previous step, $0 \le \int_0^\infty g_1(s,\tilde{s})\, d\tilde{s}\, ds \le \int_0^\infty H_0(s)H_2(\tilde{s})M\, d\tilde{s}$. That is, the function $g_2(s) = \int_0^\infty g_1(s,\tilde{s})\, d\tilde{s}$ is bounded between 0 and $\int_0^\infty \bar{g}(s,\tilde{s})\, d\tilde{s} = \hat{g}(s)$. Given that $\hat{g}(s)$ is integrable in $\mathbb{R}_+$, it provides an integrable upper bound, and it follows that the iterated integral, $\int_0^\infty \int_0^\infty g_1(s,\tilde{s})\, d\tilde{s}\, ds$, exists.

A similar argument shows that the iterated integral in the definition of $g_0(\tau)$ exists.

Let B denote the space of continuous functions $f\colon \mathbb{R}_+ \to [\overline{q}, 1]$ with the sup norm. Note that this is a complete metric space. We make the following two observations about the operator T:

Observation 1: T maps B into itself. We have already shown that for any bounded nonnegative and continuous f, $T\{f\}(\tau)$ exists. Note also that $T\{f\}(\tau) \ge \overline{q} \ge 0$ and

$$\begin{aligned} T\{f\}(\tau) &\le \overline{q} + \delta\epsilon \bigg[\int_\tau^\infty \int_0^\infty \int_0^{\tilde{s}} H_0(s-\tau) e^{-\epsilon\tilde{s}}\, dH_1(\hat{s})\, d\tilde{s}\, ds + \int_\tau^\infty \int_0^\infty H_0(s-\tau) H_2(\tilde{s})\, d\tilde{s}\, ds \bigg] \\ &= 1, \end{aligned}$$

where the inequality follows from using that $0 \le f \le 1$ and $0 \le F_s \le 1$. So $T\{f\}: \mathbb{R}_+ \to [\overline{q}, 1]$.

The continuity of $T\{f\}$ follows from the fact that $g_0(\tau)$ is continuous (as it is the sum of a constant and the product of two continuous functions) together with the fact that $\int_\tau^\infty \int_0^\infty g_1(s, \tilde{s})\, d\tilde{s}\, ds$ is an absolutely continuous function of τ.

Observation 2: T is a contraction mapping. Consider two functions f and g. Then we have that

$$T\{f\}(\tau) - T\{g\}(\tau) = \delta\epsilon \int_\tau^\infty \int_0^\infty H_0(s-\tau)H_2(\tilde{s})\big(1 - F_s(s+\tilde{s})\big)\big(f(s+\tilde{s}) - g(s+\tilde{s})\big)\, d\tilde{s}\, ds.$$

Using that $F_s(s+\tilde{s}) \in [0, 1]$, we get

$$\big|T\{f\}(\tau) - T\{g\}(\tau)\big| \le |f-g|\epsilon\delta \int_\tau^\infty \int_0^\infty H_0(s-\tau)H_2(\tilde{s})\, d\tilde{s}\, ds = \frac{\epsilon\delta}{(i+\lambda+\epsilon)(i+\lambda+\delta)}|f-g|.$$

Thus, T is a contraction mapping with modulus $\frac{\epsilon}{i+\lambda+\epsilon} \times \frac{\delta}{i+\lambda+\delta} < 1$.

It follows by the contraction mapping theorem that there exists a unique bounded and continuous function q^c such that $T\{q^c\} = q^c$ and where $q^c(\tau) \in [\overline{q}, 1]$ for all $\tau \ge 0$.

Given the existence and uniqueness of a continuous function q_c, we can substitute back in the q^o equation and obtain the existence and uniqueness of q^o. It is straightforward to show that $q^o(s) \in [0, 1]$ for all s.

APPENDIX SB: CONTINUITY OF q^o GIVEN CONSTRUCTION REQUIREMENT (16)

We have already shown above that q^c is continuous in any equilibrium. The continuity of q^o cannot be guaranteed in the same fashion (that is, independently of $\{F_\tau\}$). However, we can show that for any family $\{F_\tau\}$ that satisfies our construction requirement in (16), q^o must be continuous.

From the proof in Appendix SA, recall that q^o can be written as

$$q^o(s) = \epsilon \int_0^\infty \int_0^{\tilde{s}} \big(1 - F_s(s+\hat{s})\big)e^{-\epsilon\tilde{s}}\, dH_1(\hat{s})\, d\tilde{s} + \epsilon \int_0^\infty H_2(\tilde{s})q^c(s+\tilde{s})\big(1 - F_s(s+\tilde{s})\big)\, d\tilde{s},$$

where $H_1(s) = (1 - e^{-(i+\lambda)s})$ and $H_2(s) = e^{-(i+\lambda+\epsilon)s}$.

For a family $\{F_\tau\}$ that satisfies our construction requirement in (16), the above implies that $q_o(s) = 0$ for all $s \ge T$, as $F_s(s+\hat{s}) = 1$ for all $s \ge T$ and $\hat{s} \ge 0$.

For all $s \le T$, we have then that

$$q^o(s) = \epsilon \int_s^T \int_s^{\tilde{s}} \big(1 - F_s(\hat{s})\big)e^{-\epsilon(\tilde{s}-s)}\, dH_1(\hat{s}-s)\, d\tilde{s} + \epsilon \int_s^T H_2(\tilde{s}-s)q^c(\tilde{s})\big(1 - F_s(\tilde{s})\big)\, d\tilde{s},$$

which implies that $\lim_{s\uparrow T} q^o(s) = 0$. Thus, q^o is continuous at T.

Finally, using condition (16) and letting $\hat{x}(s) = \frac{x(s)}{1-\rho(s)}$, we have that for $s < T$,

$$q^o(s) = \epsilon \int_s^T \int_s^{\tilde{s}} e^{-\int_s^{\hat{s}} \hat{x}(\tau)\, d\tau} e^{-\epsilon(\tilde{s}-s)}\, dH_1(\hat{s}-s)\, d\tilde{s} + \epsilon \int_s^T H_2(\tilde{s}-s)q^c(\tilde{s})e^{-\int_s^{\tilde{s}} \hat{x}(\tau)\, d\tau}\, d\tilde{s},$$

which guarantees that q^o is a continuous function of s for $s \in [0, T)$.

Hence, we have shown that the function $q^o(s)$ associated with a family of default distributions that satisfy (16) must be continuous for all $s \geq 0$.

APPENDIX SC: H GIVEN BY (17) SATISFIES ASSUMPTION 1

We now show that H in equation (17) satisfies the conditions in Assumption 1 given our parameters.

For Part (i): Lipschitz Continuity. Consider two points $x_0 = (b_0, q_0)$ and $x_1 = (b_1, q_1)$ in $\mathbb{X}$. Let $H_0 = H(b_0, q_0)$ and $H_1 = H(b_1, q_1)$. Let $\tilde{r} = r + \lambda$ and $\tilde{i} = i + \lambda$. Let $[a]^+ = \max\{a, 0\}$ and for our parameters, let $\tilde{r} > \tilde{i}$. Then

$$\begin{aligned} |H_0 - H_1| &= \big|[\tilde{r} - \tilde{i}/q_0]^+(y - b_0) - [\tilde{r} - \tilde{i}/q_1]^+(y - b_1)\big| \\ &= \big|\big([\tilde{r} - \tilde{i}/q_0]^+ - [\tilde{r} - \tilde{i}/q_1]^+\big)(y - b_0) + [\tilde{r} - \tilde{i}/q_1]^+(b_1 - b_0)\big| \\ &\leq \frac{\tilde{r}^2}{\tilde{i}}|q_0 - q_1| + |\tilde{r} - i| \times |b_0 - b_1| \leq \max\big\{\tilde{r}^2/\tilde{i}, r^\star - i\big\} \times \big(|q_0 - q_1| + |b_0 - b_1|\big) \\ &\leq \sqrt{2}\max\big\{\tilde{r}^2/\tilde{i}, r^\star - i\big\}|x_0 - x_1|, \end{aligned}$$

where the first inequality follows from the facts that (i) $\tilde{r}^2/\tilde{i}$ is the highest (absolute value) slope of the function $g(q) = [\tilde{r} - \tilde{i}/q]^+$ given $\tilde{r} > \tilde{i}$ and (ii) $[\tilde{r} - \tilde{i}/q]^+ \leq \tilde{r} - \tilde{i}$ as $q \leq 1$. The second inequality follows from $a + b \leq \sqrt{2}\sqrt{a^2 + b^2}$ for $a \geq 0$, $b \geq 0$. Thus, $M \equiv \sqrt{2}\max\{\tilde{r}^2/\tilde{i}, r^\star - i\}$ is the Lipschitz constant for all all $x_0, x_1 \in \mathbb{X}$.

Parts (ii) and (iii). These are immediate.

Part (iv). In this case, $\underline{q} = \frac{i+\lambda}{r+\lambda}$, as $H(0, q) = 0$ for all $q \leq \underline{q}$ and $H(0, q) > 0$ for all $q > \underline{q}$. Now note that for our parameter values $\underline{q} = 0.6 < \frac{i+\lambda}{i+\lambda+\delta+\epsilon} = 0.875$.

Part (v). We have $H(\overline{B}, 1) = 0$ given that $\overline{B} = y$.

Part (vi). We have that $H > 0$ requires $q \in (\underline{q}, 1]$ and $b \in [0, y)$. In this case, $H(b, q) = (r^\star + \lambda - \frac{i+\lambda}{q})(y - b)$, which is differentiable in this domain.

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