SUPPLEMENT TO “THE EMPIRICAL CONTENT OF BINARY CHOICE MODELS”
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This online appendix contains: (i) the construction of the continuous extension of the choice probability function to a domain containing Ω, as mentioned in footnote 11 in the proof of Theorem 1, and (ii) a version of Theorem 1 (called Theorem S1) with proof that does not require the limit conditions C/C′ of Theorem 1, but involves a slight strengthening of the continuity conditions B/B′.

APPENDIX S1: CONSTRUCTION OF CONTINUOUS EXTENSION OF CHOICE PROBABILITY FUNCTION

In the proof of Theorem 1, the definition of \( q^{-1}(\cdot, a_1) \) in (12) in the main text implicitly assumes that \( \Omega_0(a_1) \) equals (or contains) \([y_L(a_1), y_H(a_1)]\). If however the support of price and income are discrete, then \( \Omega_0(a_1) \) can be a strict subset of \([y_L(a_1), y_H(a_1)]\). Then \( q^{-1}(\cdot, \cdot) \) is not defined at the points “in between” the points of support and, therefore, \( q^{-1}(\cdot, a_1) \) in (12) is not well-defined. To cover this case, one can extend \( q(\cdot, \cdot) \) to a continuous function \( q^c(\cdot, \cdot) \) defined on a rectangle \( \Omega^c \) containing \( \Omega \) such that (i) \( q^c(\cdot, \cdot) \) equals \( q(\cdot, \cdot) \) on \( \Omega \), (ii) \( q^c(\cdot, \cdot) \) satisfies the same shape restrictions on \( \Omega^c \) that are satisfied by \( q(\cdot, \cdot) \) on \( \Omega \), and (iii) \( q^c(\cdot, \cdot) \) satisfies the limit conditions C of Theorem 1. The proof of Theorem 1 then holds with \( \Omega, \Omega_0(\cdot) \) and \( q(\cdot, \cdot) \) equalling their corresponding extensions in the case where \((P,Y)\) have discrete support. Here, we provide an explicit construction that achieves this extension.¹

Suppose the support of \((P,Y)\) is the discrete set \( \tilde{\Omega} = \{p_1, \ldots, p_M\} \times \{y_1, \ldots, y_N\} \), with \( p_1 < p_2 < \cdots < p_M \) and \( y_1 < y_2 < \cdots < y_N \). Suppose the choice probability \( q(y, y-p) \), which is defined for \((p,y) \in \tilde{\Omega}\), satisfies the shape constraints (A) of Theorem 1, i.e. \( q(\cdot, \cdot) \) is nonincreasing in the first and nondecreasing in the second argument. We want to construct an extension of \( q(\cdot, \cdot) \), denoted by \( q^c(y, y-p) \), which (i) defined for all \((y, y-p)\) with \( p_1 \leq p \leq p_M \) and \( y_1 \leq y \leq y_N \), (ii) equals \( q(y, y-p) \) for \((p,y) \in \tilde{\Omega}\), and (iii) satisfies all three conditions A, B, C of Theorem 1. The construction proceeds in three steps.

**Step 1:** First, we extend \( q(\cdot, \cdot) \) to the rectangular grid

\[
T = \{y_1, \ldots, y_N\} \times \bigcup_{j=1}^{N} \bigcup_{k=1}^{M} \{y_j - p_k\}.
\]

To do this, define \( \tilde{q}(\cdot, \cdot) : T \rightarrow [0,1] \) as

\[
\tilde{q}(y, y-p) = \lambda \tilde{L}(y, y-p) + (1-\lambda) \tilde{U}(y, y-p),
\]

(S1)

¹Alternatively, one can construct \( q^c(\cdot, \cdot) \) as a smooth, tensor-product polynomial spline with coefficients chosen to satisfy the shape restrictions and a high enough degree to guarantee that \( q^c(\cdot, \cdot) \) passes through the interpolating points \((y_i^l, y_i^l - p^l), q(y_i^l, y_i^l - p^l) : (y^l, y_i^l - p^l) \in \tilde{\Omega})\), along the lines of Costantini and Fontanella (1990).

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where $\lambda \in [0, 1]$ is arbitrary, and for any $(y, y - p) \in T$,

$$
\tilde{L}(y, y - p) = \begin{cases} 
\sup \{q(y', y' - p') \mid (p', y') \in \tilde{\Omega} : y' \geq y, y' - p' \leq y - p \} & \text{if } \{(p', y') \in \tilde{\Omega} : y' \geq y, y' - p' \leq y - p \} \neq \emptyset, \\
0 & \text{if } \{(p', y') \in \tilde{\Omega} : y' \geq y, y' - p' \leq y - p \} = \emptyset,
\end{cases}
$$

$$
\tilde{U}(y, y - p) = \begin{cases} 
\inf \{q(y', y' - p') \mid (p', y') \in \tilde{\Omega} : y' \leq y, y' - p' \geq y - p \} & \text{if } \{(p', y') \in \tilde{\Omega} : y' \leq y, y' - p' \geq y - p \} \neq \emptyset, \\
1 & \text{if } \{(p', y') \in \tilde{\Omega} : y' \leq y, y' - p' \geq y - p \} = \emptyset.
\end{cases}
$$

Note that $\tilde{q}(\cdot, \cdot)$, which is well-defined on all of $T$, satisfies the shape constraints (A) of Theorem 1. This is because the set $\{(p', y') \in \tilde{\Omega} : y' \geq y, y' - p' \leq y - p \}$ is decreasing in $y$ for fixed $y - p$, and increasing in $y - p$ for fixed $y$, so $\tilde{L}(\cdot, \cdot)$ is decreasing in the first and increasing in the second argument; an analogous argument works for $\tilde{U}(\cdot, \cdot)$. Furthermore, if $(p, y) \in \tilde{\Omega}$, then

$$(p, y) \in \{(p', y') \in \tilde{\Omega} : y' \geq y, y' - p' \leq y - p \},$$

whence the shape restrictions on $q(\cdot, \cdot)$ imply that $\tilde{L}(y, y - p) = q(y, y - p) = \tilde{U}(y, y - p)$, and hence $\tilde{q}(y, y - p) = q(y, y - p)$. Note, however, that $\tilde{q}(\cdot, \cdot)$ does not satisfy the continuity condition (B) and the limit conditions (C) of Theorem 1.

**Step 2:** The second step is to extend $\tilde{q}(\cdot, \cdot)$ to a continuous function $q^*(\cdot, \cdot)$ on the entire rectangle $[y_1, y_N] \times [y_1 - p_M, y_N - p_1]$, satisfying the shape constraints (A) of Theorem 1, while also satisfying the interpolation conditions $q^*(y, y - p) = q(y, y - p)$ for $(p, y) \in \tilde{\Omega}$. This is done using bilinear shape-preserving interpolation as follows.

Recall $y_1 < y_2 < \cdots < y_N$, and define $w_1 < w_2 < \cdots < w_J$ with $J \leq MN$ to be the ordered values of the set $\{y_1 - p_1, \ldots, y_1 - p_M, \ldots, y_N - p_1, \ldots, y_N - p_M\}$. We can have $J < MN$ if for some $(j, k) \neq (l, m)$, it holds that $y_j - p_k = y_l - p_m$. For each $i = 1, \ldots, N - 1$, $j = 1, \ldots, J - 1$, and for $(y, y - p) \in [y_i, y_{i+1}] \times [w_j, w_{j+1}]$, let

$$
\alpha_i(y) = \frac{y - y_i}{y_{i+1} - y_i}, \quad \beta_j(w) = \frac{w - w_j}{w_{j+1} - w_j},
$$

$$
q^*(y, y - p) = (1 - \alpha_i(y)) \times (1 - \beta_j(w)) \times \tilde{q}(y_i, w_j)
$$

$$
+ \alpha_i(y) \times (1 - \beta_j(w)) \times \tilde{q}(y_i, w_{j+1})
$$

$$
+ (1 - \alpha_i(y)) \times \beta_j(w) \times \tilde{q}(y_{i+1}, w_j)
$$

$$
+ \alpha_i(y) \times \beta_j(w) \times \tilde{q}(y_{i+1}, w_{j+1}),
$$

(S2)

where $\tilde{q}(\cdot, \cdot)$ is defined in (S1).

**Step 3:** The last step in the construction is to extend $q^*(\cdot, \cdot)$ beyond $[y_1, y_N] \times [y_1 - p_M, y_N - p_1]$ to ensure that the limit conditions (C) of Theorem 1 are satisfied. To do this, choose any pair of real numbers $y_L, y_H$ s.t. $y_L < y_1$ and $y_H > y_N$. Let

$$
D = [y_L, y_H] \times [y_1 - p_M, y_N - p_1].
$$
For any $w \in [y_1 - p_M, y_N - p_1]$, define

$$q^c(y, w) = \begin{cases} 
\frac{y - y_L}{y_i - y_L} \times q^c(y_1, w) + \frac{y_1 - y}{y_i - y_L} & \text{if } y \in [y_L, y_1], \\
\frac{y_{H} - y}{y_{H} - y_N + p_1} q^c(y_N - p_1, w) & \text{if } y \in [y_N - p_1, y_H].
\end{cases}$$

(S3)

That is for $y \in [y_L, y_1]$, $q^c(y, w)$ is the negatively sloped straight line joining $q^c(y_1, w)$ to $1 = q^c(y_L, w)$, and for $y \in [y_N - p_1, y_H]$, $q^c(y, w)$ is the negatively sloped straight line joining $q^c(y_N - p_1, w)$ to $0 = q^c(y_H, w)$.

Proof that $q^c(\cdot, \cdot) : D \rightarrow [0, 1]$ equals $q(y, y - p)$ for $(p, y) \in \tilde{\Omega}$ and satisfies conditions (A), (B), (C) of Theorem 1. To see the first assertion, observe that at the grid points $y = y_i$, $y - p = w_j$, we get from (S2) that $\alpha_i(y) = 0 = \beta_j(w)$, and that $q^c(y, w) = \tilde{q}(y_i, w_j)$. We have already seen that for $(p, y) \in \tilde{\Omega}$, $q(y, y - p) = \tilde{q}(y, y - p)$. Now, since $(p, y) \in \tilde{\Omega}$ implies $(y, y - p) \in T$, putting these two conclusions together, we get that for $(p, y) \in \tilde{\Omega}$, it holds that $q^c(y, y - p) = \tilde{q}(y, y - p)$.

As for the continuity condition (B) of Theorem 1, observe that holding fixed $w$, as $y \in [y_i, y_{i+1})$ and $y_{i+1}$, we have that $\alpha_i(y) \not\to 1$ whence from (S2), it follows that

$$q^c(y, w) \downarrow (1 - \beta_j(w)) \times \tilde{q}(y_{i+1}, w_j) + \beta_j(w) \times \tilde{q}(y_{i+1}, w_{j+1}).$$

(S4)

On the other hand, for the same $w$ and for $y \in [y_{i+1}, y_{i+2}]$, we have that $\alpha_i(y) = \frac{y - y_{i+1}}{y_{i+2} - y_{i+1}}$ which at $y = y_{i+1} \in [y_{i+1}, y_{i+2})$ equals $0$, whence from (S2) with $i$ replaced by $i + 1$ and $i + 1$ replaced by $i + 2$, we get

$$q^c(y, w) = (1 - \beta_j(w)) \times \tilde{q}(y_{i+1}, w_j) + \beta_j(w) \times \tilde{q}(y_{i+1}, w_{j+1}),$$

which equals (S4). Therefore, for fixed $w$, $\tilde{q}(y, w)$ is simply a piecewise linear function of $y$ joined at the end-points $y_{2i}, \ldots, y_{N-1}$ and, therefore, continuous in $y$ for $y \in [y_1, y_N]$. For $y \in [y_L, y_H] \cap [y_1, y_N]$, continuity is obvious from (S3) and the fact that

$$\lim_{y \to y_1} q^c(y, w) = q^c(y_1, w) = \lim_{y \to y_{i+1}} q^c(y, w) \quad \text{and} \quad \lim_{y \to y_{N-1}} q^c(y, w) = q^c(y_N - p_1, w) = \lim_{y \to y_{N-1}} q^c(y, w).$$

An analogous argument shows that $q^c(y, w)$ is also continuous in $w$ for fixed $y$ (this property is not needed to prove Theorem 1 but is used in Theorem S1, the alternative version of Theorem 1 without the limiting condition, which appears below).

The limiting conditions (C) of Theorem 1 are satisfied, since (S3) implies that $q^c(y_L, w) = 1$ and $q^c(y_H, w) = 0$ for each $w \in [y_1 - p_M, y_N - p_1]$.

Finally, to see that the shape restrictions (A) of Theorem 1 hold on $[y_1, y_N] \times [y_1 - p_M, y_N - p_1]$, note from (S2) that the coefficient of $y$ in $q^c(y, w)$ equals

$$\frac{1}{y_{i+1} - y_i} \times \begin{cases} 
(1 - \beta_j(w)) \times \tilde{q}(y_{i+1}, w_j) - \tilde{q}(y_i, w_j) & \geq 0, \\
-\beta_j(w) \times \tilde{q}(y_i, w_{j+1}) - \tilde{q}(y_{i+1}, w_{j+1}) & \leq 0, \text{ since } y_i \leq y_{i+1}
\end{cases} \leq 0,$$

since

$$\beta_j(w) \leq 0, \text{ since } y_i \leq y_{i+1}.$$
Similarly, the coefficient of $w$ in $q^c(y, w)$ equals

$$\frac{1}{w_{j+1} - w_j} \left[ \begin{array}{c} (1 - \alpha_i(y)) \times \left[ \frac{q(y_i, w_{j+1}) - \tilde{q}(y_i, w_j)}{\partial w_{j+1}} \right] \\ + \alpha_i(y) \times \left[ \frac{q(y_{i+1}, w_{j+1}) - \tilde{q}(y_{i+1}, w_j)}{\partial w_{j+1}} \right] \end{array} \right] \geq 0.$$ 

From (S3), it follows that the shape restrictions also hold on $[y_L, y_1] \times [y_1 - p_M, y_N - p_1]$ and on $[y_N, y_H] \times [y_1 - p_M, y_N - p_1]$, and thus condition (A) of Theorem 1 holds on all of $[y_L, y_H] \times [y_1 - p_M, y_N - p_1]$.

Thus $q^c(\cdot, \cdot)$ satisfies all three conditions of Theorem 1.

**APPENDIX S2: MAIN RESULT WITHOUT CONDITION (C/C')**

The following is a version of Theorem 1 that does not require the technical conditions C and C' of Theorem 1, but involves a slight strengthening of the technical condition B. The proof of this version is considerably longer than that of Theorem 1. The proof works by constructing an extension $Q(\cdot, \cdot)$ of $q(\cdot, \cdot)$ which satisfies properties (A)–(C) of Theorem 1 although $q(\cdot, \cdot)$ itself does not satisfy property (C).²

Suppose the support of price $P$ and income $Y$ in the population is $[p_1, p_u] \times [y_1, y_u]$. Correspondingly, the support of $Y - P$ is $\Omega_1 = [y_1 - p_u, y_u - p_1]$. Pick any $a_1 \in \Omega_1$. Corresponding to $Y - P = a_1$, the support of $Y = a_1 + P$ is therefore

$$\Omega_0(a_1) \overset{\text{def}}{=} \left\{ \max\{p_1 + a_1, y_1\}, \min\{p_u + a_1, y_u\} \right\}.$$ 

Note that by definition, $L(\cdot)$ and $U(\cdot)$ are nondecreasing and continuous. Let $\Omega = \bigcup_{a_1 \in \Omega_1} \bigcup_{a_0 \in \Omega_0(a_1)} \{a_0, a_1\}$.

**THEOREM S1:** For binary choice under general heterogeneity, the following two statements are equivalent:

(I) The choice probabilities $q(y, y - p)$, defined above, satisfy that (A) $q(\cdot, y - p)$ is nonincreasing, and $q(y, \cdot)$ is nondecreasing; (B) $q(\cdot, \cdot)$ is continuous.

(II) There exists a pair of utility functions $W_0(\cdot, \eta)$ and $W_1(\cdot, \eta)$, where the first argument denotes the amount of numeraire, and $\eta$ denotes unobserved heterogeneity, and a distribution $G(\cdot)$ of $\eta$ such that for any $(y - p) \in \Omega_1$ and correspondingly $y \in \Omega_0(y - p)$,

$$q(y, y - p) = \int 1\{W_0(y, \eta) \leq W_1(y - p, \eta)\} dG(\eta),$$ 

where (A') for each fixed $\eta$, $W_0(\cdot, \eta)$ and $W_1(\cdot, \eta)$ are nondecreasing; (B') for each fixed $\eta$, $W_0(\cdot, \eta)$ and $W_1(\cdot, \eta)$ are continuous, and for any $(a_0, a_1) \in \Omega$, it holds that $\int 1\{W_0(a_0, \eta) \leq W_1(a_1, \eta)\} dG(\eta)$ is continuous in $(a_0, a_1)$.

²The case where $(P, Y)$ have a discrete support is handled in exactly the same way as in Theorem 1 with two small modifications: (a) Step 3 in the construction immediately above is not required, and (b) continuity of $q^c(\cdot, \cdot)$ in the second argument is guaranteed by the construction in Step 2.
Discussion of assumptions: Relative to Theorem 1, conditions (C/C') are omitted, and condition (B/B') is strengthened to continuity in both arguments. Note that under monotonicity in any one argument, the joint continuity of \( g(\cdot, \cdot) \) is equivalent to coordinate wise continuity; cf. Kruse and Deely (1969).

To prove Theorem S1, we will utilize several lemmas.

**LEMMA S1**—Apostol (1974, Ex 4.19): Suppose \( r(\cdot) : [c, b] \to \mathbb{R} \) is continuous on \([c, b]\). For \( x \in [c, b] \), define \( g(x) = \sup\{r(z) : x \leq z \leq b\} \), and \( h(x) = \sup\{r(z) : c \leq z \leq x\} \). Then \( g(\cdot) \) and \( h(\cdot) \) are continuous on \([c, b]\).

**PROOF OF LEMMA S1:** Fix any \( x \in [c, a_1] \).

First, suppose \( g(x) > r(x) \). Choose \( \varepsilon = g(x) - r(x) > 0 \). Now by continuity of \( r(\cdot) \), there must exist \( \delta > 0 \) s.t. for any \( z \in [x - \delta, x + \delta] \), we have that \( r(z) > r(x) + \varepsilon = r(x) + g(x) - r(x) = g(x) \). Therefore, \( \sup\{r(z) : x - \delta \leq z \leq x + \delta\} < g(x) \). Therefore, \( g(x - \delta) = g(x) = g(x + \delta) \), implying continuity of \( g(\cdot) \) at \( x \).

Next, suppose the sum is at \( x, \) i.e. \( g(x) = r(x) \). By continuity, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \), s.t. for all \( u \in [x - \delta, x + \delta] \), we have that \( r(u) + \varepsilon \geq r(u) \geq r(x) - \varepsilon \). For \( u \in [x, x + \delta] \), \( g(u) = \sup\{r(z) : u \leq z \leq a_1\} \geq r(u) \geq r(x) - \varepsilon = g(x) - \varepsilon \), since \( g(x) = r(x) \), by assumption. But \( g(u) \leq g(x) \) by definition. Therefore, for all \( u \in [x, x + \delta] \), we have that \( g(u) \geq g(u) > g(x) - \varepsilon \). Next, for all \( u \in [x - \delta, x] \), \( r(u) \leq r(x) + \varepsilon = g(x) + \varepsilon \) implying
\[
g(u) = \sup\{r(z) : u \leq z \leq a_1\} \\
\leq \sup\{r(z) : x - \delta \leq z \leq a_1\} \\
= \max\{\sup\{r(z) : x - \delta \leq z \leq x\}, \sup\{r(z) : x \leq z \leq a_1\}\} \\
\leq g(x) + \varepsilon.
\]
Thus for all \( u \in [x - \delta, x + \delta] \), we have that \( g(x) + \varepsilon \geq g(u) > g(x) - \varepsilon \). Therefore, \( g(\cdot) \) is continuous at \( x \).

An exactly similar proof works for \( h(x) = \sup\{r(z) : c \leq z \leq x\} \).

**Q.E.D.**

**LEMMA S2**—Taylor (1955), Chapter 15.7, Theorem VII: Suppose the function \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous, and the function \( g(\cdot) : \mathbb{R} \to \mathbb{R} \) is continuous w.r.t. the L1-norm. Then the function \( h(x) = f(g(x), x) \) is continuous on \( \mathbb{R} \).

**PROOF OF LEMMA S2:** Pick any \( x_0 \in \mathbb{R} \), and \( \varepsilon > 0 \). Continuity of \( f(\cdot, \cdot) \) implies that there exists \( \delta > 0 \) s.t. \( \|f(g(x), x) - f(g(x_0), x_0)\| \leq \varepsilon \), whenever \( \|g(x), x - (g(x_0), x_0)\| \leq \delta \). Now, continuity of \( g(\cdot) \) implies that given the above \( \delta > 0 \), there exists \( \delta_1 > 0 \) s.t. \( |g(x) - g(x_0)| \leq \delta/2 \) whenever \( |x - x_0| \leq \delta_1 \). Choose \( \delta^* = \min\{\delta/2, \delta_1\} \). Then whenever \( |x - x_0| \leq \delta^* \), we have that \( |g(x) - g(x_0)| \leq \delta/2 \) and \( |x - x_0| \leq \delta_2/2 \), and thus \( \|g(x), x - (g(x_0), x_0)\| = |g(x) - g(x_0)| + |x - x_0| \leq \delta \) and, therefore,
\[
|h(x) - h(x_0)| = |f(g(x), x) - f(g(x_0), x_0)| < \varepsilon.
\]

**Q.E.D.**

Construction: The following construction will be used to prove the theorem. Pick \( a_1 \in \Omega_1 \). Recall the definitions \( L(a_1) = \max\{p_i + a_1, y_i\} \), and \( U(a_1) = \min\{p_u + a_1, y_u\} \). Let \( a_{0L}, \)
$a_{0L}$ be any pair of real numbers satisfying $a_{0L} < y_1$ and $a_{0H} > y_u$. For any $a_0 < L(a_1)$ and $a_0 > U(a_1)$, respectively, define

$$H(a_0, a_1) = \sup \{ q(L(x), x) : L(x) \in [a_0, L(a_1)] \},$$
$$h(a_0, a_1) = \inf \{ q(U(x), x) : U(x) \in [U(a_1), a_0] \}.$$

Note that as $a_0$ decreases with $a_1$ fixed, or $a_1$ increases with $a_0$ fixed, the set $[a_0, L(a_1)]$ expands and, therefore, the sup over it weakly increases; thus $H(\cdot, a_1)$ is nonincreasing and $H(a_0, \cdot)$ is nondecreasing. Similarly, $h(\cdot, a_1)$ is nonincreasing and $h(a_0, \cdot)$ is nondecreasing. Now, define the function $Q(\cdot, \cdot) : [a_{0L}, a_{0H}] \times \Omega_1$, (S5)

$$Q(a_0, a_1) = \begin{cases} 
H(y_1, a_1) + \left(1 - H(y_1, a_1)\right) \frac{y_l - a_0}{y_l - a_{0L}} & \text{if } a_{0L} \leq a_0 < y_1, \\
H(a_0, a_1) & \text{if } y_l \leq a_0 < L(a_1), \\
q(a_0, a_1) & \text{if } a_0 \in [L(a_1), U(a_1)], \\
h(a_0, a_1) & \text{if } U(a_1) < a_0 \leq y_u, \\
\frac{a_{0H} - a_0}{a_{0H} - y_u} h(y_u, a_1) & \text{if } y_u < a_0 \leq a_{0H}.
\end{cases}$$

CLAIM S1: Suppose $q(\cdot, \cdot)$ satisfies (A) and (B) of Theorem S1. Then the function $Q(\cdot, \cdot)$ defined in (S5) satisfies the following properties:

1. $Q(\cdot, a_1)$ is nonincreasing, and $Q(a_0, \cdot)$ is nondecreasing for all $(a_0, a_1) \in [a_{0L}, a_{0H}] \times \Omega_1$.
2. $Q(\cdot, \cdot)$ is continuous in each argument, holding the other argument fixed.
3. For any $a_1 \in \Omega_1$, there exist real numbers $a_{0L}$ and $a_{0H}$ such that $\lim_{a_0 \to a_{0L}} Q(a_0, a_1) = 1$ and $\lim_{a_0 \to a_{0H}} Q(a_0, a_1) = 0$.

PROOF: Property (3) is obvious because $Q(a_{0L}, a_1) = 1$ and $Q(a_{0H}, a_1) = 0$, by construction. To show (1) and (2), fix $a_1 \in \Omega_1$. Since $q(\cdot, \cdot)$ satisfies (A) and (B) on $a_0 \in [L(a_1), U(a_1)]$, we only need to establish the properties over the range $a_0 < L(a_1)$ and $a_0 > U(a_1)$.

Property (1): First, we show that the shape restrictions hold for $Q(\cdot, \cdot)$. We have already noted that $H(\cdot, a_1)$ and $h(\cdot, a_1)$ are both nonincreasing; further since $H(y_1, a_1) \leq 1$ and $h(y_u, a_1) \geq 0$, we have that $H(y_1, a_1) + (1 - H(y_1, a_1)) \frac{y_l - a_0}{y_l - a_{0L}}$ is nonincreasing in $a_0$ for $a_{0L} \leq a_0 < y_1$, and $\frac{a_{0H} - a_0}{a_{0H} - y_u} h(y_u, a_1)$ is nonincreasing in $a_0$ for $y_u < a_0 \leq a_{0H}$. Thus $Q(a_0, a_1)$ is nonincreasing in $a_0$ for all $a_0 < L(a_1)$ and $a_0 > U(a_1)$.

Next, pick $a_0 \in [a_{0L}, a_{0H}]$, and consider monotonicity of $Q(a_0, \cdot)$. Let $a_1^1, a_1^2 \in \Omega_1$ with $a_1^1 < a_1^2$, implying $L(a_1^1) \leq L(a_1^2)$ and $U(a_1^1) \leq U(a_1^2)$. Now there are 10 cases to consider, labeled (a)–(j) below, depending on the ordering of $L(a_1^1)$ and $U(a_1^1)$, and where $a_0$ lies. Case (a) $a_{0L} \leq a_0 < y_1$, then

$$Q(a_0, a_1^1) = H(a_0, a_1^1),$$
$$= \frac{y_l - a_0}{y_l - a_{0L}} + H(y_1, a_1^1) \frac{a_0 - a_{0L}}{y_l - a_{0L}},$$
$$\leq \frac{y_l - a_0}{y_l - a_{0L}} + H(y_1, a_1^1) \frac{a_0 - a_{0L}}{y_l - a_{0L}},$$
$$= Q(a_0, a_1^2).$$
Case (b) $y_l \leq a_0 \leq L(a_1)$, that is, $[a_0, L(a_1)] \subseteq [a_0, L(a_1)]$, and so $H(a_0, a_1) \leq H(a_0, a_1)$ and, therefore, $Q(a_0, a_1) = H(a_0, a_1) \leq H(a_0, a_1) = Q(a_0, a_1)$. Case (c): $y_u < a_0 \leq a_0 H$, and Case (d) $U(a_1) < a_0 \leq y_u$, the proofs are exactly analogous to respectively (a) and (b) above.

So we are left with the following cases, where Cases (e)–(g) correspond to $U(a_1) < L(a_1)$, and (h)–(j) to $U(a_1) \geq L(a_1)$.

For Case (e) $L(a_1) \leq a_0 \leq U(a_1) < L(a_1)$, since $L(a_1) < a_0 \leq L(a_1)$, by continuity of $L(\cdot)$ and the intermediate value theorem, there exists $c \in [a_1, a_1]$ s.t. $a_0 = L(c)$. Therefore,

$$Q(a_0, a_1) = q(a_0, a_1) = q(L(c), a_1)$$

$$(1) \leq q(L(c), c)$$

$$(2) \leq \sup\{q(L(x), x) : L(x) \in [L(c), L(a_1)]\}$$

$$= \sup\{q(L(x), x) : L(x) \in [a_0, L(a_1)]\}, \text{ since } a_0 = L(c)$$

$$= Q(a_0, a_1),$$

where $(1)$ holds because $a_1 \leq c$ and condition (A) of Theorem 1, and $(2)$ holds by definition of sup. Next, suppose case (f) $L(a_1) \leq U(a_1) \leq a_0 < L(a_1) \leq U(a_2)$, then by continuity of $L(\cdot)$ and the intermediate value theorem, there exists $c \in [a_1, a_1]$ s.t. $a_0 = L(c)$; and by continuity of $U(\cdot)$ and the intermediate value theorem, there exists $d \in [a_1, a_1]$ s.t. $a_0 = U(d)$, with $d \leq c$. Then

$$Q(a_0, a_1)$$

$$= \inf\{q(U(x), x) : U(a_1) \leq U(x) \leq a_0\}, \text{ by (S5)}$$

$$= \inf\{q(U(x), x) : U(a_1) \leq U(x) \leq U(d)\}, \text{ by } a_0 = U(d)$$

$$\leq q(U(d), d), \text{ since } d \in \{x : U(a_1) \leq U(x) \leq U(d)\}$$

$$\leq q(L(c), c), \text{ by (AII) since } U(d) = a_0 = L(c) \text{ and } d \leq c$$

$$\leq \sup\{q(L(x), x) : L(c) \leq L(x) \leq L(a_1)\}, \text{ since } c \in \{x : L(c) \leq L(x) \leq L(a_1)\}$$

$$= \sup\{q(L(x), x) : a_0 \leq L(x) \leq L(a_1)\}, \text{ since } a_0 = L(c)$$

$$= Q(a_0, a_1), \text{ by definition (S5)}.$$
= Q(a_0, a_1).

Next, consider case (h) \( L(a_1^+) \leq a_0 \leq L(a_1^-) \leq U(a_1^-) \). Since \( L(a_1^+) \leq a_0 \leq L(a_1^-) \), by continuity and the intermediate value theorem, we have that \( a_0 = L(c) \) for some \( c \in [a_1^+, a_1^-] \), whence we have

\[
Q(a_0, a_1^+) = q(a_0, a_1^+) = q(L(c), a_1^+)
\leq q(L(c), c), \quad \text{since } c \geq a_1^+
\leq \sup \{q(L(x), x) : L(c) \leq L(x) \leq L(a_1^-)\}
= Q(L(c), a_1^-)
= Q(a_0, a_1^-).
\]

Next, if case (i) \( L(a_1^-) \leq L(a_1^+) \leq a_0 \leq U(a_1^+) \), we have that \( Q(a_0, a_1^+) = q(a_0, a_1^+) \leq q(a_0, a_1^-) = Q(a_0, a_1^-) \).

Finally, for the Case (j) \( L(a_1^-) \leq L(a_1^+) \leq U(a_1^-) \leq a_0 \leq U(a_1^-) \), the same argument as in (g) applies.

This establishes the requisite shape restrictions, that is, Property (1).

**Property (2):** First, consider continuity of \( Q(\cdot, a_1) \). Note that \( H(y_i, a_1) + (1 - H(y_i, a_1)) \times \frac{y - y_i}{y - y_i} \) is obviously continuous at \( a_0 \) for \( a_{0L} \leq a_0 < y_i \); next, at \( a_0 = y_i \), \( Q(a_0, a_1) = H(y_i, a_1) + (1 - H(y_i, a_1)) \times \frac{y - y_i}{y - y_i} = H(y_i, a_1) \), while at \( a_0 = L(a_1) > y_i \),

\[
Q(a_0, a_1) = \sup \{q(L(x), x) : L(x) \in [L(a_1), L(a_1)]\} = q(L(a_1), a_1),
\]

and thus \( Q(\cdot, a_1) \) is continuous at \( a_0 = y_i \) and at \( a_0 = L(a_1) \). Finally, if \( a_0 \in (y_i, L(a_1)) \), then we can have \( L(x) \in [a_0, L(a_1)] \) only if \( L(x) > y_i \) in which case \( L(x) = x + p_i \) and thus \( q(L(x), x) = q(x + p_i, x) \) implying

\[
Q(a_0, a_1) = \sup \{q(L(x), x) : a_0 \leq L(x) \leq L(a_1)\}
= \sup \{q(x + p_i, x) : x + p_i \in [a_0, L(a_1)]\}
= \sup \{q(x + p_i, x) : x \in [a_0 - p_i, L(a_1) - p_i]\}.
\]

By Lemma S3, \( q(x + p_i, x) \) is continuous in \( x \), and therefore, by Lemma S2, \( Q(a_0, a_1) \) is continuous in \( a_0 \) for fixed \( a_1 \). Thus we have that \( Q(\cdot, a_1) \) is continuous on all of \([a_{0L}, U(a_1)].\) An exactly analogous argument works for \( a_0 > U(a_1).\)

Finally, consider continuity in \( a_1 \) for fixed \( a_0 \). If (a) \( a_1 \leq y_i - p_i \), then \( L(a_1) = y_i \) and, therefore,

\[
H(a_0, a_1) = \sup \{q(L(x), x) : L(x) \in [a_0, y_i]\},
\]

which does not depend on \( a_1 \) and, therefore, trivially continuous in \( a_1 \). So consider (b) \( a_1 > y_i - p_i \), so that \( L(a_1) = a_1 + p_i \). Therefore, at \( a_0 = y_i \), \( H(a_0, a_1) = H(y_i, a_1) \) equals

\[
\sup \{q(L(x), x) : a_0 \leq L(x) \leq L(a_1)\}
= \sup \{q(L(x), x) : L(x) \in [y_i, a_1 + p_i]\}
\overset{(2)}{=} \sup \{q(L(x), x) : x \in [y_i - p_i, a_1]\}.
\]
The last equality \((2)\) follows because 
\[
L(x) = \max\{p_1 + x, y_i\} 
\]
in \([y_i, a_1 + p_i]\) if and only if \(x \in [y_i - p_u, a_1]\).

Now, since \(L(\cdot)\) is continuous, and so is \(q(\cdot, \cdot)\), the function \(x \mapsto q(L(x), x)\) is continuous in \(x\) (see Lemma S3 above), and therefore, it follows from Lemma S2 that
\[
\sup\{q(L(x), x) : x \in [y_i - p_u, a_1]\} \quad \text{is continuous in } a_1.
\]
In particular, as \(a_1 \downarrow (y_i - p_i)_{\text{r}}\), 
\(L(a_1)\) approaches \(y_i\) and so (S8) tends to (S7).

Finally, for any \(a_0 > y_i\), (recall \(a_1 > y_i - p_i\), so that \(L(a_1) = a_1 + p_i\)), we have that
\[
H(a_0, a_1) = \sup\{q(L(x), x) : L(x) \in [a_0, a_1 + p_i]\} = \sup\{q(L(x), x) : x \in [a_0 - p_i, a_1]\},
\]
which is continuous in \(a_1\) by Lemmas S2 and S3. Exactly analogous arguments hold for \((a')\) \(a_1 \geq y_u - p_u\), and \((b')\) \(a_1 < y_u - p_u\), respectively. Thus, we have that \(Q(a_0, \cdot)\) is continuous at
\(\text{each } a_0\).

**Q.E.D.**

**Lemma S3:** Suppose the function \(Q(\cdot, \cdot) : [a_{0L}, a_{0H}] \times \Omega_1 \subseteq \mathbb{R}^2 \to [0, 1]\) satisfies on its domain that
\(\text{(1) } Q(\cdot, a_1) \text{ is nonincreasing, and } Q(a_0, \cdot) \text{ is nondecreasing; (2) } Q(\cdot, a_1) \text{ is continuous, and (3) for any } a_1 \in \Omega_1, \lim_{a_0 \downarrow a_{0L}} Q(a_0, a_1) = 1 \text{ and } \lim_{a_0 \uparrow a_{0H}} Q(a_0, a_1) = 0.\)

For any fixed \(a_1 \in \Omega_1\), define for each \(u \in [0, 1]\),
\[
Q^{-1}(u, a_1) \overset{\text{defn}}{=} \sup\{a_0 \in [a_{0L}, a_{0H}] : Q(a_0, a_1) \geq u\}. \quad (S9)
\]

Then we must have that \(Q(Q^{-1}(v, a_1), a_1) = v\), for any \(v \in [0, 1]\).

**Proof of Lemma S3:** Since \(Q(\cdot, \cdot)\) satisfies the same properties as \(q(\cdot, \cdot)\) of Theorem 1(A)–(C), the proof of this lemma is identical to the proof of Claim (i) used to prove Theorem 1 in the main paper.

**Q.E.D.**

**Proof of Theorem S1:** That (II) implies (I) is straightforward, since
\[
q(y, y - p) = \int 1\{W_0(y, \eta) \leq W_1(y - p, \eta)\} \, dG(\eta)
\]
whence \((b')\) implies (B), and \((A')\) implies (A).

We now show that (I) implies (II). To do so, recall the definition of \(Q^{-1}(v, a_1)\) in (S9). Now, consider a random variable \(V \sim \text{Uniform}(0, 1)\). Define \(W_0(a_0, V) \overset{\text{defn}}{=} a_0\) and
\(W_1(a_1, V) \overset{\text{defn}}{=} Q^{-1}(V, a_1)\). We will now show that for \(y - p \in \Omega_1\) and correspondingly, \(y \in [L(y - p), U(y - p)]\), the functions \(W_0(y, V)\) and \(W_1(y - p, V)\) will rationalize the choice-probabilities \(q(y, y - p)\).

To prove this, note that for any \(v \in [0, 1]\), and \((a_0, a_1) \in \Omega,\)
\[
a_0 \leq Q^{-1}(v, a_1) \quad \text{by \(Q^{-1}(\cdot, a_1)\) nonr}\implies Q(a_0, a_1) \geq Q(Q^{-1}(v, a_1), a_1) \quad \implies Q(a_0, a_1) \geq v.
\]

Also, by definition of \(Q^{-1}(\cdot, a_1)\) as the supremum in (S9), we have that
\[
Q(a_0, a_1) \geq v \implies a_0 \leq Q^{-1}(v, a_1). \quad (S11)
\]
Therefore, by (S10) and (S11), we have that \( Q(a_0, a_1) \geq v \iff a_0 \leq Q^{-1}(v, a_1) \). Thus, for \( V \simeq U(0, 1) \), it follows that

\[
\Pr(Q^{-1}(V, a_1) \geq a_0) = \Pr(V \leq Q(a_0, a_1)) = Q(a_0, a_1).
\]

(S12)

Recall that for \( y - p \in \Omega_1 \) and correspondingly \( y \in [L(y - p), U(y - p)] \), we have that \( Q(y, y - p) = q(y, y - p) \) by definition. Therefore, it follows from (S12) that the utility functions \( W_0(y, V) \equiv y \) and \( W_1(y - p, V) \equiv Q^{-1}(V, y - p) \) with heterogeneity \( V \simeq \text{Uniform}(0, 1) \) rationalize the choice probability function \( q(\cdot, \cdot) \) on its domain.

Next, note that \( Q^{-1}(v, a'_1) \leq Q^{-1}(v, a_1) \) whenever \( a'_1 < a_1 \). To see this, suppose \( a_1 > a'_1 \) and yet \( Q^{-1}(v, a_1) < Q^{-1}(v, a'_1) \). Choose \( c \) s.t. \( Q^{-1}(v, a_1) < c < Q^{-1}(v, a'_1) \). Then by conclusion (i) of the previous lemma and by definition (S9) of \( Q^{-1}(v, \cdot) \), we must have \( Q(c, a_1) < v \leq Q(c, a'_1) \). But since \( a_1 > a'_1 \), this contradicts conclusion (1) of the Claim S1.

Next, it follows from (A) and (B) that \( Q^{-1}(v, \cdot) \) is continuous. To see this, fix \( v \in [0, 1] \), and suppose to the contrary that \( Q^{-1}(v, \cdot) \) is discontinuous at \( a_1 \); suppose there exists \( \varepsilon > 0 \) such that for any \( \delta > 0 \), \( Q^{-1}(v, a_1) > Q^{-1}(v, a'_1) + \varepsilon \) for all \( a'_1 < a_1 < a'_1 + \delta \). For any such \( a'_1 \) satisfying \( Q^{-1}(v, a_1) > Q^{-1}(v, a'_1) + \varepsilon \), it follows from the definition of \( Q^{-1}(\cdot, a'_1) \) that there exists \( \varepsilon' = \varepsilon(\varepsilon) > 0 \) s.t.

\[
Q(Q^{-1}(v, a_1), a'_1) \leq Q(Q^{-1}(v, a'_1), a'_1) - \varepsilon',
\]

by Lemma S3

\[
= Q(Q^{-1}(v, a_1), a'_1) - \varepsilon'.
\]

(S13)

Inequality (1) follows because \( Q(Q^{-1}(v, a'_1), a'_1) \leq Q(Q^{-1}(v, a_1), a'_1) \) since \( Q^{-1}(v, a_1) > Q^{-1}(v, a'_1) \), and if \( Q(Q^{-1}(v, a'_1), a'_1) = Q(Q^{-1}(v, a_1), a'_1) \) with \( Q^{-1}(v, a_1) > Q^{-1}(v, a'_1) + \varepsilon \), then that contradicts the definition of \( Q^{-1}(v, a'_1) \) as the sup. Therefore, it follows from (S13) that

\[
Q(Q^{-1}(v, a_1), a_1) - Q(Q^{-1}(v, a_1), a'_1) \geq \varepsilon',
\]

which contradicts that \( Q(\cdot, \cdot) \) is continuous in its second argument for fixed value of its first argument (see property (2) in Claim S1 above), since \( a'_1 \) can be made arbitrarily close to \( a_1 \) by choosing \( \delta \) small enough.

Finally, \( W_0(y, \eta) = y \) is obviously continuous and strictly increasing in \( y \), thus (A) holds. Finally, (B) ensures that (B’) is satisfied.

Q.E.D.

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