Econometrica Supplementary Material

SUPPLEMENT TO "A THEORY OF INPUT–OUTPUT ARCHITECTURE"

(*Econometrica*, Vol. 86, No. 2, March 2018, 559–589)

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APPENDIX A: STABLE EQUILIBRIA

A.1. *Notation*

THIS SUBSECTION SHOWS that it is without loss of generality to associate each bilateral contract with a technique and restrict the arrangement so that there is one contract for each technique.

A coalition is *connected* if, for any entrepreneurs i and j in the coalition, there is a sequence of entrepreneurs beginning with i and ending with j, all of whom are members of the coalition, such that for each consecutive pair there exists a technique in Φ for which one entrepreneur is the buyer and the other is the supplier.

LEMMA 1: *If there is a coalition of size/cardinality N or smaller with a dominating deviation, then there is a connected coalition of size/cardinality N or smaller with a dominating deviation.*

PROOF: Suppose that there is a coalition J with a dominating deviation that can be divided into two subsets that are not connected, J' and J'', so that $J' \cup J'' = J$ and $J' \cap J'' = \emptyset$. The deviation would leave every member of J at least as well off and at least one member of J strictly better off. Without loss of generality, suppose that the member who is strictly better off is in J'. Then J' has a dominating deviation: Since no member of J' is able to produce using intermediate inputs from members of J'', J' has a dominating deviation in which members of J' drop all contracts for which there are positive payments to members of J'' and otherwise deviate according to the original deviation; every member of J' is at least as well off as under the original deviation. *Q.E.D.*

LEMMA 2: *It is without loss of generality to use notation that associates each bilateral contract with a technique.*

PROOF: We first show that if j has no technique to use i's good as an input, then there is no equilibrium in which i provides goods to j or in which there is a payment between them. After that, we will show that if a coalition has a dominating deviation, then there is always an alternative dominating deviation in which each pairwise payment and trade of goods can be associated with a technique.

Toward a contradiction, suppose first that there is an equilibrium in which entrepreneur i provides goods for entrepreneur j, but j has no technique that would use good i as an input. Since j cannot resell good i, if the payment to j is positive, then i would be strictly better off dropping the contract (setting the payment and the quantity of goods to zero). If the payment is weakly negative, i would be strictly better off dropping the contract. Thus this cannot be an equilibrium. Suppose second that there is an equilibrium in which

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 https://doi.org/10.3982/ECTA10731

there is a payment from i to j but no goods are provided. Then, unless the payment is zero, one of the two would be strictly better off dropping the contract.

Next, suppose the connected coalition J has a dominating deviation in which there is a payment from i to j. Then because the coalition is connected, there is another deviation with identical payoffs where the payment from i to j is intermediated by those on the path from i to j. *Q.E.D.*

LEMMA 3: *It is without loss of generality to use notation that associates each technique with a single bilateral contract.*

PROOF: Suppose there is an arrangement in which there may be multiple bilateral contracts associated with each technique. Suppose further that the arrangement is stable with respect to deviations by coalitions of size/cardinality N. Then the alternative arrangement in which all contracts for each technique are combined into a single contract delivers the same allocation and payoffs and must also be stable with respect to deviations by coalitions of size/cardinality N because those deviations were available for the original arrangement. *Q.E.D.*

A.2. *A Supply Chain Representation*

This section describes notation that decomposes the allocation into production within the many supply chains available to produce the various goods. Such a mapping can be constructed because each technique exhibits constant returns to scale.

For a supply chain $\omega \in \Omega_j$ available to produce good j, we can summarize production at each step in the chain in the eventual production of j for consumption. Towards this, we will construct $c(\omega)$ to be consumption of j produced using the supply chain ω and the variables $\{y^n(\omega), x^n(\omega), l^n(\omega)\}$ to be the quantities of output, intermediate input, and labor used with technique n (i.e., n stages away from consumption) in the supply chain ω in the production of good j for consumption.

An entrepreneur may produce using multiple techniques and provide its output for multiple uses (e.g., for consumption and for intermediate use for several buyers). To construct the supply chain representation, we must assign the output from each source to particular destinations. We define an *assignment* as follows:

Entrepreneur j's output using technique $\phi \in U_j$ is $y(\phi)$. j's output used as an intermediate input for technique $\phi' \in D_j$ is $x(\phi')$, and its output for the household's consumption is c_j. An *assignment* defines, for each j, a set of numbers

$$\{Y_c(\phi), \{Y_{\phi'}(\phi)\}_{\phi' \in D_j}\}_{\phi \in U_j}$$

that are non-negative and satisfy

$$\sum_{\phi \in U_j} Y_c(\phi) = c_j,$$

$$\sum_{\phi \in U_j} Y_{\phi'}(\phi) = x(\phi'),$$

$$Y_c(\phi) + \sum_{\phi' \in D_j} Y_{\phi'}(\phi) = y(\phi).$$

For each $\phi \in U_j$, the assignment defines how much of the ouput from that technique gets assigned to each destination. Of the total amount of good j produced using ϕ, $Y_c(\phi)$ is the amount consumed and $Y_{\phi'}(\phi)$ is the amount used as an intermediate input by the buyer of $\phi' \in D_j$. The first constraint says that the quantity of good j that the total amount of j that is consumed must sum to the output tha tis consumed from each of j's techniques. Similarly, the second constraint says that the total amount of good j that is delivered for use an intermediate input for technique ϕ' must sum to the amount produced for this purpose cross all of j's techniques. The third constraint says that the total output from each technique must sum to output from the technique that is used for each purpose. There may be multiple assignments consistent with an allocation.[62]

Each distinct assignment generates a distinct supply chain representation. Consider a supply chain $\omega \in \Omega_j$. Let $\phi^n(\omega)$ denote the nth technique in the chain so that, for example, $\phi^0(\omega)$ is the most downstream technique, and let $\mathrm{j}^n(\omega)$ be the identity of the buyer associated with that technique so that $\mathrm{j}^0(\omega) = j$ and $\mathrm{j}^n(\omega) = b(\phi^n(\omega)) = s(\phi^{n-1}(\omega))$. In the representation, let $c(\omega)$ be the total amount of good j produced for consumption using supply chain ω.[63] Thus total consumption of good j is $c_j = \sum_{\omega \in \Omega_j} c(\omega)$. Given the assignment, $c(\omega)$ is well-defined:

$$c(\omega) \equiv Y_c\big(\phi^0(\omega)\big) \prod_{k=0}^{\infty} \frac{Y_{\phi^k(\omega)}\big(\phi^{k+1}(\omega)\big)}{x\big(\phi^k(\omega)\big)}. \tag{17}$$

$Y_c(\phi^0(\omega))$ is the output of good j from technique $\phi^0(\omega)$ that goes to the household for consumption, while $\frac{Y_{\phi^k(\omega)}(\phi^{k+1}(\omega))}{x(\phi^k(\omega))}$ is the fraction of production using technique $\phi^k(\omega)$ that is produced using technique $\phi^{k+1}(\omega)$. Note that $c(\omega)$ is not the same as $Y_c(\phi^0(\omega))$ because there may be multiple supply chains that end up in $\phi^0(\omega)$; $Y_c(\phi^0(\omega))$ sums up the output from all of those supply chains, of which $c(\omega)$ is only a part.[64] Note that summing (17) across all supply chains implies that $\sum_{\omega \in \Omega_j} c(\omega) = c_j$.

We next construct $\{l^n(\omega), x^n(\omega), y^{n+1}(\omega)\}_{n=0}^{\infty}$ iteratively beginning with $y^0(\omega) \equiv c(\omega)$ and using the following equalities:

$$\frac{y^n(\omega)}{y\big(\phi^n(\omega)\big)} = \frac{l^n(\omega)}{l\big(\phi^n(\omega)\big)} = \frac{x^n(\omega)}{x\big(\phi^n(\omega)\big)}$$

along with $x^n(\omega) = y^{n+1}(\omega)$. The first two equalities simply indicate that the fraction of total output from $\phi^n(\omega)$ that is used in chain ω equals the corresponding fractions of labor and intermediate inputs. The final equality simply states that the output at one stage in a chain is the intermediate input used in the subsequent stage.

It will be useful to define the efficiency of a supply chain to be $q(\omega) \equiv \prod_{n=0}^{\infty} z^n(\omega)^{\alpha^n}$. For this to be well defined, it must be that $\lim_{N\to\infty} \prod_{n=0}^{N-1} z^n(\omega)^{\alpha^n}$ exists. The following claim shows that, under Assumption 1, the limit always exists.

[62]If an entrepreneur produces using a single technique or if it sells its good only to the household, there is a unique assignment; otherwise, there are many that would generate the same allocation.

[63]This is distinct from the total quantity of good j produced using the supply chain ω; the latter includes production for intermediate use by other entrepreneurs.

[64]It will be shown later that in equilibrium, generically each entrepreneur uses only a single technique. This means that generically an entrepreneur's output will be produced using a single supply chain; for the supply chain that is actually used, $c(\omega) = c_j$, while for the supply chains that are not used, $c(\omega) = 0$.

LEMMA 4: *Assume $z_0 > 0$. Then, for each $\omega \in \Omega_j$, $\lim_{N\to\infty} \prod_{n=0}^{N-1} z^n(\omega)^{\alpha^n}$ exists.*

PROOF: For each n, $z^n(\omega)$ can be decomposed into $z^{n+}(\omega)z^{n-}(\omega)$, where $z^{n+}(\omega) = \max\{z^n(\omega), 1\}$ and $z^{n-}(\omega) = \min\{z^n(\omega), 1\}$, so that $\prod_{n=0}^{N-1} z^n(\omega)^{\alpha^n} = (\prod_{n=0}^{N-1} z^{n+}(\omega)^{\alpha^n}) \times (\prod_{n=0}^{N-1} z^{n-}(\omega)^{\alpha^n})$. $\prod_{n=0}^{N-1} z^{n+}(\omega)^{\alpha^n}$ is a monotone sequence so it converges to a (possibly infinite) limit. $\prod_{n=0}^{N-1} z^{n-}(\omega)^{\alpha^n}$ is a monotone sequence bounded below by $z_0^{\frac{1}{1-\alpha}}$ so it converges to a limit in the range $[z_0^{\frac{1}{1-\alpha}}, 1]$. Thus $\prod_{n=0}^{N-1} z^n(\omega)^{\alpha^n}$ converges to a (possibly infinite) limit. *Q.E.D.*

A.3. *Proof of Proposition 1*

Given an arrangement, entrepreneur j's payoff is

$$\Pi_j = \max_{p_j, c_j, \{l(\phi)\}_{\phi\in U_j}} p_j c_j + \sum_{\phi\in D_j} T(\phi) - \sum_{\phi\in U_j} \big[T(\phi) + wl(\phi)\big] \tag{18}$$

subject to satisfying the household's demand $c_j \leq C(p_j/P)^{-\varepsilon}$ and a technological constraint that total usage of good j cannot exceed total production of good j,

$$c_j + \sum_{\phi\in D_j} x(\phi) \leq \sum_{\phi\in U_j} \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z(\phi)x(\phi)^\alpha l(\phi)^{1-\alpha}. \tag{19}$$

LEMMA 5: *For any arrangement, the following hold for each j and $\phi \in U_j$:*

$$p_j = \frac{\varepsilon}{\varepsilon - 1}\lambda_j,$$

$$\frac{l(\phi)}{1-\alpha} = \left[\frac{\lambda_j z(\phi)}{w}\right]^{\frac{1}{\alpha}} \frac{x(\phi)}{\alpha}, \tag{20}$$

$$\lambda_j = w\left[\frac{c_j + \sum_{\phi\in D_j} x(\phi)}{\sum_{\phi\in U_j} \frac{1}{\alpha} z(\phi)^{\frac{1}{\alpha}} x(\phi)}\right]^{\frac{\alpha}{1-\alpha}}, \tag{21}$$

$$l_j = (1-\alpha)\left[c_j + \sum_{\phi\in D_j} x(\phi)\right]^{\frac{1}{1-\alpha}} \left[\sum_{\phi\in U_j} \frac{1}{\alpha} z(\phi)^{\frac{1}{\alpha}} x(\phi)\right]^{-\frac{\alpha}{1-\alpha}}. \tag{22}$$

PROOF: Together, the FOCs with respect to p_j and y_j imply $p_j = \frac{\varepsilon}{\varepsilon-1}\lambda_j$. Individual rationality guarantees that $x(\phi) = 0$ implies $l(\phi) = 0$. If $x(\phi) > 0$, the FOC with respect to $l(\phi)$ is $w = \lambda_j \frac{1-\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z(\phi)x(\phi)^\alpha l(\phi)^{-\alpha}$, which can be rearranged as

$$l(\phi) = \left[\frac{\lambda_j z(\phi)}{w}\right]^{\frac{1}{\alpha}} \frac{1-\alpha}{\alpha} x(\phi).$$

Substituting this into (19) and solving for λ_j yields

$$c_j + \sum_{\phi \in D_j} x(\phi) = \sum_{\phi \in U_j} \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} z(\phi) x(\phi)^\alpha \left[\frac{\lambda_j \frac{1-\alpha}{\alpha^\alpha (1-\alpha)^{1-\alpha}} z(\phi) x(\phi)^\alpha}{w} \right]^{\frac{1-\alpha}{\alpha}},$$

which can be rearranged as

$$\lambda_j = w \left[\frac{c_j + \sum_{\phi \in D_j} x(\phi)}{\sum_{\phi \in U_j} \frac{1}{\alpha} z(\phi)^{\frac{1}{\alpha}} x(\phi)} \right]^{\frac{\alpha}{1-\alpha}}.$$

Plugging this into (20) yields

$$l(\phi) = \frac{1-\alpha}{\alpha} \left[\frac{c_j + \sum_{\phi \in D_j} x(\phi)}{\sum_{\phi \in U_j} \frac{1}{\alpha} z(\phi)^{\frac{1}{\alpha}} x(\phi)} \right]^{\frac{1}{1-\alpha}} z(\phi)^{\frac{1}{\alpha}} x(\phi),$$

and summing across techniques yields (22). *Q.E.D.*

LEMMA 6: *If an arrangement is pairwise stable, then, for each* $\phi \in \Phi$, $z(\phi) q_{s(\phi)}^\alpha \le q_{b(\phi)}$ *with equality if* $x(\phi) > 0$.

PROOF: Note first that if $q_j = 0$, then it must be that either U_j is empty or $x(\phi) = 0$ for all $\phi \in U_j$, and hence $y_j = 0$. We now proceed in three cases. First, if $q_{s(\phi)} = 0$, then, as just argued, it must be that $x(\phi) = 0$, and hence the conclusion of the lemma is true because $0 \le q_{b(\phi)}$.

Second, suppose that $q_{s(\phi)} > 0$ and $q_{b(\phi)} > 0$. The envelope theorem implies that, to a first order, an increase in $x(\phi)$ reduces $s(\phi)$'s profit by $\lambda_{s(\phi)}$. To assess the impact on the buyer's profit, it will be useful to plug in (22) to the buyer's problem so it can be written as

$$\pi_j = \max_{p_j, y_j} p_j y_j - w(1-\alpha) \left[c_j + \sum_{\phi \in D_j} x(\phi) \right]^{\frac{1}{1-\alpha}} \left[\sum_{\phi \in U_j} \frac{1}{\alpha} z(\phi)^{\frac{1}{\alpha}} x(\phi) \right]^{-\frac{\alpha}{1-\alpha}}$$
$$- \sum_{\phi \in U_j} T(\phi) + \sum_{\phi \in D_j} T(\phi)$$

subject to $c_j \le (p_j/P)^{-\varepsilon} C$. Then the envelope theorem implies that, to a first order, an increase in $x(\phi)$ raises $b(\phi)$'s profit by

$$w \left[c_j + \sum_{\phi \in D_j} x(\phi) \right]^{\frac{1}{1-\alpha}} \left[\sum_{\phi \in U_j} \frac{1}{\alpha} z(\phi)^{\frac{1}{\alpha}} x(\phi) \right]^{-\frac{\alpha}{1-\alpha}-1} z(\phi)^{\frac{1}{\alpha}}.$$

Using (21), this equals

$$w\left(\frac{\lambda_{b(\phi)}}{w}\right)^{\frac{1}{\alpha}} z(\phi)^{\frac{1}{\alpha}}.$$

Pairwise stability implies that there can be no better contract between $b(\phi)$ and $s(\phi)$. If $x(\phi) > 0$, there can be no gains from either increasing or reducing $x(\phi)$, which implies $w(\frac{\lambda_{b(\phi)}}{w})^{\frac{1}{\alpha}} z(\phi)^{\frac{1}{\alpha}} = \lambda_{s(\phi)}$, or $q_{b(\phi)} = z(\phi)q^{\alpha}_{s(\phi)}$. If $x(\phi) = 0$, there can be no gains from increasing $x(\phi)$, so it must be that $w(\frac{\lambda_{b(\phi)}}{w})^{\frac{1}{\alpha}} z(\phi)^{\frac{1}{\alpha}} \leq \lambda_{s(\phi)}$, or $q_{b(\phi)} \geq z(\phi)q^{\alpha}_{s(\phi)}$.

Finally, suppose that $q_{s(\phi)} > 0$ but $q_{b(\phi)} = 0$. The latter implies that $y_{b(\phi)} = 0$. Consider a deviation in which $\tilde{x}(\phi) = \eta$. If the buyer chooses (suboptimally) to use $\tilde{l}(\phi) = \frac{1-\alpha}{\alpha}\eta\frac{\lambda_{s(\phi)}}{w}$ units of labor with the technique, its output would be $\frac{1}{\alpha}\frac{z(\phi)}{q^{1-\alpha}_{s(\phi)}}\eta$. Since $y_{b(\phi)} = 0$, the deviation implies $\tilde{c}_{b(\phi)} = \frac{1}{\alpha}\frac{z(\phi)}{q^{1-\alpha}_{s(\phi)}}\eta$ and $\tilde{p}_{b(\phi)} = P(\tilde{c}_{b(\phi)}/C)^{-\frac{1}{\varepsilon}}$. $\tilde{p}_{b(\phi)}\tilde{c}_{b(\phi)}$ is proportional to $\eta^{\frac{\varepsilon-1}{\varepsilon}}$ while $w\tilde{l}(\phi)$ is proportional to η. For η small enough, the cost to the supplier is of order η. Thus there is a contract with η small enough that would increase the joint surplus. *Q.E.D.*

LEMMA 7: *Pairwise stability implies that for all* ϕ, $\frac{wl(\phi)}{1-\alpha} = \frac{\lambda_{s(\phi)}x(\phi)}{\alpha} = \lambda_{b(\phi)}y(\phi)$ *and for each* j,

$$(1-\alpha)\lambda_j y_j = \sum_{\phi\in U_j} wl(\phi), \tag{23}$$

$$\alpha\lambda_j y_j = \sum_{\phi\in U_j} \lambda_{s(\phi)}x(\phi) \tag{24}$$

and for each $\omega \in \Omega_j$,

$$wl^n(\omega) = \alpha^n(1-\alpha)\lambda_j c(\omega). \tag{25}$$

PROOF: Individual rationality guarantees that $x(\phi) = 0$ implies $l(\phi) = 0$. If $x(\phi) > 0$, then individual rationality implies $\frac{l(\phi)}{1-\alpha} = [\frac{\lambda_{b(\phi)}z(\phi)}{w}]^{\frac{1}{\alpha}}\frac{x(\phi)}{\alpha}$ and pairwise stability implies $z(\phi)q^{\alpha}_{s(\phi)} = q_{b(\phi)}$. Together, these imply $\frac{wl(\phi)}{1-\alpha} = \frac{\lambda_{s(\phi)}x(\phi)}{\alpha}$. These along with the production function imply $\frac{\lambda_{s(\phi)}x(\phi)}{\alpha} = \lambda_{b(\phi)}y(\phi)$. Summing over techniques $\phi \in U_j$ gives (23) and (24). Finally, for each chain $\omega \in \Omega_j$, it must be that $\lambda_j c(\omega) = \lambda_j y^0(\omega) = (1-\alpha)wl^0(\omega)$ and that $(1-\alpha)wl^n(\omega) = \alpha\lambda_{j^{n+1}(\omega)}x^n(\omega) = \alpha(1-\alpha)wl^{n+1}(\omega)$ (these are true whether or not $y^0(\omega)$, $l^n(\omega)$, and $x^n(\omega)$ are positive). Together, these imply (25). *Q.E.D.*

With this in hand we turn to the implications of stability for entrepreneurs' efficiencies. Proposition 1 gives the implications of pairwise and countable stability.

PROPOSITION 1—1 of 6: *In any pairwise stable equilibrium, if* U_j *is non-empty, then* $q_j = \max_{\phi\in U_j} z(\phi)q^{\alpha}_{s(\phi)}$.

PROOF: This follows immediately from Lemma 6. *Q.E.D.*

PROPOSITION 1—2 of 6: *In any pairwise stable equilibrium,* $C = QL$ *and* $c_j = q^{\varepsilon}_j Q^{1-\varepsilon}L$.

PROOF: First, $p_j = \frac{\varepsilon}{\varepsilon-1}\frac{w}{q_j}$ implies $P = (\int_0^1 p_j^{1-\varepsilon}\,dj)^{\frac{1}{1-\varepsilon}} = \frac{\varepsilon}{\varepsilon-1}\frac{w}{Q}$, so that the quantity of j sold to the household is $c_j = (q_j/Q)^\varepsilon C$. Second, summing over the labor used in all steps in each supply chain in Ω_j and using (25), we have

$$L = \int_0^1 \sum_{\omega\in\Omega_j}\sum_{n=0}^{\infty} l^n(\omega)\,dj = \int_0^1 \sum_{\omega\in\Omega_j}\sum_{n=0}^{\infty} \alpha^n(1-\alpha)\frac{1}{q_j}c(\omega)\,dj$$
$$= \int_0^1 \frac{1}{q_j}c_j\,dj = \int_0^1 \frac{1}{q_j}(q_j/Q)^\varepsilon C\,dj = \frac{1}{Q}C.$$

Plugging this back into $c_j = (q_j/Q)^\varepsilon C$ gives the result. *Q.E.D.*

PROPOSITION 1—3 of 6: *In any pairwise stable equilibrium*, $q_j \le \sup_{\omega\in\Omega_j} q(\omega)$.

PROOF: If $q_j = 0$, the conclusion is immediate. If $q_j > 0$, chain feasibility and (25) imply that for each $\omega \in \Omega_j$,

$$c(\omega) \le \prod_{n=0}^{\infty}\left(\frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z^n(\omega) l^n(\omega)^{1-\alpha}\right)^{\alpha^n}$$
$$\le \prod_{n=0}^{\infty}\left(\frac{1}{\alpha^\alpha} z^n(\omega)\left\{\alpha^n\frac{1}{q_j}c(\omega)\right\}^{1-\alpha}\right)^{\alpha^n}.$$

Noting that

$$\prod_{n=0}^{\infty}\left(\frac{\{\alpha^n\}^{1-\alpha}}{\alpha^\alpha}\right)^{\alpha^n} = \frac{\prod_{n=0}^{\infty}(\alpha^{1-\alpha})^{n\alpha^n}}{\prod_{n=0}^{\infty}(\alpha^\alpha)^{\alpha^n}} = \frac{\alpha^{(1-\alpha)\sum_{n=0}^{\infty} n\alpha^n}}{\alpha^{\alpha\sum_{n=0}^{\infty}\alpha^n}} = \frac{\alpha^{(1-\alpha)\frac{\alpha}{(1-\alpha)^2}}}{\alpha^{\alpha\frac{1}{1-\alpha}}} = 1$$

and defining $q(\omega) \equiv \prod_{n=0}^{\infty} z^n(\omega)^{\alpha^n}$, the chain feasibility condition becomes

$$c(\omega) \le q(\omega)\prod_{n=0}^{\infty}\left(\left\{\frac{1}{q_j}c(\omega)\right\}^{1-\alpha}\right)^{\alpha^n} = \frac{q(\omega)}{q_j}c(\omega). \tag{26}$$

Towards a contradiction, suppose that $\eta q_j = \sup_{\omega\in\Omega_j} q(\omega)$ with $\eta < 1$. Then (26) implies $c(\omega) \le \eta c(\omega)$. Summing across all chains $\omega \in \Omega_j$ gives $c_j \le \eta c_j$. $q_j > 0$ implies $c_j > 0$, and hence a contradiction. *Q.E.D.*

PROPOSITION 1—4 of 6: *In any countably-stable arrangement, for each* j,

$$q_j = \sup_{\omega\in\Omega_j} q(\omega).$$

PROOF: Towards a contradiction, suppose there is a j and a $\omega \in \Omega_j$ such that $q(\omega) > q_j$. For any integer $n \ge 0$, define $q^n(\omega) \equiv \prod_{k=0}^{\infty} z^{k+n}(\omega)^{\alpha^k}$ so that $q^0(\omega) = q(\omega)$ and $q^1(\omega)$ is the maximum feasible efficiency for the next to last entrepreneur in the chain ω, etc. Since the arrangement is pairwise stable, j's total spending on labor is $wl_j = \sum_{\phi \in U_j} wl(\phi) = (1-\alpha)\lambda_j y_j$.

We will show that there is a countable coalition with a dominating deviation. The deviation has two parts. Let $\eta \in (0,1)$. For the first part, j lowers its spending on labor used with each technique, so that $\tilde{l}(\phi) = \eta^{\frac{1}{1-\alpha}} l(\phi)$. This reduces j's spending on wages by $(1-\eta^{\frac{1}{1-\alpha}})(1-\alpha)\lambda_j y_j$ and reduces its output to ηy_j.

For the second part, the entire supply chain ω increases production to make up for j's lost output. The deviation will leave each of the suppliers in the supply chain equally well off, but j better off. For each integer $n \geq 0$, let $\tilde{x}^n(\omega) = x^n(\omega) + \frac{q^{n+1}(\omega)}{q(\omega)}\alpha^{n+1}(1-\eta)y_j$ and $\tilde{T}^n(\omega) = T^n(\omega) + [\tilde{x}^n(\omega) - x^n(\omega)]\frac{w}{q^{n+1}(\omega)}$. For $n \geq 1$, entrepreneur $\mathrm{j}^n(\omega)$ could attain the same payoff as before the deviation choosing $\tilde{l}^n(\omega) = l^n(\omega) + \alpha^n(1-\alpha)(1-\eta)y_j\frac{1}{q(\omega)}$, $\tilde{p}_{\mathrm{j}^n(\omega)} = p_{\mathrm{j}^n(\omega)}$, $\tilde{c}_{\mathrm{j}^n(\omega)} = c_{j^n(\omega)}$. Entrepreneur j, on the other hand, could increase labor by $(1-\alpha)(1-\eta)y_j\frac{1}{q(\omega)}$.

For j, the cost savings from the first part would be $(1-\eta^{\frac{1}{1-\alpha}})(1-\alpha)\lambda_j y_j$, while the increased spending from the second part would be $(1-\eta)y_j\frac{w}{q(\omega)}$ (which accounts for both the increased spending on labor and the payment to the supplier). The change in j's payoff is thus

$$\left(1-\eta^{\frac{1}{1-\alpha}}\right)(1-\alpha)\lambda_j y_j - (1-\eta)y_j\frac{w}{q(\omega)}.$$

Noting that $\lim_{\eta\to 1}\frac{(1-\alpha)(1-\eta^{\frac{1}{1-\alpha}})}{1-\eta} = 1$ and $\frac{q_j}{q(\omega)} < 1$, we have that the change in j's payoff is strictly positive for η close enough to 1. *Q.E.D.*

The next two claims study efficiency and uniqueness of allocations consistent with countable stability.

PROPOSITION 1—5 of 6: *Every countably-stable equilibrium is efficient.*

PROOF: Consider the problem of a planner that, taking the set of techniques Φ as given, makes production decisions and allocates labor to maximize the utility of the representative household. For each producer $j \in [0,1]$, the planner chooses the quantity of consumption, c_j. In addition, for each of j's upstream techniques $\phi \in U_j$, the planner chooses a quantity of labor, $l(\phi)$, and a quantity of good $s(\phi)$ for j to use as an intermediate input, $x(\phi)$.

Alternatively, we can formulate the planner's problem with the supply chain representation: For each good j and for each supply chain $\omega \in \Omega_j$, the planner chooses the labor, intermediate inputs, and output used at each step in those chains to produce consumption of good j. Following the logic of Section 2.2, if the planner used supply chain $\omega \in \Omega_j$ to produce good j for consumption, its indirect production function would be $c(\omega) = q(\omega)l$. Since the planner would choose to produce good j in the least costly way possible, it would use the most efficient supply chain, so that $c_j = q_j^{\text{planner}}\bar{l}_j$, where $q_j^{\text{planner}} = \sup_{\omega\in\Omega_j} q(\omega)$ and $\bar{l}_j$ is the total labor the planner uses across all steps in all chains to produce good j for the household. Thus the planner's problem can be restated as $\max_{C,\{c_j,\bar{l}_j\}_{j\in[0,1]}} C$ subject to $C = (\int_0^1 c_j^{\frac{\varepsilon-1}{\varepsilon}}\,dj)^{\frac{\varepsilon}{\varepsilon-1}}$ and $c_j \leq q_j^{\text{planner}}\bar{l}_j, \forall j$. This yields

$C = Q^{\text{planner}} L$, where $Q^{\text{planner}} = (\int_0^1 (q_j^{\text{planner}})^{\varepsilon-1}\, dj)^{\frac{1}{\varepsilon-1}}$. In any countably-stable equilibrium, $q_j = q_j^{\text{planner}}$, so it must be that $Q = Q^{\text{planner}}$ and the equilibrium is efficient. *Q.E.D.*

Given Φ, there may be multiple allocations consistent with countable stability. Let $J^{\text{unique}}(\Phi)$ be the set of entrepreneurs for whom all production variables (i.e., $\{x(\phi), l(\phi), y(\phi)\}_{\phi\in U_j}$, $\{x(\phi)\}_{\phi\in D_j}$, p_j, c_j) are the same across all countably-stable equilibria. The following proposition shows that, generically, almost all entrepreneurs are in the set $J^{\text{unique}}(\Phi)$.

PROPOSITION 1—6 of 6: *Suppose H is atomless. Then, with probability* 1, *Φ is such that $J^{\text{unique}}(\Phi)$ has unit measure.*

PROOF: We first show that the probability that an entrepreneur has two techniques that deliver the same efficiency is zero. This follows from the fact that, given each potential supplier's efficiency, the efficiency delivered by the technique is $z(\phi)q^{\alpha}_{s(\phi)}$. Since H is atomless, the probability that any finite set of techniques has two that deliver the same efficiency is zero. Second, the probability that entrepreneur j or any entrepreneur downstream from j has two techniques that deliver the same efficiency is zero. To see this, note that since the number of downstream techniques is countable, they can be ordered. For any N, the probability that any of the first N entrepreneurs have two techniques that deliver the same efficiency is 0^N. Thus the probability that any downstream entrepreneur has two such techniques is $\lim_{N\to\infty} 0^N = 0$. Third, if no entrepreneurs downstream from j have two techniques that deliver the same efficiency, then $j \in J^{\text{unique}}(\Phi)$. To see this, consider some entrepreneur $\tilde{j}$ that is downstream from j. $\tilde{j}$ sells $c_{\tilde{j}} = q_{\tilde{j}}^{\varepsilon-1} Q^{\varepsilon} C$ to the household for consumption. If j is the nth supplier in $\tilde{j}$'s best supply chain, then, in every countably-stable equilibrium, j produces $\frac{1}{\lambda_j}\alpha^n \lambda_{\tilde{j}} c_{\tilde{j}}$ to be used in the supply chain to produce $\tilde{j}$ for consumption, using $\frac{1}{w}(1-\alpha)\alpha^n \lambda_{\tilde{j}} c_{\tilde{j}}$ units of labor and $\frac{1}{\lambda_s}\alpha^{n+1}\lambda_{\tilde{j}} c_{\tilde{j}}$ units of intermediate inputs (where λ_s is the marginal cost of the supplier used by j). And, of course, if j is not in $\tilde{j}$'s best supply chain, all of these quantities are zero. Total labor, intermediate inputs, and output of entrepreneur j is simply the sum of these quantities over all entrepreneurs weakly downstream from j. *Q.E.D.*

A.4. *Payoffs*

To simplify further derivations, we can separate the payment for any technique into two parts. Given the arrangement and individual choices, we can define

$$\tau(\phi) \equiv T(\phi) - \lambda_{s(\phi)} x(\phi),$$

$$\pi_j \equiv (p_j - \lambda_j) c_j = \frac{PC}{\varepsilon}\left(\frac{q_j}{Q}\right)^{\varepsilon-1} = \frac{1}{\varepsilon-1}\left(\frac{q_j}{Q}\right)^{\varepsilon-1} wL.$$

$\tau(\phi)$ is the value of the payment above the value of the intermediate inputs when those inputs are priced at the supplier's marginal cost. π_j is the profit from sales of good j to the household when good j is valued at j's marginal cost.

LEMMA 8: *In any pairwise stable arrangement, entrepreneur j's profit equals*

$$\Pi_j = \pi_j + \sum_{\phi\in D_j} \tau(\phi) - \sum_{\phi\in U_j} \tau(\phi).$$

PROOF: j's profit can be written as

$$\begin{aligned}
\Pi_j &= (p_j - \lambda_j)c_j + \lambda_j c_j - wl_j + \sum_{\phi\in D_j} T(\phi) - \sum_{\phi\in U_j} T(\phi) \\
&= (p_j - \lambda_j)c_j + \lambda_j\left(y_j - \sum_{\phi\in D_j} x(\phi)\right) - wl_j + \sum_{\phi\in D_j} T(\phi) - \sum_{\phi\in U_j} T(\phi) \\
&= \pi_j + \lambda_j y_j - wl_j - \sum_{\phi\in U_j} \lambda_{s(\phi)} x(\phi) + \sum_{\phi\in D_j} \tau(\phi) - \sum_{\phi\in U_j} \tau(\phi).
\end{aligned}$$

The conclusion follows from (23) and (24). *Q.E.D.*

Recall that J^* is the set of acyclic entrepreneurs and Φ^* is the set of techniques for which the buyer and supplier are members of J^*.

PROPOSITION 2—1 of 3: *In any countably-stable equilibrium, for any* $\phi \in \Phi^*$, $\tau(\phi) \leq \sum_{j\in\mathcal{B}(\phi)}(\pi_j - \pi_{j\setminus\phi})$.

PROOF: Towards a contradiction, suppose there is a ϕ such that $\tau(\phi) > \sum_{j\in\mathcal{B}(\phi)}(\pi_j - \pi_{j\setminus\phi}) + \eta$ with $\eta > 0$. Then there is a profitable deviation in which $b(\phi)$ drops technique ϕ and each entrepreneur downstream from $s(\phi)$ increases production within her best alternative supply chain that does not pass through ϕ. The set $\mathcal{B}(\phi)$—entrepreneurs with chains that go through ϕ—is countable. Label these entrepreneurs by the natural number k. Each such entrepreneur has a supply chain that does not pass through ϕ that would deliver efficiency marginal cost $\tilde{\lambda}_{j^k}$ which is arbitrarily close to $\lambda_{j^k\setminus\phi}$, that satisfies $\pi_{j^k\setminus\phi} - \frac{1}{\varepsilon}PC(\frac{w/\tilde{\lambda}_{j^k}}{Q})^{\varepsilon-1} \leq \frac{\eta}{2^k}$. In the deviation, each entrepreneur in $\mathcal{B}(\phi)$ reduces production using the chain that passes through ϕ to zero, and increases production in her alternative chain in order to generate profit from sales to the household of $\frac{1}{\varepsilon}PC(\frac{w/\tilde{\lambda}_{j^k}}{Q})^{\varepsilon-1}$. The deviation within each chain is the same as described in the proof that $q_j = \sup_{\omega\in\Omega_j} q(\omega)$. Among entrepreneurs in the deviation, the change in profit includes the recovery of the payment $\tau(\phi)$ minus the loss in profit from sales to the household, which is bounded below by

$$\begin{aligned}
&\tau(\phi) - \sum_{k=0}^{\infty}\left\{\pi_{j^k} - \frac{1}{\varepsilon}PC\left(\frac{w/\tilde{\lambda}_{j^k}}{Q}\right)^{\varepsilon-1}\right\} \\
&\quad = \tau(\phi) - \sum_{k=1}^{\infty}(\pi_{j^k} - \pi_{j^k\setminus\phi}) - \sum_{k=1}^{\infty}\left(\pi_{j^k\setminus\phi} - \frac{1}{\varepsilon}PC\left(\frac{w/\tilde{\lambda}_{j^k}}{Q}\right)^{\varepsilon-1}\right) \\
&\quad \geq \tau(\phi) - \sum_{k=1}^{\infty}(\pi_{j^k} - \pi_{j^k\setminus\phi}) - \sum_{k=1}^{\infty}\frac{\eta}{2^k} \\
&\quad > 0.
\end{aligned}$$

The entire deviation involves a countable set of entrepreneurs because a countable union of countable sets is countable. *Q.E.D.*

PROPOSITION 2—2 of 3: *In any countably-stable equilibrium, for any* $\phi \in \Phi^*$, $\tau(\phi) \geq 0$.

PROOF: If $T(\phi) < \lambda_{s(\phi)} x(\phi)$, then the supplier would gain by dropping the contract and reducing production throughout its supply chain. The cost savings to the chain would be $\lambda_{s(\phi)} x(\phi)$, which is larger than the payment $T(\phi)$ from $b(\phi)$. *Q.E.D.*

A.5. *Existence and Bargaining Power*

For any coalition J, let $\mathcal{U}(J)$ and $\mathcal{D}(J)$ be the set of techniques that are respectively directly upstream and directly downstream from members of J, so that $\mathcal{U}(J) = \{\phi | b(\phi) \in J, s(\phi) \notin J\}$ and $\mathcal{D}(J) = \{\phi | s(\phi) \in J, b(\phi) \notin J\}$. The sum of the payoffs to members of J is

$$\sum_{j \in J} \pi_j - \sum_{\phi \in \mathcal{U}(J)} \tau(\phi) + \sum_{\phi \in \mathcal{D}(J)} \tau(\phi). \tag{27}$$

If a coalition J deviates, there is a subset of techniques in $\mathcal{U}(J)$ that are dropped. Let $\mathcal{U}^-(J)$ be the subset of techniques that are dropped and, abusing notation, let $s(\mathcal{U}^-(J))$ be entrepreneurs that are the suppliers of those techniques. We also define $\lambda_{j \setminus s(\mathcal{U}^-(J))}$ and $\pi_{j \setminus s(\mathcal{U}^-(J))}$ to be what j's marginal cost and profit from sales to the household would be if it were unable to use chains that passed through any of those entrepreneurs in $s(\mathcal{U}^-(J))$.

LEMMA 9: *A contracting arrangement that generates a feasible allocation and satisfies* $q_j = \sup_{\omega \in \Omega_j} q(\omega)$ *for each* j *is countably-stable if and only if, for each coalition* J *and each subset of upstream techniques* $\mathcal{U}^-(J)$, *the following equation holds*:

$$\sum_{j \in J} \pi_j - \pi_{j \setminus s(\mathcal{U}^-(J))} + \sum_{\phi \in \mathcal{D}(J)} \min\bigl\{\tau(\phi), -(\lambda_{s(\phi)} - \lambda_{s(\phi) \setminus s(\mathcal{U}^-(J))}) x(\phi)\bigr\} \geq \sum_{\phi \in \mathcal{U}^-(J)} \tau(\phi). \tag{28}$$

PROOF: Consider an arrangement such that the resulting allocation is feasible and $q_j = \sup_{\omega \in \Omega_j} q(\omega)$ for all j.

Suppose first that the resulting allocation is countably-stable. Consider a coalition J and suppose there is a deviation where $\mathcal{U}^-(J)$ are the upstream contracts that are dropped. For any technique downstream from the coalition ($\phi \in \mathcal{D}(J)$), the coalition could either drop the contract or continue to supply those inputs. Following the deviation, the sum of the payoffs to the members of J is no better than

$$\sum_{j \in J} \pi_{j \setminus s(\mathcal{U}^-(J))} - \sum_{\phi \in \mathcal{U}(J) \setminus \mathcal{U}^-(J)} \tau(\phi) + \sum_{\phi \in \mathcal{D}(J)} \max\bigl\{0, T(\phi) - \lambda_{s(\phi) \setminus s(\mathcal{U}^-(J))} x(\phi)\bigr\}. \tag{29}$$

Further, there is a larger coalition (the union of the coalition J and those entrepreneurs involved in the alternative supply chains) with a deviation that would attain payoffs (29) for those in J and leave the other entrepreneurs in that larger deviation no worse off. Countable stability means that (27) is weakly greater than (29), which implies (28).

Next suppose (28) holds for all J and $\mathcal{U}^-(J)$. Then, following any deviation by any coalition J that drops techniques $\mathcal{U}^-(J)$, the payoff following the deviation is no greater than (29). Since (28) implies that this is weakly less than (29), the deviation cannot dominate. *Q.E.D.*

LEMMA 10: *In any contracting arrangement that generates a feasible allocation that satisfies $q_j = \sup_{\omega\in\Omega_j} q(\omega)$ for all j, for any ϕ, $\phi' \in \Phi^*$ such that ϕ' is downstream of ϕ,*

$$-[\lambda_{s(\phi')} - \lambda_{s(\phi')\setminus\phi}]x(\phi') \geq \sum_{j\in\mathcal{B}(\phi')} (\pi_j - \pi_{j\setminus s(\phi)}). \tag{30}$$

PROOF: Suppose first that $\lambda_{s(\phi')} = \lambda_{s(\phi')\setminus\phi}$. Then it must be that for each $j \in \mathcal{B}(\phi')$ that $\pi_j = \pi_{j\setminus s(\phi)}$, which implies (30).

Suppose instead that $\lambda_{s(\phi')} < \lambda_{s(\phi')\setminus\phi}$. Consider a supply chain representation of the allocation. For any $j \in \mathcal{B}(\phi')$, there is the subset of chains Ω_j that pass through ϕ' which we label $\Omega_j(\phi')$, and a k_j such that ϕ' is the k_jth technique in every $\omega \in \Omega_j(\phi')$. Then

$$x(\phi') = \sum_{j\in\mathcal{B}(\phi')} \sum_{\omega\in\Omega_j(\phi')} x^{k_j}(\omega). \tag{31}$$

If $\sum_{\omega\in\Omega_j(\phi')} c(\omega) < c_j$, then it must be that $\pi_j = \pi_{j\setminus s(\phi')}$, which implies $\pi_j = \pi_{j\setminus s(\phi)}$. If, on the other hand, $\sum_{\omega\in\Omega_j(\phi')} c(\omega) = c_j$, then

$$\pi_j = p_j c_j - \sum_{\omega\in\Omega_j(\phi')} \left[w \sum_{n=0}^{k_j} l^n(\omega) - \lambda_{s(\phi')} x^{k_j}(\omega) \right],$$

because $\lambda_j c(\omega) = wl^0(\omega) + \lambda_{s(\phi^0(\omega))} x^0(\omega) = wl^0(\omega) + wl^1(\omega) + \lambda_{s(\phi^1(\omega))} x^1(\omega) = \cdots$. We can also define $\tilde{\pi}_j$ to be

$$\tilde{\pi}_j \equiv p_j c_j - \sum_{\omega\in\Omega_j(\phi')} \left[w \sum_{n=0}^{k_j} l^n(\omega) - \lambda_{s(\phi')\setminus\phi} x^{k_j}(\omega) \right].$$

Note first that $\pi_j - \tilde{\pi}_j = \sum_{\omega\in\Omega_j(\phi')} -[\lambda_s - \lambda_{s(\phi')\setminus\phi}] x^{k_j}(\omega_j)$. Note second that $\tilde{\pi}_j \leq \pi_{j\setminus s(\phi)}$, because if j could not use chains that passed through ϕ, it could reoptimize its use of labor or choose alternative chains. Together, these imply that

$$\pi_j - \pi_{j\setminus s(\phi)} \leq \sum_{\omega\in\Omega_j(\phi')} -[\lambda_s - \lambda_{s(\phi')\setminus\phi}] x^{k_j}(\omega_j).$$

Summing over $j \in \mathcal{B}(\phi')$ and using (31) gives

$$\begin{aligned} -[\lambda_{s(\phi')} - \lambda_{s(\phi')\setminus\phi}]x(\phi') &= \sum_{j\in\mathcal{B}(\phi')} \sum_{\omega\in\Omega_j(\phi')} -[\lambda_{s(\phi')} - \lambda_{s(\phi')\setminus\phi}] x^n(\omega) \\ &\geq \sum_{j\in\mathcal{B}(\phi')} \pi_j - \pi_{j\setminus s(\phi)}. \end{aligned}$$

Q.E.D.

LEMMA 11: *Suppose there is a coalition in J^* with a dominating deviation. Then there is a dominating deviation in which at most a single technique is dropped.*

PROOF: Suppose first that there is a technique $\phi \in \Phi^*$ such that $\tau(\phi) < 0$. Then by the argument of Proposition 2(2), there is a dominating deviation in which no suppliers are dropped. Suppose instead that $\tau(\phi) \geq 0$ for all $\phi \in \Phi^*$. Toward a contradiction, suppose that there is no dominating deviation in which a single technique is dropped. We will show that this implies there is no dominating deviation in which multiple suppliers are dropped. If there are no deviations in which a single contract is dropped, it must be that, for each $\tilde{\phi} \in \mathcal{U}^-(J)$,

$$\sum_{j\in J} \pi_j - \pi_{j\setminus\tilde{\phi}} + \sum_{\phi\in\mathcal{D}(J)} \min\{\tau(\phi), -(\lambda_{s(\phi)} - \lambda_{s(\phi)\setminus\tilde{\phi}})x(\phi)\} \geq \tau(\tilde{\phi}).$$

Summing across $\tilde{\phi} \in \mathcal{U}^-(J)$,

$$\sum_{\tilde{\phi}\in\mathcal{U}^-(J)} \sum_{j\in J} \pi_j - \pi_{j\setminus\tilde{\phi}} + \sum_{\tilde{\phi}\in\mathcal{U}^-(J)} \sum_{\phi\in\mathcal{D}(J)} \min\{\tau(\phi), -(\lambda_{s(\phi)} - \lambda_{s(\phi)\setminus\tilde{\phi}})x(\phi)\} \geq \sum_{\tilde{\phi}\in\mathcal{U}^-(J)} \tau(\phi). \quad (32)$$

Consider any $j \in J$. Define $\hat{\phi}_j \in \arg\max_{\tilde{\phi}\in\mathcal{U}^-(J)} \lambda_{j\setminus\tilde{\phi}}$. That is, of all the techniques in $\mathcal{U}^-(J)$, if j could not use chains that passed through $\hat{\phi}_j$, its marginal cost would rise the most (if there are multiple such techniques, select one at random). Note that since $j \in J^*$ and J is connected set, any supply chain that goes through $\hat{\phi}_j$ does not go through any other technique in $\mathcal{U}^-(J)$. Thus, while $\pi_j > \pi_{j\setminus\hat{\phi}_j}$ and $\lambda_{j\setminus\hat{\phi}_j} \geq \lambda_j$, for any other technique $\phi' \in \mathcal{U}^-(J) \setminus \{\hat{\phi}_j\}$, $\pi_j = \pi_{j\setminus\phi'}$ and $\lambda_{j\setminus\phi'} = \lambda_j$. Therefore, summing across all techniques in $\mathcal{U}^-(J)$ gives two relationships. For each $j \in J$,

$$\sum_{\tilde{\phi}\in\mathcal{U}^-(J)} \pi_j - \pi_{j\setminus\tilde{\phi}} = \pi_j - \pi_{j\setminus\hat{\phi}_j} \leq \pi_j - \pi_{j\setminus\mathcal{U}^-(J)}, \quad (33)$$

and for each $\phi \in \mathcal{D}(J)$,

$$\begin{aligned}\sum_{\tilde{\phi}\in\mathcal{U}^-(J)} \min\{\tau(\phi), -(\lambda_{s(\phi)} - \lambda_{s(\phi)\setminus\tilde{\phi}})x(\phi)\} &= \min\{\tau(\phi), -(\lambda_{s(\phi)} - \lambda_{s(\phi)\setminus\hat{\phi}_{s(\phi)}})x(\phi)\} \\ &\leq \min\{\tau(\phi), -(\lambda_{s(\phi)} - \lambda_{s(\phi)\setminus\mathcal{U}^-(J)})x(\phi)\}, \quad (34)\end{aligned}$$

where, in each case, the inequality follows because $\pi_{j\setminus\hat{\phi}_j} \geq \pi_{j\setminus\mathcal{U}^-(J)}$ and $\lambda_{s(\phi)\setminus\hat{\phi}_{s(\phi)}} \leq \lambda_{s(\phi)\setminus\mathcal{U}^-(J)}$, respectively; the latter is more constrained. Summing (33) across $j \in J$ and (34) across $\phi \in \mathcal{D}(J)$, and then reversing the order of each summation, gives

$$\sum_{\tilde{\phi}\in\mathcal{U}^-(J)} \sum_{j\in J} \pi_j - \pi_{j\setminus\tilde{\phi}} \leq \sum_{j\in J} \pi_j - \pi_{j\setminus\mathcal{U}^-(J)}, \quad (35)$$

$$\begin{aligned}&\sum_{\tilde{\phi}\in\mathcal{U}^-(J)} \sum_{\phi\in\mathcal{D}(J)} \min\{\tau(\phi), -(\lambda_{s(\phi)} - \lambda_{s(\phi)\setminus\tilde{\phi}})x(\phi)\} \\ &\quad\leq \sum_{\phi\in D(J)} \min\{\tau(\phi), -(\lambda_{s(\phi)} - \lambda_{s(\phi)\setminus U^-(J)})x(\phi)\}.\end{aligned} \quad (36)$$

These and (32) imply (28) so that there is no dominating deviation for J. *Q.E.D.*

PROPOSITION 2—3 of 3: *For any $\beta \in [0, 1]$, there exists a countably-stable equilibrium in which $\tau(\phi) = \beta\mathcal{S}(\phi), \forall\phi \in \Phi^*$ and $\tau(\phi) = 0, \forall\phi \notin \Phi^*$.*

PROOF: For each coalition J such that no member of J is in J^*, $\tau(\phi)=0$ implies that (28) holds. Consider now a deviation by a coalition $J \subset J^*$ in which at most one technique is dropped. Divide J into two disjoint groups: J_1 are those that are downstream from the technique that is dropped, or the empty set if no technique is dropped, and J_2 are those that are not downstream from the technique that is dropped (or the empty set if all of $J_1 = J$). Similarly, divide $\mathcal{D}(J)$ into $\mathcal{D}_1$ and $\mathcal{D}_2$, those techniques in $\mathcal{D}(J)$ downstream from entrepreneurs in J_1 and J_2, respectively.

For each technique $\phi' \in \mathcal{D}_1$, we have from (30) that if ϕ is the technique that is dropped,

$$-[\lambda_{s(\phi')} - \lambda_{s(\phi')\setminus\phi}]x(\phi') \geq \sum_{j\in\mathcal{B}(\phi')} \pi_j - \pi_{j\setminus s(\phi)} \geq \beta \sum_{j\in\mathcal{B}(\phi')} \pi_j - \pi_{j\setminus s(\phi)}.$$

We also know that for each $\phi' \in \mathcal{D}_1$, we have that

$$\beta \sum_{j\in\mathcal{B}(\phi')} \pi_j - \pi_{j\setminus s(\phi)} \leq \beta \sum_{j\in\mathcal{B}(\phi')} \pi_j - \pi_{j\setminus s(\phi')} = \tau(\phi').$$

Together, these imply that for each $\phi' \in \mathcal{D}_1$,

$$\min\{\tau(\phi'), -[\lambda_{s(\phi')} - \lambda_{s(\phi')\setminus\phi}]x(\phi')\} \geq \beta \sum_{j\in\mathcal{B}(\phi')} \pi_j - \pi_{j\setminus s(\phi)}. \tag{37}$$

Equation (37) also holds for each $\phi' \in \mathcal{D}_2$ because $\pi_j = \pi_{j\setminus s(\phi)}$ for each $j \in \mathcal{B}(\phi')$. Putting these pieces together, we have that

$$\begin{aligned}\tau(\phi) &= \beta \sum_{j\in J} \pi_j - \pi_{j\setminus\phi} + \beta \sum_{\phi'\in\mathcal{D}(J)} \sum_{j\in\mathcal{B}(\phi')} \pi_j - \pi_{j\setminus\phi} \\ &\leq \sum_{j\in J} \pi_j - \pi_{j\setminus\phi} + \sum_{\phi'\in\mathcal{D}(J)} \min\{\tau(\phi'), -[\lambda_{s(\phi')} - \lambda_{s(\phi)\setminus\phi}]x(\phi')\}.\end{aligned}$$

This is equivalent to (28) for the case in which $\mathcal{U}(J)$ is a singleton, so there is no coalition $J \in J^*$ with a dominating deviation. *Q.E.D.*

APPENDIX B: PROOF OF PROPOSITION 3

The strategy begins with defining a sequence of random variables $\{X_N\}_{N=0}^{\infty}$ with the property that the maximum feasible efficiency of an entrepreneur is given by the limit of this sequence, if such a limit exists. We then show that X_N converges to a random variable X^* in $L^{\varepsilon-1}$. Next we show that the CDF of X^* is the unique fixed point of T in $\mathcal{F}$, a subset of $\bar{\mathcal{F}}$ (and that such a fixed point exists). Letting F^* be this fixed point, the law of large numbers implies the CDF of the cross-sectional distribution of efficiencies is F^* and that aggregate productivity is $\|X^*\|_{\varepsilon-1}$.

B.1. *Existence of a Fixed Point of Equation (12)*

We begin by defining three functions, $\bar{f}$, f^1, and $\underline{f}$, in $\bar{\mathcal{F}}$. To do so, we define several objects that will parameterize these functions. Let $\rho \in (0,1]$ be the smallest root of $\rho = e^{-M(1-\rho)}$. In the definition of $\bar{f}$, let $\beta > \varepsilon - 1$ be such that $\lim_{z\to\infty} z^{\beta}[1-H(z)] = 0$. Then there exists a $z_2 > 1$ such that $z > z_2$ implies $z^{\beta}[1-H(z)] < (1-\alpha)$. With this, let q_2 be a number large enough so that $q_2^{(1-\alpha)\beta} > M(z_2^{\beta}+1)$ and $q_2^{1-\alpha} > z_2$. In the definition of $\underline{f}$, $q_0 = z_0^{\frac{1}{1-\alpha}}$:

$$\bar{f}(q) \equiv \begin{cases} \rho, & q < q_2, \\ 1 - (1-\rho)\left(\dfrac{q}{q_2}\right)^{-\beta}, & q \geq q_2, \end{cases}$$

$$f^1(q) \equiv \begin{cases} \rho, & q < 1, \\ 1, & q \geq 1, \end{cases}$$

$$\underline{f}(q) \equiv \begin{cases} \rho, & q < q_0, \\ 1, & q \geq q_0. \end{cases}$$

On $\bar{\mathcal{F}}$, the set of right-continuous, weakly increasing functions $f : \mathbb{R}^+ \to [0, 1]$, consider the partial order given by the binary relation $\preceq$: $f_1 \preceq f_2 \Leftrightarrow f_1(q) \leq f_2(q), \forall q \geq 0$. Clearly, $\bar{f} \preceq f^1 \preceq \underline{f}$. Let $\mathcal{F} \subset \bar{\mathcal{F}}$ be the subset of set of right-continuous, non-decreasing functions $f : \mathbb{R}^+ \to [0, 1]$ that satisfy $\bar{f} \preceq f \preceq \underline{f}$.

LEMMA 12: *$T\underline{f} \preceq \underline{f}$ and $\bar{f} \preceq T\bar{f}$.*

PROOF: We first show $T\underline{f} \preceq \underline{f}$. For $q \geq q_0$, $T\underline{f}(q) \leq 1 = \underline{f}(q)$. For $q < q_0$,

$$T\underline{f}(q) = e^{-M\int_0^\infty [1-\underline{f}((q/z)^{1/\alpha})]\,dH(z)} = e^{-M\int_{q/q_0^\alpha}^\infty [1-\rho]\,dH(z)} \leq e^{-M[1-\rho](1-H(q_0^{1-\alpha}))} = \rho = \underline{f}(q).$$

We proceed to $\bar{f}$. First, for $q < q_2$, we have $T\bar{f}(q) = e^{-M\int_0^\infty (1-\bar{f})\,dH(z)} \geq e^{-M(1-\rho)} = \rho = f(q)$. Next, as an intermediate step, we will show that, for $q \geq q_2$,

$$\int_{z_0}^{q/q_2^\alpha} z^{\frac{\beta}{\alpha}}\,dH(z) + \left(q/q_2^\alpha\right)^{\frac{\beta}{\alpha}}\left[1 - H\left(q/q_2^\alpha\right)\right] < \left(z_2^\beta + 1\right)\left(q/q_2^\alpha\right)^{\frac{1-\alpha}{\alpha}\beta}. \tag{37}$$

To see this, note that we can integrate by parts to get

$$\int_{z_2}^{q/q_2^\alpha} z^{\beta/\alpha}\,dH(z) = \left[1 - H(z_2)\right]z_2^{\beta/\alpha} - \left(q/q_2^\alpha\right)^{\beta/\alpha}\left[1 - H\left(q/q_2^\alpha\right)\right] + \int_{z_2}^{q/q_2^\alpha} \frac{\beta}{\alpha} z^{\beta/\alpha-1}\left[1 - H(z)\right]dz.$$

Rearranging this gives

$$H(z_2)z_2^{\beta/\alpha} + \int_{z_2}^{q/q_2^\alpha} z^{\beta/\alpha}\,dH(z) + \left(q/q_2^\alpha\right)^{\beta/\alpha}\left[1 - H\left(q/q_2^\alpha\right)\right] = z_2^{\beta/\alpha} + \frac{\beta}{\alpha}\int_{z_2}^{q/q_2^\alpha} z^{\beta/\alpha-1}\left[1 - H(z)\right]dz.$$

Since $q/q_2^\alpha > z_2$, equation (37) follows from this and three inequalities:

(i) $\int_{z_0}^{z_2} z^{\beta/\alpha}\,dH(z) \leq H(z_2)z_2^{\beta/\alpha}$;

(ii) $z_2^{\beta/\alpha} \leq z_2^\beta (q/q_2^\alpha)^{\beta/\alpha-\beta}$; and

(iii) $\int_{z_2}^{q/q_2^\alpha} z^{\beta/\alpha-1}[1 - H(z)]\,dz \leq \int_0^{q/q_2^\alpha} z^{\beta/\alpha-1}[(1-\alpha)z^{-\beta}]\,dz$.

Next, beginning with $1 - T\bar{f}(q) \le -\ln T\bar{f}(q)$, we have

$$\begin{aligned}
\frac{1 - T\bar{f}(q)}{1-\rho} &\le M \int_{z_0}^{\infty} \frac{1 - \bar{f}\big((q/z)^{1/\alpha}\big)}{1-\rho}\, dH(z) \\
&= M \int_{z_0}^{q/q_2^{\alpha}} \left(\frac{(q/z)^{1/\alpha}}{q_2}\right)^{-\beta} dH(z) + M\big[1 - H\big(q/q_2^{\alpha}\big)\big] \\
&= \left(\frac{q}{q_2}\right)^{-\beta} \frac{M\big(z_2^{\beta}+1\big)}{q_2^{(1-\alpha)\beta}} \left\{ \frac{\int_{z_0}^{q/q_2^{\alpha}} z^{\frac{\beta}{\alpha}}\, dH(z) + \big(q/q_2^{\alpha}\big)^{\frac{\beta}{\alpha}}\big[1 - H\big(q/q_2^{\alpha}\big)\big]}{\big(z_2^{\beta}+1\big)\big(q/q_2^{\alpha}\big)^{\frac{1-\alpha}{\alpha}\beta}} \right\} \\
&\le \left(\frac{q}{q_2}\right)^{-\beta} \\
&= \frac{1 - \bar{f}(q)}{1-\rho}.
\end{aligned}$$

This then gives, for $q \ge q_2$, $T\bar{f}(q) \ge \bar{f}(q)$. *Q.E.D.*

LEMMA 13: *There exist least and greatest fixed points of the operator T in $\mathcal{F}$, given by* $\lim_{N\to\infty} T^N \bar{f}$ *and* $\lim_{N\to\infty} T^N \underline{f}$, *respectively.*

PROOF: The operator T is order preserving, and $\mathcal{F}$ is a complete lattice. By the Tarski fixed point theorem, the set of fixed points of T in $\mathcal{F}$ is also a complete lattice, and hence has a least and a greatest fixed point given by $\lim_{N\to\infty} T^N \bar{f}$ and $\lim_{N\to\infty} T^N \underline{f}$, respectively. *Q.E.D.*

B.2. *Existence of a Limit*

For any chain $\omega \in \Omega_j$, define $\mathfrak{q}_N(\omega) \equiv \prod_{n=0}^{N-1} z^n(\omega)^{\alpha^n}$ for $N \ge 1$ and $\mathfrak{q}_0(\omega) \equiv 1$. For any chain in Ω_j, a subchain of length N is the segment of techniques of length N that is most downstream. Let $\Omega_{j,N}$ be the set of *distinct* subchains of length N. I will use ω to denote both a supply chain and a subchain; the usage will be clear from context.

With these, we will define three sequences of random variables for each entrepreneur, $\{X_{j,N}, \bar{Y}_{j,N}, \underline{Y}_{j,N}\}$ so that their respective CDFs are $T^N f^1$, $T^N \bar{f}$, and $T^N \underline{f}$. The construction is guided by the following lemma.

LEMMA 14: *Given Φ, define the random variables $\{\hat{q}(\omega)\}_{\forall \omega \in \bigcup_{N=0}^{\infty} \Omega_{j,N}, \forall j}$ to be i.i.d. random variables with CDF $\frac{\hat{f}-\rho}{1-\rho}$. For each j, N, and for each $\omega \in \Omega_{j,N}$, let $\hat{\mathfrak{q}}_N(\omega) = \mathfrak{q}_N(\omega)\hat{q}(\omega)^{\alpha^N}$. Let $\hat{Y}_{j,N} = \max_{\omega \in \Omega_{j,N}} \hat{\mathfrak{q}}_N(\omega)$. Then the CDF of $\hat{Y}_{j,N}$ is $T^N \hat{f}$.*

PROOF: We proceed by induction. We first derive an expression for $\Pr(\hat{Y}_{j,1} \le q)$. Consider a single technique ϕ in U_j. A standard result from the theory of branching processes is that the probability $s(\phi)$ has at least one supply chain is $1-\rho$ (see Appendix B.4 for a derivation). If the supplier does have a supply chain, the subchain consisting of the technique ϕ is in the set $\Omega_{j,N}$. In that case, the probability that $z(\phi)\hat{q}(\omega)^{\alpha} \le q$ is

$\int \frac{\hat{f}((q/z)^{1/\alpha})-\rho}{1-\rho} \, dH(z)$. Thus the probability that ϕ is either not part of a subchain or that $z(\phi)\hat{q}(\omega)^\alpha \leq q$ is

$$\rho + (1-\rho)\int \frac{\hat{f}\big((q/z)^{1/\alpha}\big)-\rho}{1-\rho} \, dH(z) = \int \hat{f}\big((q/z)^{1/\alpha}\big) \, dH(z).$$

Since $\hat{Y}_{j,1} = \max_{\omega\in\Omega_{j,1}} \hat{\mathfrak{q}}_1(\omega)$, summing over the possible realizations of U_j gives

$$\begin{aligned}\Pr(\hat{Y}_{j,1} \leq q) &= \sum_{k=0}^{\infty} \frac{e^{-M}M^k}{k!}\left[\int \hat{f}\big((q/z)^{1/\alpha}\big) \, dH(z)\right]^k \\ &= e^{-M[1-\int \hat{f}((q/z)^{1/\alpha}) \, dH(z)]} = T\hat{f}.\end{aligned}$$

Now suppose the CDF of $Y_{s(\phi),N}$ is $T^N\hat{f}$. Using the logic behind equation (12), for any $\phi \in U_j$, $\Pr(z(\phi)\hat{Y}^\alpha_{s(\phi),N} \leq q) = \int T^N\hat{f}((q/z)^{1/\alpha}) \, dH(z)$, so that integrating over realizations of U_j gives

$$\Pr(\hat{Y}_{j,N+1} \leq q) = \Pr\Big(\max_{\phi\in U_j} z(\phi)\hat{Y}^\alpha_{s(\phi),N} \leq q\Big) = e^{-M[1-\int T^N\hat{f}((q/z)^{1/\alpha}) \, dH(z)]} = T^{N+1}\hat{f}.$$

Q.E.D.

If Ω_j is empty, then define $X_{j,N} = \bar{Y}_{j,N} = \underline{Y}_{j,N} = 0$ for all $N \geq 0$. If Ω_j is non-empty, we define $X_{j,N} = \sup_{\omega\in\Omega_j} \mathfrak{q}_N(\omega)$. Roughly, the remainder of this subsection shows that $X_{j,N}$ converges to q_j. Since $q_j = \sup_{\omega\in\Omega_j} \lim_{N\to\infty} \mathfrak{q}_N(\omega)$, we are essentially proving that the limit can be passed through the sup.

One consequence of Lemma 14 is that the CDF of $X_{j,N}$ is $T^N f^1$ (the variables $\hat{q}(\omega)$ are all degenerate and equal to 1). To construct $\bar{Y}_{j,N}$, given a realization of Φ, let $\{\bar{q}(\omega)\}_{\forall\omega\in\bigcup_{N=0}^{\infty}\Omega_{j,N},\forall j}$ be i.i.d. random variables, each with CDF $\frac{\bar{f}-\rho}{1-\rho}$. With this, we define $\bar{\mathfrak{q}}_N(\omega) \equiv \mathfrak{q}_N(\omega)\bar{q}(\omega)^{\alpha^N}$ and $\underline{\mathfrak{q}}_N(\omega) \equiv \mathfrak{q}_N(\omega)q_0^{\alpha^N}$ (q_0 is the same as a random variable with CDF $\frac{\underline{f}-\rho}{1-\rho}$). Last, for $N \geq 1$, let $\bar{Y}_{j,N} \equiv \max_{\omega\in\Omega_{j,N}} \bar{\mathfrak{q}}_N(\omega)$ and $\underline{Y}_{j,N} \equiv \max_{\omega\in\Omega_{j,N}} \underline{\mathfrak{q}}_N(\omega)$. Also let $X_{j,0} = 1$, $\underline{Y}_{j,0} = q_0$, and $\bar{Y}_{j,0}$ to have CDF $\frac{\bar{f}-\rho}{1-\rho}$.

To improve readability, the argument j will be suppressed when not necessary.

LEMMA 15: *For each $N \geq 0$, X_N, $\bar{Y}_N$, and $\underline{Y}_N$ are uniformly integrable in $L^{\varepsilon-1}$.*

PROOF: First, recall that $\bar{Y}_0$ is defined so that its CDF is $\bar{f}$. Since T is order preserving, the relations $T^N\underline{f} \succeq T^Nf^1 \succeq T^N\bar{f}$ and $T^N\bar{f} \succeq T^{N-1}\bar{f}$ imply that $T^N\underline{f} \succeq T^Nf^1 \succeq \bar{f}$. As a consequence, $\bar{Y}_0$ first-order stochastically dominates each $\underline{Y}_N$, X_N, and $\bar{Y}_N$. Therefore, $\mathbb{E}|\bar{Y}_0|^{\varepsilon-1} = \frac{q_2^{\varepsilon-1}}{1-\frac{\varepsilon-1}{\beta}} < \infty$ serves as a uniform bound on each $\mathbb{E}|X_N|^{\varepsilon-1}$, $\mathbb{E}|\bar{Y}_N|^{\varepsilon-1}$, and $\mathbb{E}|\underline{Y}_N|^{\varepsilon-1}$. *Q.E.D.*

LEMMA 16: *There exists a random variable X^* such that X_N converges to X^* almost surely and in $L^{\varepsilon-1}$.*

PROOF: Let $P_N \equiv \frac{X_N}{\prod_{n=0}^{N}\mu_n}$, where $\mu_n \equiv M\int_{z_0}^{\infty} w^{\alpha^n}\rho^{1-H(w)} \, dH(w)$. We first show that $\{P_N\}$ is a submartingale with respect to $\{\Omega_N\}$.

If Ω is empty, then $\mathbb{E}[P_N|\Omega_{N-1}] = P_{N-1} = 0$. Otherwise, define a set $\mathcal{D}_N$ as follows: Let $\omega_N^* \in \arg\max_{\omega\in\Omega_N} \mathfrak{q}_N(\omega)$ so that $X_N = \mathfrak{q}_N(\omega_N^*)$. Let $\mathcal{D}_N \subseteq \Omega_N$ be the set of subchains in Ω_N for which the first $N-1$ links are ω_{N-1}^*. In other words, all subchains in $\mathcal{D}_N$ are of the form $\omega_{N-1}^*\phi$.

Define the random variable $D_N = \max_{\omega\in\mathcal{D}_N} \mathfrak{q}_N(\omega)$. Since $\mathcal{D}_N \subseteq \Omega_N$, it must be that $X_N \geq D_N$. We now show that $\mathbb{E}[D_N|\Omega_{N-1}] \geq \mu_N X_{N-1}$.

The probability that $|\mathcal{D}_N| = k$ is $\frac{e^{-M[1-\rho]}[M(1-\rho)]^k}{[1-e^{-M[1-\rho]}]k!}$ for $k \geq 1$. To see this, note that for any entrepreneur—specifically the one most upstream in the subchain ω_{N-1}^*—the number of techniques follows a Poisson distribution with mean M. Each of those has probability $1-\rho$ of being part of a chain that continues indefinitely, and we are conditioning on having at least one chain continuing indefinitely.

Each of those techniques has a productivity drawn from H. For any ϕ such that $\omega_{N-1}^*\phi \in \mathcal{D}_N$, we have that

$$\Pr\bigl(\mathfrak{q}_N\bigl(\omega_{N-1}^*\phi\bigr) < x|\Omega_{N-1}\bigr) = \Pr\bigl(z(\phi)^{\alpha^n} < x/X_{N-1}\bigr) = H\bigl((x/X_{N-1})^{\alpha^{-N}}\bigr).$$

Given X_{N-1}, if $\mathcal{D}_N$ consists of k subchains, the probability that $D_N < x$ is

$$\Pr\bigl(D_N < x\big|\Omega_{N-1}, |\mathcal{D}_N| = k\bigr) = H\bigl((x/X_{N-1})^{\alpha^{-N}}\bigr)^k.$$

With this, the CDF of D_N, given Ω_{N-1}, is

$$\begin{aligned}
\Pr(D_N < x|\Omega_{N-1}) &= \sum_{k=1}^{\infty} \Pr\bigl(D_N < x\big|X_{N-1}, |\mathcal{D}_N| = k\bigr)\Pr\bigl(|\mathcal{D}_N| = k\bigr) \\
&= \sum_{k=1}^{\infty} H\bigl((x/X_{N-1})^{\alpha^{-N}}\bigr)^k \frac{e^{-M[1-\rho]}\bigl[M(1-\rho)\bigr]^k}{\bigl[1-e^{-M[1-\rho]}\bigr]k!} \\
&= \frac{e^{-M[1-\rho][1-H((x/X_{N-1})^{\alpha^{-N}})]} - e^{-M[1-\rho]}}{1-e^{-M[1-\rho]}} \\
&= \frac{\rho^{[1-H((x/X_{N-1})^{\alpha^{-N}})]} - \rho}{1-\rho}.
\end{aligned}$$

We can now compute the conditional expectation of D_N (using the change of variables $w = (x/X_{N-1})^{\alpha^{-N}}$):

$$\mathbb{E}[D_N|\Omega_{N-1}] = X_{N-1}\int_{z_0}^{\infty} w^{\alpha^N} \log\rho^{-1}\frac{\rho^{1-H(w)}}{1-\rho}\,dH(w) = \mu_N X_{N-1}.$$

Putting this together, we have

$$\mathbb{E}[P_N|\Omega_{N-1}] = \frac{1}{\prod_{n=0}^{N}\mu_n}\mathbb{E}[X_N|\Omega_{N-1}] \geq \frac{1}{\prod_{n=0}^{N}\mu_n}\mathbb{E}[D_N|\Omega_{N-1}] = \frac{1}{\prod_{n=0}^{N}\mu_n}\mu_N X_{N-1} = P_{N-1}.$$

We next show that $\{P_N\}$ is uniformly integrable, that is, that $\sup_N \mathbb{E}[P_N] < \infty$. Since $\sup_N \mathbb{E}[X_N] < \infty$, it suffices to show a uniform lower bound on $\{\prod_{n=0}^{N}\mu_n\}$. Since each $\mu_n \geq z_0^{\alpha^n}$ and $z_0 < 1$, we have that $\prod_{n=0}^{N}\mu_n \geq \prod_{n=0}^{N} z_0^{\alpha^n} \geq \prod_{n=0}^{\infty} z_0^{\alpha^n} = z_0^{\frac{1}{1-\alpha}}$.

We have therefore established that $\{P_N\}_{N\in\mathbb{N}}$ is a uniformly integrable (in L^1) submartingale, so by the martingale convergence theorem, there exists a P such that P_N converges to P almost surely. By the continuous mapping theorem, there exists an X^* such that X_N converges to X^* almost surely. Since each $X_N^{\varepsilon-1}$ is dominated by the integrable random variable $\bar{Y}_0^{\varepsilon-1}$, by dominated convergence we have that X_N converges to X^* in $L^{\varepsilon-1}$. *Q.E.D.*

LEMMA 17: *If Ω is non-empty, then with probability* 1, $X^* = \sup_{\omega\in\Omega} q(\omega)$.

PROOF: We first show that $X^* \geq \sup_{\omega\in\Omega} q(\omega)$ with probability 1. Consider any realization of techniques, Φ. For any $\nu > 0$, there exists a $\hat{\omega} \in \Omega$ such that $q(\hat{\omega}) > \sup_{\omega\in\Omega} q(\omega) - \nu$. There also exists an N_1 such that $N > N_1$ implies $\mathfrak{q}_N(\hat{\omega}) > q(\hat{\omega}) - \nu$. Last, there exists an N_2 such that $N > N_2$ implies $X_N < X^* + \nu$ with probability 1. We then have for $N > \max\{N_1, N_2\}$ that

$$X^* > X_N - \nu = \max_{\omega\in\Omega_N} \mathfrak{q}_N(\omega) - \nu \geq \mathfrak{q}_N(\hat{\omega}) - \nu > q(\hat{\omega}) - 2\nu > \sup_{\omega\in\Omega} q(\omega) - 3\nu, \quad \text{w.p.1.}$$

This is true for any $\nu > 0$, so $X^* \geq \sup_{\omega\in\Omega} q(\omega)$. We next show the opposite inequality. For any N, we have

$$\sup_{\omega\in\Omega} q(\omega) \geq \sup_{\omega\in\Omega} \mathfrak{q}_N(\omega) z_0^{\frac{\alpha^N}{1-\alpha}} = X_N z_0^{\frac{\alpha^N}{1-\alpha}}.$$

Since this is true for any N and $\lim_{N\to\infty} z_0^{\frac{\alpha^N}{1-\alpha}} = 1$, we can take the limit to get $\sup_{\omega\in\Omega} q(\omega) \geq X^*$ with probability 1. *Q.E.D.*

B.3. *Characterization of the Limit*

We will show below that $\log \bar{Y}_N - \log \underline{Y}_N$ converges to 0 in probability. Since $X_N \in [\underline{Y}_N, \bar{Y}_N]$, it must be that both $\bar{Y}_N$ and $\underline{Y}_N$ converge to X^* in probability. Convergence in probability implies convergence in distribution, which gives two implications. First, $T^N \bar{f}$ and $T^N \underline{f}$ converge to the same limiting function. Since these are the least and greatest fixed points of T in $\mathcal{F}$, this limiting function, F^*, is the unique fixed point of T in $\mathcal{F}$. Second, since $T^N \bar{f} \preceq T^N f^1 \preceq T^N \underline{f}$, F^* is the CDF of X^*.

We first show that $\log \bar{Y}_N - \log \underline{Y}_N$ converges to zero in probability.

LEMMA 18: *If Ω is non-empty, then for any $\eta > 1$*, $\lim_{N\to\infty} \Pr(\bar{Y}_N / \underline{Y}_N > \eta) = 0$.

PROOF: Let $\mathfrak{S}_{j,N}$ be the set of chain stubs: sequences of N techniques $\{\phi_n\}_{n=0}^{N-1}$ such that $s(\phi_n) = b(\phi_{n+1})$ and $b(\phi_0) = j$. Note that each $\omega \in \Omega_{j,N}$ is a chain stub so that $\Omega_{j,N} \subseteq \mathfrak{S}_{j,N}$, but not the other way around because each subchain $\omega \in \Omega_{j,N}$ must satisfy the additional requirement that there is a chain in Ω_j for which it is the N most downstream techniques.

A standard result from the theory of branching processes (see Appendix B.4 for a derivation) is that for any x, $\mathbb{E}[x^{|\mathfrak{S}_N|}] = \varphi^{(N)}(x)$ where $\varphi^{(N)}$ is the N-fold composition of $\varphi(x) \equiv e^{-M(1-x)}$, the probability generating function for $|\mathfrak{S}_1|$, and expectations are taken over realizations of Φ.

Given the set of techniques, Φ, we have that for each subchain ω in Ω_N that $\bar{\mathfrak{q}}_N(\omega)/\underline{\mathfrak{q}}_N(\omega) = (\frac{\bar{q}(\omega)}{q_0})^{\alpha^N}$. We therefore have

$$\Pr\bigl(\bar{\mathfrak{q}}_N(\omega)/\underline{\mathfrak{q}}_N(\omega) \leq \eta|\Phi\bigr) = \Pr\biggl(\biggl(\frac{\bar{q}(\omega)}{q_0}\biggr)^{\alpha^N} \leq \eta\Big|\Phi\biggr) = \frac{\bar{f}\bigl(\eta^{\alpha^{-N}} q_0\bigr) - \rho}{1-\rho}.$$

If $|\Omega_N|$ is the number of distinct subchains of length N, the probability that every subchain $\omega \in \Omega_N$ satisfies $\bar{\mathfrak{q}}_N(\omega)/\underline{\mathfrak{q}}_N(\omega) \leq \eta$ is $(\frac{\bar{f}(\eta^{\alpha^{-N}} q_0)-\rho}{1-\rho})^{|\Omega_N|}$ so that

$$\Pr(\bar{Y}_N/\underline{Y}_N \leq \eta|\Phi) = \biggl(\frac{\bar{f}\bigl(\eta^{\alpha^{-N}} q_0\bigr) - \rho}{1-\rho}\biggr)^{|\Omega_N|} \geq \biggl(\frac{\bar{f}\bigl(\eta^{\alpha^{-N}} q_0\bigr) - \rho}{1-\rho}\biggr)^{|\mathfrak{S}_N|}.$$

Taking expectations over Φ, this implies

$$\begin{aligned}\Pr(\bar{Y}_N/\underline{Y}_N \leq \eta) &= \mathbb{E}\bigl[\Pr(\bar{Y}_N/\underline{Y}_N \leq \eta|\Phi)\bigr]\\ &\geq \mathbb{E}\biggl[\biggl(\frac{\bar{f}\bigl(\eta^{\alpha^{-N}} q_0\bigr) - \rho}{1-\rho}\biggr)^{|\mathfrak{S}_N|}\biggr] = \varphi^{(N)}\biggl(\frac{\bar{f}\bigl(\eta^{\alpha^{-N}} q_0\bigr) - \rho}{1-\rho}\biggr).\end{aligned}$$

Put differently, $\lim_{N\to\infty}\Pr(\bar{Y}_N/\underline{Y}_N > \eta) \leq \lim_{N\to\infty} 1 - \varphi^{(N)}(\frac{\bar{f}(\eta^{\alpha^{-N}} q_0)-\rho}{1-\rho})$. We complete the proof by showing $\lim_{N\to\infty} 1 - \varphi^{(N)}(\frac{\bar{f}(\eta^{\alpha^{-N}} q_0)-\rho}{1-\rho}) = 0$.

To do this, we first show that for $x \in [0,1]$, $\frac{d}{dx}\varphi^{(N)}(x) \leq M^N$. To see this, note that φ is convex and $\varphi'(1) = M$, so that $\varphi'(x) \leq M$ for $x \leq 1$. In addition, if $x \in [0,1]$, then $\varphi(x) \in (0,1]$, which implies $\varphi^{(N)}(x) \in (0,1]$ for each N. We then have

$$\frac{d}{dx}\varphi^{(N)}(x) = \prod_{n=1}^{N}\varphi'\bigl(\varphi^{(n-1)}(x)\bigr) \leq M^N.$$

With this, for any x, we can bound $\varphi^{(N)}(x)$ by

$$\varphi^{(N)}(x) = \varphi^{(N)}(1) - \int_x^1 \varphi^{(N)\prime}(w)\,dw \geq 1 - M^N\int_x^1 dw \geq 1 - M^N[1-x].$$

Last, $\lim_{N\to\infty} M^N[1 - \frac{\bar{f}(\eta^{\alpha^{-N}} q_0)-\rho}{1-\rho}] = \lim_{N\to\infty} M^N q_2^{\beta}(\eta^{-\beta\alpha^{-N}} q_0^{-\beta}) = 0$. *Q.E.D.*

We now come to the main result.

PROPOSITION 3: *There is a unique fixed point of T on $\mathcal{F}$, F^*. F^* is the CDF of X^*. Aggregate productivity is $Q = (\int_0^\infty q^{\varepsilon-1}\,dF^*(q))^{\frac{1}{\varepsilon-1}}$ with probability* 1.

PROOF: If Ω is non-empty, the combination of $\log\bar{Y}_N - \log\underline{Y}_N \xrightarrow{p} 0$, $\bar{Y}_N \geq X_N \geq \underline{Y}_N$, and $X_N \xrightarrow{p} X^*$ implies that $\bar{Y}_N \xrightarrow{p} X^*$ and $\underline{Y}_N \xrightarrow{p} X^*$. If Ω is empty, then $X^* = 0$, so that $\bar{Y}_N \xrightarrow{p} X^*$ and $\underline{Y}_N \xrightarrow{p} X^*$. Together, these imply that $\bar{Y}_N \xrightarrow{p} X^*$ and $\underline{Y}_N \xrightarrow{p} X^*$ unconditionally.

We first show that there is a unique fixed point, which is also the CDF of X^*. The CDFs of $\bar{Y}_N$ and $\underline{Y}_N$ are $T^N\bar{f}$ and $T^N\underline{f}$, respectively. The least and greatest fixed points of T

in $\mathcal{F}$ are $\lim_{N\to\infty} T^N \bar{f}$ and $\lim_{N\to\infty} T^N \underline{f}$, respectively. Convergence in probability implies convergence in distribution, so the least and greatest fixed point are the same and this fixed point is the CDF of X^*. Call this fixed point F^*.

Since $\{\bar{Y}_N\}$ and $\{\underline{Y}_N\}$ are uniformly integrable in $L^{\varepsilon-1}$, we have by Vitali's convergence theorem that $\bar{Y}^N \to X^*$ in $L^{\varepsilon-1}$ and $\underline{Y}^N \to X^*$ in $L^{\varepsilon-1}$.

Putting all of these pieces together, we have that the CDF of q_j is F^*. We next show that aggregate productivity is $Q = (\int_0^\infty q^{\varepsilon-1}\, dF^*(q))^{\frac{1}{\varepsilon-1}}$. For this, we simply apply the law of large numbers for a continuum economy of Uhlig (1996). To do this, we must verify that the efficiencies are pairwise uncorrelated. This is trivial: consider two entrepreneurs, j and i. Since the set of entrepreneurs in any of j's supply chains is countable, the probability that i and j have overlapping supply chains is zero. The theorem in Uhlig (1996) also requires that the variable in question has a finite variance, and if it does, then the L^2 integral exists. Here we are interested in the $L^{\varepsilon-1}$ norm, so we require that X^* is $L^{\varepsilon-1}$ integrable. Therefore, we have that $Q = (\int_0^\infty q^{\varepsilon-1}\, dF^*(q))^{\frac{1}{\varepsilon-1}}$ with probability 1. *Q.E.D.*

B.4. *The Number of Supply Chains*

For completeness, we show the derivation of several results from the theory of branching processes that are used in this paper (see, e.g., Athreya and Ney (1972)).

Let $B_{j,N} \equiv |\mathfrak{S}_{j,N}|$ be the number of distinct chain stubs of length N. Recall that a chain stub for j is a finite sequence of techniques such that j is the buyer of the most downstream technique and the supplier of each technique is the buyer of the next-most upstream technique. Let $p(k)$ be the probability that an entrepreneur has exactly k techniques, in this case equal to $\frac{e^{-M}M^k}{k!}$, and let $P_N(l, k)$ be the probability that, in total, l different entrepreneurs have among them k chain stubs of length N (i.e., $k = \sum_{i=1}^l B_{j_i,N}$). Note that $P_N(1, k)$ is the probability an entrepreneur has exactly k chain stubs of length N (i.e., the probability that $B_{j,N} = k$). We will suppress the argument j when not needed for clarity.

Define $\varphi(x) = \sum_{k=0}^\infty p(k)x^k$ to be the probability generating function for the random variable B_1. If the arrival of techniques follows a Poisson distribution, then $\varphi(x) = e^{-M(1-x)}$. Also, for each N, let $\varphi_N(\cdot)$ be the probability generating function associated with B_N. If $\varphi^{(N)}$ is the N-fold composition of φ, then we have the convenient result:

LEMMA 19: $\varphi_N(x) = \varphi^{(N)}(x)$.

PROOF: We proceed by induction. By definition, the statement is true for $N = 1$. Noting that $\sum_{k=0}^\infty P_1(l, k)x^k = \varphi(x)^l$, we have

$$\varphi_{N+1}(x) = \sum_{l=0}^\infty P_{N+1}(1, l)x^l = \sum_{l=0}^\infty \sum_{k=0}^\infty P_N(1, k)P_1(k, l)x^l = \sum_{k=0}^\infty P_N(1, k) \sum_{l=0}^\infty P_1(k, l)x^l$$

$$= \sum_{k=0}^\infty P_N(1, k)\varphi(x)^k = \varphi_N\bigl(\varphi(x)\bigr).$$

Q.E.D.

We immediately have the following:

CLAIM 1: *For any* x, $\mathbb{E}[x^{B_N}] = \varphi^{(N)}(x)$.

PROOF: $\mathbb{E}[x^{B_N}] = \sum_{k=0}^{\infty} P_N(1,k)x^k = \varphi_N(x) = \varphi^{(N)}(x)$. *Q.E.D.*

We next study the probability that an entrepreneur has no supply chains.

CLAIM 2: *The probability that a single entrepreneur has no supply chains is the smallest root*, ρ, *of* $y = \varphi(y)$.

PROOF: The probability that an entrepreneur has no chain stubs greater than length N is $P_N(1,0)$, which is equal to $\varphi_N(0)$ and hence $\varphi^{(N)}(0)$. Then the probability that a single entrepreneur has no supply chains is $\lim_{N\to\infty} \varphi^{(N)}(0)$. Next, note that φ is increasing and convex, $\varphi(1) = 1$, and $\varphi(0) \geq 0$. This implies that in the range $[0, 1]$, the equation $\varphi(y) = y$ has either a unique root at $y = 1$ or two roots, $y = 1$ and a second in $(0, 1)$.

Let ρ be the smallest root. Note that $y \in [0, \rho)$ implies $\varphi(y)(y, \rho)$ and that $y \in (\rho, 1)$ (if any such y exist) implies $\varphi(y) \in (\rho, y)$. Together these imply that if $y \in [0, 1)$, the sequence $\{\varphi^{(N)}(y)\}$ is monotone and bounded, and therefore has a limit. We have $\varphi^{(N+1)}(0) = \varphi(\varphi^{(N)}(0))$. Taking limits of both sides (and noting that φ is continuous) gives $\lim_{N\to\infty} \varphi^{(N+1)}(0) = \varphi(\lim_{N\to\infty} \varphi^{(N)}(0))$. Therefore, the limit is a root of $y = \varphi(y)$, and therefore must be ρ. In other words, $\lim_{N\to\infty} \varphi^{(N)}(0) = \rho$. *Q.E.D.*

CLAIM 3: *If* $M \leq 1$, *then with probability* 1 *an entrepreneur has no supply chains. If* $M > 1$, *then there is a strictly positive probability the entrepreneur has at least one supply chain.*

PROOF: In this case, we have $\varphi(x) = e^{-M(1-x)}$. If $M \leq 1$, then the smallest root of $y = \varphi(y)$ is $y = 1$. If $M > 1$, the smallest root is strictly less than 1. *Q.E.D.*

APPENDIX C: CROSS-SECTIONAL PATTERNS

C.1. *Distribution of Customers*

PROPOSITION 4—1 of 2: *Suppose Assumption* 2 *holds. Among entrepreneurs with efficiency* q, *the number of actual customers follows a Poisson distribution with mean* $\frac{m}{\theta} q^{\alpha\zeta}$.

PROOF: We first derive an expression for $\tilde{F}(x)$. This is the probability that a potential buyer has no alternative techniques that deliver efficiency better than x. The potential buyer will have $n-1$ other techniques with probability $\frac{e^{-M}M^n}{n!(1-e^{-M})}$. The probability that a single alternative delivers efficiency no greater than x is $G(x)$. Therefore, the probability that none of the potential buyer's alternatives deliver efficiency better than x is

$$\tilde{F}(x) = \frac{\sum_{n=1}^{\infty} \frac{e^{-M}M^n}{n!} G(x)^{n-1}}{1 - e^{-M}} = \frac{1}{G(x)\big(1 - e^{-M}\big)} \left[\sum_{n=0}^{\infty} \frac{e^{-M}M^n}{n!} G(x)^n - e^{-M} \right]$$
$$= \frac{F(x) - e^{-M}}{G(x)\big(1 - e^{-M}\big)}.$$

Consider an entrepreneur with efficiency q_s. If a single downstream technique has productivity z, the technique delivers efficiency zq_s^{α} to the potential customer, and will be selected by that customer with probability $\tilde{F}(zq_s^{\alpha})$. Integrating over possible productivities, the probability that a single downstream technique is used by the customer is

$\int_{z_0}^{\infty} \tilde{F}(zq_s^{\alpha})\,dH(z)$. Since the number of downstream techniques follows a Poisson distribution with mean M, the number of downstream techniques that are used follows a Poisson distribution with mean $M\int_{z_0}^{\infty} \tilde{F}(zq_s^{\alpha})\,dH(z)$.

Using the functional form for H and taking the limit as $z_0 \to 0$, this can be simplified considerably. Since $\lim_{z_0\to 0} e^{-mz_0^{-\zeta}} = 0$ and $\lim_{z_0\to 0} G(q) = 1$, we have that

$$\lim_{z_0\to 0} mz_0^{-\zeta}\int_{z_0}^{\infty} \frac{F(zq_s^{\alpha}) - e^{-mz_0^{-\zeta}}}{G(zq_s^{\alpha}) - e^{-mz_0^{-\zeta}}}\zeta z_0^{\zeta} z^{-\zeta-1}\,dz = m\int_0^{\infty} e^{-\theta(zq_s^{\alpha})^{-\zeta}}\zeta z^{-\zeta-1}\,dz = \frac{m}{\theta}q_s^{\alpha\zeta}.$$

Q.E.D.

PROPOSITION 4—2 of 2: *Suppose Assumption* 2 *holds*. *Let* p_k *be the fraction of entrepreneurs with* k *customers*. *Among all entrepreneurs*, *the fraction of entrepreneurs with at least* n *customers asymptotically follows a power law with exponent* $1/\alpha$: $\sum_{k=n}^{\infty} p_k \sim \frac{1}{\Gamma(1-\alpha)^{1/\alpha}}n^{-1/\alpha}$.

PROOF: With the functional forms, among firms with efficiency q, the distribution of customers is Poisson with mean $\frac{m}{\theta}q^{\alpha\zeta}$. If $p_n(q)$ is the probability that an entrepreneur with efficiency q has n customers, $p_n(q) = \frac{(\frac{m}{\theta}q^{\alpha\zeta})^n e^{-\frac{m}{\theta}q^{\alpha\zeta}}}{n!}$. Integrating across efficiencies, the unconditional probability that an entrepreneur has n customers is

$$p_n = \int_0^{\infty} p_n(q)\,dF(q) = \int_0^{\infty} \frac{\left(\frac{m}{\theta}q^{\alpha\zeta}\right)^n e^{-\frac{m}{\theta}q^{\alpha\zeta}}}{n!}\,dF(q).$$

We will make the change of variables $u = \frac{m}{\theta}q^{\alpha\zeta}$. Noting that $\theta = \Gamma(1-\alpha)m\theta^{\alpha}$, this means that $\theta q^{-\zeta} = (\frac{m}{\theta^{1-\alpha}})^{1/\alpha}(\frac{m}{\theta}q^{\alpha\zeta})^{-1/\alpha} = [\Gamma(1-\alpha)u]^{-1/\alpha}$ and $\zeta\frac{dq}{q} = \frac{1}{\alpha}\frac{du}{u}$. Together, these imply that $dF(q) = \zeta\theta q^{-\zeta-1}e^{-\theta q^{-\zeta}}\,dq = \frac{e^{-[\Gamma(1-\alpha)u]^{-1/\alpha}}}{\Gamma(1-\alpha)^{\frac{1}{\alpha}}\alpha}u^{-\frac{1}{\alpha}-1}\,du$, so that p_n can be written as

$$p_n = \int_0^{\infty} \frac{u^n e^{-u}}{n!}\frac{e^{-[\Gamma(1-\alpha)u]^{-1/\alpha}}}{\Gamma(1-\alpha)^{\frac{1}{\alpha}}\alpha}u^{-\frac{1}{\alpha}-1}\,du.$$

Theorem 2.1 of Willmot (1990) states that if the probabilities of a mixed Poisson distribution are given by $p_n = \int \frac{(\lambda x)^n e^{-\lambda x}}{n!}f(x)\,dx$, then, if $f(x) \sim C(x)x^{\gamma}e^{-\beta x}$, $x \to \infty$ where $C(x)$ is a locally bounded function on $(0, \infty)$ which varies slowly at infinity, $\beta \geq 0$, and $-\infty < \gamma < \infty$ (with $\gamma < -1$ if $\beta = 0$), then $p_n \sim \frac{C(n)}{(\lambda+\beta)^{\gamma+1}}(\frac{\lambda}{\lambda+\beta})^n n^{\gamma}$ as $n \to \infty$. Since $\lim_{u\to\infty} e^{-[\Gamma(1-\alpha)u]^{-1/\alpha}} = 1$, this theorem implies $\lim_{n\to\infty}\frac{\frac{1}{\Gamma(1-\alpha)^{1/\alpha}\alpha}n^{-\frac{1}{\alpha}-1}}{p_n} = 1$. Then Theorem 1 of Section VIII.9 of Feller (1971) implies that $\lim_{n\to\infty}\frac{np_n}{\sum_{k=n}^{\infty}p_k} = \frac{1}{\alpha}$, giving the desired result. *Q.E.D.*

CLAIM 4: *Under Assumption* 2, $\frac{\mathrm{Cov}(\log q, \#\ customers)}{St.\,Dev.(\log q)} = \frac{\sqrt{6}}{\pi}\int_0^1 \frac{x^{-\alpha}-1}{1-x}\,dx$.

PROOF: First, note that $\Gamma'(t) = \frac{d}{dt}\int_0^{\infty} x^{t-1}e^{-u}\,du = \int_0^{\infty}\log u u^{t-1}e^{-u}\,du$. Second, under Assumption 2, letting γ be the Euler–Mascheroni constant and using $\log q = \frac{1}{\zeta}[\log\theta -$

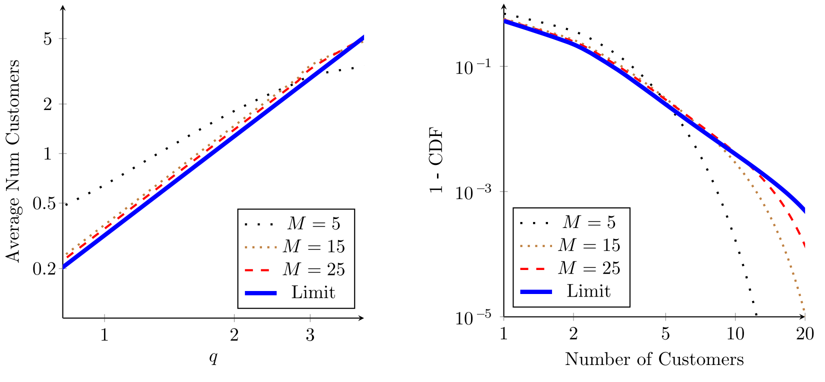
$\log(\theta q^{-\zeta})]$, we have

$$\begin{aligned}
\mathbb{E}[\log q] &= \int_0^\infty \log q \, dF(q) = \frac{1}{\zeta}\left[\log\theta - \int_0^\infty \log\bigl(\theta q^{-\zeta}\bigr)\, dF(q)\right] \\
&= \frac{1}{\zeta}\left[\log\theta - \int_0^\infty \log u e^{-u}\, du\right] = \frac{1}{\zeta}[\log\theta + \gamma], \\
\mathbb{E}\bigl[(\log q)^2\bigr] &= \int_0^\infty \frac{1}{\zeta^2}\bigl[\log\theta - \log\bigl(\theta q^{-\zeta}\bigr)\bigr]^2 dF(q) = \int_0^\infty \frac{1}{\zeta^2}[\log\theta - \log u]^2 e^{-u}\, du \\
&= \frac{1}{\zeta^2}\bigl[(\log\theta)^2 + 2\log\theta\gamma + \gamma^2 + \pi^2/6\bigr], \\
\mathbb{E}\left[\log q \frac{m}{\theta} q^{\alpha\zeta}\right] &= \int_0^\infty \frac{1}{\zeta}\bigl[\log\theta - \log\bigl(\theta q^{-\zeta}\bigr)\bigr]\left(\frac{m}{\theta^{1-\alpha}}\bigl(\theta q^{-\zeta}\bigr)^{-\alpha}\right) dF(q) \\
&= \int_0^\infty \frac{1}{\zeta}[\log\theta - \log u]\left(\frac{1}{\Gamma(1-\alpha)} u^{-\alpha}\right) e^{-u}\, du \\
&= \frac{1}{\zeta}\left[\log\theta - \frac{\int_0^\infty \log u u^{-\alpha} e^{-u}\, du}{\Gamma(1-\alpha)}\right] = \frac{1}{\zeta}\left[\log\theta - \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)}\right].
\end{aligned}$$

The first two imply that the standard deviation of $\log q$ is $\frac{\pi}{\zeta\sqrt{6}}$. The first and third (and $\mathbb{E}[\frac{m}{\theta}q^{\alpha\zeta}] = 1$) imply that $\mathrm{Cov}(\log q, \#\text{ customers}) = \frac{1}{\zeta}[\frac{-\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} - \gamma] = \frac{1}{\zeta}\int_0^1 \frac{x^{-\alpha}-1}{1-x}\, dx$. *Q.E.D.*

C.1.1. *Sequence of Economies*

As discussed in Section 4, Assumption 2 can be interpreted as the limit of a sequence of economies in which $H(z) = 1 - (z/z_0)^{-\zeta}$ and the limit as $z_0 \to 0$ is taken as $m \equiv M z_0^{-\zeta}$ is held fixed. Figure 4 shows the behavior of the distribution of customers as the sequence converges to this limit. Panel (a) shows the expected number of customers among entrepreneurs with a given level of efficiency while panel (b) plots the right CDF of the



(a) Conditional means (b) Unconditional distribution

FIGURE 4.—Distribution of customers, sequence of economies. This figure plots features of the distribution of customers for several economies with $H(z) = 1 - (z/z_0)^{-\zeta}$ with $\zeta = 4$ and $\alpha = 0.5$. The line labeled "Limit" corresponds to an economy that satisfies Assumption 2.

overall distribution, each on a log-log plot. As can be seen in each panel, these features of the distribution of customers converge pointwise to their respective limits, with the tail converging most slowly.

C.2. *The Distribution of Employment*

Because employment is the sum of several components (labor used to produce inputs for each customer and for the household), rather than working with the CDF of the size distribution, $\mathcal{L}(\cdot)$, it will be easier to work with its Laplace–Stieltjes transform, $\hat{\mathcal{L}}(s) \equiv \int_0^\infty e^{-sl}\, d\mathcal{L}(l)$. Similarly, if $\mathcal{L}(\cdot|q)$ is the CDF of the conditional size distribution among entrepreneurs with efficiency q, its transform is $\hat{\mathcal{L}}(s|q) \equiv \int_0^\infty e^{-sl}\, d\mathcal{L}(l|q)$. These are related in that $\mathcal{L}(l) = \int_0^\infty \mathcal{L}(l|q)\, dF(q)$ and $\hat{\mathcal{L}}(s) = \int_0^\infty \hat{\mathcal{L}}(s|q)\, dF(q)$. This section characterizes these transforms and then studies their implications for the size distribution.

We first derive a relationship between the conditional size distributions among entrepreneurs with different efficiencies. Recall that $\tilde{F}(q) \equiv \frac{F(q)-e^{-M}}{G(q)(1-e^{-M})}$ describes the CDF of a potential buyer's best alternative technique.

LEMMA 20: *The transforms* $\{\hat{\mathcal{L}}(\cdot|q)\}$ *satisfy*

$$\hat{\mathcal{L}}(s|q) = e^{-s(1-\alpha)(q/Q)^{\varepsilon-1}L} e^{-M\int_0^\infty \tilde{F}(zq^\alpha)[1-\hat{\mathcal{L}}(\alpha s|zq^\alpha)]\, dH(z)}.$$

PROOF: Total labor used by an entrepreneur is the sum of labor used to make output for consumption and for use as an intermediate input by others. We use the fact that the Laplace–Stieltjes transform of a sum of random variables is the product of the transforms of each.

An entrepreneur with efficiency q uses $(1-\alpha)(q/Q)^{\varepsilon-1}L$ units of labor in making goods for the household. The transform of this is $e^{-s(1-\alpha)(q/Q)^{\varepsilon-1}L}$.

We next consider labor used to make intermediate inputs. Recall that if j uses l_j units of labor, j's supplier will use αl_j units of labor to make the inputs for j. Thus, if the transform of labor used by a buyer with efficiency q_b is $\hat{\mathcal{L}}(s|q_b)$, then the transform of labor used by its supplier to make intermediates is

$$\int_0^\infty \frac{1}{\alpha}\Pr\left(l_j = \frac{l}{\alpha}\right)e^{-sl}\, dl = \int_0^\infty \Pr\left(l_j = \frac{l}{\alpha}\right)e^{-(\alpha s)\frac{l}{\alpha}}\, d\left(\frac{l}{\alpha}\right) = \hat{\mathcal{L}}(\alpha s|q_b).$$

For an entrepreneur with efficiency q, consider a single downstream technique with productivity z, so that the technique delivers efficiency to the buyer of zq^α. With probability $\tilde{F}(zq^\alpha)$ it is the buyer's best technique, in which case the transform of labor used to create intermediates for that customer is $\hat{\mathcal{L}}(\alpha s|zq^\alpha)$. With probability $1-\tilde{F}(zq^\alpha)$ the potential buyer uses an alternative supplier, in which case the transform of labor used to create intermediates for that customer is simply 1. Putting these together and integrating over possible realizations of productivity, the transform of labor used to make intermediates for a single *potential* customer is

$$\int_0^\infty \{[1-\tilde{F}(zq^\alpha)] + \tilde{F}(zq^\alpha)\hat{\mathcal{L}}(\alpha s|zq^\alpha)\}\, dH(z) = 1 - \int_0^\infty \tilde{F}(zq^\alpha)[1-\hat{\mathcal{L}}(\alpha s|zq^\alpha)]\, dH(z).$$

Each entrepreneur has n potential customers with probability $\frac{M^n e^{-M}}{n!}$, so the transform over labor used to create all intermediate goods (summing across all potential customers) is

$$\sum_{n=0}^{\infty} \frac{M^n e^{-M}}{n!}\left(1 - \int_0^{\infty} \tilde{F}\left(zq^{\alpha}\right)\left[1 - \hat{\mathcal{L}}\left(\alpha s|zq^{\alpha}\right)\right] dH(z)\right)^n = e^{-M\int_0^{\infty} \tilde{F}(zq^{\alpha})[1-\hat{\mathcal{L}}(\alpha s|zq^{\alpha})]\,dH(z)}.$$

$\hat{\mathcal{L}}(s|q)$ is simply the product of the transforms of labor used to make final consumption and labor used to make intermediate inputs. *Q.E.D.*

Under Assumption 2, the overall size distribution can be characterized without the intermediate step of solving for the conditional size distributions.

LEMMA 21: *Define $v \equiv \frac{\varepsilon-1}{\zeta}$. Under Assumption 2, the transforms $\hat{\mathcal{L}}(\cdot)$ and $\hat{\mathcal{L}}(\cdot|q)$ satisfy*

$$\hat{\mathcal{L}}(s) = \int_0^{\infty} e^{-s(1-\alpha)\frac{t^{-v}}{\Gamma(1-v)}L} e^{-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}[1-\hat{\mathcal{L}}(\alpha s)]} e^{-t}\, dt, \tag{38}$$

$$\hat{\mathcal{L}}(s|q) = e^{-s(1-\alpha)\frac{(\theta q^{-\zeta})^{-v}}{\Gamma(1-v)}L} e^{-\frac{(\theta q^{-\zeta})^{-\alpha}}{\Gamma(1-\alpha)}[1-\hat{\mathcal{L}}(\alpha s)]}. \tag{39}$$

PROOF: First, using the functional forms, the term $M\int_0^{\infty} \tilde{F}(zq^{\alpha})[1 - \hat{\mathcal{L}}(\alpha s|zq^{\alpha})]\,dH(z)$ can be written as $mz_0^{-\zeta}\int_{z_0}^{\infty} \tilde{F}(zq^{\alpha})[1 - \hat{\mathcal{L}}(\alpha s|zq^{\alpha})]\zeta z_0^{\zeta} z^{-\zeta-1}\,dz$. Since $\tilde{F}(zq^{\alpha}) \to e^{-\theta(zq^{\alpha})^{-\zeta}}$, this becomes (using the change of variables $w = zq^{\alpha}$):

$$\frac{mq^{\alpha\zeta}}{\theta}\int_0^{\infty} e^{-\theta w^{-\zeta}}\left[1 - \hat{\mathcal{L}}(\alpha s|w)\right]\zeta\theta w^{-\zeta-1}\,dw = \frac{mq^{\alpha\zeta}}{\theta}\left[1 - \hat{\mathcal{L}}(\alpha s)\right],$$

where the last step follows because $e^{-\theta w^{-\zeta}}\zeta\theta w^{-\zeta-1}\,dw = dF(w)$. We use this to express $\hat{\mathcal{L}}(\cdot|q)$ and $\hat{\mathcal{L}}(\cdot)$:

$$\hat{\mathcal{L}}(s|q) \to e^{-s(1-\alpha)(q/Q)^{\varepsilon-1}L} e^{-\frac{mq^{\alpha\zeta}}{\theta}[1-\hat{\mathcal{L}}(\alpha s)]},$$

$$\hat{\mathcal{L}}(s) = \int_0^{\infty} \hat{\mathcal{L}}(s|q)\,dF(q) \to \int_0^{\infty} e^{-s(1-\alpha)(q/Q)^{\varepsilon-1}L} e^{-\frac{mq^{\alpha\zeta}}{\theta}[1-\hat{\mathcal{L}}(\alpha s)]}\zeta\theta q^{-\zeta-1}e^{-\theta q^{-\zeta}}\,dq.$$

The conclusion follows from the substitutions $Q^{\varepsilon-1} = \Gamma(1-v)\theta^{v}$ and $m = \frac{\theta^{1-\alpha}}{\Gamma(1-\alpha)}$, and the change of variables $t = \theta q^{-\zeta}$. *Q.E.D.*

C.3. *Tail Behavior*

Let $\rho = \min\{\alpha^{-1}, v^{-1}\}$ and let N be the greatest integer that is strictly less than ρ. For any integer n, let $\mu_n \equiv \int_0^{\infty} l^n\,d\mathcal{L}(l) = (-1)^n\hat{\mathcal{L}}^{(n)}(0)$ be the nth moment of the size distribution, where $\hat{\mathcal{L}}^{(n)}$ denotes the nth derivative of $\hat{\mathcal{L}}$. The strategy is to show that $\mu_N - (-1)^N\hat{\mathcal{L}}^{(N)}(s)$ is regularly varying with index $\rho - N$ as $s \searrow 0$. Using the Tauberian theorem of Bingham and Doney (1974), this will imply that $1 - \mathcal{L}(l)$ is regularly varying with index $-\rho$ as $l \to \infty$. The theorem gives this implication only when $0 < \rho - N < 1$, so we restrict attention to that case.

Define $\varphi(s;t) \equiv s\frac{(1-\alpha)}{\Gamma(1-v)}t^{-v} + \frac{1-\hat{\mathcal{L}}(\alpha s)}{\Gamma(1-\alpha)}t^{-\alpha}$ so that $\hat{\mathcal{L}}(s) = \int_0^\infty e^{-\varphi(s;t)}e^{-t}\,dt$. Since we will be interested in $\hat{\mathcal{L}}^{(n)}$, it will be useful to derive an expression for $\frac{d^n}{ds^n}[e^{-\varphi(s;t)}]$. By Faa di Bruno's formula (a generalization of the chain rule to higher derivatives), we have that

$$\frac{d^n}{ds^n}\big[e^{-\varphi(s;t)}\big] = e^{-\varphi(s;t)}\sum_{\iota\in I_n}\frac{n!}{\iota_1!(1!)^{\iota_1}\cdots\iota_n!(n!)^{\iota_n}}\prod_{j=1}^{n}\big[-\varphi^{(j)}(s;t)\big]^{\iota_j},$$

where I_n is the set of all n-tuples of nonnegative integers $\iota = (\iota_1,\dots,\iota_n)$ such that $1\iota_1 + \cdots + n\iota_n = n$.

Since $\varphi^{(1)}(s;t) = \frac{(1-\alpha)}{\Gamma(1-v)}t^{-v} + \frac{-\alpha\hat{\mathcal{L}}^{(1)}(\alpha s)}{\Gamma(1-\alpha)}t^{-\alpha}$ and $\varphi^{(j)}(s;t) = \frac{-\alpha^j\hat{\mathcal{L}}^{(j)}(\alpha s)}{\Gamma(1-\alpha)}t^{-\alpha}$ for $j\geq 2$, we have for each $\iota\in I_n$ that

$$\prod_{j=1}^{n}\big[-\varphi^{(j)}(s;t)\big]^{\iota_j} = \left[\sum_{k=0}^{\iota_1}\binom{\iota_1}{k}\left[\frac{-(1-\alpha)}{\Gamma(1-v)}t^{-v}\right]^k\left[\frac{\alpha\hat{\mathcal{L}}^{(1)}(\alpha s)}{\Gamma(1-\alpha)}t^{-\alpha}\right]^{\iota_1-k}\right]\prod_{j=2}^{n}\left[\frac{\alpha^j\hat{\mathcal{L}}^{(j)}(\alpha s)}{\Gamma(1-\alpha)}t^{-\alpha}\right]^{\iota_j},$$

where the first term is simply the binomial expansion of $[\frac{-(1-\alpha)}{\Gamma(1-v)}t^{-v} + \frac{\alpha\hat{\mathcal{L}}^{(1)}(\alpha s)}{\Gamma(1-\alpha)}t^{-\alpha}]^{\iota_1}$. This can be rearranged as

$$\prod_{j=1}^{n}\big[-\varphi^{(j)}(s;t)\big]^{\iota_j} = \sum_{k=0}^{\iota_1}\binom{\iota_1}{k}\left[\frac{-(1-\alpha)}{\Gamma(1-v)}\frac{\Gamma(1-\alpha)}{\alpha\hat{\mathcal{L}}^{(1)}(\alpha s)}\right]^k\prod_{j=1}^{n}\left[\frac{\alpha^j\hat{\mathcal{L}}^{(j)}(\alpha s)}{\Gamma(1-\alpha)}\right]^{\iota_j}t^{-[\alpha(\sum_{j=1}^n\iota_j-k)+vk]}.$$

Thus we can write

$$\frac{d^n}{ds^n}\big[e^{-\varphi(s;t)}\big] = e^{-\varphi(s;t)}\sum_{\iota\in I_n}\sum_{k=0}^{\iota_1}B_{n,\iota,k}(s)t^{-\beta(\iota,k)}, \tag{40}$$

where

$$B_{n,\iota,k}(s) \equiv \frac{n!}{\iota_1!(1!)^{\iota_1}\cdots\iota_n!(n!)^{\iota_n}}\binom{\iota_1}{k}\left[\frac{-(1-\alpha)}{\Gamma(1-v)}\frac{\Gamma(1-\alpha)}{\alpha\hat{\mathcal{L}}^{(1)}(\alpha s)}\right]^k\prod_{j=1}^{n}\left[\frac{\alpha^j\hat{\mathcal{L}}^{(j)}(\alpha s)}{\Gamma(1-\alpha)}\right]^{\iota_j},$$

$$\beta(\iota,k) \equiv \alpha\left(\sum_{j=1}^{n}\iota_j - k\right) + vk.$$

Note that if $n<\rho$, then each $\beta(\iota,k)\in(0,1)$ because

$$\beta(\iota,k) = \alpha\left(\sum_{j=1}^{n}\iota_j - k\right) + vk \leq \frac{1}{\rho}\left(\sum_{j=1}^{n}\iota_j - k\right) + \frac{1}{\rho}k = \frac{1}{\rho}\sum_{j=1}^{n}\iota_j \leq \frac{n}{\rho} < 1.$$

We first show that the nth derivative can be taken inside the integral.

LEMMA 22: *For any integer $n<\rho$,*

$$\hat{\mathcal{L}}^{(n)}(s) = \int_0^\infty\frac{d^n}{ds^n}\big[e^{-\varphi(s;t)}\big]e^{-t}\,dt = \sum_{\iota\in I_n}\sum_{k=0}^{\iota_1}B_{n,\iota,k}(s)\int_0^\infty e^{-\varphi(s;t)}t^{-\beta(\iota,k)}e^{-t}\,dt.$$

PROOF: We first show that for each $\iota \in I_{n-1}$,

$$\frac{d}{ds}\int_0^\infty e^{-\varphi(s;t)} t^{-\beta(\iota,k)} e^{-t}\,dt = \int_0^\infty \frac{d}{ds}\big[e^{-\varphi(s;t)}\big] t^{-\beta(\iota,k)} e^{-t}\,dt.$$

Since $\frac{d}{ds}[e^{-\varphi(s;t)}] = (\frac{(1-\alpha)}{\Gamma(1-v)} t^{-v} + \frac{-\alpha\hat{\mathcal{L}}^{(1)}(\alpha s)}{\Gamma(1-\alpha)} t^{-\alpha}) e^{-\varphi(s;t)}$ and $e^{-\varphi(s;t)} \leq 1$ for $s, t \geq 0$, the integrand on the RHS is dominated by $(\frac{(1-\alpha)}{\Gamma(1-v)} t^{-v} + \frac{-\alpha\hat{\mathcal{L}}^{(1)}(\alpha s)}{\Gamma(1-\alpha)} t^{-\alpha}) t^{-\beta(\iota,k)} e^{-t}$. This is integrable for each s because $\beta(\iota,k) \leq \frac{n-1}{\rho}$ for $\iota \in I_{n-1}$, which implies both $\alpha + \beta(\iota,k) < 1$ and $v + \beta(\iota,k) < 1$.

With this, we proceed by induction. Trivially, the conclusion holds for $n = 0$. Now for $n < \rho$, assume that $\hat{\mathcal{L}}^{(n-1)}(s) = \int_0^\infty \frac{d^{n-1}}{ds^{n-1}}[e^{-\varphi(s;t)}] e^{-t}\,dt$. Equation (40) implies

$$\hat{\mathcal{L}}^{(n-1)}(s) = \sum_{\iota \in I_{n-1}} \sum_{k=0}^{\iota_1} B_{n-1,\iota,k}(s) \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(\iota,k)} e^{-t}\,dt.$$

Differentiating each side gives

$$\begin{aligned}
\hat{\mathcal{L}}^{(n)}(s) &= \sum_{\iota \in I_{n-1}} \sum_{k=0}^{\iota_1} \frac{dB_{n-1,\iota,k}(s)}{ds} \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(\iota,k)} e^{-t}\,dt \\
&\quad + B_{n-1,\iota,k}(s) \frac{d}{ds} \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(\iota,k)} e^{-t}\,dt \\
&= \int_0^\infty \sum_{\iota \in I_{n-1}} \sum_{k=0}^{\iota_1} \left[\frac{dB_{n-1,\iota,k}(s)}{ds} e^{-\varphi(s;t)} + B_{n-1,\iota,k}(s) \frac{de^{-\varphi(s;t)}}{ds}\right] t^{-\beta(\iota,k)} e^{-t}\,dt \\
&= \int_0^\infty \frac{d}{ds}\left\{\sum_{\iota \in I_{n-1}} \sum_{k=0}^{\iota_1} \big[B_{n-1,\iota,k}(s) e^{-\varphi(s;t)}\big] t^{-\beta(\iota,k)}\right\} e^{-t}\,dt \\
&= \int_0^\infty \frac{d}{ds}\left\{\frac{d^{n-1}}{ds^{n-1}}\big[e^{-\varphi(s;t)}\big]\right\} e^{-t}\,dt \\
&= \int_0^\infty \frac{d^n}{ds^n}\big[e^{-\varphi(s;t)}\big] e^{-t}\,dt.
\end{aligned}$$

Q.E.D.

LEMMA 23: *For any integer $n < \rho$, $\mu_n < \infty$.*

PROOF: Again, we proceed by induction. $\mu_0 = 1$. Now assume that $\mu_0, \ldots, \mu_{n-1} < \infty$. We begin with the expression for $\hat{\mathcal{L}}^{(n)}(s)$:

$$\hat{\mathcal{L}}^{(n)}(s) = \sum_{\iota \in I_n} \sum_{k=0}^{\iota_1} B_{n,\iota,k}(s) \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(\iota,k)} e^{-t}\,dt.$$

Define $\tilde{I}_n \equiv I_n \setminus (0, \dots, 0, 1)$. Then, pulling out from the sum the term for $\iota = (0, \dots, 0, 1)$, we have

$$\hat{\mathcal{L}}^{(n)}(s) = \frac{\alpha^n \hat{\mathcal{L}}^{(n)}(\alpha s)}{\Gamma(1-\alpha)} \int_0^\infty e^{-\varphi(s;t)} t^{-\alpha} e^{-t}\, dt + \sum_{\iota \in \tilde{I}_n} \sum_{k=0}^{\iota_1} B_{n,\iota,k}(s) \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(\iota,k)} e^{-t}\, dt.$$

We can take the limit as $s \searrow 0$ of each side. Since $e^{-\varphi(s;t)}$ is dominated by 1, the limit can be taken inside of each integral, and since $\lim_{s \searrow 0} \varphi(s; t) = 0$, we have

$$\hat{\mathcal{L}}^{(n)}(0) = \alpha^n \hat{\mathcal{L}}^{(n)}(0) + \sum_{\iota \in \tilde{I}_n} \sum_{k=0}^{\iota_1} B_{n,\iota,k}(0) \Gamma\{1 - \beta(\iota, k)\}.$$

For each $\iota \in \tilde{I}_n$, and for each k, $B_{n,\iota,k}(0)$ is proportional to a product of derivatives of $\hat{\mathcal{L}}$, and each of those derivatives is of order less than n. Since all of these are finite, $\hat{\mathcal{L}}^{(n)}(0) < \infty$. *Q.E.D.*

With this, we show that $\mu_N - (-1)^N \hat{\mathcal{L}}^{(N)}(s)$ is regularly varying as $s \searrow 0$. First, we have

$$\begin{aligned}
&\mu_N - (-1)^N \hat{\mathcal{L}}^{(N)}(s) \\
&\quad = (-1)^N \big[\hat{\mathcal{L}}^{(N)}(0) - \hat{\mathcal{L}}^{(N)}(s)\big] \\
&\quad = (-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{\iota_1} \bigg[B_{N,\iota,k}(0) \int_0^\infty t^{-\beta(\iota,k)} e^{-t}\, dt - B_{N,\iota,k}(s) \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(\iota,k)} e^{-t}\, dt\bigg].
\end{aligned}$$

We can decompose the object of interest into three terms:

$$\frac{\mu_N - (-1)^N \hat{\mathcal{L}}^{(N)}(s)}{s^{\rho - N}} = A_1(s) + A_2(s) + A_3(s),$$

where A_1, A_2, and A_3 are defined as

$$\begin{aligned}
A_1(s) &\equiv s^{N-\rho} (-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{\iota_1} \big[B_{N,\iota,k}(0) - B_{N,\iota,k}(s)\big] \int_0^\infty t^{-\beta(\iota,k)} e^{-t}\, dt, \\
A_2(s) &\equiv s^{N-\rho} (-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{\iota_1} B_{N,\iota,k}(s) \int_0^\infty t^{-\beta(\iota,k)} \big[1 - e^{-\varphi(s;t)}\big]\, dt, \\
A_3(s) &\equiv s^{N-\rho} (-1)^{N+1} \sum_{\iota \in I_N} \sum_{k=0}^{\iota_1} B_{N,\iota,k}(s) \int_0^\infty t^{-\beta(\iota,k)} \big[1 - e^{-\varphi(s;t)}\big]\big[1 - e^{-t}\big]\, dt.
\end{aligned}$$

This particular decomposition is useful because it will allow for the use of the monotone convergence theorem in characterizing the limiting behavior of A_2 and A_3.

LEMMA 24: *If* $\rho \notin \mathbb{N}$, $\lim_{s \searrow 0} A_1(s) = \alpha^\rho \lim_{s \to 0} \frac{\mu_N - (-1)^N \hat{\mathcal{L}}^{(N)}(s)}{s^{\rho - N}}$.

PROOF: As above, we can separate the term for $\iota = (0, \dots, 0, 1)$ from $\tilde{I}_N$, to write

$$\begin{aligned}
\lim_{s\searrow 0} A_1(s) &= \lim_{s\searrow 0} s^{N-\rho}(-1)^N \sum_{\iota\in I_N}\sum_{k=0}^{\iota_1}\left[B_{N,\iota,k}(0) - B_{N,\iota,k}(s)\right]\Gamma\bigl(1-\beta(\iota,k)\bigr) \\
&= \lim_{s\searrow 0}(-1)^N \frac{\left[\dfrac{\alpha^N \hat{\mathcal{L}}^{(N)}(0)}{\Gamma(1-\alpha)} - \dfrac{\alpha^N \hat{\mathcal{L}}^{(N)}(\alpha s)}{\Gamma(1-\alpha)}\right]}{s^{\rho-N}}\Gamma(1-\alpha) \\
&\quad + \lim_{s\searrow 0}(-1)^N \sum_{\iota\in \tilde{I}_N}\sum_{k=0}^{\iota_1} \frac{\left[B_{N,\iota,k}(0) - B_{N,\iota,k}(s)\right]}{s^{\rho-N}}\Gamma\bigl(1-\beta(\iota,k)\bigr).
\end{aligned}$$

The first term is simply $\alpha^\rho \lim_{s\searrow 0} \frac{\mu_N - (-1)^N \hat{\mathcal{L}}^{(N)}(\alpha s)}{(\alpha s)^{\rho-N}}$. The second term equals zero: After using L'Hospital's rule, the numerator becomes a multinomial of derivatives of $\hat{\mathcal{L}}$ of order no greater than N (so all of these are finite), while the denominator becomes $(\rho - N)s^{\rho-N-1}$, which goes to infinity. *Q.E.D.*

To characterize the limiting behavior of A_2 and A_3, it will be useful to define

$$\begin{aligned}
\kappa &\equiv \frac{1}{\Gamma\bigl(1-\rho^{-1}\bigr)}\bigl((1-\alpha)\mathbb{I}_{v\geq\alpha} + \alpha\mathbb{I}_{\alpha\geq v}\bigr), \\
\tilde{\varphi}(s; w) &\equiv \varphi\bigl(s; \left[1-\hat{\mathcal{L}}(\alpha s)\right]^\rho \kappa^\rho \alpha^{-\rho} w^{-\rho}\bigr).
\end{aligned}$$

$\tilde{\varphi}$ is defined this way so that $\lim_{s\to 0}\tilde{\varphi}(s; w) = w$ (this can be easily verified for each of the three cases: $\alpha > v$, $\alpha = v$, and $\alpha < v$).

LEMMA 25: *$\tilde{\varphi}(s; w)$ is non-decreasing in s in the neighborhood of zero.*

PROOF: Using the definitions of $\tilde{\varphi}$ and φ, we have

$$\tilde{\varphi}(s; w) = s^{1-\rho v}(1-\alpha)\frac{\kappa^{-\rho v}\left[\dfrac{1-\hat{\mathcal{L}}(\alpha s)}{\alpha s}\right]^{-\rho v}}{\Gamma(1-v)} w^{\rho v} + \frac{(\kappa/\alpha)^{-\rho\alpha}}{\Gamma(1-\alpha)}\left[1-\hat{\mathcal{L}}(\alpha s)\right]^{1-\alpha\rho} w^{\alpha\rho}.$$

We first show that $\frac{1-\hat{\mathcal{L}}(s)}{s}$ is non-increasing in a neighborhood of zero. Since $\mu_2 > 0$,

$$\lim_{s\searrow 0}\frac{d}{ds}\left(\frac{1-\hat{\mathcal{L}}(s)}{s}\right) = \lim_{s\searrow 0}\frac{-s\hat{\mathcal{L}}^{(1)}(s) - \left[1-\hat{\mathcal{L}}(s)\right]}{s^2} = -\hat{\mathcal{L}}^{(2)}(0) < 0. \tag{41}$$

Next, since $1 - \hat{\mathcal{L}}(\alpha s)$ is non-decreasing in s, and since both $1 - \rho v$ and $1 - \rho\alpha$ are non-negative, $\tilde{\varphi}(s; w)$ is non-decreasing in the neighborhood of zero. *Q.E.D.*

LEMMA 26: $\lim_{s\searrow 0} A_2(s) = \kappa^\rho \frac{\rho}{\rho - N}\Gamma(1-\rho+N)$.

PROOF: Using the change of variables $w = [1 - \hat{\mathcal{L}}(\alpha s)]\kappa\alpha^{-1}t^{-1/\rho}$, we have

$$A_2(s) = (-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{\iota_1} s^{N-\rho\beta(\iota,k)} B_{N,\iota,k}(s) \left\{ \frac{[1 - \hat{\mathcal{L}}(\alpha s)]}{\alpha s} \kappa \right\}^{\rho-\rho\beta(\iota,k)} \times \int_0^\infty [1 - e^{-\tilde{\varphi}(s;w)}] \rho w^{-\rho[1-\beta(\iota,k)]-1}\, dw.$$

We next take the limit of each side as s goes to zero. Lemma 25 and the monotone convergence theorem imply that the limit can be brought inside the integral to yield

$$\lim_{s \searrow 0} A_2(s) = (-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{\iota_1} \Bigl(\lim_{s \searrow 0} s^{N-\rho\beta(\iota,k)} \Bigr) B_{N,\iota,k}(0) \kappa^{\rho-\rho\beta(\iota,k)} \times \int_0^\infty [1 - e^{-w}] \rho w^{-\rho[1-\beta(\iota,k)]-1}\, dw.$$

Noting that $N \geq \rho\beta(\iota,k)$, the term $\lim_{s\to 0} s^{N-\rho\beta(\iota,k)}$ is zero unless $N = \rho\beta(\iota,k)$. Thus $\lim_{s\searrow 0} A_2(s)$ can be written as

$$\lim_{s \searrow 0} A_2(s) = (-1)^N \kappa^{\rho-N} \int_0^\infty [1 - e^{-w}] \rho w^{-(\rho-N)-1}\, dw \sum_{\iota \in I_N} \sum_{k=0}^{\iota_1} B_{N,\iota,k}(0) \mathbb{I}_{N=\rho\beta(\iota,k)}.$$

The integral is $\int_0^\infty [1 - e^{-w}] \rho w^{-(\rho-N)-1}\, dw = \frac{\rho}{\rho-N}\Gamma(1-\rho+N)$. To finish the proof, we show that

$$(-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{\iota_1} B_{N,\iota,k}(0) \mathbb{I}_{N=\rho\beta(\iota,k)} = \kappa^N. \tag{42}$$

To see this, note first that $N = \rho\beta(\iota,k)$ requires $\iota = (N, 0, \dots, 0)$. If $\alpha > v$, $N = \rho\beta(\iota,k)$ also requires $k = 0$, whereas if $\alpha < v$, it requires $k = N$. If $\alpha = v$, $N = \rho\beta(\iota,k)$ for each $k \in \{0, \dots, N\}$. For each of these three cases, one can compute each nonzero term in the sum and verify equation (42). *Q.E.D.*

LEMMA 27: $\lim_{s\searrow 0} A_3(s) = 0$.

PROOF: The strategy is the same as in the previous lemma. Using the same change of variables $w = [1 - \hat{\mathcal{L}}(\alpha s)]\kappa\alpha^{-1}t^{-1/\rho}$, we have

$$A_3(s) = (-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{\iota_1} s^{N-\rho\beta(\iota,k)} B_{N,\iota,k}(s) \left\{ \frac{[1 - \hat{\mathcal{L}}(\alpha s)]}{\alpha s} \kappa \right\}^{\rho-\rho\beta(\iota,k)} \times \int_0^\infty [1 - e^{-\tilde{\varphi}(s;w)}][1 - e^{-[1-\hat{\mathcal{L}}(\alpha s)]^\rho (\kappa/\alpha)^\rho w^{-\rho}}] \rho w^{-\rho[1-\beta(\iota,k)]-1}\, dw.$$

We can take a limit of each side. Since both $\tilde{\varphi}(s; w)$ and $[1 - \hat{\mathcal{L}}(\alpha s)]^\rho (\kappa/\alpha)^\rho w^{-\rho}$ are nondecreasing in s in the neighborhood of $s = 0$, we can use the monotone convergence theorem to bring the limit inside the integral. Thus we have

$$\begin{aligned}\lim_{s\searrow 0} A_3(s) = (-1)^N \sum_{\iota\in I_N}\sum_{k=0}^{\iota_1}\Big(\lim_{s\searrow 0} s^{N-\rho\beta(\iota,k)}\Big) B_{N,\iota,k}(0)\kappa^{\rho-\rho\beta(\iota,k)} \\ \times \int_0^\infty \lim_{s\searrow 0}\big[1-e^{-\tilde{\varphi}(s;w)}\big]\big[1-e^{-[1-\hat{\mathcal{L}}(\alpha s)]^\rho(\kappa/\alpha)^\rho w^{-\rho}}\big]\rho w^{-\rho[1-\beta(\iota,k)]-1}\,dw.\end{aligned}$$

Since the limit of the integrand is zero for each integral, we have $\lim_{s\searrow 0} A_3(s) = 0$. *Q.E.D.*

We finally come to the main result.

PROPOSITION 5—1 of 2: *Suppose that Assumption* 2 *holds and that* $\rho \equiv \min\{\frac{1}{\alpha}, \frac{1}{(\varepsilon-1)/\zeta}\}$ *is not an integer. Then* $1-\mathcal{L}(l) \sim \frac{\kappa^\rho}{1-\alpha^\rho} l^{-\rho}$.

PROOF: The previous lemmas imply that

$$\lim_{s\searrow 0}\frac{\mu_N-(-1)^N\hat{\mathcal{L}}^{(N)}(s)}{s^{\rho-N}} = \alpha^\rho \lim_{s\searrow 0}\frac{\mu_N-(-1)^N\hat{\mathcal{L}}^{(N)}(s)}{s^{\rho-N}} + \kappa^\rho\frac{\rho}{\rho-N}\Gamma(1-\rho+N)+0$$

or

$$\lim_{s\searrow 0}\frac{\mu_N-(-1)^N\hat{\mathcal{L}}^{(N)}(s)}{s^{\rho-N}} = \frac{\kappa^\rho}{1-\alpha^\rho}\frac{\rho}{\rho-N}\Gamma(1-\rho+N).$$

By Theorem A in Bingham and Doney (1974), we therefore have that

$$\lim_{l\to\infty}\frac{1-\mathcal{L}(l)}{l^{-\rho}} = \frac{\kappa^\rho}{1-\alpha^\rho}\bigg[(-1)^N\frac{\Gamma(\rho-N)}{\Gamma(\rho)}\frac{\Gamma(1-\rho+N)}{\Gamma(1-\rho)}\bigg].$$

Since $\Gamma(\rho) = \Gamma(\rho-N)\prod_{k=1}^N(\rho-k)$ and $(-1)^N\Gamma(1-\rho+N) = \Gamma(1-\rho)\prod_{k'=1}^N(-1)\times(1-\rho+N-k') = \Gamma(1-\rho)\prod_{k=1}^N(\rho-k)$, the term in brackets equals unity, completing the proof. *Q.E.D.*

Next, we turn to the tail behavior of the conditional size distribution, $\mathcal{L}(\cdot|q)$.

PROPOSITION 5—2 of 2: *Suppose that Assumption* 2 *holds and that* $\rho \equiv \min\{\frac{1}{\alpha}, \frac{1}{(\varepsilon-1)/\zeta}\}$ *is not an integer. Then* $1-\mathcal{L}(l|q) \sim \frac{mq^{\alpha\zeta}}{\theta}\alpha^\rho[1-\mathcal{L}(l)]$.

PROOF: Lemma 21 and the definition of φ imply that the transform of $\mathcal{L}(\cdot|q)$ can be written as $\hat{\mathcal{L}}(s|q) = e^{-\varphi(s;\theta q^{-\zeta})}$, with derivatives

$$\hat{\mathcal{L}}^{(n)}(s|q) = e^{-\varphi(s;\theta q^{-\zeta})}\sum_{\iota\in I_n}\sum_{k=0}^{\iota_1} B_{n,\iota,k}(s)\big(\theta q^{-\zeta}\big)^{-\beta(\iota,k)}.$$

For $k \le n < \rho$, $B_{n,\iota,k}(0)$ is finite, so $\mu_n(q) = \hat{\mathcal{L}}^{(n)}(0|q) < \infty$ for each $n < \rho$. Then, using $t(q) \equiv \theta q^{-\zeta}$, we have

$$
\begin{aligned}
&\lim_{s\searrow 0} \frac{\hat{\mathcal{L}}^{(N)}(0|q) - \hat{\mathcal{L}}^{(N)}(s|q)}{s^{\rho-N}} \\
&= \lim_{s\searrow 0} \frac{\sum_{\iota\in I_N}\sum_{k=0}^{\iota_1} B_{N,\iota,k}(0)t(q)^{-\beta(\iota,k)} - e^{-\varphi(s;t(q))}\sum_{\iota\in I_N}\sum_{k=0}^{\iota_1} B_{N,\iota,k}(s)t(q)^{-\beta(\iota,k)}}{s^{\rho-N}} \\
&= \lim_{s\searrow 0} \sum_{\iota\in I_N}\sum_{k=0}^{\iota_1} \frac{B_{N,\iota,k}(0) - B_{N,\iota,k}(s)}{s^{\rho-N}} t(q)^{-\beta(\iota,k)} \\
&\quad + \lim_{s\searrow 0} \frac{1-e^{-\varphi(s;t(q))}}{s^{\rho-N}} \sum_{\iota\in I_N}\sum_{k=0}^{\iota_1} B_{N,\iota,k}(s)t(q)^{-\beta(\iota,k)}.
\end{aligned}
\tag{43}
$$

Using the logic of Lemma 24, the first term is

$$
\begin{aligned}
\lim_{s\searrow 0}\sum_{\iota\in I_N}\sum_{k=0}^{\iota_1}\frac{B_{N,\iota,k}(0)-B_{N,\iota,k}(s)}{s^{\rho-N}}t^{-\beta(\iota,k)} &= \lim_{s\searrow 0}\frac{\frac{\alpha^N\hat{\mathcal{L}}^{(N)}(0)}{\Gamma(1-\alpha)} - \frac{\alpha^N\hat{\mathcal{L}}^{(N)}(\alpha s)}{\Gamma(1-\alpha)}}{s^{\rho-N}}t^{-\alpha} \\
&\quad + \lim_{s\searrow 0}\sum_{\iota\in\tilde{I}_N}\sum_{k=0}^{\iota_1}\frac{\left[B_{N,\iota,k}(0)-B_{N,\iota,k}(s)\right]}{s^{\rho-N}}t^{-\beta(\iota,k)} \\
&= \frac{\alpha^\rho t^{-\alpha}}{\Gamma(1-\alpha)}\lim_{s\searrow 0}\frac{\hat{\mathcal{L}}^{(N)}(0)-\hat{\mathcal{L}}^{(N)}(s)}{s^{\rho-N}} + 0.
\end{aligned}
$$

The second term of (43) is zero because using L'Hospital's rule gives

$$
\lim_{s\searrow 0}\frac{1-e^{-\varphi(s;t)}}{s^{\rho-N}} = \lim_{s\searrow 0}\frac{\varphi^{(1)}(s;t)e^{-\varphi(s;t)}}{(\rho-N)s^{\rho-N-1}} = 0.
$$

We thus have

$$
\lim_{s\searrow 0}\frac{\hat{\mathcal{L}}^{(N)}(0|q)-\hat{\mathcal{L}}^{(N)}(s|q)}{s^{\rho-N}} = \frac{\alpha^\rho\left(\theta q^{-\zeta}\right)^{-\alpha}}{\Gamma(1-\alpha)}\lim_{s\searrow 0}\frac{\hat{\mathcal{L}}^{(N)}(0)-\hat{\mathcal{L}}^{(N)}(s)}{s^{\rho-N}}.
$$

The result follows from Theorem A in Bingham and Doney (1974) and noting that $\frac{m}{\theta} = \frac{\theta^{-\alpha}}{\Gamma(1-\alpha)}$. *Q.E.D.*

C.4. *The Cost Share of Intermediate Inputs*

Here we fill in the missing step from the proof of Proposition 7. We show that

$$
\rho \equiv \theta \int_0^\infty \int_0^\infty \left[V\left(\max\{u,\tilde{q}\}\right) - V(\tilde{q})\right] dF(\tilde{q})\zeta u^{-\zeta-1}\,du
$$

and $V(q) = \frac{1}{\varepsilon-1}(q/Q)^{\varepsilon-1}wL + \frac{m}{\theta}q^{\alpha\zeta}\rho$ jointly imply that $\rho = \frac{1}{1-\alpha}\frac{1}{\zeta}wL$.

To see this, starting with the expression for ρ, we can integrate the inner integral by parts and switch the order of integration to get

$$\begin{aligned}
\rho &= \theta \int_0^\infty \int_0^\infty \big[V\big(\max\{u, \tilde{q}\}\big) - V(\tilde{q})\big]\, dF(\tilde{q}) \zeta u^{-\zeta-1}\, du \\
&= \theta \int_0^\infty \int_0^u \big[V(u) - V(\tilde{q})\big]\, dF(\tilde{q}) \zeta u^{-\zeta-1}\, du \\
&= \theta \int_0^\infty \bigg[\big[V(u) - V(\tilde{q})\big] F(\tilde{q})\big|_0^u - \int_0^u -V'(\tilde{q}) F(\tilde{q})\, d\tilde{q}\bigg] \zeta u^{-\zeta-1}\, du \\
&= \theta \int_0^\infty \int_0^u V'(\tilde{q}) F(\tilde{q})\, d\tilde{q} \zeta u^{-\zeta-1}\, du \\
&= \theta \int_0^\infty \int_{\tilde{q}}^\infty \zeta u^{-\zeta-1}\, du V'(\tilde{q}) F(\tilde{q})\, d\tilde{q} \\
&= \theta \int_0^\infty \tilde{q}^{-\zeta} V'(\tilde{q}) F(\tilde{q})\, d\tilde{q}.
\end{aligned}$$

Using the expressions for V and F and the change of variables $x = \theta q^{-\zeta}$, we have

$$\begin{aligned}
\rho &= \theta \int_0^\infty \tilde{q}^{-\zeta} V'(\tilde{q}) F(\tilde{q})\, d\tilde{q} \\
&= \theta \int_0^\infty \tilde{q}^{-\zeta} \bigg[\frac{q^{\varepsilon-2}}{Q^{\varepsilon-1}} wL + \frac{m}{\theta} \alpha \zeta \tilde{q}^{\alpha\zeta-1} \rho\bigg] e^{-\theta \tilde{q}^{-\zeta}}\, d\tilde{q} \\
&= \frac{1}{\zeta} \int_0^\infty \bigg[\frac{\theta^{\frac{\varepsilon-1}{\zeta}} x^{-\frac{\varepsilon-1}{\zeta}}}{Q^{\varepsilon-1}} wL + \frac{m}{\theta^{1-\alpha}} \alpha \zeta x^{-\alpha} \rho\bigg] e^{-x}\, dx \\
&= \frac{1}{\zeta} \Bigg[\frac{\theta^{\frac{\varepsilon-1}{\zeta}} \Gamma\Big(1 - \dfrac{\varepsilon-1}{\zeta}\Big)}{Q^{\varepsilon-1}} wL + \frac{m}{\theta^{1-\alpha}} \Gamma(1-\alpha) \alpha \zeta \rho\Bigg] \\
&= \frac{1}{\zeta} [wL + \alpha \zeta \rho].
\end{aligned}$$

Solving for ρ gives $\rho = \frac{1}{\zeta(1-\alpha)} wL$.

C.5. *Identification of ζ*

This section presents an extended model in which entrepreneurs are divided into K groups indexed by k with measures $\mu_1, \dots, \mu_K$, respectively. The sales of any entrepreneur in group k are subject to the proportional tax of $\delta_k - 1$. This section shows that the ratio of intermediate input spending on inputs from suppliers in group k relative to intermediate input spending on inputs from suppliers in group k' is $\frac{\int_{j \in J_k} \sum_{\phi \in D_j} T(\phi)}{\int_{j \in J_{k'}} \sum_{\phi \in D_j} T(\phi)} = \frac{\mu_k \delta_k^{-\zeta}}{\mu_{k'} \delta_{k'}^{-\zeta}}$.

Let $F_k(q)$ be the fraction of entrepreneurs in group k with efficiency no greater than q. Among entrepreneurs in group k, let $F_{kk'}(q)$ be the fraction who have no techniques associated with suppliers in group k' that would deliver efficiency greater than q. Thus $F_k(q) = \prod_{k'=1}^K F_{kk'}(q)$.

CLAIM 5: *Suppose that Assumption 2 holds. Among entrepreneurs in each group, the fraction with efficiency no greater than q is $F_k(q) = e^{-\theta q^{-\zeta}}$, where θ satisfies*

$$\theta = m\Gamma(1-\alpha)\theta^\alpha \sum_{k=1}^{K} \mu_k \delta_k^{-\zeta} \tag{44}$$

and

$$F_{kk'}(q) = e^{m\Gamma(1-\alpha)\theta^\alpha \mu_{k'} \delta_{k'}^{-\zeta} q^{-\zeta}}. \tag{45}$$

PROOF: We follow the same steps as Section 3, first deriving expressions for the distribution of efficiencies under Assumption 1 and then taking the limit $z_0 \to 0$. The efficiency delivered by a technique with match-specific efficiency z that uses a supplier in group k' with efficiency q_s is $\frac{1}{\delta_{k'}} z q_s^\alpha$. For an entrepreneur in group k, the number of techniques that use suppliers in group k' follows a Poisson distribution with mean $M\mu_{k'}$. For each of those techniques, the probability that the technique delivers efficiency better than q is $\int_{z_0}^\infty [1 - F_{k'}((q\delta_{k'}/z)^{1/\alpha})]\, dH(z)$. We therefore have $F_{kk'}(q) = e^{-M\mu_{k'} \int_{z_0}^\infty [1-F_{k'}((q\delta_{k'}/z)^{1/\alpha})]\, dH(z)}$. Using $M = mz_0^{-\zeta}$ and $H(z) = 1 - (z/z_0)^{-\zeta}$, taking the limit as $z_0 \to 0$, and the change of variables $x = (q\delta_{k'}/z)^{1/\alpha}$ yields

$$\begin{aligned} F_{kk'}(q) &\to e^{-m\mu_{k'} \int_0^\infty [1-F_{k'}((q\delta_{k'}/z)^{1/\alpha})]\zeta z^{-\zeta-1}\, dz} \\ &= e^{-q^{-\zeta} \delta_{k'}^{-\zeta} m\mu_{k'} \int_0^\infty [1-F_{k'}(x)]\alpha\zeta x^{-\alpha\zeta-1}\, dx}. \end{aligned}$$

Multiplying over k' gives

$$F_k(q) = \prod_{k'} F_{kk'}(q) = e^{-q^{-\zeta} \sum_{k'} \delta_{k'}^{-\zeta} m\mu_{k'} \int_0^\infty [1-F_{k'}(x)]\alpha\zeta x^{-\alpha\zeta-1}\, dx}.$$

Define $\theta = \sum_{k'} \delta_{k'}^{-\zeta} m\mu_{k'} \int_0^\infty [1 - F_{k'}(x)]\alpha\zeta x^{-\alpha\zeta-1}\, dx$; we have that $F_k(q) = e^{-\theta q^{-\zeta}}$ for all k. Using this, we can integrate to get $\int_0^\infty [1 - F_{k'}(x)]\alpha\zeta x^{-\alpha\zeta-1}\, dx = \Gamma(1-\alpha)\theta^\alpha$. Substituting this back into the expressions for θ and $F_{kk'}$ gives (44) and (45). *Q.E.D.*

CLAIM 6: *Among entrepreneurs in each group with efficiency q, the fraction that use suppliers in group k is*

$$\frac{\mu_k \delta_k^{-\zeta}}{\sum_{k'=1}^{K} \mu_{k'} \delta_{k'}^{-\zeta}}. \tag{46}$$

PROOF: Consider an entrepreneur in group k_0. The probability that the entrepreneur has efficiency q and uses a supplier from group k is $dF_{k_0 k}(q) \prod_{k'' \neq k} F_{k_0 k''}(q)$, the product of the density of the efficiency delivered by the best technique that uses a supplier from k evaluated at q and the probability that all other techniques associated with suppliers in all other groups deliver efficiency no greater than q. The overall probability that the entrepreneur has efficiency q is the sum of these probabilities over all groups of suppliers, $\sum_{k'}[dF_{k_0k'}(q)\prod_{k''\neq k}F_{k_0k''}(q)]$. Thus, for an entrepreneur in group k_0, the conditional probability of using a supplier in group k, conditioning on having efficiency q, is

$$\frac{\prod_{k''\neq k}F_{k_0k''}(q)\,dF_{k_0k}(q)}{\sum_{k'}\prod_{k''\neq k}F_{k_0k''}(q)\,dF_{k_0k'}(q)}.$$

Using the expression for $F_{kk'}$ from (45) gives the desired result. *Q.E.D.*

CLAIM 7: *The fraction of payments for intermediate inputs to suppliers in group k is*

$$\frac{\mu_k\delta_k^{-\zeta}}{\sum_{k'=1}^{K}\mu_{k'}\delta_{k'}^{-\zeta}}.$$

PROOF: Among buyers in group k_0 with efficiency q, the transfer associated with a technique that gets used is independent of any characteristics of the supplier. This happens because both the surplus of the technique and the employment of the buyer depend only on techniques that are downstream from the buyer and the buyer's second best technique, both of which are independent of the supplier's characteristics. Thus the fraction of payments to suppliers in group k equals the probability that the supplier used is in group k. Since this probability equals (46) for each q, the fraction of total payments across all entrepreneurs in group k_0 of all efficiencies also equals (46). *Q.E.D.*

APPENDIX D: EVIDENCE FROM PRODUCER-LEVEL DATA

One prediction of the model is that if α is higher, then, all else equal, the size distribution will have a weakly thicker right tail. This section provides some preliminary evidence on the relationship between intermediate input shares and right tails of size distributions. In particular, I ask whether industries with larger intermediate input shares have size distributions with thicker tails.

I first study French firms. Di Giovanni, Levchenko, and Ranciere (2011) computed power law exponents for the tails of the distribution of revenue among French firms in each industry in 2006 and reported these in their Table A2. I compare these to intermediate input shares from the French Input Output Tables in 2006 taken from the World Input Output Database (Timmer, Dietzenbacher, Los, Stehrer, and Vries (2015)). Figure 5 plots each, while Table I shows regressions of the log of the tail index on the log of the intermediate input share. Di Giovanni, Levchenko, and Ranciere (2011) separated industries into tradeable and non-tradeable, and reported tail coefficients for the distribution of both domestic and total sales. In line with the theory, all four columns of Table I indicate that industries with higher intermediate input shares tend to have thicker right tails (lower exponent). This holds for the distributions of both total and domestic sales, and both across all industries and within categories.

Second, I study U.S. establishments. I compute power law tail exponents for establishment size distributions for each industry-year in the United States using an extract from

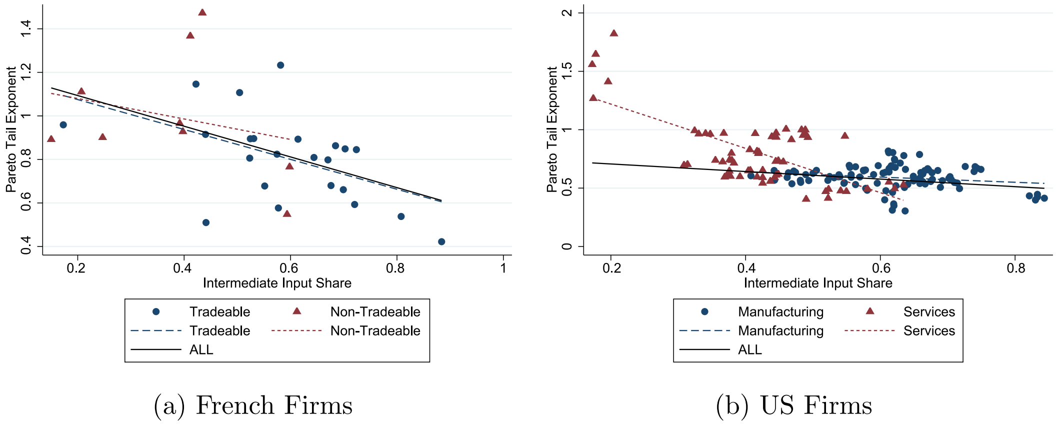


FIGURE 5.—Distribution of customers. For French and US, this figure shows the estimated power law tail exponent for each industry plotted against its intermediate input share. The figures also plot separate regression lines for all industries and for industries within each broad sector (tradeable vs non-tradeable for French firms, manufacturing vs. services for US firms).

the Statistics of U.S. Businesses made available by Rossi-Hansberg and Wright (2007) that corresponds to the distribution of employment across establishments in the years 1990, 1992, 1994, 1995, and 1997.[65] The extracts report the number of firms and number of establishments in 45 size categories for various two-digit SIC industries. Within each industry-year, I compute for each size category the fraction of firms/establishments above the minimum size of the category. I thus have 45 points of the counter-cumulative size distribution. To compute tail coefficients, I choose a size cutoff. I then regress the log of the counter-cumulative distribution on the log of the employment level among employment levels above the cutoff. The baseline results use a size cutoff of 150 employees, but results are similar when using cutoffs of 100, 200, or 400 employees.

TABLE I

FRENCH FIRMS[a]

Variables	All Sales (1)	Domestic Sales (2)	All Sales (3)	Domestic Sales (4)
log(Intermediate Input Share)	−0.297***	−0.234**	−0.253**	−0.197
	(0.0986)	(0.0973)	(0.123)	(0.122)
Observations	33	33	33	33
R-squared	0.200	0.135	0.210	0.143
Tradeable FE	No	No	Yes	Yes

[a] $^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$. Robust standard errors are in parentheses. In each regression, the log of the tail index is regressed on the log of the intermediate input share. The first and third columns use the tail index from the distribution of total revenue, whereas the second and fourth use domestic revenue. The third and fourth use fixed effects indicating whether the industry produces a tradeable good.

[65] Rossi-Hansberg and Wright (2007) also reported an extract for 2000, but I omit these because of the switch from SIC to NAICS.

TABLE II

U.S. FIRMS AND ESTABLISHMENTS[a]

	Firms		Establishments	
	All	Within Sector	All	Within Sector
Variables	(1)	(2)	(3)	(4)
log(Intermediate Input Share)	−0.144	−0.265	−0.122	−0.235
	(0.197)	(0.218)	(0.104)	(0.151)
Observations	280	280	280	280
R-squared	0.019	0.393	0.020	0.114
Sector FE	No	Yes	No	Yes

[a] $^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$. Robust standard errors clustered by industry are in parentheses. In each regression, the log of the tail index is regressed on the log of the intermediate input share. The second and fourth columns use fixed effects for broad SIC sector.

Table II and Figure 5(b) show the relationship between the tail coefficients and intermediate input shares from the same industry-year from the BEA. The results are suggestive but less than conclusive.[66] While the point estimates indicate that industries with higher intermediate input shares have both firm and establishment size distributions with thicker tails, none of the estimates are precise enough to distinguish statistically from zero.

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Co-editor Daron Acemoglu handled this manuscript.

Manuscript received 16 April, 2012; final version accepted 7 September, 2017; available online 8 September, 2017.

[66]The analogous figure for U.S. establishments is very similar to Figure 5(b).