Econometrica Supplementary Material

SUPPLEMENT TO "THE IDENTIFICATION OF BELIEFS FROM ASSET DEMAND"

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APPENDIX A: AFRIAT INEQUALITIES AND REVEALED BELIEFS

Our theoretical result about the possibility of identification has implications for revealed preference analysis. While with a finite number of observations beliefs obviously cannot be identified, we show that, as the number of observations goes to infinity and the observations become dense, the underlying characteristics can be uniquely recovered.

We assume that the number of states is fixed at some S, and one observes a collection of pairs $\{((x_0^i, y^i), (q^i, e_0^i))\}_{i=1}^I$, of choices of date 0 consumption and holdings of assets, $\{x_0^i, y^i\}$, at prices and incomes $\{q^i, e_0^i\}$. We assume throughout that $x_0^i + q^i y^i = e_0^i$, and that $x_0^i > 0$, for all $i = 1, \dots, I$.

We define unobservable characteristics to be $(u(\cdot), \xi) = (u(\cdot), (\beta, \pi, R, e))$ (we omit S, since it is fixed), and define demand, as a function of prices, income, and these characteristics, as

$$\begin{aligned} &(x_0, y)\big(q, e_0; u(\cdot), \xi\big) \\ &\qquad = \arg\max_{x\geq 0, y} u(x_0) + \beta \sum_{s=1}^{S} \pi_s u(x_s), \\ &\text{s.t.} \quad x_0 + qy \leq e_0, \\ &\qquad\quad x_s - R_s y \leq e_s, \quad s = 1, \dots, S. \end{aligned}$$

The following lemma gives necessary and sufficient conditions for the observations to be consistent with expected utility maximization.

LEMMA A.1: *The following two statements are equivalent*:

1 *There exist fundamentals* $(u(\cdot), \xi)$ *such that, for all* $i = 1, \dots, I$,

$$\big(x_0^i, y^i\big) \in (x_0, y)\big(q^i, e_0^i; u(\cdot), \xi\big),$$

and such that $e_s + R_s y^i > 0$ *for all* $s = 1, \dots, S$.

2 *There exists* $(\{m^i\}_{i=1}^I, \xi) = (\{m^i\}_{i=1}^I, (\beta, \pi, e, R))$, *with* $m^i \in \mathbb{R}_{++}^{S+1}$, *for* $i = 1, \dots, I$, *such that*

– *for all* $i = 1, \dots, I$,

$$m_0^i q^i = \beta \sum_s \pi_s R_s m_s^i; \tag{A.1}$$

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– *for all* $i, j = 1, \ldots, I$ *and all* $s, s' = 0, \ldots, S$,

$$\left(m_s^i - m_{s'}^j\right)\left(c_s^i - c_{s'}^j\right) \leq 0, \quad \textit{and} \quad < 0, \quad \textit{if } c_s^i \neq c_{s'}^j, \tag{A.2}$$

where $c_s^i = e_s + R_s y > 0$, *for* $s = 1, \ldots, S$ *and* $c_0^i = e_0^i - q^i y = x_0^i$.

The proof follows directly from Varian (1983). Note that, since asset demand has to satisfy the Strong Axiom of Revealed Preference, these conditions are not vacuous: there exist observations of asset demands and date 0 consumption that cannot be rationalized by any characteristics.

Lemma A.1 gives a method[1] for the construction of beliefs and attitudes towards risk that rationalize a finite data set. Evidently, with a finite number of observations, these characteristics are not unique, and recovered beliefs need not coincide with the true beliefs of the decision maker. To address this issue, we complement the revealed preference argument with a convergence result: it guarantees that recovered characteristics, in particular, beliefs, converge to the true ones as the data points become dense.

We want to prove that when the number of observations goes to infinity, a set of solutions to the Afriat inequalities from Lemma A.1 converges to a singleton which only contains the true underlying characteristic. In order to do so, we consider a nested sequence of sets of observations. We denote by $\mathcal{D}^n$ a set of n observed prices and date 0 endowments. We denote by $\mathcal{D}$ an open set of prices and date 0 endowments for which the demand function is well defined and invertible, and as above, denote by $\mathcal{X}_0$ the projection of the open set of observed choices on date 0 consumption.

PROPOSITION A.1: *Let* $(\mathcal{D}^n \subset \mathcal{D} : n = 1, \ldots)$ *be an increasing sequence of finite sets of observed prices and date* 0 *endowments with* $\ldots, \mathcal{D}^n \subset \mathcal{D}^{n+1}, \ldots,$ *and with* $\bigcup_n \mathcal{D}^n$ *dense in an open set* $\mathcal{D} \subset \mathbb{R}^A \times \mathbb{R}_+$. *Let* $(\hat{x}_0, \hat{y})(q, e_0)$ *be a continuous function on* $\mathcal{D}$, *let*

$$\left(x_0^{i,n}, y^{i,n}\right) = (\hat{x}_0, \hat{y})\left(q^{i,n}, e_0^{i,n}\right), \quad i = 1, \ldots, n,$$

and suppose that for each n *there exist* $(\{m^{i,n}\}_{i=1}^n, \xi^n)$ *satisfying* (A.1) *and* (A.2) *for the observations* $\{(x_0^{i,n}, y^{i,n}), (q^{i,n}, e_0^{i,n})\}_{i=1}^n$ *and satisfying* $(m^{i,n}, \xi^n) \in \mathcal{K}$ *for each* i, n, *and some compact set* $\mathcal{K}$.

Then there exist fundamentals $(u^*(\cdot), \xi^*)$ *such that*

$$(\hat{x}_0, \hat{y})(q, e_0) = (x_0, y)\left(q, e_0; u^*(\cdot), \xi^*\right) \quad \textit{for all } (q, e_0) \in \mathcal{D}.$$

Moreover, if these fundamentals satisfy the sufficient conditions of Theorem 1, *then* $\xi^n \to \xi^*$ *and* $u^n(\cdot) \to u^*(\cdot)$, *where* $u^n(\cdot)$ *is the piecewise linear function with slopes* $\alpha m_s^{i,n}$ *at* $c_s^{i,n}$ *for all* (i, n), s, *with an* $\alpha > 0$ *that ensures the normalization* $u^n(1) = 0$ *and* $u^{n\prime}(1) = 1$.

PROOF: Consider the sequence $((u^n(\cdot), \xi^n) : n = 1, \ldots)$, and note that, by compactness of $\mathcal{K}$, the $u^n(\cdot)$ are equicontinuous and there exists an accumulation point $(\bar{u}(\cdot), \bar{\xi})$. Since $\bar{u}$ must be concave, it must be continuous on $\mathcal{X}_0$. Note that each $(u^n(\cdot), \xi^n)$ as well as $(\bar{u}(\cdot), \bar{\xi})$ correspond to continuous, increasing, and concave indirect utility functions, $v^n(x_0, y)$ and $\bar{v}(x_0, y)$, over date 0 consumption and assets.

[1]It is beyond the scope of this paper to investigate the existence of efficient algorithms for the determination of a solution to the Afriat inequalities.

We first argue that the limit characteristics must generate a demand function that is identical to $(\hat{x}_0, \hat{y})(\cdot)$; that is, for all $(q, e_0) \in \mathcal{D}$,

$$(x_0, y)\big(q, e_0; \bar{u}(\cdot), \bar{\xi}\big) = (\hat{x}_0, \hat{y})(q, e_0).$$

If not, there exist $(q^*, e_0^*) \in \mathcal{D}$ and $(x_0^*, y^*) = (\hat{x}_0, \hat{y})(q^*, e_0^*)$ as well as $(\bar{x}_0, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}^A$ such that $\bar{v}(\bar{x}_0, \bar{y}) > \bar{v}(x_0^*, y^*)$ while $\bar{x}_0 + q^*\bar{y} \leq e_0^*$ and, by the continuity and concavity of $\bar{u}$, without loss of generality,

$$\bar{x}_0 + q^*\bar{y} < e_0^*.$$

Since $\bigcup_n \mathcal{D}^n \subset \mathcal{D}$ is dense, there exists a sequence $(q^n, e_0^n) \in \mathcal{D}^n : n = 1, \dots)$ such that $(q^n, e_0^n) \to (q^*, e_0^*)$. By continuity of $(\hat{x}_0, \hat{y})(\cdot)$, there is an associated sequence $(x_0^n, y^n) \to (x_0^*, y^*)$.

Since $\bar{v}(\cdot)$ is continuous, there is an n sufficiently large such that

$$\bar{v}\big(x_0^n, y^n\big) < \bar{v}(\bar{x}_0, \bar{y})$$

and

$$\bar{x}_0 + q^n\bar{y} < e_0^n.$$

But since the sets $\mathcal{D}^n$ are nested, for all $m \geq n$,

$$v^m\big(x_0^n, y^n\big) \geq v^m(\bar{x}_0, \bar{y}),$$

which contradicts the fact that $v^m \to \bar{v}$.

To prove the second part of the result, note that Lemma 1 together with the fact that $\bar{u}(\cdot)$ must be differentiable almost everywhere imply that $\bar{u}$ is the unique cardinal utility that generates the observed demand. The Identification Theorem implies that fundamentals are unique and that therefore the accumulation point must be the unique limit of the sequence $((u^n(\cdot), \xi^n) : n = 1, \dots)$. *Q.E.D.*

Mas-Colell (1977) made the argument in a different setting. Our proof differs from his in that we work in the space of utility functions while he showed convergence in preferences. It is not clear how to directly apply his proof strategy and show that the limiting preferences over assets can be represented by expected utility over consumption.

APPENDIX B: WRONSKIANS AND POWER SERIES

While polynomials are linearly independent, functions that can be written as power series might not be. It is instructive to consider the case of CARA cardinal utility,

$$u(x) = -e^{-x} = -1 + x + \cdots + (-1)^{(k+1)}\frac{1}{k!}x^k + \cdots + (-1)^{(n+1)}\frac{1}{n!}x^n, \dots.$$

In order to simplify the exposition, we restrict attention to the case $S = 2$. The polynomial approximation of $u(x)$ of order n is

$$u_n(x) = -1 + x + \cdots + (-1)^{(k+1)}\frac{1}{k!}x^k + \cdots + (-1)^{(n+1)}\frac{1}{n!}x^n;$$

evidently,

$$u_n^{(1)}(x) = 1 - x + \cdots + (-1)^{(k+1)} \frac{1}{(k-1)!} x^{(k-1)} + \cdots + (-1)^{(n+1)} \frac{1}{(n-1)!} x^{(n-1)},$$

and

$$u_n^{(2)}(x) = -1 + x + \cdots + (-1)^{(k+1)} \frac{1}{(k-2)!} x^{(k-2)} + \cdots + (-1)^{(n+1)} \frac{1}{(n-2)!} x^{(n-2)}.$$

It follows that, if

$$A^{2,n} = \begin{pmatrix} 1 & -1 & \dots & (-1)^{k+1}\frac{1}{(k-1)!} & \dots & \dots & (-1)^{(n+1)}\frac{1}{(n-1)!} \\ -1 & 1 & \dots & (-1)^{k+2}\frac{1}{(k-1)!} & \dots & (-1)^{n+1}\frac{1}{(n-2)!} & 0 \end{pmatrix},$$

and

$$B_n^2 = \begin{pmatrix} 1 & 1 \\ (e_1+x) & (e_2+x) \\ \vdots & \vdots \\ (e_1+x)^k & (e_2+x)^k \\ \vdots & \vdots \\ (e_1+x)^{(n-1)} & (e_2+x)^{(n-1)} \end{pmatrix},$$

the Wronskian of the family of functions $\{u^{(n-S+1)}(e_s+x)\}$, that is, of the derivatives of order $(n-S+1)$ of the functions $\{u(e_s+x)\}$, is

$$\begin{aligned} W_{2,n} &= A_n^2 B_n^2 \\ &= \begin{pmatrix} 1 & -1 & \dots & \dots & (-1)^{(n+1)}\frac{1}{(n-1)!} \\ -1 & 1 & \dots & (-1)^{n+1}\frac{1}{(n-2)!} & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 1 \\ (e_1+x) & (e_2+x) \\ \vdots & \vdots \\ (e_1+x)^k & (e_2+x)^k \\ \vdots & \vdots \\ (e_1+x)^{(n-1)} & (e_2+x)^{(n-1)} \end{pmatrix}. \end{aligned}$$

For finite n, Proposition 1(3) implies that the rank of $W_{2,n} = A_n^2 B_n^2$ is full, even if this is not clear from the expressions above. But, as $n \to \infty$, the matrix A_n^2 converges to a matrix of row rank 1, which implies that the Wronskian is singular; this accounts for the failure of identification of CARA cardinal utility.

Alternatively, for CRRA cardinal utility, and, in particular,

$$u(x) = \ln x,$$

the power series expansion at $\bar{x}=1$ is

$$u(x)=\ln x=0+(x-1)+\cdots+(-1)^{(k-1)}\frac{1}{k}(x-1)^k+\cdots+(-1)^{(n-1)}\frac{1}{n}(x-1)^n,\ldots.$$

In order to simplify the exposition, we restrict attention to the case $S=2$. The polynomial approximation of $u(x)$ of order n is

$$u_n(x)=0+(x-1)+\cdots+(-1)^{(k-1)}\frac{1}{k}(x-1)^k+\cdots+(-1)^{(n-1)}\frac{1}{n}(x-1)^n;$$

evidently,

$$u_n^{(1)}(x)=1-x+\cdots+(-1)^k(x-1)^k+\cdots+(-1)^{(n-1)}x^{(n-1)},$$

and

$$u_n^{(2)}(x)=-1+x+\cdots+(-1)^{(k+1)}(k+1)(x-1)^k+\cdots+(-1)^{(n-1)}(n-1)x^{(n-2)}.$$

It follows that

$$A_n^2=\begin{pmatrix}1 & -1 & \ldots & (-1)^k & \ldots & \ldots & (-1)^{(n-1)}\\ -1 & 2 & \ldots & (-1)^{k+1}(k+1) & \ldots & (-1)^{n-1} & 0\end{pmatrix},$$

$$B_n^2=\begin{pmatrix}1 & 1\\ (e_1+x-1) & (e_2+x-1)\\ \vdots & \vdots\\ (e_1+x-1)^k & (e_2+x-1)^k\\ \vdots & \vdots\\ (e_1+x-1)^{(n-1)} & (e_2+x-1)^{(n-1)}\end{pmatrix},$$

and the Wronskian of the family of functions $\{u^{(n-S+1)}(e_s+x)\}$ is

$$\begin{aligned}W_{2,n}&=A_n^2B_n^2\\ &=\begin{pmatrix}1 & -1 & \ldots & \ldots & (-1)^{(n+1)}\dfrac{1}{(n-1)!}\\ -1 & 2 & \ldots & (-1)^{n+1}\dfrac{1}{(n-2)!} & 0\end{pmatrix}\\ &\quad\times\begin{pmatrix}1 & 1\\ (e_1+x-1) & (e_2+x-1)\\ \vdots & \vdots\\ (e_1+x-1)^k & (e_2+x-1)^k\\ \vdots & \vdots\\ (e_1+x-1)^{(n-1)} & (e_2+x-1)^{(n-1)}\end{pmatrix}.\end{aligned}$$

For all n, and as $n\to\infty$, the matrix A_n^2 remains of rank 2; this is in contrast to the CARA case.

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