Econometrica Supplementary Material

SUPPLEMENT TO "WHEN DOES PREDATION DOMINATE COLLUSION?"

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S1. PROOF FROM SECTION 4.4

PROOF OF CLAIM 1: LET $\alpha(k)$ DENOTE THE VALUE OF $\alpha_1(k, 1)$, the guaranteed probability of winning a price war for firm 1 when it is in state k and firm 2 is in state 1. Observe that $\alpha(K) = \alpha^*(2)$, so Condition 1 is equivalent to $\alpha(K) > R/2$.

The function $\alpha(k)$ satisfies the following system of equations:

$$\alpha(1) = \gamma^D\big(1 - \theta\gamma^D\big) + \big(1 - \gamma^D\big)\big(1 - \theta\gamma^D\big)\alpha(1),$$
$$\alpha(k) = \gamma^D + \big(1 - \gamma^D\big)\big[\big(1 - \theta\gamma^D\big)\alpha(k) + \theta\gamma^D\alpha(k - 1)\big] \quad \text{for } k > 1.$$

That is, when both firms are in state 1, then firm 1 wins the price war if firm 2's state deteriorates and firm 1's does not. If neither firm's state changes, then firm 1 has again probability $\alpha(1)$ of winning. Similarly, when firm 1 starts at state $k > 1$, then it wins for sure if firm 2's state moves down, it wins with probability $\alpha(k - 1)$ if only firm 1's state moves down, and it wins with probability $\alpha(k)$ if neither state changes. Recall that in the example, a firm's state cannot move up during a price war.

Rearranging yields

$$\alpha(1) = \frac{1 - \theta\gamma^D}{1 + \theta\big(1 - \gamma^D\big)}, \qquad \alpha(k) = 1 - \beta + \beta\alpha(k - 1) \quad \text{for } k > 1,$$

where

$$\beta \equiv \frac{\theta\big(1 - \gamma^D\big)}{1 + \theta\big(1 - \gamma^D\big)} \in (0, 1).$$

The dependence of β on θ and γ^D is suppressed for readability. Taking differences yields

$$\Delta(2) = (1 - \beta)\big(1 - \alpha(1)\big), \qquad \Delta(k) = \beta\Delta(k - 1) \quad \text{for } k > 2,$$

where $\Delta(k) \equiv \alpha(k) - \alpha(k - 1)$. Thus,

$$\begin{aligned}\alpha(k) &= \alpha(1) + \Delta(2) + \Delta(3) + \cdots + \Delta(k)\\ &= \alpha(1) + \Delta(2) + \beta\Delta(2) + \cdots + \beta^{k-2}\Delta(2)\\ &= \alpha(1) + \frac{1 - \beta^{k-1}}{1 - \beta}\Delta(2)\\ &= \alpha(1) + \big(1 - \alpha(1)\big)\big(1 - \beta^{k-1}\big),\end{aligned}$$

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so $\alpha(K) > R/2$ if

$$\frac{R}{2} < \alpha(1) + \bigl(1-\alpha(1)\bigr)\bigl(1-\beta^{K-1}\bigr)$$

$$\Leftrightarrow$$

$$\frac{\theta^K\bigl(1-\gamma^D\bigr)^{K-1}}{\bigl[1+\theta\bigl(1-\gamma^D\bigr)\bigr]^K} = \beta^{K-1}\bigl(1-\alpha(1)\bigr) < 1-\frac{R}{2}.$$

The left-hand side is maximized at $\gamma^D = (\theta - K + 1)/\theta$, so plugging in that value gives the sufficient condition

$$\frac{\theta/(K-1)}{\bigl[K/(K-1)\bigr]^K} < 1-\frac{R}{2}.$$

At $K = 2$, that condition becomes

$$\frac{\theta}{4} < 1-\frac{R}{2}.$$

Rearranging yields the second part of the claim.

For the first part of the claim, note that $[K/(K-1)]^K > e$, so $\alpha(K) > R/2$ if

$$\frac{\theta/(K-1)}{\bigl[K/(K-1)\bigr]^K} < \frac{\theta/(K-1)}{e} < 1-\frac{R}{2}$$

$$\Leftrightarrow$$

$$K > \frac{\theta}{\left(1-\dfrac{R}{2}\right)e} + 1,$$

as desired. *Q.E.D.*

S2. PROOFS FROM SECTION 5

All the examples in this section are from the Bertrand case, and in all the following propositions, "sustainable" means that there is an SPE that generates the specified outcome, and "unsustainable" means that there is no Nash equilibrium that yields that outcome.

S2.1. *Patience and Cartel Size*

EXAMPLE S1: There are two, three, or four firms ($N \in \{2, 3, 4\}$) and two non-bankruptcy states ($S = \{1, 2\}$). The discount factor is either $\delta = 0.8$ or $\delta = 0.82$. Transition rates satisfy the following:

- $\Gamma(\pi, 2)[1] = 0.1$ and $\Gamma(\pi, 2)[0] = 0$ for all $\pi \geq 0$;
- $\Gamma(\pi, 1)[1] > 0.1$ for all $\pi \geq 0$ and $\Gamma(0, 1)[1] = 1$; and
- $\Gamma(0, 1)[0] = 0.09$ and $\Gamma(\pi, 1)[0] = 0$ for all $\pi > 0$.

That is, the probability that a strong firm becomes weak is 0.1 for any level of profits. The probability that a weak firm becomes strong is less than 0.9 for any profit and 0 given no profit. Only a weak firm earning zero profit can go bankrupt, and the probability that such a firm goes bankrupt is 0.09.

EXAMPLE S2: There are two, three, or four firms ($N \in \{2, 3, 4\}$) and two non-bankruptcy states ($S = \{1, 2\}$). The discount factor is either $\delta = 0.9$ or $\delta = 0.95$. Transition rates satisfy the following:

- $\Gamma(\pi, 2)[1] = 0.15$ and $\Gamma(\pi, 2)[0] = 0$ for all $\pi \geq 0$;
- $\Gamma(\pi, 1)[1] > 0.15$ for all $\pi \geq 0$ and $\Gamma(0, 1)[1] = 1$; and
- $\Gamma(0, 1)[0] = 0.1$ and $\Gamma(\pi, 1)[0] = 0$ for all $\pi > 0$.

PROPOSITION S1: *In Example S1, when $N = 4$, collusion on the monopoly price π^M is unsustainable when $\delta = 0.8$ but sustainable when $\delta = 0.82$. In Example S2, when $N = 3$, collusion on the monopoly price π^M is sustainable when $\delta = 0.9$ but unsustainable when $\delta = 0.95$.*

PROPOSITION S2: *In Example S1, when $\delta = 0.8$, collusion on the monopoly price π^M is sustainable when $N = 2$ or $N = 3$ but unsustainable when $N = 4$. In Example S2, when $\delta = 0.9$, collusion on the monopoly price π^M is unsustainable when $N = 2$ or $N = 4$ but sustainable when $N = 3$.*

PROOF OF PROPOSITIONS S1 AND S2: Note that a price war (where all firms price at 0 until only one is left) is always an SPE, and it gives all players their minmax payoffs. Thus, a given on-path strategy profile is sustainable (in either Nash or subgame perfect equilibrium) if and only if no firm can gain by a one-shot deviation followed by a price war.

First consider Example S1, and suppose that there are two active firms ($n = 2$). Let $\underline{v}_2^{ss'}(\delta)$ denote the expected continuation payoff after a deviation (i.e., the payoff from a price war) to a firm in state s whose rival is in state s'. In particular,

$$\underline{v}_2^{21}(\delta) = (1-\delta)0 + \delta\big[0.09\pi^M + (1-0.09)0.1\underline{v}_2^{11}(\delta) + (1-0.09)(1-0.1)\underline{v}_2^{21}(\delta)\big],$$
$$\underline{v}_2^{11}(\delta) = (1-\delta)0 + \delta\big[0.09 \cdot 0 + (1-0.09)0.09\pi^M + (1-0.09)^2\underline{v}_2^{11}(\delta)\big].$$

Collusion on the monopoly price p^M is part of an SPE if a firm prefers collusive profits ($\pi^M/2$) to the profit from an optimal deviation: undercutting its rival and starting a price war. Note that if a firm prefers not to deviate when it is strong (state 2) and its rival is weak (state 1), then it cannot gain from deviating in any other state vector. The reason is that the equilibrium payoff is the same for any state vector, and so is the one-shot profit from undercutting, while the expected continuation payoff in a price war $\underline{v}_2^{ss'}(\delta)$ is increasing in the firm's state s tomorrow and decreasing in its rival's s'; the condition in both examples that $\Gamma(\pi, 1)[1] > \Gamma(\pi, 2)[1]$ for all π implies that tomorrow's state is positively correlated with today's for each firm.

Let $\Delta_2(\delta)$ denote the payoff from that optimal deviation:

$$\Delta_2(\delta) \equiv (1-\delta)\pi^M + \delta\big[0.09\pi^M + (1-0.09)0.1\underline{v}_2^{11}(\delta) + (1-0.09)(1-0.1)\underline{v}_2^{21}(\delta)\big].$$

At $\delta = 0.8$, solving yields $\underline{v}_2^{21}(0.8) \approx 0.250\pi^M$, $\underline{v}_2^{11}(0.8) \approx 0.194\pi^M$, and $\Delta_2(0.8) \approx 0.450\pi^M$. Since $\Delta_2(0.8) < \frac{1}{2}\pi^M$, no deviation is profitable, and collusion is sustainable.

When there are three active firms, again the best time to deviate is when a firm is strong and its rivals are weak. The relevant price war payoffs in that case are

$$\begin{aligned}\underline{v}_3^{211}(\delta) &= (1-\delta)0 + \delta\big[(0.09)^2\pi^M + 2\cdot 0.09(0.91)\big[0.1\underline{v}_2^{11}(\delta)+0.9\underline{v}_2^{21}(\delta)\big] \\ &\quad + (0.91)^2\big[0.1\underline{v}_3^{111}(\delta)+0.9\underline{v}_3^{211}(\delta)\big]\big], \\ \underline{v}_3^{111}(\delta) &= (1-\delta)0 \\ &\quad + \delta\big[0.09\cdot 0 + (0.09)^2(0.91)\pi^M + 2\cdot 0.09(0.91)^2\underline{v}_2^{11}(\delta) + (0.91)^3\underline{v}_3^{111}(\delta)\big].\end{aligned}$$

The payoff from the optimal deviation, $\Delta_3(\delta)$, is

$$\begin{aligned}&(1-\delta)\pi^M + \delta\big[(0.09)^2\pi^M + 2\cdot 0.09(0.91)\big[0.1\underline{v}_2^{11}(\delta)+0.9\underline{v}_2^{21}(\delta)\big] \\ &\quad + (0.91)^2\big[0.1\underline{v}_3^{111}(\delta)+0.9\underline{v}_3^{211}(\delta)\big]\big].\end{aligned}$$

At $\delta = 0.8$, solving yields $\underline{v}_3^{211}(0.8) \approx 0.107\pi^M$, $\underline{v}_3^{111}(0.8) \approx 0.073\pi^M$, and $\Delta_3(0.8) \approx 0.307\pi^M$. Since $\Delta_3(0.8) < \frac{1}{3}\pi^M$, no deviation is profitable, and collusion is again sustainable. Repeating that analysis for the case of four active firms, however, yields $\underline{v}_4^{2111}(0.8) \approx 0.056\pi^M$, $\underline{v}_{34}^{1111}(0.8) \approx 0.034\pi^M$, and $\Delta_4(0.8) \approx 0.256\pi^M$. Since $\Delta_4(0.8) > \frac{1}{4}\pi^M$, a firm gains by undercutting when it is strong and its rivals are all weak. Thus, collusion is not sustainable.

In summary: in Example S1, collusion on the monopoly price is sustainable at $\delta = 0.8$ when there are two or three active firms, but not when there are four. Applying the same analysis for $\delta = 0.82$ yields $\Delta_4(0.82) \approx 0.247\pi^M$. Since $\Delta_4(0.82) < \frac{1}{4}\pi^M$, in that setting collusion is sustainable.

In Example S2, similar calculations yield $\Delta_2(0.9) \approx 0.506\pi^M$, $\Delta_3(0.9) \approx 0.3328\pi^M$, $\Delta_4(0.9) \approx 0.252\pi^M$, and $\Delta_3(0.95) \approx 0.395\pi^M$. Thus, at $\delta = 0.9$, collusion on the monopoly price is sustainable by three firms but not by two or four. At $\delta = 0.95$, such collusion is not sustainable by three firms, either. *Q.E.D.*

S2.2. *Unequal Market Shares*

EXAMPLE S3: There are two firms ($N = 2$) and two non-bankruptcy states ($S = \{1, 2\}$). The discount factor is $\delta = 0.95$. Transition rates satisfy the following:

- $\Gamma(\pi, 2)[1] = 0.17$ for $\pi \in [0, 0.75\pi^M]$, $\Gamma(\pi, 2)[1] = 0.05$ for $\pi > 0.75\pi^M$, and $\Gamma(\pi, 2)[0] = 0$ for all $\pi \geq 0$;
- $\Gamma(\pi, 1)[2] = 0.05$ for $\pi \in (0, 0.25\pi^M)$, $\Gamma(\pi, 1)[2] = 0.1$ for $\pi \geq 0.25\pi^M$, and $\Gamma(0, 1)[2] = 0$; and
- $\Gamma(0, 1)[0] = 0.1$ and $\Gamma(\pi, 1)[0] = 0$ for all $\pi > 0$.

PROPOSITION S3: *In Example S3, collusion on the monopoly price π^M where the firms split the monopoly profit π^M each period is unsustainable. If the firms can divide market demand (or, equivalently, if they can transfer money to each other), then there is a collusive SPE in which* (i) *the firms set the monopoly price p^M each period,* (ii) *when $s_1 = s_2$, both firms get profit $\pi^M/2$, and* (iii) *when $s_i > s_j$, firm i gets profit $0.9\pi^M$ and firm j gets profit $0.1\pi^M$.*

PROOF: The proof is similar to the proofs of Propositions S1 and S2. First, calculating the value of $\underline{v}^{21}$, the expected continuation payoff from a price war to a firm in state 2 whose rival is in state 1 yields $\underline{v}^{21} \approx 0.513\pi^M > \frac{1}{2}\pi^M$, so equal sharing is not sustainable: the strong firm would prefer to start a price war.

As before, a price war is always an SPE. Therefore, again as before, to prove that the specified unequal sharing strategy is an SPE, it is sufficient to show that no firm can gain by a one-shot, on-path deviation followed by a price war. Let $\hat{v}^{ss'}$ denote the expected continuation payoff from the specified strategy to a firm in state s whose rival is in state s'. In particular,

$$\hat{v}^{21} = (1-\delta)0.9\pi^M + \delta\big[(0.05)^2\hat{v}^{12} + (0.05)(0.95)\big(\hat{v}^{22} + \hat{v}^{11}\big) + (0.95)^2\hat{v}^{21}\big],$$
$$\hat{v}^{12} = (1-\delta)0.1\pi^M + \delta\big[(0.05)^2\hat{v}^{21} + (0.05)(0.95)\big(\hat{v}^{22} + \hat{v}^{11}\big) + (0.95)^2\hat{v}^{12}\big],$$
$$\hat{v}^{22} = (1-\delta)0.5\pi^M + \delta\big[(0.17)^2\hat{v}^{11} + (0.17)(0.83)\big(\hat{v}^{21} + \hat{v}^{12}\big) + (0.83)^2\hat{v}^{22}\big],$$
$$\hat{v}^{11} = (1-\delta)0.5\pi^M + \delta\big[(0.1)^2\hat{v}^{22} + (0.1)(0.9)\big(\hat{v}^{21} + \hat{v}^{12}\big) + (0.9)^2\hat{v}^{11}\big].$$

Solving yields $\hat{v}^{21} \approx 0.638\pi^M$, $\hat{v}^{12} \approx 0.362\pi^M$, and $\hat{v}^{22} = \hat{v}^{11} = 0.5\pi^M$. Next, let $\Delta^{ss'}$ denote the expected continuation payoff from the optimal deviation (undercutting the price p^M and starting a price war) to a firm in state s whose rival is in state s':

$$\Delta^{21} = (1-\delta)\pi^M + \delta\big[0.1\pi^M + 0.9\big(0.05\underline{v}^{11} + 0.95\underline{v}^{21}\big)\big],$$
$$\Delta^{12} = (1-\delta)\pi^M + \delta\big[(0.1)(0.17)\underline{v}^{21} + (0.1)(0.83)\underline{v}^{22} + (0.9)(0.17)\underline{v}^{11} + (0.9)(0.83)\underline{v}^{12}\big],$$
$$\Delta^{22} = (1-\delta)\pi^M$$
$$+ \delta\big[(0.05)(0.17)\underline{v}^{11} + (0.05)(0.83)\underline{v}^{12} + (0.95)(0.17)\underline{v}^{21} + (0.95)(0.83)\underline{v}^{22}\big],$$
$$\Delta^{11} = (1-\delta)\pi^M + \delta\big[0.1\pi^M + 0.9\big(0.1\underline{v}^{21} + 0.9\underline{v}^{11}\big)\big].$$

Solving yields $\Delta^{21} \approx 0.577\pi^M$, $\Delta^{12} \approx 0.268\pi^M$, $\Delta^{22} \approx 0.364\pi^M$, and $\Delta^{11} \approx 0.474\pi^M$. Since $\Delta^{ss'} < \hat{v}^{ss'}$ for every state vector ss', deviating is never profitable, and the specified strategy is an SPE. *Q.E.D.*

S2.3. *Recurring Collusion*

EXAMPLE S4: There are two firms ($N = 2$) and two non-bankruptcy states ($S = \{1, 2\}$). The discount factor is $\delta = 0.95$. Transition rates satisfy the following:

- $\Gamma(\pi, 2)[1] = 0.05$ and $\Gamma(\pi, 2)[0] = 0$ for all $\pi \geq 0$;
- $\Gamma(\pi, 1)[1] = 0.99$ for all $\pi \geq \pi^M/2$ and $\Gamma(0, 1)[1] = 1$; and
- $\Gamma(0, 1)[0] = 0.06$ and $\Gamma(\pi, 1)[0] = 0$ for all $\pi > 0$.

PROPOSITION S4: *In Example S4, collusion on the monopoly price π^M is unsustainable. There is an SPE in which, when both firms are active, they set the monopoly price p^M in any period in which $s_1 = s_2$, and set price* 0 *otherwise.*

PROOF: Define the strategy σ^* as follows: in the first period, and after any history in which neither firm has deviated, actions are as specified in the proposition. After a deviation, both firms set price 0 until one firm goes bankrupt. For states $s, s' \in \{1, 2\}^2$, let $v^{ss'}$ denote the expected payoff to a firm in state s whose rival is in state s' if both follow σ^*. Those values satisfy the following equations:

$$v^{21} = (1-\delta)0 + \delta\big[0.06\pi^M + (1-0.06)0.05v^{11} + (1-0.06)(1-0.05)v^{21}\big],$$

$$v^{12} = (1-\delta)0 + \delta\big[0.06 \cdot 0 + (1-0.06)0.05v^{11} + (1-0.06)(1-0.05)v^{12}\big],$$

$$v^{22} = (1-\delta)\frac{1}{2}\pi^M + \delta\big[(1-0.05)0.05\big[v^{21} + v^{12}\big] + 0.05^2 v^{11} + (1-0.05)^2 v^{22}\big],$$

$$v^{11} = (1-\delta)\frac{1}{2}\pi^M + \delta\big[(1-0.01)0.01\big[v^{21} + v^{12}\big] + 0.01^2 v^{22} + (1-0.01)^2 v^{11}\big].$$

Solving yields $v^{21} \approx 0.509\pi^M$, $v^{12} \approx 0.133\pi^M$, $v^{22} \approx 0.386\pi^M$, and $v^{11} \approx 0.451\pi^M$. Similarly, let $\underline{v}^{ss'}$ denote the expected continuation payoff after a deviation (i.e., the payoff from a price war) to a firm in state s whose rival is in state s':

$$\underline{v}^{21} = (1-\delta)0 + \delta\big[0.06\pi^M + (1-0.06)0.05\underline{v}^{11} + (1-0.06)(1-0.05)\underline{v}^{21}\big],$$

$$\underline{v}^{12} = (1-\delta)0 + \delta\big[0.06 \cdot 0 + (1-0.06)0.05\underline{v}^{11} + (1-0.06)(1-0.05)\underline{v}^{12}\big],$$

$$\underline{v}^{22} = (1-\delta)0 + \delta\big[(1-0.05)0.05\big[\underline{v}^{21} + \underline{v}^{12}\big] + 0.05^2 \underline{v}^{11} + (1-0.05)^2 \underline{v}^{22}\big],$$

$$\underline{v}^{11} = (1-\delta)0 + \delta\big[0.06 \cdot 0 + (1-0.06)0.06\pi^M + (1-0.06)^2 \underline{v}^{11}\big].$$

Solving yields $\underline{v}^{21} \approx 0.474\pi^M$, $\underline{v}^{12} \approx 0.098\pi^M$, $\underline{v}^{22} \approx 0.187\pi^M$, and $\underline{v}^{11} \approx 0.334\pi^M$.

To verify that σ^* is an SPE, it is necessary to check only that neither firm wants to deviate first. (After the first deviation, the price that a firm sets does not affect its payoff.) When both firms are in state 2, a firm's best deviation would be to a price just below p^M, yielding an expected payoff of

$$(1-\delta)\pi^M + \delta\big[(1-0.05)0.05\big[\underline{v}^{21} + \underline{v}^{12}\big] + 0.05^2 \underline{v}^{11} + (1-0.05)^2 \underline{v}^{22}\big] \approx 0.237\pi^M.$$

Since 0.239 is less than the payoff from σ^*, $v^{22} \approx 0.386\pi^M$, a firm cannot gain from deviating. Similarly, when both firms are in state 1, undercutting yields

$$(1-\delta)\pi^M + \delta\big[0.06\pi^M + (1-0.06)0.01\underline{v}^{21} + (1-0.06)(1-0.01)\underline{v}^{11}\big] \approx 0.406\pi^M.$$

The payoff from σ^*, $v^{11} \approx 0.451\pi^M$, is higher, so again the deviation is not profitable.

Finally, note that when the firms have asymmetric strengths (and are setting price 0 under σ^*), deviating to a different price has no effect on today's profit and lowers continuation payoffs (since $\underline{v}^{ss'} < v^{ss'}$ for all $s, s' \in \{1, 2\}^2$). Thus, σ^* is an SPE.

To see that there is no Nash equilibrium in which the two firms always collude on the monopoly price, observe that, in that case, a strong firm would gain by undercutting a weak rival: the resulting payoff,

$$(1-\delta)\pi^M + \delta\big[0.06\pi^M + (1-0.06)0.05\underline{v}^{11} + (1-0.06)(1-0.05)\underline{v}^{21}\big] \approx 0.524\pi^M,$$

strictly exceeds $\frac{1}{2}\pi^M$. *Q.E.D.*

In the equilibrium of Proposition S4, on average a price war lasts for 9.3 periods. It ends either when the weaker firm goes bankrupt (probability 0.56) or when the stronger firm's state declines and the now-evenly-matched firms start to collude again (probability 0.44). On average, then, there are 1.8 distinct price wars before one firm goes bankrupt. When both firms are initially strong, the interval of collusion lasts on average for 11.5 periods; when both are weak, collusion lasts for 50.3 periods on average. Thus, from an initial state vector where both firms are weak, the expected time until one firm goes bankrupt is 106 periods. By comparison, if both firms priced at 0 until one went bankrupt, the expected time till bankruptcy would be only 8.6 periods. Significant collusion can occur, although that collusion is only temporary.

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