Econometrica Supplementary Material

SUPPLEMENT TO "ESTIMATION OF NONPARAMETRIC MODELS WITH SIMULTANEITY"

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THIS SUPPLEMENT PRESENTS THE STATEMENTS AND PROOFS OF THE LEMMAS that are used in the Appendix of the paper. Lemmas A.1–A.6 are used in the proofs of Theorems 2.4 and 3.2. Lemma B.1 is used in the proof of Theorem 3.1.

LEMMAS

LEMMA A.1: *Let $\overline{M}^y$ and $\overline{M}^x$ be compact and convex sets such that $\overline{M} = \{y\} \times \overline{M}^t$ is strictly included in the interior of $\overline{M}^y \times \overline{M}^x$. Let $\mathtt{F}$ denote the set of bounded and continuously differentiable functions g on the extension of the support of (Y, X) to the whole space and such that the function $\widetilde{g}$ defined by $\widetilde{g}(x) = \int g(y, x)\,dy$ is bounded and continuously differentiable. For all functions g in $\mathtt{F}$, denote by $\|g\|$ the sum of the sup norms of the values and derivatives of g and of $\widetilde{g}$ on $\overline{M}^y \times \overline{M}^x$. For any $(y, x) \in \overline{M}^y \times \overline{M}^x$, define the functionals $\alpha_{y_j}(\cdot)$ and $\beta_{x_s}(\cdot)$ on $\mathtt{F}$ by $\alpha_{y_j}(g) = \partial \log g_{Y|X=x}(y)/\partial y_j$ and $\beta_{x_s}(g) = \partial \log g_{Y|X=x}(y)/\partial x_s$. For simplicity we leave the argument (y, x) implicit. Assume that the density $f_{Y,X}$ belongs to $\mathtt{F}$ and it is such that, for $\delta > 0$ and all $(y, x) \in \overline{M}^y \times \overline{M}^x$, $f_{Y,X}(y, x) > \delta$ and $f_X(x) > \delta$. Then, there exist finite $a > 0$ and $\widetilde{\delta}_0 > 0$ such that, for all $h \in \mathtt{F}$ with $\|h\| \leq \widetilde{\delta}_0$ and all $(y, x) \in \overline{M}^y \times \overline{M}^x$,*

$$\alpha_{y_j}(f + h) - \alpha_{y_j}(f) = D\alpha_{y_j}(f; h) + R\alpha_{y_j}(f; h) \quad \textit{and}$$
$$\beta_{x_s}(f + h) - \beta_{x_s}(f) = D\beta_{x_s}(f; h) + R\beta_{x_s}(f; h),$$

where

$$D\alpha_{y_j}(f; h) = \frac{[h_{y_j} f - f_{y_j} h]}{f^2};$$

$$R\alpha_{y_j}(f; h) = -\frac{h[h_{y_j} f - f_{y_j} h]}{f^2(f + h)};$$

$$D\beta_{x_s}(f; h) = \left[\frac{[h_{x_s} f - f_{x_s} h]}{f^2} - \frac{[\widetilde{h}_{x_s} \widetilde{f} - \widetilde{f}_{x_s} \widetilde{h}]}{\widetilde{f}^2}\right];$$

$$R\beta_{x_s}(f; h) = -\left[\frac{h[h_{x_s} f - f_{x_s} h]}{f^2(f + h)} - \frac{\widetilde{h}[\widetilde{h}_{x_s} \widetilde{f} - \widetilde{f}_{x_s} \widetilde{h}]}{\widetilde{f}^2(\widetilde{f} + \widetilde{h})}\right].$$

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Moreover,

$$\left|D\alpha_{y_j}(f;h)\right| \le a\|h\|; \quad \left|D\beta_{x_s}(f;h)\right| \le a\|h\|;$$
$$\left|R\alpha_{y_j}(f;h)\right| \le a\|h\|^2; \quad \textit{and} \quad \left|R\beta_{x_s}(f;h)\right| \le a\|h\|^2.$$

PROOF: To simplify notation, we will denote $f_{Y,X}(y,x)$ by f, $f_X(x)$ by $\widetilde{f}$, $\partial f_{Y,X}(y,x)/\partial y_j$ by f_{y_j}, $\partial f_{Y,X}(y,x)/\partial x_s$ by f_{x_s}, and $\partial f_X(x)/\partial x_s$ by $\widetilde{f}_{x_s}$, with similar shorthands for functions g and h. By the definitions of $\alpha_{y_j}(g)$ and $\beta_{x_s}(g)$,

$$\alpha_{y_j}(g) = \frac{\dfrac{\partial g_{Y,X}(y,x)}{\partial y_j}}{g_{Y,X}(y,x)} = \frac{g_{y_j}}{g} \quad \text{and}$$
$$\beta_{x_s}(g) = \left(\frac{\dfrac{\partial g_{Y,X}(y,x)}{\partial x_s}}{g_{Y,X}(y,x)} - \frac{\dfrac{\partial g_X(x)}{\partial x_s}}{g_X(x)} \right) = \frac{g_{x_s}}{g} - \frac{\widetilde{g}_{x_s}}{\widetilde{g}}.$$

Let $\widetilde{\delta}_0 = \min\{\delta/2, 1\}$. Then, for all $h \in \mathrm{F}$ such that $\|h\| \le \widetilde{\delta}_0$, one has that $|h| \le \delta/2$ and $|\widetilde{h}| \le \delta/2$. Since $f > \delta$ and $\widetilde{f} > \delta$, it follows that $(f+h) > \delta/2$ and $(\widetilde{f} + \widetilde{h}) > \delta/2$. By rearranging terms, it follows that

$$\begin{aligned}
&\alpha_{y_j}(f+h) - \alpha_{y_j}(f) \\
&\quad = \left[\frac{f_{y_j} + h_{y_j}}{f+h} - \frac{f_{y_j}}{f} \right] \\
&\quad = \left[\frac{[h_{y_j} f - f_{y_j} h]}{f^2} - \frac{h[h_{y_j} f - f_{y_j} h]}{f^2(f+h)} \right] \\
&\quad = \frac{[h_{y_j} f - f_{y_j} h]}{f^2} - \frac{h[h_{y_j} f - f_{y_j} h]}{f^2(f+h)} \\
&\quad = D\alpha_{y_j}(f;h) + R\alpha_{y_j}(f;h) \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\beta_{x_s}(f+h) - \beta_{x_s}(f) &= \left[\frac{f_{x_s} + h_{x_s}}{f+h} - \frac{f_{x_s}}{f} \right] - \left[\frac{\widetilde{f}_{x_s} + \widetilde{h}_{x_s}}{\widetilde{f} + \widetilde{h}} - \frac{\widetilde{f}_{x_s}}{\widetilde{f}} \right] \\
&= \left[\frac{[h_{x_s} f - f_{x_s} h]}{f^2} - \frac{h[h_{x_s} f - f_{x_s} h]}{f^2(f+h)} \right] \\
&\quad - \left[\frac{[\widetilde{h}_{x_s} \widetilde{f} - \widetilde{f}_{x_s} \widetilde{h}]}{\widetilde{f}^2} - \frac{\widetilde{h}[\widetilde{h}_{x_s} \widetilde{f} - \widetilde{f}_{x_s} \widetilde{h}]}{\widetilde{f}^2(\widetilde{f} + \widetilde{h})} \right] \\
&= \left[\frac{[h_{x_s} f - f_{x_s} h]}{f^2} - \frac{[\widetilde{h}_{x_s} \widetilde{f} - \widetilde{f}_{x_s} \widetilde{h}]}{\widetilde{f}^2} \right]
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \left[\frac{h[h_{x_s} f - f_{x_s} h]}{f^2(f+h)} - \frac{\widetilde{h}[\widetilde{h}_{x_s}\widetilde{f} - \widetilde{f}_{x_s}\widetilde{h}]}{\widetilde{f}^2(\widetilde{f}+\widetilde{h})} \right] \\
&= D\beta_{x_s}(f;h) + R\beta_{x_s}(f;h),
\end{aligned}
$$

where the last equalities in each of the two expressions above follow by the definitions of $D\alpha_{y_j}(f;h)$, $R\alpha_{y_j}(f;h)$, $D\beta_{x_s}(f;h)$, and $R\beta_{x_s}(f;h)$ in the statement of the lemma.

It remains to show that for some finite $a > 0$, which does not depend on (y, x),

$$
\begin{aligned}
&\left|D\alpha_{y_j}(f;h)\right| \le a\|h\|; \quad \left|D\beta_{x_s}(f;h)\right| \le a\|h\|; \\
&\left|R\alpha_{y_j}(f;h)\right| \le a\|h\|^2; \quad \text{and} \quad \left|R\beta_{x_s}(f;h)\right| \le a\|h\|^2.
\end{aligned}
$$

Let $a = \max\{4\|f\|/(\delta^2), 4\|f\|/(\delta^3)\}$. By the definition of $\|f\|$ and $\|h\|$, it follows that $|f|$, $|f_{y_j}|$, $|f_{x_s}|$, $|\widetilde{f}|$, and $|\widetilde{f}_{x_s}|$ are not larger than $\|f\|$ and that $|h|$, $|h_{y_j}|$, $|h_{x_s}|$, $|\widetilde{h}|$, and $|\widetilde{h}_{x_s}|$ are not larger than $\|h\|$. Moreover, as was shown above, for all h such that $\|h\| < \delta/2$, $(f+h)$ and $(\widetilde{f}+\widetilde{h})$ are larger than $\delta/2$. By the assumption that $f, \widetilde{f} > \delta$, it then follows that

$$
\begin{aligned}
\left|D\alpha_{y_j}(f;h)\right| &= \left| \frac{[h_{y_j} f - f_{y_j} h]}{f^2} \right| \le \frac{2\|f\|}{\delta^2}\|h\| \le a\|h\|, \\
\left|R\alpha_{y_j}(f;h)\right| &= \left| \frac{h[h_{y_j} f - f_{y_j} h]}{f^2(f+h)} \right| \le \frac{2\|f\|}{\delta^3}\|h\|^2 \le a\|h\|^2, \\
\left|D\beta_{x_s}(f;h)\right| &= \left| \frac{[h_{x_s} f - f_{x_s} h]}{f^2} - \frac{[\widetilde{h}_{x_s}\widetilde{f} - \widetilde{f}_{x_s}\widetilde{h}]}{\widetilde{f}^2} \right| \\
&\le \left(\frac{4\|f\|}{\delta^2}\right)\|h\| \le a\|h\|, \quad \text{and} \\
\left|R\beta_{x_s}(f;h)\right| &= \left| \frac{h[h_{x_s} f - f_{x_s} h]}{f^2(f+h)} - \frac{\widetilde{h}[\widetilde{h}_{x_s}\widetilde{f} - \widetilde{f}_{x_s}\widetilde{h}]}{\widetilde{f}^2(\widetilde{f}+\widetilde{h})} \right| \\
&\le \frac{4\|f\|}{\delta^3}\|h\|^2 \le a\|h\|^2.
\end{aligned}
$$

This completes the proof of the lemma. *Q.E.D.*

LEMMA A.2: *Let $\overline{M}^y$, $\overline{M}^x$, $\overline{M}$, $\overline{M}^t$, $\mathtt{F}$, $\|\cdot\|$, $\alpha_{y_j}(\cdot)$, and $\beta_{x_s}(\cdot)$ be as in the statement of Lemma A.1. Assume that the density $f_{Y,X}$ belongs to $\mathtt{F}$ and it is such that for $\delta > 0$ and all $(y, x) \in \overline{M}^y \times \overline{M}^x$, $f_{Y,X}(y, x) > \delta$ and $f_X(x) > \delta$. Let μ denote a bounded, nonnegative, continuously differentiable function, with values equal to zero when any coordinate of $x \in \overline{M}^t$ is on the boundary of $\overline{M}^t$,* *positive values when x is in the interior of $\overline{M}^t$, and such that $\int_{\overline{M}} \mu(y,x)\,dx = 1$. For simplicity we will leave the argument (y,x) implicit. Define the functional Φ_{y_j,x_s} on* $\mathtt{F}$ *by*

$$\Phi_{y_j,x_s}(g) = \int_{\overline{M}} \alpha_{y_j}(g)\beta_{x_s}(g)\mu(y,x)\,dx - \left(\int_{\overline{M}} \alpha_{y_j}(g)\mu(y,x)\,dx\right)\left(\int_{\overline{M}} \beta_{x_s}(g)\mu(y,x)\,dx\right).$$

Then, there exist finite $b_1 > 0$ and $\widetilde{\delta}_1 > 0$ such that, for all $h \in \mathtt{F}$ with $\|h\| \le \widetilde{\delta}_1$,

$$\Phi_{y_j,x_s}(f+h) - \Phi_{y_j,x_s}(f) = D\Phi_{y_j,x_s}(f;h) + R\Phi_{y_j,x_s}(f;h),$$

where, employing the notation of Lemma A.1 for the definitions of $D\alpha_{y_j}(f;h)$, $R\alpha_{y_j}(f;h)$, $D\beta_{x_s}(f;h)$, and $R\beta_{x_s}(f;h)$,

$$\begin{aligned} &D\Phi_{y_j,x_s}(f;h) \\ &\quad = \int_{\overline{M}} D\alpha_{y_j}(f;h)\left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f)\mu\,dx\right)\mu\,dx \\ &\qquad + \int_{\overline{M}} D\beta_{x_s}(f;h)\left(\alpha_{y_j}(f) - \int_{\overline{M}} \alpha_{y_j}(f)\mu\,dx\right)\mu\,dx \quad \text{and} \\ &R\Phi_{y_j,x_s}(f;h) \\ &\quad = \int_{\overline{M}} R\alpha_{y_j}(f;h)\left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f)\mu\,dx\right)\mu\,dx \\ &\qquad + \int_{\overline{M}} R\beta_{x_s}(f;h)\left(\alpha_{y_j}(f) - \int_{\overline{M}} \alpha_{y_j}(f)\mu\,dx\right)\mu\,dx \\ &\qquad + \int_{\overline{M}} \big(D\alpha_{y_j}(f;h) + R\alpha_{y_j}(f;h)\big)\big(D\beta_{x_s}(f;h) + R\beta_{x_s}(f;h)\big)\mu\,dx \\ &\qquad - \left(\int_{\overline{M}} \big(D\alpha_{y_j}(f;h) + R\alpha_{y_j}(f;h)\big)\mu\,dx\right) \\ &\qquad \times \left(\int_{\overline{M}} \big(D\beta_{x_s}(f;h) + R\beta_{x_s}(f;h)\big)\mu\,dx\right). \end{aligned}$$

Moreover,

$$\big|D\Phi_{y_j,x_s}(f;h)\big| \le b_1\|h\| \quad \text{and} \quad \big|R\Phi_{y_j,x_s}(f;h)\big| \le b_1\|h\|^2.$$

PROOF: Let $\widetilde{\delta}_1 = \min\{\widetilde{\delta}_0/2, 1\}$. To show that for all h such that $\|h\| \le \widetilde{\delta}_1$

$$\Phi_{y_j,x_s}(f+h) - \Phi_{y_j,x_s}(f) = D\Phi_{y_j,x_s}(f;h) + R\Phi_{y_j,x_s}(f;h),$$

we note that by the definitions of Φ_{y_j,x_s}, α_{y_j}, and β_{x_s}, it can be shown by adding and subtracting terms that

$$\begin{aligned}
&\Phi_{y_j,x_s}(f+h)-\Phi_{y_j,x_s}(f)\\
&\quad=\int_{\overline{M}}\alpha_{y_j}(f+h)\beta_{x_s}(f+h)\mu\,dx\\
&\qquad-\left(\int_{\overline{M}}\alpha_{y_j}(f+h)\mu\,dx\right)\left(\int_{\overline{M}}\beta_{x_s}(f+h)\mu\,dx\right)\\
&\qquad-\int_{\overline{M}}\alpha_{y_j}(f)\beta_{x_s}(f)\mu\,dx+\left(\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\left(\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\\
&\quad=\int_{\overline{M}}\big(\alpha_{y_j}(f+h)-\alpha_{y_j}(f)\big)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\mu\,dx\\
&\qquad+\int_{\overline{M}}\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\big(\beta_{x_s}(f+h)-\beta_{x_s}(f)\big)\mu\,dx\\
&\qquad+\int_{\overline{M}}\big(\alpha_{y_j}(f+h)-\alpha_{y_j}(f)\big)\big(\beta_{x_s}(f+h)-\beta_{x_s}(f)\big)\mu\,dx\\
&\qquad-\left(\int_{\overline{M}}\big(\alpha_{y_j}(f+h)-\alpha_{y_j}(f)\big)\mu\,dx\right)\\
&\qquad\times\left(\int_{\overline{M}}\big(\beta_{x_s}(f+h)-\beta_{x_s}(f)\big)\mu\,dx\right).
\end{aligned}$$

Employing the expressions of $(\alpha_{y_j}(f+h)-\alpha_{y_j}(f))$ and of $(\beta_{x_s}(f+h)-\beta_{x_s}(f))$ in terms of $D\alpha_{y_j}(f;h)$, $R\alpha_{y_j}(f;h)$, $D\beta_{x_s}(f;h)$, and $R\beta_{x_s}(f;h)$, guaranteed by Lemma A.1, it follows from the last equality that

$$\begin{aligned}
&\Phi_{y_j,x_s}(f+h)-\Phi_{y_j,x_s}(f)\\
&\quad=\int_{\overline{M}}D\alpha_{y_j}(f;h)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\mu\,dx\\
&\qquad+\int_{\overline{M}}D\beta_{x_s}(f;h)\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\mu\,dx\\
&\qquad+\int_{\overline{M}}R\alpha_{y_j}(f;h)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\mu\,dx\\
&\qquad+\int_{\overline{M}}R\beta_{x_s}(f;h)\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\mu\,dx\\
&\qquad+\int_{\overline{M}}\big(D\alpha_{y_j}(f;h)+R\alpha_{y_j}(f;h)\big)\big(D\beta_{x_s}(f;h)+R\beta_{x_s}(f;h)\big)\mu\,dx
\end{aligned}$$

$$- \left(\int_{\overline{M}} \big(D\alpha_{y_j}(f;h) + R\alpha_{y_j}(f;h) \big) \mu \, dx \right)$$
$$\times \left(\int_{\overline{M}} \big(D\beta_{x_s}(f;h) + R\beta_{x_s}(f;h) \big) \mu \, dx \right).$$

Note that the sum of the first two terms is $D\Phi_{y_j,x_s}(f;h)$ and the sum of the last four terms is $R\Phi_{y_j,x_s}(f;h)$. Hence,

$$\Phi_{y_j,x_s}(f+h) - \Phi_{y_j,x_s}(f) = D\Phi_{y_j,x_s}(f;h) + R\Phi_{y_j,x_s}(f;h).$$

It remains to show that for some finite $b_1 > 0$,

$$\big| D\Phi_{y_j,x_s}(f;h) \big| \le b_1 \|h\| \quad \text{and} \quad \big| R\Phi_{y_j,x_s}(f;h) \big| \le b_1 \|h\|^2.$$

Let $a = \max\{4\|f\|/(\delta^2), 4\|f\|/(\delta^3)\}$, as defined in the proof of Lemma A.1, and let $b_1 = \max\{a\frac{16\|f\|}{\delta}, 16a^2\}$. Since by assumption $f > \delta$ and $\widetilde{f} > \delta$, it follows by the definitions of $\alpha_{y_j}(f)$ and $\beta_{x_s}(f)$, the assumption that $\int_{\overline{M}} \mu(y,x)\,dx = 1$, and the conclusion of Lemma A.1 that

$$\left| \int_{\overline{M}} D\alpha_{y_j}(f;h) \left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f) \mu \, dx \right) \mu \, dx \right|$$
$$\le \int_{\overline{M}} \big| D\alpha_{y_j}(f;h) \big| \left| \left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f) \mu \, dx \right) \right| \mu \, dx$$
$$\le \int_{\overline{M}} a\|h\| \left| \left(\left| \frac{4\|f\|}{\delta} \right| \right) \right| \mu \, dx \le a\|h\| \frac{4\|f\|}{\delta} \le \frac{b_1}{2} \|h\|$$

and that

$$\left| \int_{\overline{M}} D\beta_{x_s}(f;h) \left(\alpha_{y_j}(f) - \int_{\overline{M}} \alpha_{y_j}(f) \mu \, dx \right) \mu \, dx \right|$$
$$\le \int_{\overline{M}} \big| D\beta_{x_s}(f;h) \big| \left| \left(\alpha_{y_j}(f) - \int_{\overline{M}} \alpha_{y_j}(f) \mu \, dx \right) \right| \mu \, dx$$
$$\le \int_{\overline{M}} a\|h\| \left| \left(\left| \frac{2\|f\|}{\delta} \right| \right) \right| \mu \, dx \le a\|h\| \frac{2\|f\|}{\delta} \le \frac{b_1}{2} \|h\|.$$

Hence, $|D\Phi_{y_j,x_s}(f;h)| \le b_1\|h\|$. Analogously, for the four terms of $R\Phi_{y_j,x_s}(f;h)$,

$$\left| \int_{\overline{M}} R\alpha_{y_j}(f;h) \left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f) \mu \, dx \right) \mu \, dx \right|$$
$$\le \int_{\overline{M}} a\|h\|^2 \left| \left(\left| \frac{4\|f\|}{\delta} \right| \right) \right| \mu \, dx \le \frac{b_1}{4} \|h\|^2,$$

$$\left|\int_{\overline{M}} R\beta_{x_s}(f;h)\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\mu\,dx\right|$$
$$\leq \int_{\overline{M}} a\|h\|^2\left|\left(\left|\frac{2\|f\|}{\delta}\right|\right)\right|\mu\,dx \leq \frac{b_1}{4}\|h\|^2,$$
$$\left|\int_{\overline{M}}\big(D\alpha_{y_j}(f;h)+R\alpha_{y_j}(f;h)\big)\big(D\beta_{x_s}(f;h)+R\beta_{x_s}(f;h)\big)\mu\,dx\right|$$
$$\leq \int_{\overline{M}} a^2\big(\|h\|+\|h\|^2\big)^2\mu\,dx \leq a^2\big(\|h\|+\|h\|^2\big)^2 \leq 4a^2\|h\|^2 \leq \frac{b_1}{4}\|h\|^2,$$
$$\left|\left(\int_{\overline{M}}\big(D\alpha_{y_j}(f;h)+R\alpha_{y_j}(f;h)\big)\mu\,dx\right)\right.$$
$$\left.\times\left(\int_{\overline{M}}\big(D\beta_{x_s}(f;h)+R\beta_{x_s}(f;h)\big)\mu\,dx\right)\right|$$
$$\leq a^2\big(\|h\|+\|h\|^2\big)^2 \leq \frac{b_1}{4}\|h\|^2.$$

Hence, $|R\Phi_{y_j,x_s}(f;h)| \leq b_1\|h\|^2$. This completes the proof. *Q.E.D.*

LEMMA A.3: *Let* $\overline{M}^y$, $\overline{M}^x$, $\overline{M}$, $\overline{M}^t$, $\mathtt{F}$, $\|\cdot\|$, $\alpha_{y_j}(\cdot)$, $\beta_{x_s}(\cdot)$, *and* μ *be as in the statement of Lemma A.1. Assume that the density* $f_{Y,X}$ *belongs to* $\mathtt{F}$ *and it is such that, for* $\delta > 0$ *and all* $(y,x) \in \overline{M}^y \times \overline{M}^x$, $f_{Y,X}(y,x) > \delta$ *and* $f_X(x) > \delta$. *For simplicity we will leave the argument* (y,x) *implicit. Define the functional* Φ_{y_j,y_s} *and* Φ_{x_j,x_s} *on* $\mathtt{F}$ *by*

$$\Phi_{y_j,y_s}(g) = \int_{\overline{M}}\alpha_{y_j}(g)\alpha_{y_s}(g)\mu(y,x)\,dx - \left(\int_{\overline{M}}\alpha_{y_j}(g)\mu(y,x)\,dx\right)\left(\int_{\overline{M}}\alpha_{y_s}(g)\mu(y,x)\,dx\right),$$
$$\Phi_{x_j,x_s}(g) = \int_{\overline{M}}\beta_{x_j}(g)\beta_{x_s}(g)\mu(y,x)\,dx - \left(\int_{\overline{M}}\beta_{x_j}(g)\mu(y,x)\,dx\right)\left(\int_{\overline{M}}\beta_{x_s}(g)\mu(y,x)\,dx\right).$$

Then, there exist finite $b_2, b_3 > 0$ *and* $\widetilde{\delta}_2, \widetilde{\delta}_3 > 0$ *such that, for all* $h \in \mathtt{F}$ *with* $\|h\| \leq \min\{\widetilde{\delta}_2, \widetilde{\delta}_3\}$,

$$\Phi_{y_j,y_s}(f+h) - \Phi_{y_j,y_s}(f) = D\Phi_{y_j,y_s}(f;h) + R\Phi_{y_j,y_s}(f;h),$$
$$\Phi_{x_j,x_s}(f+h) - \Phi_{x_j,x_s}(f) = D\Phi_{x_j,x_s}(f;h) + R\Phi_{x_j,x_s}(f;h),$$

where, employing the notation of Lemma A.1 for the definitions of $D\alpha_{y_j}(f;h)$, $R\alpha_{y_j}(f;h)$, $D\alpha_{y_s}(f;h)$, *and* $R\alpha_{y_s}(f;h)$,

$$\begin{aligned}
&D\Phi_{y_j,y_s}(f;h)\\
&\quad= \int_{\overline{M}} D\alpha_{y_j}(f;h)\left(\alpha_{y_s}(f) - \int_{\overline{M}} \alpha_{y_s}(f)\mu\,dx\right)\mu\,dx\\
&\qquad + \int_{\overline{M}} D\alpha_{y_s}(f;h)\left(\alpha_{y_j}(f) - \int_{\overline{M}} \alpha_{y_j}(f)\mu\,dx\right)\mu\,dx \quad \textit{and}\\
&R\Phi_{y_j,y_s}(f;h)\\
&\quad= \int_{\overline{M}} R\alpha_{y_j}(f;h)\left(\alpha_{y_s}(f) - \int_{\overline{M}} \alpha_{y_s}(f)\mu\,dx\right)\mu\,dx\\
&\qquad + \int_{\overline{M}} R\alpha_{y_s}(f;h)\left(\alpha_{y_j}(f) - \int_{\overline{M}} \alpha_{y_j}(f)\mu\,dx\right)\mu\,dx\\
&\qquad + \int_{\overline{M}} \big(D\alpha_{y_j}(f;h) + R\alpha_{y_j}(f;h)\big)\big(D\alpha_{y_s}(f;h) + R\alpha_{y_s}(f;h)\big)\mu\,dx\\
&\qquad - \left(\int_{\overline{M}} \big(D\alpha_{y_j}(f;h) + R\alpha_{y_j}(f;h)\big)\mu\,dx\right)\\
&\qquad \times \left(\int_{\overline{M}} \big(D\alpha_{y_s}(f;h) + R\alpha_{y_s}(f;h)\big)\mu\,dx\right),\\
&D\Phi_{x_j,x_s}(f;h)\\
&\quad= \int_{\overline{M}} D\beta_{x_j}(f;h)\left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f)\mu\,dx\right)\mu\,dx\\
&\qquad + \int_{\overline{M}} D\beta_{x_s}(f;h)\left(\beta_{x_j}(f) - \int_{\overline{M}} \beta_{x_j}(f)\mu\,dx\right)\mu\,dx \quad \textit{and}\\
&R\Phi_{x_j,x_s}(f;h)\\
&\quad= \int_{\overline{M}} R\beta_{x_j}(f;h)\left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f)\mu\,dx\right)\mu\,dx\\
&\qquad + \int_{\overline{M}} R\beta_{x_s}(f;h)\left(\beta_{x_j}(f) - \int_{\overline{M}} \beta_{x_j}(f)\mu\,dx\right)\mu\,dx\\
&\qquad + \int_{\overline{M}} \big(D\beta_{x_j}(f;h) + R\beta_{x_j}(f;h)\big)\big(D\beta_{x_s}(f;h) + R\beta_{x_s}(f;h)\big)\mu\,dx
\end{aligned}$$

$$- \left(\int_{\overline{M}} \big(D\beta_{x_j}(f;h) + R\beta_{x_j}(f;h) \big) \mu \, dx \right) \times \left(\int_{\overline{M}} \big(D\beta_{x_s}(f;h) + R\beta_{x_s}(f;h) \big) \mu \, dx \right).$$

Moreover,

$$\left| D\Phi_{y_j,y_s}(f;h) \right| \le b_2 \|h\| \quad \textit{and} \quad \left| R\Phi_{y_j,y_s}(f;h) \right| \le b_2 \|h\|^2,$$
$$\left| D\Phi_{x_j,x_s}(f;h) \right| \le b_3 \|h\| \quad \textit{and} \quad \left| R\Phi_{x_j,x_s}(f;h) \right| \le b_3 \|h\|^2.$$

PROOF: The proof follows similarly to the proof of Lemma A.2, after replacing β_{x_s} with α_{x_s} in the results related to $\Phi_{y_j,y_s}(g)$ and replacing α_{x_j} with β_{x_j} in the results related to $\Phi_{x_j,x_s}(g)$, with the obvious modification of the upperbounds. *Q.E.D.*

LEMMA A.4: *Let* $\overline{M}^y$, $\overline{M}^x$, $\overline{M}$, $\overline{M}^t$, $\mathtt{F}$, $\|\cdot\|$, $\alpha_{y_j}(\cdot)$, $\beta_{x_s}(\cdot)$, μ, $\Phi_{y_j,x_s}(\cdot)$, $\Phi_{y_j,y_s}(\cdot)$, *and* $\Phi_{x_j,x_s}(\cdot)$ *be as in the statements of Lemmas* A.2 *and* A.3. *Assume that the density* $f_{Y,X}$ *belongs to* $\mathtt{F}$ *and it is such that, for* $\delta > 0$ *and all* $(y,x) \in \overline{M}^y \times \overline{M}^x$, $f_{Y,X}(y,x) > \delta$ *and* $f_X(x) > \delta$. *For simplicity we will leave the argument* (y,x) *implicit. Define the functional* Ξ *on* $\mathtt{F}$ *by*

$$\Xi(g) = \frac{\Phi_{y_1,y_2}(g)\Phi_{x,x}(g) - \Phi_{y_1,x}(g)\Phi_{y_2,x}(g)}{\Phi_{y_1,y_1}(g)\Phi_{x,x}(g) - \Phi^2_{y_1,x}(g)},$$

and assume that, for some $\delta^* > 0$, $[\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)] > \delta^*$. *Then, there exist finite* $c > 0$ *and* $\widetilde{\delta} > 0$ *such that, for all* $h \in \mathtt{F}$ *with* $\|h\| \le \widetilde{\delta}$,

$$\Xi(f+h) - \Xi(f) = D\Xi(f;h) + R\Xi(f;h),$$

where

$$\begin{aligned} &D\Xi(f;h) \\ &\quad = \frac{[D\Phi_{y_1,y_2}(f;h)\Phi_{x,x}(f) - D\Phi_{y_1,x}(f;h)\Phi_{y_2,x}(f)]}{\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)} \\ &\qquad + \frac{[\Phi_{y_1,y_2}(f)D\Phi_{x,x}(f;h) - \Phi_{y_1,x}(f)D\Phi_{y_2,x}(f;h)]}{\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)} \\ &\qquad - \left[\Phi_{y_1,y_2}(f)\Phi_{x,x}(f) - \Phi_{y_1,x}(f)\Phi_{y_2,x}(f)\right] \\ &\qquad \times \frac{[D\Phi_{y_1,y_1}(f;h)\Phi_{x,x} + \Phi_{y_1,y_1}(f)D\Phi_{x,x}(f;h)]}{[\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)]^2} \end{aligned}$$

$$+\left[2D\Phi_{y_1,x}(f;h)\Phi_{y_1,x}(f)\right]$$
$$\times\frac{[\Phi_{y_1,y_2}(f)\Phi_{x,x}(f)-\Phi_{y_1,x}(f)\Phi_{y_2,x}(f)]}{[\Phi_{y_1,y_1}(f)\Phi_{x,x}(f)-\Phi^2_{y_1,x}(f)]^2},$$
$$\left|D\Xi_{y_j,y_s}(f;h)\right|\le c\|h\|,\quad \textit{and}$$
$$\left|R\Xi_{y_j,y_s}(f;h)\right|\le c\|h\|^2.$$

PROOF: Let $b = \max\{b_1, b_2, b_3, (8\|f\|^2/\delta^2)\}$ and $\widetilde{\delta} = \min\{\widetilde{\delta}_1, \widetilde{\delta}_2, \widetilde{\delta}_3, \delta^*/(32b^2)\}$, where $\widetilde{\delta}_1$ and b_1 are defined in Lemma A.2 and $\widetilde{\delta}_2$, $\widetilde{\delta}_3$, b_2, b_3 are defined in Lemma A.3. Since the assumptions of Lemmas A.2 and A.3 are satisfied, it follows from those lemmas that, for all h in $\mathtt{F}$ with $\|h\|\le\widetilde{\delta}$, and for $j,s=1,2$,

$$\Phi_{y_j,y_s}(f+h)-\Phi_{y_j,y_s}(f)=D\Phi_{y_j,y_s}(f;h)+R\Phi_{y_j,y_s}(f;h),$$
$$\Phi_{y_j,x}(f+h)-\Phi_{y_j,x}(f)=D\Phi_{y_j,x}(f;h)+R\Phi_{y_j,x}(f;h),\quad\text{and}$$
$$\Phi_{x,x}(f+h)-\Phi_{x,x}(f)=D\Phi_{x,x}(f;h)+R\Phi_{x,x}(f;h),$$

where $D\Phi_{y_j,x}(f;h)$, $R\Phi_{y_j,x}(f;h)$, $D\Phi_{y_j,y_s}(f;h)$, $R\Phi_{y_j,y_s}(f;h)$, $D\Phi_{x,x}(f;h)$, and $R\Phi_{x,x}(f;h)$ are as defined in Lemmas A.2 and A.3. By those lemmas, these terms satisfy

$$\left|D\Phi_{y_j,x_s}(f;h)\right|\le b\|h\|\quad\text{and}\quad\left|R\Phi_{y_j,x_s}(f;h)\right|\le b\|h\|^2,$$
$$\left|D\Phi_{y_j,y_s}(f;h)\right|\le b\|h\|\quad\text{and}\quad\left|R\Phi_{y_j,y_s}(f;h)\right|\le b\|h\|^2,\quad\text{and}$$
$$\left|D\Phi_{x,x}(f;h)\right|\le b\|h\|\quad\text{and}\quad\left|R\Phi_{x,x}(f;h)\right|\le b\|h\|^2.$$

We seek a first order expansion

$$\Xi(f+h)-\Xi(f)=D\Xi(f;h)+R\Xi(f;h),$$

with

$$\left|D\Xi(f;h)\right|\le c\|h\|\quad\text{and}\quad\left|R\Xi(f;h)\right|\le c\|h\|^2.$$

Denote D', N', D, and N by

$$N'=\Phi_{y_1,y_2}(f+h)\Phi_{x,x}(f+h)-\Phi_{y_1,x}(f+h)\Phi_{y_2,x}(f+h),$$
$$N=\Phi_{y_1,y_2}(f)\Phi_{x,x}(f)-\Phi_{y_1,x}(f)\Phi_{y_2,x}(f),$$
$$D'=\Phi_{y_1,y_1}(f+h)\Phi_{x,x}(f+h)-\left[\Phi_{y_1,x}(f+h)\right]^2,$$
$$D=\Phi_{y_1,y_1}(f)\Phi_{x,x}(f)-\left[\Phi_{y_1,x}(f)\right]^2.$$

Then,

$$
\begin{aligned}
&\Xi(f+h) - \Xi(f) \\
&\quad = \frac{\Phi_{y_1,y_2}(f+h)\Phi_{x,x}(f+h) - \Phi_{y_1,x}(f+h)\Phi_{y_2,x}(f+h)}{\Phi_{y_1,y_1}(f+h)\Phi_{x,x}(f+h) - [\Phi_{y_1,x}(f+h)]^2} \\
&\qquad - \frac{\Phi_{y_1,y_2}(f)\Phi_{x,x}(f) - \Phi_{y_1,x}(f)\Phi_{y_2,x}(f)}{\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - [\Phi_{y_1,x}(f)]^2} \\
&\quad = \frac{N'}{D'} - \frac{N}{D}.
\end{aligned}
$$

We will make use of the equality

$$
\frac{N'}{D'} - \frac{N}{D} = \frac{N'D - ND'}{D^2} - \frac{(D'-D)(N'D - ND')}{D'D^2}.
$$

Lemmas A.2 and A.3 and the definitions of N', D', N, and D imply that

$$
\begin{aligned}
N' &= \Phi_{y_1,y_2}(f+h)\Phi_{x,x}(f+h) - \Phi_{y_1,x}(f+h)\Phi_{y_2,x}(f+h) \\
&= \Phi_{y_1,y_2}(f)\Phi_{x,x}(f) - \Phi_{y_1,x}(f)\Phi_{y_2,x}(f) \\
&\quad + D\Phi_{y_1,y_2}(f;h)\Phi_{x,x}(f) + \Phi_{y_1,y_2}(f)D\Phi_{x,x}(f;h) \\
&\quad - D\Phi_{y_1,x}(f;h)\Phi_{y_2,x}(f) - \Phi_{y_1,x}(f)D\Phi_{y_2,x}(f;h) \\
&\quad + R_N,
\end{aligned}
$$

where

$$
\begin{aligned}
R_N = {}& \Phi_{y_1,y_2}(f;h)\big[R\Phi_{x,x}(f+h)\big] \\
&+ D\Phi_{y_1,y_2}(f;h)\big[D\Phi_{x,x}(f+h) + R\Phi_{x,x}(f+h)\big] \\
&+ R\Phi_{y_1,y_2}(f;h)\big[\Phi_{x,x}(f+h) + D\Phi_{x,x}(f+h) + R\Phi_{x,x}(f+h)\big] \\
&- \Phi_{y_1,x}(f;h)\big[R\Phi_{y_2,x}(f+h)\big] \\
&- D\Phi_{y_1,x}(f;h)\big[D\Phi_{y_2,x}(f+h) + R\Phi_{y_2,x}(f+h)\big] \\
&- R\Phi_{y_1,x}(f;h)\big[\Phi_{y_2,x}(f+h) + D\Phi_{y_2,x}(f+h) + R\Phi_{y_2,x}(f+h)\big]
\end{aligned}
$$

and

$$
\begin{aligned}
D' &= \Phi_{y_1,y_1}(f+h)\Phi_{x,x}(f+h) - \Phi^2_{y_1,x}(f+h) \\
&= \Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f) \\
&\quad + D\Phi_{y_1,y_1}(f;h)\Phi_{x,x} + \Phi_{y_1,y_1}(f)D\Phi_{x,x}(f;h) \\
&\quad - 2D\Phi_{y_1,x}(f;h)\Phi_{y_1,x}(f) \\
&\quad + R_D,
\end{aligned}
$$

where

$$\begin{aligned}
R_D = {} & \Phi_{y_1,y_1}(f;h)\big[R\Phi_{x,x}(f+h)\big] \\
& + D\Phi_{y_1,y_1}(f;h)\big[D\Phi_{x,x}(f+h) + R\Phi_{x,x}(f+h)\big] \\
& + R\Phi_{y_1,y1}(f;h)\big[\Phi_{x,x}(f+h) + D\Phi_{x,x}(f+h) + R\Phi_{x,x}(f+h)\big] \\
& - \Phi_{y_1,x}(f;h)\big[R\Phi_{y_1,x}(f+h)\big] \\
& - D\Phi_{y_1,x}(f;h)\big[D\Phi_{y_1,x}(f+h) + R\Phi_{y_1,x}(f+h)\big] \\
& - R\Phi_{y_1,x}(f;h) \\
& \times \big[\Phi_{y_1,x}(f+h) + D\Phi_{y_1,x}(f+h) + R\Phi_{y_1,x}(f+h)\big].
\end{aligned}$$

Then,

$$\begin{aligned}
& N'D - ND' \\
& \quad = \big[\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\big] \\
& \qquad \times \big[D\Phi_{y_1,y_2}(f;h)\Phi_{x,x}(f) + \Phi_{y_1,y_2}(f)D\Phi_{x,x}(f;h)\big] \\
& \qquad - \big[\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\big] \\
& \qquad \times \big[D\Phi_{y_1,x}(f;h)\Phi_{y_2,x}(f) + \Phi_{y_1,x}(f)D\Phi_{y_2,x}(f;h)\big] \\
& \qquad - \big[\Phi_{y_1,y_2}(f)\Phi_{x,x}(f) - \Phi_{y_1,x}(f)\Phi_{y_2,x}(f)\big] \\
& \qquad \times \big[D\Phi_{y_1,y_1}(f;h)\Phi_{x,x} + \Phi_{y_1,y_1}(f)D\Phi_{x,x}(f;h)\big] \\
& \qquad + 2\big[\Phi_{y_1,y_2}(f)\Phi_{x,x}(f) - \Phi_{y_1,x}(f)\Phi_{y_2,x}(f)\big]D\Phi_{y_1,x}(f;h)\Phi_{y_1,x}(f) \\
& \qquad + \big[\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\big]R_N \\
& \qquad - \big[\Phi_{y_1,y_2}(f)\Phi_{x,x}(f) - \Phi_{y_1,x}(f)\Phi_{y_2,x}(f)\big]R_D
\end{aligned}$$

and

$$\begin{aligned}
D' - D = {} & D\Phi_{y_1,y_1}(f;h)\Phi_{x,x} + \Phi_{y_1,y_1}(f)D\Phi_{x,x}(f;h) \\
& - 2D\Phi_{y_1,x}(f;h)\Phi_{y_1,x}(f) + R_D.
\end{aligned}$$

Denote $D\Xi(f;h)$ and $R\Xi(f;h)$ by

$$\begin{aligned}
& D\Xi(f;h) \\
& \quad = \frac{[D\Phi_{y_1,y_2}(f;h)\Phi_{x,x}(f) + \Phi_{y_1,y_2}(f)D\Phi_{x,x}(f;h)]}{\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)} \\
& \qquad - \frac{[D\Phi_{y_1,x}(f;h)\Phi_{y_2,x}(f) + \Phi_{y_1,x}(f)D\Phi_{y_2,x}(f;h)]}{\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)}
\end{aligned}$$

$$
\begin{aligned}
&- \left[D\Phi_{y_1,y_1}(f;h)\Phi_{x,x} + \Phi_{y_1,y_1}(f)D\Phi_{x,x}(f;h)\right] \\
&\times \frac{[\Phi_{y_1,y_2}(f)\Phi_{x,x}(f) - \Phi_{y_1,x}(f)\Phi_{y_2,x}(f)]}{[\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)]^2} \\
&+ \frac{[2D\Phi_{y_1,x}(f;h)\Phi_{y_1,x}(f)][\Phi_{y_1,y_2}(f)\Phi_{x,x}(f) - \Phi_{y_1,x}(f)\Phi_{y_2,x}(f)]}{[\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)]^2}
\end{aligned}
$$

and

$$
R\Xi(f;h) = \frac{DR_N - NR_D}{D^2} - \frac{[D' - D][N'D - ND']}{D^2 D'}.
$$

Then,

$$
\Xi(f+h) - \Xi(f) = D\Xi(f;h) + R\Xi(f;h).
$$

It remains to show that, for some $c > 0$,

$$
\left|D\Xi(f;h)\right| \leq c\|h\| \quad \text{and} \quad \left|R\Xi(f;h)\right| \leq c\|h\|^2.
$$

By Lemmas A.2 and A.3, for all h in $\mathtt{F}$ with $\|h\| \leq \widetilde{\delta}$,

$$
\begin{aligned}
&\Phi_{y_1,y_1}(f+h)\Phi_{x,x}(f+h) - \Phi^2_{y_1,x}(f+h) \\
&\quad = \left[\Phi_{y_1,y_1}(f) + D\Phi_{y_1,y_1}(f;h) + R\Phi_{y_1,y_1}(f;h)\right] \\
&\qquad \times \left[\Phi_{x,x}(f) + D\Phi_{x,x}(f;h) + R\Phi_{x,x}(f;h)\right] \\
&\qquad - \left[\Phi_{y_1,x}(f) + D\Phi_{y_1,x}(f;h) + R\Phi_{y_1,x}(f;h)\right]^2,
\end{aligned}
$$

and, by the definition of b, for all $w, z \in \{y_1, y_2, x\}$, $|D\Phi_{w,z}(f;h)| \leq b\|h\|$ and $|R\Phi_{w,z}(f;h)| \leq b\|h\|^2$. Moreover, by the assumptions on $f_{Y,X}$ and μ, for all such w, z, $|\Phi_{w,z}(f)| \leq (8\|f\|^2/\delta^2) \leq b$. Hence,

$$
\begin{aligned}
&\Phi_{y_1,y_1}(f+h)\Phi_{x,x}(f+h) - \Phi^2_{y_1,x}(f+h) \\
&\quad = \Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f) + T
\end{aligned}
$$

for T such that

$$
|T| \leq 16b^2\|h\| \leq 16b^2\widetilde{\delta} \leq \delta^*/2.
$$

Then, since

$$
\Phi_{y_1,y_1}(f)\Phi_{x,x}(f) - \Phi^2_{y_1,x}(f) > \delta^*,
$$

it follows that

$$
\Phi_{y_1,y_1}(f+h)\Phi_{x,x}(f+h) - \Phi^2_{y_1,x}(f+h) > \delta^*/2,
$$

and then, by the definitions of D and D',

$$D^2 D' > \left(\delta^*\right)^3/2.$$

Let $c = \max\{16b^2/(\delta^*)^2, 3{,}776 b^6/(\delta^*)^3\}$. Then, for all h such that $\|h\| \le \widetilde{\delta}$,

$$\left|D\Xi(f;h)\right| \le \frac{16b^2}{(\delta^*)^2} \le c\|h\|,$$

and since for all h such that $\|h\| \le \widetilde{\delta}$, $|D'| \le 18b^2$, $|D| \le 2b^2$, $|N| \le 2b^2$, $|R_N| \le 12b^2\|h\|^2$, $|R_D| \le 12b^2\|h\|^2$, $|D' - D| \le 16b^2\|h\|$, and $|N'D - ND'| \le 64b^4\|h\|$, it follows that

$$\begin{aligned}\left|R\Xi(f;h)\right| &\le \frac{18b^2(24b^4\|h\|^2 + 24b^4\|h\|^2) + (16b^2\|h\|)(64b^4\|h\|)}{(\delta^*)^3/2}\\ &\le c\|h\|^2.\end{aligned}$$

This completes the proof of Lemma A.4. *Q.E.D.*

LEMMA A.5: *Let* $\overline{M}^y$, $\overline{M}^x$, $\overline{M}$, $\overline{M}^t$, $\mathtt{F}$, $\|\cdot\|$, $\alpha_{y_j}(\cdot)$, $\beta_{x_s}(\cdot)$, μ, $\Phi_{y_j,x_s}(\cdot)$, $\Phi_{y_j,y_s}(\cdot)$, *and* $\Phi_{x_j,x_s}(\cdot)$ *be as in the statements of Lemmas* A.2 *and* A.3. *Suppose that either Assumptions* 2.7–2.10 *or Assumptions* 2.7, 3.4, 3.5, 2.9, *and* 2.10 *are satisfied*. *Let* G *denote the dimension of the vector of observable endogenous variables* Y *and let* $\overline{M} = \{y\} \times \overline{M}^t$. *Then*,

$$\begin{aligned}&\sqrt{N\sigma^{G+2}}\, D\Phi_{y_j,x_s}(f, \widehat{f} - f)\\ &\quad = \sqrt{N\sigma^{G+2}} \int_{\overline{M}} \left(\widehat{f}_{y_j}(y,x) - f_{y_j}(y,x)\right)\\ &\qquad \times \left[\frac{\mu\left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f)\mu\, dx\right)}{f}\right] dx + o_p(1),\\ &\sqrt{N\sigma^{G+2}}\left[\Phi_{x_j,x_s}(\widehat{f}) - \Phi_{x_j,x_s}(f)\right]\\ &\quad = \sqrt{N\sigma^{G+2}}\, D\Phi_{x_j,x_s}(f, \widehat{f} - f) + \sqrt{N\sigma^{G+2}}\, R\Phi_{x_j,x_s}(f, \widehat{f} - f)\\ &\quad = o_p(1),\\ &\sqrt{N\sigma^{G+2}}\, D\Phi_{y_j,y_s}(f; \widehat{f} - f)\\ &\quad = \sqrt{N\sigma^{G+2}} \int_{\overline{M}} \left(\widehat{f}_{y_j}(y,x) - f_{y_j}(y,x)\right)\end{aligned}$$

$$\times\left[\frac{\mu\left(\alpha_{y_s}(f)-\int_{\overline{M}}\alpha_{y_s}(f)\mu\,dx\right)}{f}\right]dx$$

$$+\sqrt{N\sigma^{G+2}}\int_{\overline{M}}\big(\widehat{f}_{y_s}(y,x)-f_{y_s}(y,x)\big)$$

$$\times\left[\frac{\mu\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)}{f}\right]dx$$

$$+o_p(1),\quad \textit{and}$$

$$\sqrt{N\sigma^{G+2}}\|\widehat{f}-f\|^2\to 0\quad \textit{in probability}.$$

PROOF: First note that

$$\int_{\overline{M}}D\beta_{x_s}(f;h)\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\mu\,dx \tag{T.2}$$

$$=\int_{\overline{M}}\frac{[h_{x_s}f-f_{x_s}h]}{f^2}\frac{f_{y_j}}{f}\mu\,dx-\int_{\overline{M}}\frac{[\widetilde{h}_{x_s}\widetilde{f}-\widetilde{f}_{x_s}\widetilde{h}]}{\widetilde{f}^2}\frac{f_{y_j}}{f}\mu\,dx$$

$$-\left(\int_{\overline{M}}\frac{[h_{x_s}f-f_{x_s}h]}{f^2}\mu\,dx\right)\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\mu\,dx\right)$$

$$+\left(\int_{\overline{M}}\frac{[\widetilde{h}_{x_s}\widetilde{f}-\widetilde{f}_{x_s}\widetilde{h}]}{\widetilde{f}^2}\mu\,dx\right)\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\mu\,dx\right).$$

Since, by Assumption 2.8, the values and derivatives of the continuously differentiable function μ when any coordinate of x is on the boundary of $\overline{M}^t$ are equal to zero and f and $\widetilde{f}$ are larger than δ everywhere on $\overline{M}$ and $\overline{M}^t$, it follows that the continuously differentiable functions $[h\mu]/f$, $[h\mu f_{y_j}]/f^2$, $[\widetilde{h}\mu]/\widetilde{f}$, and $[\widetilde{h}\mu f_{y_j}]/[\widetilde{f}f]$ vanish when any coordinate of x is on the boundary of $\overline{M}^t$. Hence, integration by parts of the terms in (T.2) containing h_{x_s} or $\widetilde{h}_{x_s}$ gives

$$\int_{\overline{M}}D\beta_{x_s}(f;h)\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\mu\,dx$$

$$=-\int_{\overline{M}}h\left[\frac{\partial}{\partial x_s}\left(\frac{\mu f_{y_j}}{f^2}\right)+\frac{f_{x_s}f_{y_j}\mu}{f^3}\right]dx$$

$$+\int_{\overline{M}}\widetilde{h}\left[\frac{\partial}{\partial x_s}\left(\frac{\mu f_{y_j}}{\widetilde{f}f}\right)+\frac{\widetilde{f}_{x_s}f_{y_j}\mu}{\widetilde{f}^2 f}\right]dx$$

$$+\left(\int_{\overline{M}} h\left[\frac{\partial}{\partial x_s}\left(\frac{\mu}{f}\right)+\frac{f_{x_s}\mu}{f^2}\right]dx\right)\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\mu\,dx\right)$$
$$-\left(\int_{\overline{M}} \widetilde{h}\left[\frac{\partial}{\partial x_s}\left(\frac{\mu}{\widetilde{f}}\right)+\frac{\widetilde{f}_{x_s}\mu}{\widetilde{f}^2}\right]dx\right)\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\mu\,dx\right).$$

Let $h=\widehat{f}-f$,

$$\omega_a(x)=-\frac{f_{y_j}\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\mu}{f^2}$$
$$-\left[\frac{\partial}{\partial x_s}\left(\frac{\mu f_{y_j}}{f^2}\right)+\frac{f_{x_s}f_{y_j}\mu}{f^3}\right]$$
$$+\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\mu\,dx\right)\left[\frac{\partial}{\partial x_s}\left(\frac{\mu}{f}\right)+\frac{f_{x_s}\mu}{f^2}\right],$$
$$\omega_b(x)=\left[\frac{\partial}{\partial x_s}\left(\frac{\mu f_{y_j}}{\widetilde{f}f}\right)+\frac{\widetilde{f}_{x_s}f_{y_j}\mu}{\widetilde{f}^2 f}\right.$$
$$\left.-\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\mu\,dx\right)\left[\frac{\partial}{\partial x_s}\left(\frac{\mu}{\widetilde{f}}\right)+\frac{\widetilde{f}_{x_s}\mu}{\widetilde{f}^2}\right]\right].$$

Then, by Lemmas A.2 and A.3 and by (T.2),

$$D\Phi_{y_j,x_s}(f;\widehat{f}-f)$$
$$=\int_{\overline{M}} D\alpha_{y_j}(f;\widehat{f}-f)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\mu\,dx$$
$$+\int_{\overline{M}} D\beta_{x_s}(f;\widehat{f}-f)\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\mu\,dx$$
$$=\int_{\overline{M}}\left[\frac{[(\widehat{f_{y_j}}-f_{y_j})f]}{f^2}\right]\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\mu\,dx$$
$$-\int_{\overline{M}}\left[\frac{f_{y_j}(\widehat{f}-f)}{f^2}\right]\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\mu\,dx$$
$$-\int_{\overline{M}}(\widehat{f}-f)\left[\frac{\partial}{\partial x_s}\left(\frac{\mu f_{y_j}}{f^2}\right)+\frac{f_{x_s}f_{y_j}\mu}{f^3}\right]dx$$
$$+\int_{\overline{M}}(\widehat{\widetilde{f}}-\widetilde{f})\left[\frac{\partial}{\partial x_s}\left(\frac{\mu f_{y_j}}{\widetilde{f}f}\right)+\frac{\widetilde{f}_{x_s}f_{y_j}\mu}{\widetilde{f}^2 f}\right]dx$$

$$
\begin{aligned}
&+\left(\int_{\overline{M}}(\widehat{f}-f)\left[\frac{\partial}{\partial x_s}\left(\frac{\mu}{f}\right)+\frac{f_{x_s}\mu}{f^2}\right]dx\right)\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\mu\,dx\right)\\
&-\left(\int_{\overline{M}}(\widehat{\widetilde{f}}-\widetilde{f})\left[\frac{\partial}{\partial x_s}\left(\frac{\mu}{\widetilde{f}}\right)+\frac{\widetilde{f}_{x_s}\mu}{\widetilde{f}^2}\right]dx\right)\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\mu\,dx\right)\\
=&\int_{\overline{M}}(\widehat{f}_{y_j}-f_{y_j})\frac{\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)}{f}\mu\,dx\\
&+\int_{\overline{M}}(\widehat{f}-f)\omega_a(x)\,dx+\int_{\overline{M}}(\widehat{\widetilde{f}}-\widetilde{f})\omega_b(x)\,dx.
\end{aligned}
$$

By Assumptions 2.7, 2.9, and 2.10, it follows by Lemma 5.3 in Newey (1994), by letting in that lemma, $k_1=G$, $k_2=\dim(x)$, $l=0$, $t=x$, $h_0=f$, $m(\widehat{h})=(\int_{\overline{M}}\widehat{f}(y,x)\omega_a(x)\,dx)$, and $m(h_0)=(\int_{\overline{M}}f(y,x)\omega_a(x)\,dx)$, that for a finite constant matrix V_a,

$$\sqrt{N\sigma^G}\left(\int_{\overline{M}}\big(\widehat{f}(y,x)-f(y,x)\big)\omega_a(x)\,dx\right)\to N(0,V_a).$$

Similarly, letting in that same lemma, $k_1=0$, $k_2=\dim(x)$, $l=0$, $t=x$, $h_0=\widetilde{f}$, $m(\widehat{h})=(\int_{\overline{M}}\widehat{\widetilde{f}}(x)\omega_b(x)\,dx)$, and $m(h_0)=(\int_{\overline{M}}\widetilde{f}(x)\omega_b(x)\,dx)$, it follows that, for a finite constant matrix V_b,

$$\sqrt{N}\left(\int_{\overline{M}}\big(\widehat{\widetilde{f}}(x)-\widetilde{f}(x)\big)\omega_b(x)\,dx\right)\to N(0,V_b).$$

Hence,

$$
\begin{aligned}
&\sqrt{N\sigma^{G+2}}D\Phi_{y_j,x_s}(f,\widehat{f}-f)\\
&\quad=\sqrt{N\sigma^{G+2}}\int_{\overline{M}}\big(\widehat{f}_{y_j}(y,x)-f_{y_j}(y,x)\big)\\
&\qquad\times\left[\frac{\mu\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)}{f}\right]dx+o_p(1).
\end{aligned}
$$

Using analogous arguments for $D\Phi_{x_j,x_s}(f,\widehat{f}-f)$ and for $D\Phi_{y_j,y_s}(f,\widehat{f}-f)$, it follows that

$$
\begin{aligned}
&\sqrt{N\sigma^{G+2}}D\Phi_{x_j,x_s}(f,\widehat{f}-f)\\
&\quad=\sqrt{N\sigma^{G+2}}\int_{\overline{M}}D\beta_{x_j}(f;\widehat{f}-f)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\mu\,dx
\end{aligned}
$$

$$+\sqrt{N\sigma^{G+2}}\int_{\overline{M}} D\beta_{x_s}(f;\widehat{f}-f)\left(\beta_{x_j}(f)-\int_{\overline{M}}\beta_{x_j}(f)\mu\,dx\right)\mu\,dx$$
$$=o_p(1)$$

and that

$$\sqrt{N\sigma^{G+2}}D\Phi_{y_j,y_s}(f;\widehat{f}-f)$$
$$=\sqrt{N\sigma^{G+2}}\int_{\overline{M}} D\alpha_{y_j}(f;\widehat{f}-f)\left(\alpha_{y_s}(f)-\int_{\overline{M}}\alpha_{y_s}(f)\mu\,dx\right)\mu\,dx$$
$$+\sqrt{N\sigma^{G+2}}\int_{\overline{M}} D\alpha_{y_s}(f;\widehat{f}-f)\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\mu\,dx$$
$$=\sqrt{N\sigma^{G+2}}\int_{\overline{M}}(\widehat{f}_{y_j}-f_{y_j})\left(\frac{\left(\alpha_{y_s}(f)-\int_{\overline{M}}\alpha_{y_s}(f)\mu\,dx\right)\mu}{f}\right)dx$$
$$+\sqrt{N\sigma^{G+2}}\int_{\overline{M}}(\widehat{f}_{y_s}-f_{y_s})\left(\frac{\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\mu}{f}\right)dx$$
$$+o_p(1).$$

To show that

$$\sqrt{N\sigma^{G+2}}\|\widehat{f}-f\|^2\to 0 \quad \text{in probability,}$$

we note that from Appendix B.3 in Newey (1994),

$$\|\widehat{f}-f\|=O_p\left(\sqrt{\log(N)/\left(N\sigma^{2G+2}\right)}+\sigma^s\right).$$

Since by Assumption 2.10, $\sqrt{N\sigma^{G+2}}(\sqrt{\log(N)/(N\sigma^{2G+2})}+\sigma^s)^2$ converges to zero, it follows that

$$\sqrt{N\sigma^{G+2}}\|\widehat{f}-f\|^2\to 0 \quad \text{in probability.} \qquad \textit{Q.E.D.}$$

LEMMA A.6: *Under the same definitions, notations, and assumptions in Lemma A.5, for all j, s and all $w, z\in\{y_j, y_s, x_j, x_s\}$, for finite $\overline{c}>0$, and for $\|\widehat{f}-f\|$ sufficiently small,*

$$\sup_{(y,x)\in\overline{M}}\Bigg|\left(\Delta\partial_w\log\widehat{f}_{Y|X=x}(y)\right)\left(\Delta\partial_z\log\widehat{f}_{Y|X=x}(y)\right)\frac{\mu^2}{\widehat{f}(y,x)}$$
$$-\left(\Delta\partial_w\log f_{Y|X=x}(y)\right)\left(\Delta\partial_z\log f_{Y|X=x}(y)\right)\frac{\mu^2}{f(y,x)}\Bigg|\le\overline{c}\|\widehat{f}-f\|.$$

PROOF: By Lemma A.1, it follows that, for all $\widehat{f}$ such that $\|\widehat{f}-f\| \leq \widetilde{\delta}_0$ and for all $(y,x) \in \overline{M}$,

$$\left|\frac{\widehat{f}_{y_j}(y,x)}{\widehat{f}(y,x)} - \frac{f_{y_j}(y,x)}{f(y,x)}\right| = \left|\alpha_{y_j}(\widehat{f}) - \alpha_{y_j}(f)\right|$$
$$\leq \left|D\alpha_{y_j}(f;\widehat{f}-f)\right| + \left|R\alpha_{y_j}(f;\widehat{f}-f)\right|$$
$$\leq a\|\widehat{f}-f\| + a\|\widehat{f}-f\|^2.$$

Hence,

$$\sup_{(y,x)\in\overline{M}} \left|\alpha_{y_j}(\widehat{f}) - \alpha_{y_j}(f)\right| \leq a\|\widehat{f}-f\| + a\|\widehat{f}-f\|^2.$$

Since μ is bounded, $\int_{\overline{M}} \mu(y,x)\,dx = 1$, and $\overline{M}$ is compact, this implies that

$$\left|\int_{\overline{M}} \alpha_{y_j}(\widehat{f})\mu\,dx - \int_{\overline{M}} \alpha_{y_j}(f)\mu\,dx\right|$$
$$\leq \int_{\overline{M}} \left|\alpha_{y_j}(\widehat{f}) - \alpha_{y_j}(f)\right|\mu\,dx$$
$$\leq a\|\widehat{f}-f\| + a\|\widehat{f}-f\|^2,$$

and therefore, for all $(y,x) \in \overline{M}$,

$$\left|\left(\alpha_{y_j}(\widehat{f}) - \int_{\overline{M}} \alpha_{y_j}(\widehat{f})\mu\,dx\right) - \left(\alpha_{y_j}(f) - \int_{\overline{M}} \alpha_{y_j}(f)\mu\,dx\right)\right|$$
$$\leq \left|\alpha_{y_j}(\widehat{f}) - \alpha_{y_j}(f)\right| + \left|\int_{\overline{M}} \alpha_{y_j}(\widehat{f})\mu\,dx - \int_{\overline{M}} \alpha_{y_j}(f)\mu\,dx\right|$$
$$\leq 2a\|\widehat{f}-f\| + 2a\|\widehat{f}-f\|^2.$$

By an analogous argument for β_{x_s}, it can shown, employing Lemma A.1, that for all $(y,x) \in \overline{M}$,

$$\left|\left(\beta_{x_s}(\widehat{f}) - \int_{\overline{M}} \beta_{x_s}(\widehat{f})\mu\,dx\right) - \left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}\mu\,dx\right)\right|$$
$$\leq 2a\|\widehat{f}-f\| + 2a\|\widehat{f}-f\|^2.$$

Since for all $(y,x) \in \overline{M}$, $f(y,x) > \delta$ and $f(x) > \delta$, it also follows that, since by the definition of $\widetilde{\delta}_0$ in the proof of Lemma A.1, $\widetilde{\delta}_0 = \min\{\delta/2, 1\}$, for all $\widehat{f}$ such that $\|\widehat{f}-f\| \leq \widetilde{\delta}_0$,

$$\left|\frac{1}{\widehat{f}(y,x)}-\frac{1}{f(y,x)}\right| \leq \frac{|\widehat{f}(y,x)-f(y,x)|}{|\widehat{f}(y,x)||f(y,x)|} \leq \frac{2\|\widehat{f}-f\|}{\delta^2}.$$

Denote $\widehat{\alpha}=(\alpha_{y_j}(\widehat{f})-\int_{\overline{M}}\alpha_{y_j}(\widehat{f})\mu\,dx)$, $\widehat{\beta}=[\beta_{x_s}(\widehat{f})-\int_{\overline{M}}\beta_{x_s}(\widehat{f})\mu\,dx]$, $\alpha=(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx)$, and $\beta=[\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\,dx]$. Note that

$$\frac{\widehat{\alpha}\widehat{\beta}}{\widehat{f}}-\frac{\alpha\beta}{f}=\frac{(\widehat{\alpha}-\alpha)\widehat{\beta}f+(\widehat{\beta}-\beta)\alpha f-\alpha\beta(\widehat{f}-f)}{\widehat{f}f}.$$

Then,

$$\begin{aligned}\left|\frac{\widehat{\alpha}\widehat{\beta}}{\widehat{f}}-\frac{\alpha\beta}{f}\right| &= \left|\frac{(\widehat{\alpha}-\alpha)\widehat{\beta}f+(\widehat{\beta}-\beta)\alpha f-\alpha\beta(\widehat{f}-f)}{\widehat{f}f}\right| \\ &\leq \frac{|(\widehat{\alpha}-\alpha)||\widehat{\beta}|f+|\widehat{\beta}-\beta||\alpha f|+|\widehat{f}-f||\alpha\beta|}{|\widehat{f}||f|} \\ &\leq \frac{2[\|(\widehat{\alpha}-\alpha)\||\widehat{\beta}||f|+\|\widehat{\beta}-\beta\||\alpha f|+\|\widehat{f}-f\||\alpha\beta|]}{\delta^2}.\end{aligned}$$

Since $|\widehat{\beta}|=|\beta+D\beta+R\beta|\leq|\beta|+|D\beta|+|R\beta|$ and all the three last terms are either uniformly bounded, by Assumption 2.7, or bounded by a constant times $\|\widehat{f}-f\|$, by Lemma A.1, it follows that for a finite $\overline{c}>0$, for all $(y,x)\in\overline{M}$,

$$\begin{aligned}&\Big|\big(\Delta\partial_w\log\widehat{f}_{Y|X=x}(y)\big)\big(\Delta\partial_z\log\widehat{f}_{Y|X=x}(y)\big)\frac{\mu^2}{\widehat{f}(y,x)} \\ &\quad -\big(\Delta\partial_w\log f_{Y|X=x}(y)\big)\big(\Delta\partial_z\log f_{Y|X=x}(y)\big)\frac{\mu^2}{f(y,x)}\Big| \leq \overline{c}\|\widehat{f}-f\|,\end{aligned}$$

where $w=y_j$ and $z=x_s$. A similar argument holds for $w\in\{y_s,x_j,x_s\}$ and $z\in\{y_j,y_s,x_j\}$. *Q.E.D.*

LEMMA B.1: *Suppose that* (3.2) *and Assumptions* 3.1–3.3 *are satisfied*. *Then*,

$$\begin{aligned}&\big(r'_x-\widetilde{r}'_x\big(\widetilde{r}'_y\big)^{-1}r'_y\big)q_\varepsilon \\ &\quad=\left[1-\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{|r_y|}{r^1_{y_1}r^2_x}\right)\right](\mathbf{g}_x-\gamma_x) \\ &\qquad-\left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{r^1_{y_2}}{r^1_{y_1}}\right)-\widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\right](\mathbf{g}_{y_1}-\gamma_{y_1})\end{aligned}$$

and

$$-(\gamma_x - \widetilde{\gamma}_x) + \widetilde{r}'_x \big(\widetilde{r}'_y\big)^{-1}(\gamma_y - \widetilde{\gamma}_y)$$
$$= -(\gamma_x - \widetilde{\gamma}_x) - \widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)(\gamma_{y_1} - \widetilde{\gamma}_{y_1}) + \widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)(\gamma_{y_2} - \widetilde{\gamma}_{y_2}),$$

where the notation corresponds to that used in the proof of Theorem 3.1.

PROOF: Taking logs and differentiating both sides of (T.3.1), in the proof of Theorem 3.1, with respect to x, leaving the arguments implicit, we get

$$\mathbf{g}_x = r^2_x q_{\varepsilon_2} + \gamma_x, \tag{B.1.1}$$

and taking logs and differentiating both sides of (T.3.1) with respect to y_1, we get

$$\mathbf{g}_{y_1} = r^1_{y_1} q_{\varepsilon_1} + r^2_{y_1} q_{\varepsilon_2} + \gamma_{y_1}. \tag{B.1.2}$$

Since the matrix

$$\begin{pmatrix} 0 & r^2_x \\ r^1_{y_1} & r^2_{y_1} \end{pmatrix}$$

is invertible, because by assumption $r^1_{y_1}$ and r^2_x are different from zero, the unique value of the vector $(q_{\varepsilon_1}, q_{\varepsilon_2})$ that solves (B.1.1)–(B.1.2) is

$$\begin{pmatrix} q_{\varepsilon_1} \\ q_{\varepsilon_2} \end{pmatrix} = \begin{pmatrix} \dfrac{-r^2_{y_1}}{r^1_{y_1} r^2_x} & \dfrac{1}{r^1_{y_1}} \\ \dfrac{1}{r^2_x} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{g}_x - \gamma_x \\ \mathbf{g}_{y_1} - \gamma_{y_1} \end{pmatrix}. \tag{B.1.3}$$

Since $r'_x = (0, r^2_x)$, it follows by (B.1.3) that

$$r'_x q_\varepsilon = r^2_x q_{\varepsilon_2} = \mathbf{g}_x - \gamma_x. \tag{B.1.4}$$

By the definition of r'_y and (B.1.3),

$$r'_y q_\varepsilon = \begin{pmatrix} r^1_{y_1} & r^2_{y_1} \\ r^1_{y_2} & r^2_{y_2} \end{pmatrix} \begin{pmatrix} \dfrac{-r^2_{y_1}}{r^1_{y_1} r^2_x} & \dfrac{1}{r^1_{y_1}} \\ \dfrac{1}{r^2_x} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{g}_x - \gamma_x \\ \mathbf{g}_{y_1} - \gamma_{y_1} \end{pmatrix} \tag{B.1.5}$$
$$= \begin{pmatrix} 0 & 1 \\ \dfrac{-r^1_{y_2} r^2_{y_1}}{r^1_{y_1} r^2_x} + \dfrac{r^2_{y_2}}{r^2_x} & \dfrac{r^1_{y_2}}{r^1_{y_1}} \end{pmatrix} \begin{pmatrix} \mathbf{g}_x - \gamma_x \\ \mathbf{g}_{y_1} - \gamma_{y_1} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{g}_{y_1} - \gamma_{y_1} \\ \left(\frac{r^1_{y_2}}{r^1_{y_1}}\right)(\mathbf{g}_{y_1} - \gamma_{y_1}) + \left(\frac{|r_y|}{r^1_{y_1} r^2_x}\right)(\mathbf{g}_x - \gamma_x) \end{pmatrix}.$$

Note that

$$\text{(B.1.6)} \quad \widetilde{r}'_x \big(\widetilde{r}'_y\big)^{-1} = \begin{pmatrix} 0 & \frac{\widetilde{r}^2_x}{|\widetilde{r}_y|} \end{pmatrix} \begin{pmatrix} \widetilde{r}^2_{y_2} & -\widetilde{r}^2_{y_1} \\ -\widetilde{r}^1_{y_2} & \widetilde{r}^1_{y_1} \end{pmatrix}$$
$$= \begin{pmatrix} -\widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) & \widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) \end{pmatrix}.$$

By (B.1.5) and (B.1.6), it then follows that

$$\text{(B.1.7)} \quad \big[\widetilde{r}'_x \big(\widetilde{r}'_y\big)^{-1}\big]\big[r'_y q_\varepsilon\big]$$
$$= \left[\begin{pmatrix} -\widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) & \widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) \end{pmatrix}\right]$$
$$\times \left[\begin{pmatrix} \mathbf{g}_{y_1} - \gamma_{y_1} \\ \left(\frac{r^1_{y_2}}{r^1_{y_1}}\right)(\mathbf{g}_{y_1} - \gamma_{y_1}) + \left(\frac{|r_y|}{r^1_{y_1} r^2_x}\right)(\mathbf{g}_x - \gamma_x) \end{pmatrix}\right]$$
$$= \left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{r^1_{y_2}}{r^1_{y_1}}\right) - \widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\right](\mathbf{g}_{y_1} - \gamma_{y_1})$$
$$+ \left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{|r_y|}{r^1_{y_1} r^2_x}\right)\right](\mathbf{g}_x - \gamma_x).$$

By (B.1.4) and (B.1.7),

$$\big(r'_x - \widetilde{r}'_x\big(\widetilde{r}'_y\big)^{-1} r'_y\big) q_\varepsilon$$
$$= \big[r'_x q_\varepsilon\big] - \big[\widetilde{r}'_x\big(\widetilde{r}'_y\big)^{-1}\big]\big[r'_y q_\varepsilon\big]$$
$$= (\mathbf{g}_x - \gamma_x) - \left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{r^1_{y_2}}{r^1_{y_1}}\right) - \widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\right](\mathbf{g}_{y_1} - \gamma_{y_1})$$
$$- \left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{|r_y|}{r^1_{y_1} r^2_x}\right)\right](\mathbf{g}_x - \gamma_x)$$
$$= \left[1 - \widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{|r_y|}{r^1_{y_1} r^2_x}\right)\right](\mathbf{g}_x - \gamma_x)$$
$$- \left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{r^1_{y_2}}{r^1_{y_1}}\right) - \widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\right](\mathbf{g}_{y_1} - \gamma_{y_1}).$$

By (B.1.6),

$$
\begin{aligned}
&-(\gamma_x-\widetilde{\gamma}_x)+\widetilde{r}_x'(\widetilde{r}_y')^{-1}(\gamma_y-\widetilde{\gamma}_y)\\
&\quad=-(\gamma_x-\widetilde{\gamma}_x)+\begin{pmatrix}-\widetilde{r}^1_{y_2}\left(\dfrac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) & \widetilde{r}^1_{y_1}\left(\dfrac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\end{pmatrix}\begin{pmatrix}\gamma_{y_1}-\widetilde{\gamma}_{y_1}\\ \gamma_{y_2}-\widetilde{\gamma}_{y_2}\end{pmatrix}\\
&\quad=-(\gamma_x-\widetilde{\gamma}_x)-\widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)(\gamma_{y_1}-\widetilde{\gamma}_{y_1})+\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)(\gamma_{y_2}-\widetilde{\gamma}_{y_2}).
\end{aligned}
$$

Q.E.D.

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