This note points out that the proof of Theorem 1, the main theorem, in Ergin (2002) needs two corrections. We provide two counterexamples to Ergin’s (2002) proof and show that the theorem holds as it is by providing an alternative proof.

KEYWORDS: Matching, stability, efficiency, acyclicity.

1. INTRODUCTION

For allocating resources to agents, many institutions use matching mechanisms involving no monetary transfer. For instance, consider a one-to-many matching problem of school choice where each school has a quota on the number of students and a priority ranking of students, and each student has a preference order over the set of schools and remaining unmatched. Schools’ quotas and priority rankings are exogenously fixed and common knowledge among all agents, including a planner, while students’ preferences are not known to the planner. In such an environment, the planner matches each student with at most one school, so that the number of students matched with each school is within that school’s quota.

For such problems, Ergin (2002) presents a necessary and sufficient condition on priorities and quotas under which there exists a stable matching rule which associates a stable matching to each preference profile that is (i) Pareto efficient for students, (ii) group strategyproof, or (iii) consistent across outcomes for problems involving different groups of students and levels of quotas. According to his Theorem 1, a stable matching rule satisfying each of the conditions (i), (ii), and (iii) exists if and only if a profile of priority rankings and quotas of schools satisfies a condition called acyclicity. Following this result, many papers use acyclicity as a crucial condition. (See, e.g., Kesten (2006), Haeringer and Klijn (2009), and Kojima (2011).)

However, Ergin’s (2002) proof of the theorem needs two corrections. In this note, we provide two counterexamples to the proof and show that the theorem holds as it is by providing an alternative proof that accommodates these counterexamples.
2. NOTATION AND DEFINITIONS

Let \( A \) be a set of types of goods (schools in the above example) and \( N \) be a set of agents (students in the above example). \( q := (q_a)_{a \in A} \) denotes a vector of quotas where for each \( a \in A \), \( q_a \geq 1 \) represents the number of available goods of type \( a \). A priority structure is a profile of linear order over agents \( \succeq := (\succeq_a)_{a \in A} \), where for each \( a \in A \), \( \succeq_a \) ranks agents with respect to their priority for \( a \). For each \( a \in A \) and \( i \in N \), let \( U_a(i) := \{ j \in N | j \succeq_a i \} \).

Distinct \( a, b \in A \) and \( i, j, k \in N \) constitute a cycle if

- (C) Cycle condition: \( i \succ_a j \succ_a k \succ_b i \).
- (S) Scarcity condition: There are (possibly empty) disjoint sets of agents \( N_a, N_b \subset N \setminus \{i, j, k\} \) such that \( N_a \subset U_a(j), N_b \subset U_b(i) \), \( |N_a| = q_a - 1 \), and \( |N_b| = q_b - 1 \).

If \( \succeq \) and \( q \) have no cycle, they are called acyclical. More generally, distinct \( a_0, a_1, \ldots, a_{n-1} \in A \) and \( j, i_0, i_1, \ldots, i_{n-1} \in N \) with \( n \geq 2 \) constitute a generalized cycle if

- (C) \( i_0 \succ a_0 j \succ a_0 i_1 \succ a_1 i_2 \succ a_2 \cdots i_{n-1} \succ a_{n-1} i_0 \).
- (S) There are disjoint sets of agents \( N_{a_0}, N_{a_1}, \ldots, N_{a_{n-1}} \subset N \setminus \{j, i_0, i_1, \ldots, i_{n-1}\} \) such that \( N_{a_0} \subset U_{a_0}(j), N_{a_1} \subset U_{a_1}(i_0), N_{a_2} \subset U_{a_2}(i_1), \ldots, N_{a_{n-2}} \subset U_{a_{n-2}}(i_{n-3}), N_{a_{n-1}} \subset U_{a_{n-1}}(i_{n-2}) \), and for all \( l = 0, 1, \ldots, n-1 \), \( |N_{a_l}| = q_l - 1 \).

In the definition, we consider \( n \) to be the “size” of the generalized cycle. Then, it is clear that the generalized cycle of size 2 is a cycle. Ergin (2002) argues that the following proposition holds and uses this as a critical lemma for proving Theorem 1.

**LEMMA**—Ergin (2002, p. 2495): If \( \succeq \) and \( q \) have a generalized cycle, then they also have a cycle.

3. COUNTEREXAMPLE TO ERGIN’S PROOF

To prove the Lemma above, suppose that \( a_0, a_1, \ldots, a_{n-1} \in A \) and \( j, i_0, i_1, \ldots, i_{n-1} \in N \) constitute the shortest generalized cycle of size \( n > 2 \) with \( N_{a_0}, N_{a_1}, \ldots, N_{a_{n-1}} \subset N \setminus \{j, i_0, i_1, \ldots, i_{n-1}\} \) as in the above definition. In the original proof of the Lemma, Ergin (2002) utilizes the following two claims.

**CLAIM 1**: If \( i_0 \succ a_2 i_2 \), then \( a_1, a_2 \) and \( i_0, i_1, i_2 \) constitute a cycle with \( N_{a_1} \) and \( N_{a_2} \), that is, \( i_0 \succ a_2 i_2 \succ a_2 i_1 \succ a_1 i_0, N_{a_2} \subset U_{a_2}(i_2) \), and so on.

**CLAIM 2**: If \( i_2 \succ a_2 i_0 \), then \( a_0, a_2, a_3, \ldots, a_{n-1} \) and \( j, i_0, i_2, i_3, \ldots, i_{n-1} \) constitute a generalized cycle with \( N_{a_0}, N_{a_2}, N_{a_3}, \ldots, N_{a_{n-1}} \), that is, \( i_0 \succ a_0 j \succ a_0 i_{n-1} \succ a_{n-1} \cdots i_3 \succ a_3 i_2 \succ a_2 i_0, N_{a_2} \subset U_{a_2}(i_0) \), and so on.

However, both of these claims turn out to be incorrect. We provide two counterexamples, the former to Claim 1 and the latter to Claim 2. For simplicity, consider a case where for \( a_0, a_1, a_2 \in A \), \( q_{a_0} = q_{a_1} = q_{a_2} = 2 \), and suppose that \( a_0, a_1, a_2 \) and \( j, i_0, i_1, i_2 \in N \) constitute the shortest generalized cycle of size 3 with \( N_{a_0}, N_{a_1}, \) and \( N_{a_2} \), that is, \( i_0 \succ a_0 j \succ a_0 i_2 \succ a_2 i_1 \succ a_1 i_0 \) and \( N_{a_0} \subset U_{a_0}(j) \), \( N_{a_1} \subset U_{a_1}(i_0) \), \( N_{a_2} \subset U_{a_2}(i_1) \) and \( |N_{a_0}| = |N_{a_1}| = |N_{a_2}| = 1 \). Let \( k \in N \) be an agent such that \( N_{a_2} = \{k\} \).

**COUNTEREXAMPLE TO CLAIM 1**: Suppose \( i_0 \succ a_2 i_2 \succ a_2 k \succ a_2 i_1 \). (Note that \( i_0 \succ a_2 i_2 \), i.e., the assumption of Claim 1, is satisfied.) Then, \( N_{a_2} \) does not satisfy that \( N_{a_2} \subset U_{a_2}(i_2) \) because \( N_{a_2} = \{k\} \), but \( i_2 \succ a_2 k \), so \( N_{a_2} \cap U_{a_2}(i_2) = \emptyset \). Therefore, \( a_1, a_2 \), and \( i_0, i_1, i_2 \) cannot constitute a cycle with \( N_{a_1} \) and \( N_{a_2} \).
Counterexample to Claim 2: Suppose \( i_2 > a_2 \), \( i_0 > a_2 \), \( k > a_2 \) \( i_1 \). (Note that \( i_2 > a_2 \), \( i_0 \), i.e., the assumption of Claim 2, is satisfied.) Then, \( N_{a_2} \) does not satisfy \( N_{a_2} \subset U_{a_2}(i_0) \) because \( N_{a_2} = \{ k \} \), but \( i_0 > a_2 \), \( k \), so \( N_{a_2} \cap U_{a_2}(i_0) = \emptyset \). Therefore, \( a_0, a_2 \) and \( j, i_0, i_2, i_1 \) cannot constitute a generalized cycle of size 2, that is, a cycle, with \( N_{a_0} \) and \( N_{a_2} \).

Note that we do not exclude a possibility that a cycle or a generalized cycle other than those considered in Claim 1 and Claim 2 may exist. In the last section, we provide an alternative proof of the aforementioned Lemma that accommodates these counterexamples.

4. ALTERNATIVE PROOF

Suppose that \( \succeq \) and \( q \) have a generalized cycle and let the size of the shortest generalized cycle be \( n > 2 \), that is, \( a_0, a_1, \ldots, a_{n-1} \in A \); \( j, i_0, i_1, \ldots, i_{n-1} \in N \) and \( N_{a_0}, N_{a_1}, \ldots, N_{a_{n-1}} \subset N \setminus \{ j, i_0, i_1, \ldots, i_{n-1} \} \) constitute the shortest generalized cycle of size \( n > 2 \).

Case (1-α): If \( i_0 > a_2 \), \( i_2 \) and, for all \( i \in N_{a_2} \), \( i > a_2 \) \( i_2 \), then \( i_0 > a_2 \) \( i_2 \) \( i_1 \) \( a_1 \) \( i_0 \) and \( N_{a_2}, N_{a_1} \subset N \setminus \{ i_2, i_1, i_0 \} \) are disjoint sets satisfying \( N_{a_2} \subset U_{a_2}(i_2), N_{a_1} \subset U_{a_1}(i_0), |N_{a_2}| = q_{a_2} - 1 \), and \( |N_{a_1}| = q_{a_1} - 1 \). Therefore, \( a_2, a_1, i_0, i_2, i_1 \) constitute a cycle, that is, a generalized cycle of size 2, which is a contradiction.

Case (1-β): If \( i_0 > a_2 \), \( i_2 \) and there exists \( i \in N_{a_2} \) such that \( i > a_2 \), \( i > a_2 \), \( i_2 \), let \( i_* \) be the minimum element in \( N_{a_2} \) with respect to \( > a_2 \) and \( N_{a_2} = N_{a_2} \cup \{ i_2 \} \setminus \{ i_* \} \). Then, \( i_0 > a_2 \), \( i_* > a_2 \), \( i_1 > a_1 \), \( i_0 \) and \( N_{a_1}, N_{a_2} \subset N \setminus \{ i_1, i_0, i_* \} \) are disjoint sets satisfying \( N_{a_2} \subset U_{a_2}(i_*), N_{a_1} \subset U_{a_1}(i_0), |N_{a_2}| = q_{a_2} - 1 \), and \( |N_{a_1}| = q_{a_1} - 1 \). Therefore, \( a_2, a_1, i_0, i_*, i_1 \) constitute a cycle, that is, a generalized cycle of size 2, which is a contradiction.

Case (2-α): If \( i_2 > a_2 \), \( i_0 \) and, for all \( i \in N_{a_2} \), \( i > a_2 \), \( i_0 \), then \( i_0 > a_0 \), \( j > a_0 \), \( i_{n-1} > a_{n-1} \), \( \ldots \), \( i_3 > a_3 \), \( i_2 > a_2 \), \( i_0 \), and \( N_{a_0}, N_{a_2}, \ldots, N_{a_{n-1}} \subset N \setminus \{ j, i_0, i_2, \ldots, i_{n-1} \} \) are disjoint sets satisfying \( N_{a_0} \subset U_{a_0}(j), N_{a_2} \subset U_{a_2}(i_0), N_{a_1} \subset U_{a_2}(i_1), \ldots, N_{a_{n-2}} \subset U_{a_{n-2}}(i_{n-3}), N_{a_{n-1}} \subset U_{a_{n-1}}(i_{n-2}) \), and for all \( l = 0, 2, 3, \ldots, n-1 \), \( |N_{a_l}| = q_{a_l} - 1 \). Therefore, \( a_0, a_2, a_3, \ldots, a_{n-1} \) and \( j, i_0, i_2, i_3, \ldots, i_{n-1} \) constitute a generalized cycle of size \( n - 1 \), which is a contradiction.

Case (2-β): If \( i_2 > a_2 \), \( i_0 \) and there exists \( i \in N_{a_2} \) such that \( i_0 > a_2 \), \( i > a_2 \), \( i_1 \), let \( i_{**} \) be the minimum element in \( N_{a_2} \) with respect to \( > a_2 \) and \( N_{a_2} = N_{a_2} \cup \{ i_2 \} \setminus \{ i_{**} \} \). Then, \( i_0 > a_2 \), \( i_{**} > a_2 \), \( i_1 > a_1 \), \( i_0 \) and \( N_{a_1}, N_{a_2} \subset N \setminus \{ i_1, i_0, i_{**} \} \) are disjoint sets satisfying \( N_{a_2} \subset U_{a_2}(i_{**}), N_{a_1} \subset U_{a_1}(i_0), |N_{a_2}| = q_{a_2} - 1 \), and \( |N_{a_1}| = q_{a_1} - 1 \). Therefore, \( a_2, a_1, i_0, i_{**}, i_1 \) constitute a cycle, that is, a generalized cycle of size 2, which is a contradiction.

Q.E.D.

REFERENCES


[1-3]


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4Virtually, Ergin (2002) only considers two of the four cases, Case (1-α) and Case (2-α), of the alternative proof.