Econometrica Supplementary Material

SUPPLEMENT TO "THE COMPARATIVE STATICS OF CONSTRAINED OPTIMIZATION PROBLEMS" (*Econometrica*, Vol. 75, No. 2, March 2007, 401–431)

BY JOHN K.-H. QUAH

The purpose of this supplement is to discuss some of the central concepts and results of the main paper in the case where the objective function is defined on a subset of R^2. The general theory developed in the main paper in an l-dimensional context is particularly simple and intuitive in this special case. The results here are also interesting because comparative statics problems in R^2 are ubiquitous in economic theory. We give an application of our techniques to the portfolio problem with two risky assets.

KEYWORDS: Supermodular functions, single crossing property, comparative statics, portfolio problem, risk aversion.

WE BEGIN BY CONSTRUCTING a comparative statics theorem that is applicable to two-dimensional problems. We show that it is a corollary of the central comparative statics result (Theorem 2) in the main paper. In the process, we revisit the notions of $\mathcal{C}_i$-quasisupermodularity and $\mathcal{C}_i$-flexible set order and explain what they mean in a two-dimensional context.

The objective function is $f: X_1 \times X_2 \to R$, where $X_1 = (\underline{x}_1, \overline{x}_1)$ and $X_2 = (\underline{x}_2, \overline{x}_2)$ are nonempty open intervals in R. Our theorem gives conditions on the function f and the constraint sets H and G that guarantee that $\arg\max_{x\in H} f(x)$ is 2-higher than $\arg\max_{x\in g} f(x)$.[1]

We say that the function $f: X_1 \times X_2 \to R$ is *type I well behaved* if the following conditions hold: (i) f is a C^1 function with $f_2 > 0$ and (ii) the set $\{f(x_1, x_2) \in R : x_2 \in X_2\}$ does not vary with x_1, i.e., for any x_1' and x_1'' in X_1,

$$\{f(x_1', x_2) \in R : x_2 \in X_2\} = \{f(x_1'', x_2) \in R : x_2 \in X_2\}.$$

The crucial implication of these conditions is that f has well-behaved indifference curves in the sense that each curve is a differentiable function of x_1. We state this precisely in the next result.

LEMMA S1: *Suppose $f: X_1 \times X_2 \to R$ is type I well behaved. Then at each $x^* = (x_1^*, x_2^*)$ in $X_1 \times X_2$, there is a C^1 function $\psi: X_1 \to R$ such that $\psi(x_1^*) = x_2^*$ and $f(x_1, \psi(x_1)) = f(x_1^*, x_2^*)$ for all x_1 in X_1.*

PROOF: Let $f(x_1^*, x_2^*) = K$, so K is in $\{f(x_1^*, x_2) \in R : x_2 \in X_2\}$. Using condition (ii) in the definition of a type I well-behaved function, we know that at

[1] Recall (from Section 2 of the main paper) that for sets S' and S in R^2, the set S' is said to be 2-*higher* than S if, whenever both sets are nonempty, for any x in S there is x' in S' such that $x_2' \geq x_2$, and for any x' in S' there is x in S such that $x_2' \geq x_2$. When the optimal solutions are unique, this simply means that the optimal value of x_2 increases as the constraint set changes from G to H.

each $\hat{x}_1$ in X_1, there is $\hat{x}_2$ such that $f(\hat{x}_1, \hat{x}_2) = K$. Given that, by condition (i), $f_2 > 0$, we know that $\hat{x}_2$ is unique. Thus there is indeed a function $\psi : X_1 \to X_2$ such that $f(x_1, \psi(x_1)) = K$. The fact that ψ is C^1 follows from the assumption that f is C^1 with $f_2 \neq 0$. *Q.E.D.*

Similarly, we say that the function f is *type II well behaved* if the following conditions hold: (i) f is a C^1 function with $f_1 > 0$ and (ii) the set $\{f(x_1, x_2) \in R : x_1 \in X_1\}$ does not vary with x_2. Lemma S1 guarantees that, for a type II well-behaved function f, every indifference curve is a differentiable function of x_2.

Note that it is certainly possible for an objective function to be both types I and II well behaved; for example, this is true if $f : R^2_{++} \to R$ is given by $f(x_1, x_2) = x_1 x_2$. However, a type I well-behaved function (with, by definition, $f_2 > 0$) may have f_1 taking different signs at different points. If so, the indifference curves are not 1–1 functions of x_1; thus they are not expressible as functions of x_2.

We say that the indifference curves of $f : X_1 \times X_2 \to R$ have the *declining slope property* if *either* of the following statements holds:

(i) The function f is type I well behaved and $f_1(x_1, x_2)/f_2(x_1, x_2)$ is decreasing in x_1.

(ii) The function f is type II well behaved and $f_2(x_1, x_2)/f_1(x_1, x_2)$ is increasing in x_1.

The motivation for this terminology is clear when we consider the case where $f_1 > 0$ and $f_2 > 0$, so that each indifference curve must be downward sloping. Then $-f_1(x_1, x_2)/f_2(x_1, x_2)$ is the slope of the indifference curve and for this to be increasing means that the curve is getting flatter as x_1 increases (compare the curves I' and I'' in Figure S1). For simple conditions that guarantee the declining slope property, recall that we have shown in the main paper (Section 4.1) that f satisfies this property if $f_1 \geq 0$, $f_2 > 0$, $f_{11} \leq 0$, and $f_{12} \geq 0$.

We turn now to the constraint sets G and H. The constraint set G is said to be *regular* if its boundary is the graph of a decreasing and differentiable function. In formal terms, there is an open interval $I^G = (\underline{x}_2, x_2^G) \subseteq X_2$ and a differentiable function $g : I^G \to X_1$, with $g' \leq 0$, such that

$$G = \{x \in X_1 \times X_2 : x_2 \in I^G \text{ and } x_1 \leq g(x_2)\}.$$

Assume that G and H are both regular sets with the boundary of H given by $h : I^H \to X_1$, where $I^H = (\underline{x}_2, x_2^H)$. Then H is said to have a *steeper boundary* than G if the (i) $I^G \subseteq I^H$ (equivalently, $x_2^G \leq x_2^H$), (ii) for all x_2 in I^G, we have $g(x_2) \leq h(x_2)$, and (iii) for all x_2 in I^G, we have $g'(x_2) \leq h'(x_2)$. This is illustrated in Figure S1. Note that conditions (i) and (ii) imply that H contains G, while condition (iii) guarantees that the boundary of H at $(\tilde{a}_1, a_2)$ is steeper than the boundary of G at (a_1, a_2).

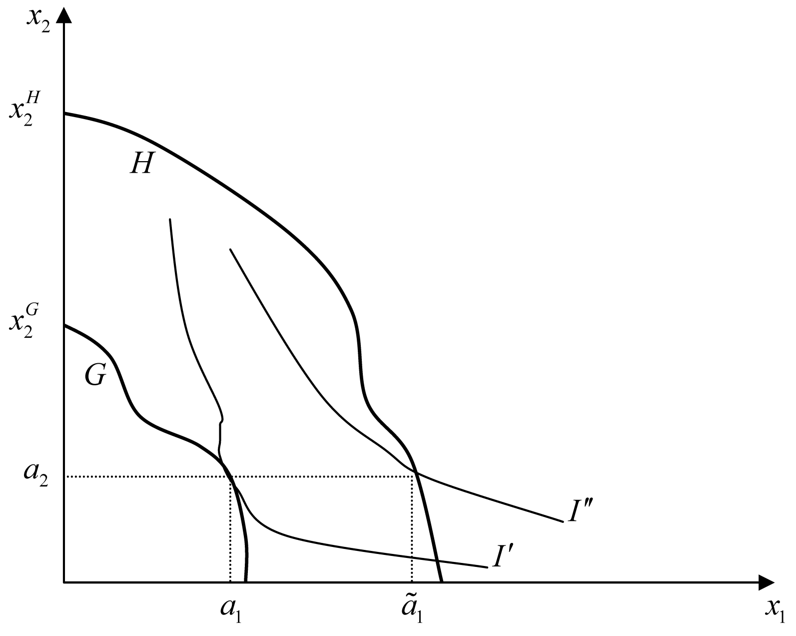

FIGURE S1.

We can now state the comparative statics result.

THEOREM S1: *Suppose that the indifference curves of* $f : X_1 \times X_2 \to R$ *obey the declining slope condition and that the constraint sets* H *and* G *are both regular with* H *having a steeper boundary than* G. *Then* $\arg\max_{x \in H} f(x)$ *is* 2*-higher than* $\arg\max_{x \in G} f(x)$.

Comparative statics problems can often be formulated in a way that conforms to the setup of Theorem S1. One instance, which we have already encountered in the main paper (Example 4), is Rybcsynski's theorem. (In that case, G and H are the production possibility sets of the economy before and after the increase in the capital stock.) The standard approach for dealing with these problems is by substitution. Suppose we are interested in how the value of x_2, which maximizes $f(x_1, x_2)$ subject to $x_1 \leq g(x_2, t)$, changes as the parameter t changes. Provided we know that f is locally nonsatiated, so that the constraint is binding, this problem can be converted into the one-dimensional problem of maximizing $f(g(x_2, t), x_2)$. This latter problem can often then be fruitfully studied using techniques already developed for studying one-dimensional problems (see, in particular, Athey, Milgrom, and Roberts (1998)). However, this does not negate the value of Theorem S1, because even when other techniques can be used, this theorem provides a particularly transparent approach to many such comparative statics problems.

Before we give a formal proof of Theorem S1, it is worth looking at Figure S1 to convince ourselves that it is completely intuitive. The function f, when subject to the constraint G, is maximized at (a_1, a_2). By the declining slope property, the indifference curve is *flatter* at $(\tilde{a}_1, a_2)$ than at (a_1, a_2). On the other hand, the slope of the constraint set H at $(\tilde{a}_1, a_2)$ is *steeper* than the slope of G at (a_1, a_2). Together they imply that f is maximized at a higher value of x_2 when the constraint set is H. Note also that this conclusion does not rely on the convexity of the indifference curves or the concavity of the constraint set boundaries.

Proof of Theorem S1

We prove this result by applying Theorem 2 from the main paper. Before we do that, it is worth recalling the main concepts in that theorem, specialized to the two-dimensional context. It requires the objective function $f : X_1 \times X_2 \to R$ to be *$\mathcal{C}_2$-quasisupermodular*; this condition says that for any two points (x'_1, x'_2) and (x''_1, x''_2) with $x'_2 > x''_2$ and $x'_1 < x''_1$, and for any k in the interval $[0, x''_1 - x'_1]$, we have

$$f(x'_1, x'_2) - f(x'_1 + k, x''_2) \geq (>)\, 0 \\ \implies \quad f(x''_1 - k, x'_2) - f(x''_1, x''_2) \geq (>)\, 0.$$

We see from Figure S2 that the points a, b, c, and d form a backward bending parallelogram; as k takes different values in the interval $[0, x''_1 - x'_1]$, we obtain a family of parallelograms. The $\mathcal{C}_2$-quasisupermodularity requires that the expression $f(d) - f(c)$ be positive whenever $f(a) - f(b)$ is positive.

On the constraint sets G and H, the theorem requires that H dominates G in the *$\mathcal{C}_2$-flexible set order*. In the case when G and H are both regular, with H containing G, this conditions requires that for any two points (x'_1, x'_2) in G and (x''_1, x''_2) in H with $x'_2 > x''_2$ and $x'_1 < x''_1$, there exists a real number K in the interval $[0, x''_1 - x'_1]$ such that $(x'_1 + K, x''_2)$ is in G and $(x''_1 - K, x'_2)$ is in H. Note that, once again, the points (x'_1, x'_2), (x''_1, x''_2), $(x'_1 + K, x''_2)$, and $(x''_1 - K, x'_2)$ form a backward-bending parallelogram.

The following result combines Theorem 2 and Proposition 3(i) from the main paper, specialized to the two-dimensional case.

THEOREM S2: *Suppose that $f : X_1 \times X_2 \to R$ is a $\mathcal{C}_2$-quasisupermodular function and that the constraint set H dominates G in the $\mathcal{C}_2$-flexible set order. Then* $\arg\max_{x \in H} f(x)$ *is* 2*-higher than* $\arg\max_{x \in G} f(x)$.

Clearly, Theorem S1 follows immediately from Theorem S2 if the next two propositions are true.

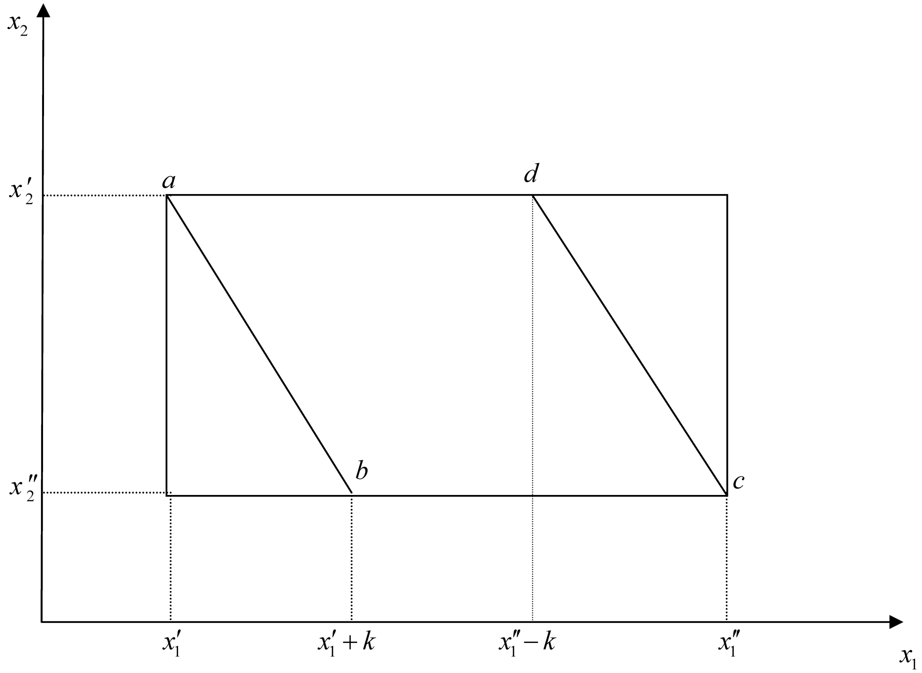

FIGURE S2.

PROPOSITION S1: *Suppose that the indifference curves of $f : X_1 \times X_2 \to R$ obey the declining slope property. Then f is $\mathcal{C}_2$-quasisupermodular.*

PROPOSITION S2: *Suppose that the sets H and G in $X_1 \times X_2$ are both regular with H having a steeper boundary than G. Then H dominates G in the $\mathcal{C}_2$-flexible set order.*

We already proved Proposition S2 when we considered an extension of Rybcsynski's theorem in the main paper (see Example 4 and Lemma A1), so we now turn to the proof of Proposition S1.

PROOF OF PROPOSITION S1: Assume that the indifference curve of f has the declining slope property; we confine ourselves to the case where f is type I well behaved and $f_1(x_1, x_2)/f_2(x_1, x_2)$ is decreasing in x_1. The other case can be proved in a similar manner. Our proof is a variation of the one given by Milgrom and Shannon (1994, Theorem 3) for the single crossing property. Suppose that $x_1' < x_1''$ and $x_2' > x_2''$ with $f(x_1', x_2') \geq f(x_1'', x_2'')$. We wish to show that $f(x_1'+k, x_2') \geq f(x_1''+k, x_2'')$ for $k > 0$. Define $I = [x_1', x_1'']$. Let the indifference curve through the point (x_1'', x_2'') be represented on I by the function $\tau : I \to R$, where $\tau(x_1'') = x_2''$. Then $f(x_1'' + k, \tau(x_1'')) - f(x_1' + k, \tau(x_1'))$ equals

$$\begin{aligned}
&\int_{x_1'+k}^{x_1''+k} \frac{df}{dt}(t, \tau(t-k))\, dt \\
&\quad = \int_{x_1'+k}^{x_1''+k} f_1(t, \tau(t-k)) + f_2(t, \tau(t-k))\tau'(t-k)\, dt \\
&\quad = \int_{x_1'+k}^{x_1''+k} \left[\frac{f_1(t, \tau(t-k))}{f_2(t, \tau(t-k))} + \tau'(t-k)\right] f_2(t, \tau(t-k))\, dt.
\end{aligned}$$

By the declining slope property, this expression is less than

$$\int_{x_1'+k}^{x_1''+k} \left[\frac{f_1(t-k, \tau(t-k))}{f_2(t-k, \tau(t-k))} + \tau'(t-k)\right] f_2(t, \tau(t-k))\, dt = 0.$$

It equals zero because $f(t, \tau(t))$ is identically constant. So $f(x_1' + k, \tau(x_1')) \geq f(x_1'' + k, \tau(x_1'')) = f(x_1'' + k, x_2'')$. Given that $f(x_1', x_2') \geq (>) f(x_1'', x_2'')$ by assumption and f is increasing in x_2, we have $x_2' \geq (>) \tau(x_1')$. The fact that f is increasing in x_2 also guarantees that $f(x_1' + k, x_2') \geq (>) f(x_1'' + k, x_2'')$. *Q.E.D.*

We have shown at the end of Section 4.1 in the main paper that when the partial derivatives of f are both positive, the declining slope property is also *necessary* for f to be $\mathcal{C}_2$-quasisupermodular. However, the condition is not necessary if the partial derivatives are not always positive. For example, if $f_2(x) > 0$ and $f_1(x) < 0$ for all x in X, then f is (trivially) $\mathcal{C}_2$-quasisupermodular because the indifference curves must all slope upward, but they need not satisfy the declining slope property.

It is possible to write down a result similar to Theorem S1 with the stronger conclusion that $\arg\max_{x\in H} f(x)$ is higher than $\arg\max_{x\in G} f(x)$.[2] Note that Theorem S1 gives us conditions that guarantee that $\arg\max_{x\in H} f(x)$ is 2-higher than $\arg\max_{x\in G} f(x)$. Clearly, we can write down analogous conditions on the indifference curves of f and the boundaries of H and G that guarantee that $\arg\max_{x\in H} f(x)$ is 1-higher than $\arg\max_{x\in G} f(x)$. If *both* sets of conditions are imposed, then, by Propositions S1 and S2, we obtain that f is $\mathcal{C}_i$-quasisupermodular and H dominates G in the $\mathcal{C}_i$-flexible set order for $i = 1$ and 2. By Theorem 2 and Proposition 3(ii) in the main paper, this guarantees that $\arg\max_{x\in H} f(x)$ is higher than $\arg\max_{x\in G} f(x)$.

[2]Recall, from Section 2 of the main paper that a set S' is higher than S if, whenever both sets are nonempty, for any x in S there is x' in S' such that $x' \geq x$ and for any x' in S' there is x in S such that $x' \geq x$.

An Application of Theorem S1

We end with an application of Theorem S1 to the portfolio problem. More two-dimensional applications can be found in Quah (2006).

Consider the standard portfolio problem of an agent who has to choose between two assets: a safe asset with constant and positive payoff r and a risky asset with payoff s governed by the density function f. The agent has the Bernoulli utility function $u : R \to R$, so that its objective function is $U(a, b) = \int u(bs + ar) f(s)\, ds$. It is well known that the agent's investment in the risky asset will increase with wealth if his coefficient of risk aversion decreases with wealth. The standard proof of this result converts the agent's problem into a single variable (the level of risky investment) problem by making a substitution using the budget identity and then establishing that some version of the single crossing property holds (see, for example, Gollier (2001) or Athey (2002)).

Another natural way to obtain this result is to use Theorem S1. The budget line before and after the increase in wealth has the same slope, so the only work that needs doing is to establish that U obeys the declining slope property. The function U is strictly increasing in a if u is strictly increasing, so we need to show that

$$\frac{\int u'(bs + ar) s f(s)\, ds}{\int u'(bs + ar) r f(s)\, ds}$$

is increasing with a. For this to hold, it is sufficient that $u'(as + br)$ be log-supermodular in (s, b).[3] The cross-derivative of $\ln u'(as + br)$ is $(\ln u')''(as + br)ar$; it is not hard to check that u has decreasing risk aversion if and only if $(\ln u')'' \geq 0$ (in other words, $\ln u'$ is convex), so $u'(as + br)$ is log supermodular if we restrict the domain of a to $a > 0$. As is well known (see Gollier (2001)), we can, if we prefer, make this last restriction nonbinding by assuming that the risky payoff has a mean return greater than r and that u is concave (until this point, the concavity of u has not been used).

We can use our approach to generalize this standard result to the case when both assets are risky.[4] Suppose that asset A has a payoff rt, where r is a positive constant and $t > 0$ is stochastic, and asset B has a payoff st, where s is also stochastic. We assume that s and t are independent and are distributed according to density functions f and h, respectively. If we wish, we can interpret this as a situation in which both assets have nominal payoffs and the price level is sto-

[3]The ratio $\int g(s)\phi(s, \theta)\, ds / \int h(s)\phi(s, \theta)\, ds$ increases with θ if $g(s)/h(s)$ increases with s and ϕ is a log-supermodular function of (s, θ) (see Athey (2002)). In our case, $\theta = a$, $\phi = u'$, $g(s) = sf(s)$, and $h(s) = rf(s)$.

[4]For other comparative statics results with two risky assets, see Jewitt (2000).

chastic, so that rt and st measure the real returns of the two assets. The agent's utility when he holds a of asset A and b of asset B is then given by

$$U(a,b) = \int u(bst + art) f(s) h(t)\, ds\, dt. \tag{S1}$$

The next proposition guarantees that U obeys the declining slope property. Thus, by Theorem S1, the demand for asset B is normal.

PROPOSITION S3: *The function U as defined by* (S1) *obeys the declining slope property if u is C^3, $u' > 0$, $u'' \leq 0$ and the coefficient of risk aversion of u is decreasing.*

PROOF: We need to show that the ratio

$$R(a,b) = \frac{\int u'(bst + art) st f(s) h(t)\, dt\, ds}{\int u'(bst + art) rt f(s) h(t)\, dt\, ds}$$

is increasing with a. Define $v(z) = \int u(tz) h(t)\, dt$. Then $v'(z) = \int u'(tz) t \times h(t)\, dt$, so that $R(a,b) = \int v'(bs + ar) sf(s)\, ds / \int v'(bs + ar) rf(s)\, ds$. From the previous argument, we know that this is increasing in a provided v exhibits decreasing risk aversion. It is not hard to check (alternatively, see Gollier (2001)) that, when v is concave (which it is because u is concave), this property holds if and only if

$$-\frac{v'''(z)}{v''(z)} \geq -\frac{v''(z)}{v'(z)} \quad \text{for all } z.$$

To show this, we set $-v''(z)/v'(z) = \lambda$ and claim that

$$v'''(z) + \lambda v''(z) = \int [t^3 u'''(tz) + \lambda t^2 u''(tz)] h(t)\, dt \geq 0.$$

Clearly, this is true if there is some number m such that

$$t^3 u'''(tz) + \lambda t^2 u''(tz) \geq m[\lambda u'(tz) t + u''(tz) t^2], \tag{S2}$$

because the integral of the right-hand side gives $\lambda v'(z) + v''(z) = 0$.[5] Denoting $a = -u''(tz)/u'(tz)$ and recalling that $-u'''(tz)/u''(tz) \geq a$ because u has diminishing risk aversion, we can check that a sufficient condition for (S2) to be true (after dividing by $t > 0$) is that $a^2 t^2 - \lambda a t \geq m[\lambda - at]$. This is true if we set $m = -\lambda$. *Q.E.D.*

Dept. of Economics, St. Hugh's College, Oxford University, Oxford OX2 6LE, United Kingdom; john.quah@economics.ox.ac.uk.

[5]In fact, the existence of m is also *necessary*. See Gollier (2001), who refers to this equivalence as *the diffidence theorem*.

REFERENCES

ATHEY, S. (2002): "Monotone Comparative Statics under Uncertainty," *The Quarterly Journal of Economics*, 117, 187–223. [7]

ATHEY, S., P. MILGROM, AND J. ROBERTS (1998): "Robust Comparative Statistics," Draft Chapters, http://www.stanford.edu/~athey/draftmonograph98.pdf. [3]

GOLLIER, C. (2001): *The Economics of Risk and Time.* Cambridge, MA: MIT Press. [7,8]

JEWITT, I. (2000): "Risk and Risk Aversion in the Two Risky Asset Problem," Mimeo, **???**. [7]

MILGROM, P., AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–180. [5]

QUAH, J. K.-H. (2006): "Additional Notes on the Comparative Statics of Constrained Optimization Problems," Working Paper 2006-W9, Nuffield College, http://www.nuffield.ox.ac.uk/economics/papers/. [7]