SUPPLEMENT TO “EFFICIENCY IN REPEATED GAMES REVISITED: THE ROLE OF PRIVATE STRATEGIES.”
TECHNICAL DETAILS FOR EXAMPLE 2

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FIRST, WE SHOW how to derive the PE payoff in Figure 2 in the main paper. The private equilibrium in Section 3.2 of the main paper relies only on the assumption \( p(Y|D,D) > 0 = p(Y|D,C) = p(Y|C,D) \); thus it also works in the present example, irrespective of the level of \( \varepsilon \). The incentive constraints (6) and (7) in Section 3.2 reduce to a quadratic equation in \( q \),

\[
(1 - \delta)(h - d)q + d = \delta q p(Y|D,D)(1 - q - qh),
\]

where \( q \) is the probability to play \( D \). In Example 2, we have \( h = 6, d = 1 \), and \( p(Y|D,D) = 1/3 \). Hence (1) becomes

\[
7\delta q^2 + (15 - 16\delta)q + 3(1 - \delta) = 0.
\]

Whereas we are interested in the most efficient equilibrium, we choose the smaller root: \( q(\delta) \equiv q = \frac{1}{14\delta}(-15 + 16\delta - \sqrt{225 - 564\delta + 340\delta^2}) \). Computation shows that this solution is real and lies in \([0, 1]\) when \( \delta \geq 0.992 \). The associated symmetric private equilibrium payoff for each player is \( v(\delta) = 1 - 7q(\delta) \), the graph of which is depicted by the solid line in Figure 2.

Next, we present the lemmas and the derivation of \( \delta(v) \) that we cited when we derived the upper bound of the PPE payoffs. Throughout this supplement, \( \overline{v} \) refers to the best symmetric payoff of the PPE payoffs.

**LEMMA 1:** When \( \overline{v} > 0 \), there exists a positive constant \( L \) independent of \( \varepsilon \) such that \( \Delta_1(\omega) + \Delta_2(\omega) \geq L \) is satisfied for all \( \omega \).

**PROOF:** When \( \overline{v} > 0 \), the first-period action profile in the best symmetric PPE lies in the set

\[
Q_+ = \{(q_1, q_2)|g_1(q_1, q_2) + g_2(q_1, q_2) > 0\},
\]

where \( q_i \) is the probability that player \( i \) chooses action \( D \). If this were not the case, so that \( g'_1 + g'_2 \leq 0 \) in the formula (16) in the paper, a continuation equilibrium would provide a better symmetric PPE with payoff \( (v'_1(\omega) + v'_2(\omega))/2 > \overline{v} \), which contradicts our premise that \( \overline{v} \) is the best symmetric PPE payoff. Whereas \( F(q_1, q_2) \equiv g_1(q_1, q_2) + g_2(q_1, q_2) = 2 - 6q_2 - 6q_1 + 10q_1q_2 \), we have

\[
(q_1, q_2) \in Q_+ \implies q_i < 1/3 \quad \text{for} \quad i = 1, 2.
\]
This is shown as follows. Note that \( F(q_1, q_2) \) is linear in \( q_1 \) and that both 
\[ F(0, q_2) = 2 - 6q_2 \text{ and } F(1, q_2) = 4(q_2 - 1) \] 
are nonpositive if \( q_2 \geq 1/3 \). Hence 
\( F(q_1, q_2) \), which is a convex combination of those values, is nonpositive if 
\( q_2 \geq 1/3 \). A symmetric argument shows that \( F \) is nonpositive if \( q_1 \geq 1/3 \). Hence 
\( F \) is positive only if \( q_1, q_2 < 1/3 \).

Note that, for any \((q_1, q_2)\), we have (i) \( p(Y|q_1, q_2) \leq p(X_k|q_1, q_2) \), \( k = 1, 2 \), and (ii) \( p(Y|q_1, q_2) \) does not depend on \( \varepsilon \). Hence, for any \((q_1, q_2) \in Q_+ \) and any \( \omega \), \( p(\omega|q_1, q_2) \) is bounded below by \( r \equiv \min_{q_1, q_2 \in [0, 1/3]} p(Y|q_1, q_2) \) 
(we used (2) here), which is a positive number independent of \( \varepsilon \). Now consider the dynamic programming equation (16) in the paper. Because 
\( \bar{v} > 0 \), \( g_1(q_1, q_2) + g_2(q_1, q_2) \leq 2 \), and 
\( \sum_{\omega}(\Delta_1(\omega) + \Delta_2(\omega))p(\omega|q_1, q_2) \leq r \min_{\omega}(\Delta_1(\omega) + \Delta_2(\omega)) \) (this is implied by \( \Delta_1(\omega) + \Delta_2(\omega) \leq 0 \) (see the main paper) and \( r \leq p(\omega|q_1, q_2) \)), we have

\[ \forall \omega, \ -L \leq \Delta_1(\omega) + \Delta_2(\omega) \]

for \( L \equiv 2/r \).

Q.E.D.

**LEMMA 2:** For any (large) constant \( K > 0 \), we can find a (small enough) \( \varepsilon > 0 \) such that \( \bar{v} > 0 \) requires

\[ (\Delta_1(\omega), \Delta_2(\omega)) \in D \]

\[ = \{(\Delta_1, \Delta_2)|-L \leq \Delta_1 + \Delta_2 \leq 0 \text{ and } \Delta_i > K \text{ for } i = 1 \text{ or } 2\} \]

for some \( \omega \) if \( \varepsilon \leq \varepsilon \).

**PROOF:** Suppose the claim does not hold. Then, for any \( K > 0 \) and any \( \varepsilon > 0 \), there must be some \( \varepsilon \leq \varepsilon \) for which \( \bar{v} > 0 \) is sustained as a symmetric PPE by \((\Delta_1(\omega), \Delta_2(\omega))\), which lies for all \( \omega \) in a compact set

\[ D' = \{(\Delta_1, \Delta_2)|-L \leq \Delta_1 + \Delta_2 \leq 0 \text{ and } \Delta_i \leq K \text{ for } i = 1, 2\} \]

Let \((q_1, q_2)\) be the first-period action to sustain \( \bar{v} \). Whereas \( \bar{v} > 0 \), the proof of 
Lemma 1 above shows that \((q_1, q_2) \in Q_+ \). In addition, the incentive compatibility condition

\[ g(D, q_j) - g(C, q_j) \leq \sum_{\omega=X_1,X_2,Y} \Delta_i(\omega)[p(\omega|C, q_j) - p(\omega|D, q_j)] \]

is satisfied for \( i, j = 1, 2 \) and \( j \neq i \), which should hold with equality if player \( i \) 
mixes \( C \) and \( D \).

Given that this is true for any \( \varepsilon > 0 \), there is a sequence \( \{\varepsilon^n, \Delta_i^n, \Delta_2^n, q_1^n, q_2^n\} \) 
such that \( \varepsilon^n \to 0 \text{ as } n \to \infty \), where (i) \( \Delta_i^n \equiv (\Delta_i^n(Y), \Delta_i^n(X_1), \Delta_i^n(X_2)) \), 
(ii) \( (\Delta_i^n, \Delta_2^n, q_1^n, q_2^n) \) satisfies incentive constraint (3), and (iii) \( (\Delta_i^n, \Delta_2^n, q_1^n, q_2^n) \)
lies in compact set \((D')^3 \times [0, 1/3]^2\) (here we used (2)). By (iii), there is a converging subsequence; let \((\Delta_1^*, \Delta_2^*, q_1^*, q_2^*)\) be its limit. Whereas both sides of incentive constraint (3) are continuous in \((\varepsilon, \Delta_1, \Delta_2, q_1, q_2)\), the limit also satisfies (3).

In the limit where \(\varepsilon = 0\), outcomes \(X_1\) and \(X_2\) always realize with an equal probability for any action profile. Hence, essentially we can regard \(\{X_1, X_2\}\) as a single outcome \(X\). This enables us to use our results in Section 3.1 of the main paper, which presumes two outcomes \(X\) and \(Y\). To this end, define \(\Delta_1^*(X) \equiv \frac{1}{2}\Delta_1^*(X_1) + \frac{1}{2}\Delta_1^*(X_2)\). Whereas the limit satisfies (3), a simple calculation shows that \((\Delta_1^*(X), \Delta_2^*(Y))\) satisfies the incentive constraint for the game with two outcomes \(X\) and \(Y\).

The limit also satisfies \(q_1^*, q_2^* \leq 1/3\), which implies that a unilateral deviation from \((q_1^*, q_2^*)\) makes \(X\) (i.e., \(\{X_1, X_2\}\)) more likely. Hence, \((q_1^*, q_2^*)\) is in set \(Q\) defined in Section 3.1 of the main paper. Then the upper bound in Lemma 1 in the main paper applies. Therefore, the payoff associated with the limit is bounded above by

\[
\max_{q \in [0,1]} g(C, q) - \frac{d(q)}{L(q) - 1} \geq \max_{q \in [0,1]} (1 - 7q) - \frac{1 + 5q}{2q^3 - 1} - 1 = 1 - \frac{1}{2} = 1/2 < 0.
\]

However, whereas the payoffs along the sequence are strictly positive, their limits should be nonnegative. This constitutes a contradiction. Q.E.D.

Finally, we show how to derive \(\delta(\bar{v})\), a lower bound of \(\delta\) to satisfy

\[
\left( \frac{1 - \delta}{\delta} D + (v_1^0, v_2^0) \right) \cap V^F \neq \emptyset,
\]

where \((v_1^0, v_2^0)\) is an equilibrium payoff profile to obtain symmetric payoff \(\bar{v}\) (possibly with public randomization). Note that if this condition (4) is satisfied for some \(\delta'\), then it is also satisfied for all \(\delta > \delta'\). Hence, any value of \(\delta\) such that \(\left( \frac{1 - \delta}{\delta} D + (v_1^0, v_2^0) \right) \cap V^F = \emptyset\) is a lower bound of discount factor to satisfy (4).

A reasonably tight lower bound is obtained by the value of \(\delta\) that is determined as in Figure S1. The two lines defined by \(v_1 + 7v_2 = 8\) and \(7v_1 + v_2 = 8\) lie on the Pareto frontier of the feasible payoff set \(V^F\), so that \(V^F\) is contained in set \(W\) in the figure. The shaded areas correspond to set \(\frac{1 - \delta}{\delta} D + v'\). We pick the point \(v'\) (such that \(2\bar{v} = v_1' + v_2'\)) off the 45° line to deal with the possibility that \((v_1^0, v_2^0)\) may not be a symmetric payoff profile. The particular choice of point \(v'\)

1Note that signal distribution \(p\) is a continuous function of \(\varepsilon\).

2This follows from the fact that the upper bound in Lemma 1 in the main paper is derived by the incentive constraint and \(q \in Q\), both of which are satisfied by the limit point.
makes sure that, if $\delta$ is determined as in Figure S1, then $\frac{1-\delta}{\delta}D + (v_1^0, v_2^0)$ always lies outside of $W$ (hence outside of $V^F$) for any possible choice of $(v_1^0, v_2^0)$ (i.e., for any $(v_1^0, v_2^0)$ in $W$ (hence in $V^F$) that satisfies $v_1^0 + v_2^0 = 2\bar{v}$). In summary, if $\delta$ is determined as in Figure S1, then we have $\frac{1-\delta}{\delta}D + (v_1^0, v_2^0) \cap V^F = \emptyset$.

Figure S1 shows that we have

$$v''_1 - v'_1 = \frac{1 - \delta}{\delta}K.$$  

The value of $v'_1$ is obtained by solving $v_1 + v_2 = 2\bar{v}$ and $v_1 + 7v_2 = 8$, and we find $v'_1 = \frac{7\bar{v} + 4K}{3}$. Similarly, $v''_1$ is determined by $v_1 + v_2 = 2\bar{v} - (\frac{1-\delta}{\delta})L$ and $7v_1 + v_2 = 8$, and we find $v''_1 = (8 - 2\bar{v} + (\frac{1-\delta}{\delta})L)/6$. By plugging these results into equation (5), we obtain a lower bound of the discount factor to support $\bar{v}$:

$$\delta(\bar{v}) = \frac{3K - \frac{L}{2}}{3K - \frac{L}{2} + 8(1 - \bar{v})}.$$  

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