SUPPLEMENT TO “SOLVING ASSET PRICING MODELS WHEN
THE PRICE–DIVIDEND FUNCTION IS ANALYTIC”

BY OVIDIU L. CALIN, YU CHEN, THOMAS F. COSIMANO,
AND ALEX A. HIMONAS

This appendix provides proofs of some results stated in our paper. First,
we prove the existence and uniqueness of the price-dividend function (Proposi-
tion 1) to the integral equation (6) in the vector space $S$ (Definition 1). Sec-
ond, we calculate the system of linear equations (22) for the coefficients of
the power series for the price-dividend function. Finally, we use Cauchy’s inte-
gral formula to bound all the derivatives of the price-expected dividend func-
tion (23) so that we can calculate the error terms (24), (25), and (26).

We use the following notation in the proofs: $C(n) = \sum_{i=1}^{n} |\phi|^i = \frac{|\phi(1-|\phi|^n)|}{1-|\phi|}$ for $n = 0, 1, 2, \ldots$, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$, for all $x \in \mathbb{R}$, $\theta = x_0 + \sigma^2(1 + \phi - \alpha)(1 - \gamma)$, $K_1 = K_0 \exp(x_0 K_1 + \frac{\sigma^2(1-\gamma)(\phi-\alpha)(2+\phi-\alpha)}{2})$, $K_3 = K_0 \exp(x_0 K_3 + \frac{|\phi K_1| |\sigma|}{1-|\phi|})$, $K_5 = K_4 \exp\left(\frac{\alpha^2 k_2^2 s^2}{2(1-|\phi|^2)} + \frac{|\phi K_1| |\alpha|}{1-|\phi|}\right)$, and $K_6 = \frac{1}{\sqrt{\pi}} K_0 \exp\left(-\frac{\alpha^2}{2}(1 - \gamma - \alpha \gamma + K_1)^2\right)$. The term $Q$ is defined in (18), which satisfies the integral equation (19).

A series of lemmas are now proved which are used in the proof of Proposition 1.

**Lemma 2:** Let $a > 0$ and $\varphi(x) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-x-a}^{x+a} e^{-s^2/2\sigma^2} ds$ for all $x \in \mathbb{R}$. Then $\varphi(x) \leq 2\Phi\left(\frac{a}{\sigma}\right)$.

**Proof:** These proof follows from the fact that $\varphi$ has a unique global maximum at $x = 0$. Q.E.D.

**Lemma 3:** For any real numbers $A$ and $k$ with $k \geq 0$, we have

$$\frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} e^{-1/(2\sigma^2)(s-A)^2+k|s|} ds \leq e^{k^2 \sigma^2/2+k|A|} [1 + 2\Phi(k\sigma)].$$

**Proof:** Note that

$$-\frac{1}{2\sigma^2}(s-A)^2 + ks = -\frac{1}{2\sigma^2}(s-A-k\sigma)^2 + \frac{k^2 \sigma^2}{2} + k A,$$

$$\frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} e^{-1/(2\sigma^2)(s-A)^2+k|s|} ds = \frac{1}{\sqrt{2\pi} \sigma} \left( \int_{-\infty}^{0} e^{-1/(2\sigma^2)(s-A)^2-ks} ds + \int_{0}^{\infty} e^{-1/(2\sigma^2)(s-A)^2+ks} ds \right)$$
By Lemma 2, we achieve
$$\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2\sigma^2}(s-A)^2 + k|s|) ds \leq \exp(k^2\sigma^2 + k|A|)(1 + 2\Phi(k\sigma)).$$

Q.E.D.

**LEMMA 4:** For any \( f \in S \), the function \( Tf \) defined by

$$ (Tf)(x) = \frac{K_3e^{\phi K_1x}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} f(s)e^{-1/(2\sigma^2)(s-(\phi x+\theta))^2} ds $$

lies in the vector space \( S \). Thus, \( T \) defines a linear transformation from the vector space \( S \) to itself.

**PROOF:** By the definition of \( S \), we can find positive constants \( M \) and \( k \) such that \( |f(x)| \leq Me^{k|x|} \). Using Lemma 3, we deduce

$$ |(Tf)(x)| \leq \frac{K_3e^{\phi K_1x}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} |f(s)|e^{-1/(2\sigma^2)(s-(\phi x+\theta))^2} ds $$

$$ \leq MK_3e^{\phi K_1x} \int_{-\infty}^{\infty} e^{-1/(2\sigma^2)(s-(\phi x+\theta))^2 + k|s|} ds $$

$$ \leq MK_3[1 + 2\Phi(k\sigma)]e^{\phi K_1x + k^2\sigma^2/2 + k(|\phi|x + |\theta|)} $$

$$ \leq MK_3e^{k^2\sigma^2/2 + k|\theta|}[1 + 2\Phi(k\sigma)]e^{\phi (|k|K_1)|x|}. $$

The continuity of \( Tf \) can be easily checked by applying the standard argument in real analysis; for example, see Theorem 56 in Kaplan (1956).

Q.E.D.

Now construct a sequence of functions \( \{Q_n\}_{n=0, 1, 2, \ldots} \) by setting \( Q_0 = 0 \) and

$$ Q_{n+1}(x) = K_0 + \frac{K_3e^{\phi K_1x}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} Q_n(s) \exp\left(-\frac{1}{2\sigma^2}[s-(\phi x + \theta)]^2\right) ds, $$

(2)
and set $Q = \sum_{n=0}^{\infty} [Q_{n+1} - Q_n]$.

Recall that $C(n) = \sum_{i=1}^{n} |\phi|^i = \frac{|\phi| (1 - |\phi|^n)}{1 - |\phi|}$ satisfies (a) $C(n + 1) = |\phi| \times (C(n) + 1)$, (b) $C(n) < \frac{|\phi|}{1 - |\phi|}$, and (c) $\lim_{n \to \infty} C(n) = \frac{|\phi|}{1 - |\phi|}$.

**Lemma 5:** For $n = 0, 1, 2, \ldots$,

$$0 \leq Q_{n+1}(x) - Q_n(x)$$

$$\leq K_0 K_4^n \exp \left( \frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i) \right) e^{C(n)|K_1||x|}.$$

**Proof:** These inequalities can be easily proven by induction on $n$. Q.E.D.

**Lemma 6:** The series $Q = \sum_{n=0}^{\infty} [Q_{n+1} - Q_n]$ is uniformly convergent on any bounded closed interval.

**Proof:** Fix a bounded closed interval $[a, b]$. For any $x \in [a, b]$, by Lemma 5 we see

$$0 \leq \sum_{n=0}^{\infty} [Q_{n+1}(x) - Q_n(x)]$$

$$\leq K_0 \sum_{n=0}^{\infty} K_4^n \exp \left( \frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i) \right) e^{C(n)|K_1||x|}.$$

The ratio test together with the Weierstrass $M$-test (see Theorem 30 of Kaplan (1956)) implies the uniform convergence of the series $Q = \sum_{n=0}^{\infty} [Q_{n+1} - Q_n]$ on $[a, b]$, provided

$$\lim_{n \to \infty} K_4 \exp \left( \frac{\sigma^2 K_1^2}{2} C(n)^2 + |K_1 \theta| C(n) \right) e^{C(n+1) - C(n)|K_1||a+b|}$$

$$= K_4 \exp \left( \frac{\sigma^2 K_1^2 \phi^2}{2(1 - |\phi|)^2} + \frac{|\phi K_1 \theta|}{1 - |\phi|} \right) = K_5 < 1.$$

Q.E.D.

**Lemma 7:** The function $Q$ lies in the vector space $S$.

**Proof:** The continuity of $Q$ follows from Theorem 31 in Kaplan (1956) and Lemma 6. By Lemma 5, we also have

$$|Q(x)| \leq K_0 \sum_{n=0}^{\infty} K_4^n \exp \left( \frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i) \right) e^{C(n)|K_1||x|}.$$
\[ \leq \left( K_0 \sum_{n=0}^{\infty} K_n^4 \exp \left( \frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i) \right) \right) \times \exp \left( \frac{| \phi K_1 |}{1 - | \phi | | x |} \right). \]

Q.E.D.

**Lemma 8:** For any \( x \in \mathbb{R} \), we have \( \lim_{n \to \infty} \frac{1}{\sqrt{2\pi \sigma}} \int_{-\infty}^{\infty} [Q(s) - Q_n(s)] \times e^{-1/(2\sigma^2)(s-(\phi x + \theta))^2} ds = 0. \)

**Proof:** This is an immediate consequence of Theorem 32 in Kaplan (1956) and Lemma 6. Q.E.D.

**Proof of Proposition 1:** Equivalently, we are going to show that Equation (19) has a unique solution \( Q \) in the vector space \( S \).

*Existence.* Applying the limit as \( n \) approaches \( \infty \) to Equation (2) shows that the function \( \tilde{Q} = \sum_{n=0}^{\infty} [Q_{n+1} - Q_n] = \lim_{n \to \infty} Q_n \) is a solution to (19) in the vector space \( S \).

*Uniqueness.* Suppose that \( \tilde{Q} \in S \) is another solution to (19). Then \( Q \) and \( \tilde{Q} \) satisfy the functional equation \( Q - \tilde{Q} = T(Q - \tilde{Q}) \). By the definition of \( S \), we can also find positive constants \( M \) and \( k \) such that \( |Q(x) - \tilde{Q}(x)| \leq Me^{k|x|} \).

Using the results in the proof of Lemma 4, we see

\[ |Q(x) - \tilde{Q}(x)| \leq (T|Q - \tilde{Q}|)(x) \leq MK_3 e^{k^2 \sigma^2/2 + k|\theta|} [1 + 2\Phi(k \sigma)] e^{(|\phi|^2 k + C(1)|K_1|)|x|} \]

for all \( x \in \mathbb{R} \). Applying this computation \( n \) times gives rise to

\[ |Q(x) - \tilde{Q}(x)| \leq M K_3^n e^{k^2 \sigma^2/2 + k|\theta|} \prod_{i=0}^{n-1} \left( [1 + 2\Phi(\sigma|\phi|^i k + \sigma C(i)|K_1|)] e^{(|\phi|^n k + C(n)|K_1|)|x|} \right). \]

Note that

\[ \lim_{n \to \infty} K_3 e^{\frac{(|\phi|^n k + C(n)|K_1|)^2 \sigma^2}{2} + (|\phi|^n k + C(n)|K_1|)|\theta|} \times [1 + 2\Phi(\sigma|\phi|^n k + \sigma C(n)|K_1|)] \]
We can find a positive integer \( N \) and a positive real number \( \delta < 1 \) so that for any \( n \geq N \),

\[
K_3 \exp \left( \frac{(|\phi|^n k + C(n)|K_1|)^2 \sigma^2}{2} + (|\phi|^n k + C(n)|K_1|)|\theta| \right) \\
\times \left[ 1 + 2\Phi(\sigma|\phi|^n k + \sigma C(n)|K_1|)|\theta| \right] \leq \delta.
\]

Then for all \( n \geq N \), we have

\[
|Q(x) - \hat{Q}(x)| \\
\leq MK_3^N \left\{ \prod_{i=0}^{N-1} \exp \left( \frac{(|\phi|^i k + C(i)|K_1|)^2 \sigma^2}{2} + (|\phi|^i k + C(i)|K_1|)|\theta| \right) \\
\times \left[ 1 + 2\Phi(\sigma|\phi|^i k + \sigma C(i)|K_1|)|\theta| \right] \right\} \delta^{n-N} e^{(|\phi|^n k + C(n)|K_1|)|x|}.
\]

Since \( \lim_{n \to \infty} \delta^{n-N} = 0 \) and \( \lim_{n \to \infty} e^{(|\phi|^n k + C(n)|K_1|)|x|} = \exp(\frac{|\phi|^n k + C(n)|K_1|)|x|) \), we obtain \( Q(x) = \hat{Q}(x) \) for all \( x \in \mathbb{R} \). This completes the proof of Proposition 1. \( Q.E.D. \)

**Solving for the Coefficients in the Analytic Price–Dividend Function**

The integral equation (6) may be written as

\[
P(x) = K_0 e^{K_1 x} + K_0 e^{K_1 x - \sigma^2/2(1-\gamma)^2} I(x),
\]

where \( I(x) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{(1-\gamma)(x-s)^2/(2\sigma^2)} P(x_0 + \phi x + s) \, ds \). To solve the integral equation we posit that \( P(x) \) is analytic, in which case it has the functional form (21). Consequently we may write \( P(x_0 + \phi x + s) \) as

\[
P(x_0 + \phi x + s) = e^{K_1(x_0 + \phi x + s)} \sum_{k=0}^{n} b_k (x_0 + \phi x + s - x_*)^k.
\]
Using the binomial theorem\textsuperscript{26} we may rewrite this equation as

\[ P(x_0 + \phi x + s) = e^{K_1(x_0 + \phi x + s)} \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} s^i \right). \]

Substituting this result into \( I(x) \) we obtain

\[ I(x) = e^{K_1(x_0 + \phi x)} \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} \gamma_i, \]

where \( \gamma_i = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp((1 - \gamma + K_1)s - \frac{s^2}{2\sigma^2}) s^i ds \). The evaluation of the integral \( \gamma_i \) entails the use of a change of variable followed by the evaluation of all the moments of the normal distribution: \( \gamma_i = K_3 a_i \), where \( K_7 = \frac{1}{\sqrt{\pi}} e^{-\sigma^2/2(1-\gamma+K_1)^2} \) and \( a_i \equiv \sum_{j=0}^{i} \binom{i}{j} (\sigma^2(1 - \gamma + K_1)^{j-1})(1+(\frac{-1}{1}))^{\frac{j+1}{2}} \Gamma[\frac{j+1}{2}]. \)

We can now write \( I(x) \) as \( K_7 e^{K_1(x_0 + \phi x)} \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} a_i \), so that the integral equation becomes

\[ e^{-K_1 x} P(x) = K_0 + K_6 e^{K_1(x_0 + \phi x)} \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} a_i \]

\[ = \sum_{k=0}^{n} b_k (x - x_*)^k. \]

Next we find the undetermined coefficients \( b_i, i = 0, \ldots, n \). To do this, we collect all the functions of \( x \) on the left-hand side of this equation and use Taylor’s theorem to take an \( n \)th order Taylor expansion around \( x_* \). This produces a system of \( (n + 1) \) linear equations in the variables \( b_i, i = 0, \ldots, n \).

Define the functions \( q(x) = K_1(x_0 + \phi x), r_{i,k}(x) = (x_0 - x_* + \phi x)^{k-i}, \) and \( w_{i,k}(x) = \exp(q(x)) r_{i,k}(x) \approx \sum_{l=0}^{n} \frac{1}{l!} w_{i,k}^{(l)}(x_*) (x - x_*)^l \). Substituting for \( w_{i,k}(x) \) into the integral equation gives us

\[ K_0 + K_6 \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} \left( \sum_{l=0}^{n} \frac{1}{l!} w_{i,k}^{(l)}(x_*) (x - x_*)^l \right) a_i \]

\[ = \sum_{k=0}^{n} b_k (x - x_*)^k. \]

\textsuperscript{26}The binomial theorem states that we may write \( (x_0 + \phi x + s)^k = \sum_{i=0}^{k} \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} s^i \).
Finally, equate the coefficients on the left- and right-hand sides of this equation to yield

\[ b_0 = K_0 + K_6 \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} w_{i,k}(x_i) a_i \quad \text{and} \]

\[ b_l = K_0 \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} \frac{1}{l!} w_{i,k}(x_i) a_i, \]

where \( l = 1, \ldots, n \). These equations are an \( n + 1 \) system of linear equations in the \( b_i \)'s.

**Analytic Error of the Taylor Polynomial Approximation**

The complex function \( Q(z) = e^{-K_1 z} P(z) \) is analytic and it is expressible as a Taylor series

\[ Q(z) = \sum_{k=0}^{\infty} \frac{O^{(k)}(z_s)}{k!} (z - z_s)^k \quad \text{for any complex number } z, \]

where \( z_s = x_s \). We may use Cauchy's integral formula to estimate the error of the Taylor polynomial approximation \( Q(z) \approx \sum_{k=0}^{n} \frac{O^{(k)}(z_s)}{k!} (z - z_s)^k \) near \( z = z_s \).

Write \( C_r \) for the circle of radius \( r > 0 \) centered at \( z = z_s \) in the complex plane. Cauchy's integral formula (see Corollary 5.9 in Conway (1973)) gives

\[ O^{(k)}(z_s) = \frac{k!}{2\pi i} \oint_{C_r} \frac{Q(z)}{(z - z_s)^{k+1}} \, dz \quad \text{for } k = 0, 1, 2, \ldots. \]

Each point \( z = x + iy \) on \( C_r \) satisfies \( x_s - r \leq x \leq x_s + r \) and \(-r \leq y \leq r\). Set

\[ A = K_0 \sum_{n=0}^{\infty} K_4^n \exp \left( \frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i) \right). \]

Using the result in the proof of Lemma 7, we obtain \( 0 \leq Q(s) \leq A \exp \left( \frac{\phi K_1}{1-|\phi|} |s| \right) \) for all \( s \in \mathbb{R} \), and

\[ |Q(z)| \leq K_0 + \frac{K_3 |e^{\phi K_1 (x+iy)}|}{\sqrt{2\pi \sigma}} \int_{-\infty}^{\infty} Q(s) e^{-1/(2\sigma^2)(s-\phi x-\theta - i\phi y)^2} \, ds \]

\[ = K_0 + \frac{K_3 e^{\phi K_1 x + \phi^2 y^2 / (2\sigma^2)}}{\sqrt{2\pi \sigma}} \int_{-\infty}^{\infty} Q(s) e^{-1/(2\sigma^2)(s-\phi x-\theta)^2} \, ds \]

\[ \leq K_0 + \frac{AK_3 e^{(\phi K_1 x + \phi^2 y^2 / (2\sigma^2)}}{\sqrt{2\pi \sigma}} \]
\[
\times \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2\sigma^2} [s - \phi x - \theta]^2 + \frac{\phi K_1}{1 - |\phi|} |s| \right) ds
\]

\[
\leq K_0 + AK_3 \exp \left( |\phi K_1| \right) \left( \frac{\phi^2 |K_1 x|}{1 - |\phi|} + \frac{\phi^2 y^2}{2\sigma^2} + \frac{\phi^2 K_1^2 \sigma^2}{2(1 - |\phi|)^2} + \frac{|\phi K_1 \theta|}{1 - |\phi|} \right)
\]

\[
\times \left[ 1 + 2\Phi \left( \frac{|\phi K_1| |\sigma|}{1 - |\phi|} \right) \right]
\]

\[
= K_0 + AK_3 \exp \left( \frac{|\phi K_1| (|x| + |\theta|)}{1 - |\phi|} + \frac{\phi^2 y^2}{2\sigma^2} + \frac{\phi^2 K_1^2 \sigma^2}{2(1 - |\phi|)^2} \right)
\]

\[
\times \left[ 1 + 2\Phi \left( \frac{|\phi K_1| |\sigma|}{1 - |\phi|} \right) \right]
\]

\[
\leq K_0 + AK_3 \exp \left( \frac{|\phi K_1| (|r + x_* + |\theta|)}{1 - |\phi|} + \frac{\phi^2 r^2}{2\sigma^2} + \frac{\phi^2 K_1^2 \sigma^2}{2(1 - |\phi|)^2} \right)
\]

\[
\times \left[ 1 + 2\Phi \left( \frac{|\phi K_1| |\sigma|}{1 - |\phi|} \right) \right].
\]

Define

\[
B_r = K_0 + AK_3 \exp \left( \frac{|\phi K_1| (|r + x_* + |\theta|)}{1 - |\phi|} + \frac{\phi^2 r^2}{2\sigma^2} + \frac{\phi^2 K_1^2 \sigma^2}{2(1 - |\phi|)^2} \right)
\]

\[
\times \left[ 1 + 2\Phi \left( \frac{|\phi K_1| |\sigma|}{1 - |\phi|} \right) \right].
\]

Cauchy’s integral formula for \(Q^{(k)}(z_*)\) together with the above estimate to \(|Q(z)|\) gives rise to

\[
|Q^{(k)}(z_*)| \leq \left| \frac{k!}{2\pi i} \oint_{C_r} \frac{Q(z)}{(z - z_*)^{k+1}} dz \right| = \frac{k!}{2\pi} \int_0^{2\pi} Q(z_* + r e^{i\theta}) r e^{i\theta} d\theta
\]

\[
\leq \frac{k!}{2\pi} \int_0^{2\pi} |Q(z_* + r e^{i\theta})| r d\theta \leq \frac{B_r k!}{r^k}.
\]

This bound allows us to estimate the analytic error (24).
Dept. of Finance, Mendoza College of Business, University of Notre Dame, Notre Dame, IN 46556, U.S.A.; cosimano.1@nd.edu, www.nd.edu/~tcosiman/, and
Dept. of Mathematics, University of Notre Dame, Notre Dame, IN 46556, U.S.A.; Alex.A.Himonas.1@nd.edu.

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