SUPPLEMENT TO "ROBUST MECHANISM DESIGN"

(*Econometrica*, Vol. 73, No. 6, November 2005, 1771–1813)

BY DIRK BERGEMANN AND STEPHEN MORRIS

In this supplement we provide a set of sufficient conditions that encompass the quasilinear environment. With these conditions, we can generalize the equivalence results presented in Proposition 4 for common prior full support payoff type spaces. The first condition replaces the compactness condition; the second and third conditions generalize the important features of the quasilinear utility model. This supplement ends with an example that is meant to illustrate that the conditions, in particular, the condition on bounded allocation differences, are not easily dispensed with.

KEYWORDS: Mechanism design, common knowledge, universal type space, interim equilibrium, ex post equilibrium, dominant strategies.

WE SUBSTITUTE THE COMPACTNESS CONDITION by essential compactness. To this end let

$$\Delta_i(\theta_{-i}, f_i) \triangleq \min_{(\theta_i,\theta_i')\in\Theta_i\times\Theta_i} \begin{bmatrix} \widetilde{u}_i(f_0(\theta_i,\theta_{-i}), f_i(\theta_i,\theta_{-i}), (\theta_i,\theta_{-i})) \\ -\widetilde{u}_i(f_0(\theta_i',\theta_{-i}), f_i(\theta_i',\theta_{-i}), (\theta_i,\theta_{-i})) \end{bmatrix}.$$

DEFINITION 1 —Essential Compactness: *Each $\widetilde{u}_i(y_0, y_i, \theta)$ is continuous with respect to y_i. For each θ and i, there is a compact set $\overline{F}_i(\theta) \subseteq F_i(\theta)$, such that for each i and θ_{-i}, there exists $f_i^*:\Theta\to Y_i$ with $f_i^*(\theta)\in\overline{F}_i(\theta)$ such that $\Delta_i(\theta_{-i}, f_i^*) \geq \Delta_i(\theta_{-i}, f_i)$ for all $f_i:\Theta\to Y_i$ with each $f_i(\theta)\in F_i(\theta)$.*

DEFINITION 2 —Transferable Utility: *For every $\psi\in\Delta(\Theta_{-i})$ and $f_i:\Theta\to Y_i$ with $f_i(\theta)\in F_i(\theta)$ for all θ, there exists $\overline{f}_i:\Theta_i\to Y_i$ with $f_i(\theta)\in F_i(\theta)$ for all θ such that*

$$\sum_{\theta_{-i}\in\Theta_{-i}} \psi_i(\theta_{-i})\widetilde{u}_i\big(f_0(\theta_i',\theta_{-i}), \overline{f}_i(\theta_i'), (\theta_i,\theta_{-i})\big)$$
$$= \sum_{\theta_{-i}\in\Theta_{-i}} \psi_i(\theta_{-i})\widetilde{u}_i\big(f_0(\theta_i',\theta_{-i}), f_i(\theta_i',\theta_{-i}), (\theta_i,\theta_{-i})\big)$$

for all θ_i and θ_i'.

DEFINITION 3 —Bounded Allocation Differences: *There exists M such that*

$$\big|\widetilde{u}_i(y_0, y_i, \theta) - \widetilde{u}_i(y_0', y_i, \theta')\big| \leq M$$
for all i, $y_0, y_0'\in Y_0$, $y_i\in Y_i$, and $\theta,\theta'\in\Theta$.

Essential compactness ensures that the problem of maximizing the minimal gains from truth-telling in the ex post incentive constraints always has a well-defined solution. Transferable utility ensures that every utility compensation for agent i can be achieved by assigning the private component $\overline{f}_i$ conditionally on the reported payoff type θ_i' of agent i only. A sufficient condition for transferable utility is that $\widetilde{u}_i(y_0, y_i, \theta)$ is continuous with respect to y_i and that the utility is positively and negatively unbounded in y_i for every y_0 and θ. Bounded allocation differences ensure that the differences in utilities due to the public component and the payoff type for every given private component are bounded.

LEMMA 1: *If F is essentially compact, separable with transferable utility, and satisfies bounded allocation differences, and F is interim implementable on every full support common prior payoff type space $\mathcal{T}$, then F is ex post incentive implementable.*

PROOF: Suppose F is not ex post compatible. By essential compactness, for each i and θ_{-i}, there exists $\delta > 0$, such that for all $f_i : \Theta \to Y_i$,

$$\min_{(\theta_i,\theta_i')\in\Theta_i\times\Theta_i}\begin{bmatrix} \widetilde{u}_i(f_0(\theta_i,\theta_{-i}), f_i(\theta_i,\theta_{-i}), (\theta_i,\theta_{-i})) \\ -\widetilde{u}_i(f_0(\theta_i',\theta_{-i}), f_i(\theta_i',\theta_{-i}), (\theta_i,\theta_{-i}))\end{bmatrix} \leq -\delta. \tag{1}$$

Now suppose that F is interim implementable on every full support common prior payoff type space. Then, for every $p \in \Delta_{++}(\Theta)$, there exists for each i, $f_i^p : \Theta \to Y_i$ such that $f_i^p(\theta) \in F_i(\theta)$ for all θ and

$$\min_{(\theta_i,\theta_i')\in\Theta_i\times\Theta_i}\begin{bmatrix} \sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_i,\theta_{-i})\widetilde{u}_i(f_0(\theta_i,\theta_{-i}), f_i^p(\theta_i,\theta_{-i}), (\theta_i,\theta_{-i})) \\ -\sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_i,\theta_{-i})\widetilde{u}_i(f_0(\theta_i',\theta_{-i}), f_i^p(\theta_i',\theta_{-i}), (\theta_i,\theta_{-i}))\end{bmatrix}$$
$$\geq 0$$

for all θ_i and θ_i'. By transferable utility, there exists $\overline{f}_i^p : \Theta_i \to Y_i$ such that $\overline{f}_i^p(\theta) \in F_i(\theta)$ for all θ and

$$\min_{(\theta_i,\theta_i')\in\Theta_i\times\Theta_i}\begin{bmatrix} \sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_i,\theta_{-i})\widetilde{u}_i(f_0(\theta_i,\theta_{-i}), \overline{f}_i^p(\theta_i), (\theta_i,\theta_{-i})) \\ -\sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_i,\theta_{-i})\widetilde{u}_i(f_0(\theta_i',\theta_{-i}), \overline{f}_i^p(\theta_i'), (\theta_i,\theta_{-i}))\end{bmatrix}$$
$$\geq 0.$$

By bounded allocation differences, for any $\theta_{-i}^* \in \Theta_{-i}$,

$$\left|\widetilde{u}_i\big(f_0(\theta_i',\theta_{-i}), \overline{f}_i^p(\theta_i'), (\theta_i,\theta_{-i})\big) - \widetilde{u}_i\big(f_0(\theta_i',\theta_{-i}^*), \overline{f}_i^p(\theta_i'), (\theta_i,\theta_{-i}^*)\big)\right|$$
$$\leq M.$$

We can express the expected utility under $p(\cdot)$ from f_0 and $\overline{f}_i^p$ as

$$\begin{aligned}
&\sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_i,\theta_{-i})\widetilde{u}_i\big(f_0(\theta_i',\theta_{-i}),\overline{f}_i^p(\theta_i'),(\theta_i,\theta_{-i})\big)\\
&\quad=\widetilde{u}_i\big(f_0(\theta_i',\theta_{-i}^*),\overline{f}_i^p(\theta_i'),(\theta_i,\theta_{-i}^*)\big)\\
&\qquad+\sum_{\theta_{-i}\neq\theta_{-i}^*} p(\theta_i,\theta_{-i})\big[\widetilde{u}_i\big(f_0(\theta_i',\theta_{-i}),\overline{f}_i^p(\theta_i'),(\theta_i,\theta_{-i})\big)\\
&\qquad\qquad-\widetilde{u}_i\big(f_0(\theta_i',\theta_{-i}^*),\overline{f}_i^p(\theta_i'),(\theta_i,\theta_{-i}^*)\big)\big],
\end{aligned}$$

so

$$\begin{aligned}
&\Bigg|\sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_i,\theta_{-i})\big(\widetilde{u}_i\big(f_0(\theta_i,\theta_{-i}),\overline{f}_i^p(\theta_i),(\theta_i,\theta_{-i})\big)\\
&\qquad\qquad-\widetilde{u}_i\big(f_0(\theta_i',\theta_{-i}),\overline{f}_i^p(\theta_i'),(\theta_i,\theta_{-i})\big)\big)\\
&\qquad-\big(\widetilde{u}_i\big(f_0(\theta_i,\theta_{-i}^*),\overline{f}_i^p(\theta_i),(\theta_i,\theta_{-i}^*)\big)\\
&\qquad\quad-\widetilde{u}_i\big(f_0(\theta_i',\theta_{-i}^*),\overline{f}_i^p(\theta_i'),(\theta_i,\theta_{-i}^*)\big)\big)\Bigg|\\
&\quad\le(1-p(\theta_i,\theta_{-i}^*))2M.
\end{aligned}$$

However, consider a sequence of priors with $p^n\to p^*$ and $p^*(\theta_{-i}^*)=1$. Since

$$\begin{aligned}
&\min_{(\theta_i,\theta_i')\in\Theta_i\times\Theta_i}\left[\begin{array}{l}\sum_{\theta_{-i}\in\Theta_{-i}} p^n(\theta_i,\theta_{-i})\widetilde{u}_i(f_0(\theta_i,\theta_{-i}),\overline{f}_i^{p^n}(\theta_i),(\theta_i,\theta_{-i}))\\ -\sum_{\theta_{-i}\in\Theta_{-i}} p^n(\theta_i,\theta_{-i})\widetilde{u}_i(f_0(\theta_i',\theta_{-i}),\overline{f}_i^{p^n}(\theta_i'),(\theta_i,\theta_{-i}))\end{array}\right]\\
&\quad\ge 0
\end{aligned}$$

for all n, we have that

$$\min_{(\theta_i,\theta_i')\in\Theta_i\times\Theta_i}\left[\begin{array}{l}\widetilde{u}_i(f_0(\theta_i,\theta_{-i}^*),\overline{f}_i^{p^n}(\theta_i),(\theta_i,\theta_{-i}^*))\\ -\widetilde{u}_i(f_0(\theta_i',\theta_{-i}^*),\overline{f}_i^{p^n}(\theta_i'),(\theta_i,\theta_{-i}^*))\end{array}\right]$$

tends to 0 as $n\to\infty$, but this contradicts (1). *Q.E.D.*

LEMMA 2: *Essential compactness, transferable utility, and bounded allocation differences are satisfied in the quasilinear environment.*

PROOF: (1) *Essential compactness*. Let

$$M = \max_{y_0, y_0'} |v_i(y_0, \theta) - v_i(y_0', \theta)|,$$

and let $F_i = \{f_i : \Theta \to \mathbb{R}\}$ and $\overline{F}_i = \{f_i : \Theta \to [-2M, 2M]\}$. To show essential compactness, it is enough to show that for all $f_i \in F_i$ and θ_{-i}, there exists $\overline{f}_i \in \overline{F}_i$ with $\Delta_i(\theta_{-i}, f_i) \leq \Delta_i(\theta_{-i}, \overline{f}_i)$. To see this, let $f_i^0(\theta_i) = 0$ for all θ_i; note that $f_i^0 \in \overline{F}_i$ and $\Delta_i(\theta_{-i}, f_i^0) \geq -M$. If

$$\max_{\theta_i, \theta_i'} |f_i(\theta_i) - f_i(\theta_i')| > 2M,$$

then $\Delta_i(\theta_{-i}, f_i) < -M \leq \Delta_i(\theta_{-i}, f_i^0)$. If

$$\max_{\theta_i, \theta_i'} |f_i(\theta_i) - f_i(\theta_i')| \leq 2M,$$

fix any $\overline{\theta}_i$ and let $\widetilde{f}_i(\theta_i) = f_i(\theta_i) - f_i(\overline{\theta}_i)$. Clearly, $\widetilde{f}_i \in \overline{F}_i$ and $\Delta_i(\theta_{-i}, f_i) \leq \Delta_i(\theta_{-i}, \widetilde{f}_i)$.

(2) *Transferable utility* is immediate: just set

$$\overline{f}_i(\theta_i) \triangleq \sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\theta_{-i}) f_i(\theta_i, \theta_{-i}).$$

(3) *Bounded allocation differences* are also immediate: set

$$M = \max_{i, \theta, \theta', y_0, y_0'} |v_i(y_0, \theta) - v_i(y_0', \theta')|,$$

which completes the proof. *Q.E.D.*

The following example satisfies essential compactness and transferable utility, but does not satisfy bounded allocation differences. Yet, it is arguably a very slight departure from the quasilinear model.

EXAMPLE 4: Consider the example with two agents, $i = 1, 2$. The payoff type space of agent 1 is $\Theta_1 = \{1, 2\}$ and of agent 2 is $\Theta_2 = \{1, 2, 3\}$. We consider an additive utility function

$$u_i(y_0, \theta) + v_i(y_i, \theta_i)$$

as a minimal extension of the quasilinear utility function. The allocation rule for the common component is

(2)

f_0	$\theta_2 = 1$	$\theta_2 = 2$	$\theta_2 = 3$
$\theta_1 = 1$	a	c	d
$\theta_1 = 2$	b	c	d

and the utility from the common component $u_1(y_0, \theta)$ for agent 1 is given by

(3)

$u_1(a,\cdot)$	1	2	3	$u_1(b,\cdot)$	1	2	3	$u_1(c,\cdot)$	1	2	3	$u_1(d,\cdot)$	1	2	3
1	1	0	0	1	2	0	0	1	0	1	0	1	0	0	1
2	2	0	0	2	1	0	0	2	0	1	0	2	0	0	1

The social choice correspondence for the private component is $F_i = \mathbb{R}_+$ and the utility from the private component is given by

$$v_i(y_i, \theta_i) = \sum_{k=1}^{\theta_i} y_i^k \tag{4}$$

or

$$v_i(y_i, 1) = y_i, \quad v_i(y_i, 2) = y_i + y_i^2.$$

It is easy to verify that the utility function is supermodular in (y_i, θ_i) and thus well behaved with respect to implementation constraints. We exclusively focus attention on the incentive problem of agent 1 at $\theta_2 = 1$ and use the additional payoff type of agent 2, $\theta_2 = 2, 3,$ as a conditioning device in the interim implementation.

By (2) and (3), the SCC F is not ex post incentive compatible for agent 1 at $\theta_2 = 1$. The ex post incentive constraints for agent 1 at $\theta_2 = 1$ are given by

$$\begin{aligned} &u_1(f_0(1,1),(1,1)) + v_1(f_1(1,1),1) \\ &\quad \geq u_1(f_0(2,1),(1,1)) + v_1(f_1(2,1),1), \\ &u_1(f_0(2,1),(2,1)) + v_1(f_1(2,1),2) \\ &\quad \geq u_1(f_0(1,1),(2,1)) + v_1(f_1(1,1),2). \end{aligned} \tag{5}$$

After inserting the payoffs from the common component and rearranging the utility from the private component, we have

$$\begin{aligned} &v_1(f_1(1,1),1) - v_1(f_1(2,1),1) \geq 1, \\ &v_1(f_1(2,1),2) - v_1(f_1(1,1),2) \geq 1, \end{aligned}$$

but both inequalities cannot be satisfied simultaneously because it follows from (4) that

$$\begin{aligned} &v_1(f_1(2,1),2) - v_1(f_1(1,1),2) > 0 \\ &\quad \Leftrightarrow \quad v_1(f_1(2,1),1) - v_1(f_1(1,1),1) > 0. \end{aligned}$$

However we can interim implement the social choice correspondence F for every full support prior. It is easiest to demonstrate this with an independent prior:

$$p_1(\theta_2=1|\cdot)=1-2\varepsilon, \quad p_1(\theta_2=2|\cdot)=p_1(\theta_2=3|\cdot)=\varepsilon \quad \forall\,\theta_1\in\Theta_1.$$

We offer different rewards for each payoff type θ_1 of agent 1 at different realizations of θ_2. We use the fact that $v_1(y_1,\theta_1)$ grows at different rates to obtain interim incentive compatibility. More precisely, the following rewards as a function of the announced type accomplish interim implementation for all ε satisfying $0<\varepsilon<\frac{1}{2}$:

$$f_1(1,1)=0, \quad f_1(1,2)=\frac{3}{\varepsilon}, \quad f_1(1,3)=\frac{5}{\varepsilon} \tag{6}$$

and

$$f_1(2,1)=0, \quad f_1(2,2)=\frac{1}{\varepsilon}, \quad f_1(2,3)=\frac{6}{\varepsilon}. \tag{7}$$

(With correlated rather than independent priors, we could use differential probabilities as well as differential rewards to guarantee interim incentive compatibility.) To verify interim incentive compatibility, it suffices to establish that for $\theta_1=1$,

$$\sum_{\theta_2\in\Theta_2} p_1(\theta_2)\big(v_1(f_1(1,\theta_2),1)-v_1(f_1(2,\theta_2),1)\big)\geq 1, \tag{8}$$

and conversely for payoff type $\theta_1=2$,

$$\sum_{\theta_2\in\Theta_2} p_1(\theta_2)\big(v_1(f_1(2,\theta_2),2)-v_1(f_1(1,\theta_2),2)\big)\geq 1. \tag{9}$$

Inserting $f_1(\theta_1,\theta_2)$ from (6) and (7) into (8) and (9), we find

$$\varepsilon\left(\left(\frac{3}{\varepsilon}+\frac{5}{\varepsilon}\right)-\left(\frac{1}{\varepsilon}+\frac{6}{\varepsilon}\right)\right)\geq 1$$

and

$$\varepsilon\left(\left(\frac{1}{\varepsilon}+\left(\frac{1}{\varepsilon}\right)^2+\frac{6}{\varepsilon}+\left(\frac{6}{\varepsilon}\right)^2\right)-\left(\frac{3}{\varepsilon}+\left(\frac{3}{\varepsilon}\right)^2+\frac{5}{\varepsilon}+\left(\frac{5}{\varepsilon}\right)^2\right)\right)\geq 1,$$

and it is easy to verify that both inequalities are satisfied.

Dept. of Economics, Yale University, 28 Hillhouse Avenue, New Haven, CT 06511, U.S.A.; dirk.bergemann@yale.edu; http://cowles.econ.yale.edu/faculty/bergemann.htm
and
Dept. of Economics, Princeton University, 211 Fisher Hall, Prospect Street, Princeton, NJ 08544, U.S.A.; smorris@princeton.edu; http://www.princeton.edu/˜smorris/.