SUPPLEMENT TO “MONOPOLISTIC COMPETITION: BEYOND THE CONSTANT ELASTICITY OF SUBSTITUTION”  
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In this appendix, we prove the various statements made in our paper. In Appendix A, we study the impact of market size on the FEE. Appendix B is devoted to the multisector economy, while Appendix C shows that equilibrium under the translog behaves like equilibrium under the CARA.

APPENDIX A: THE IMPACT OF MARKET SIZE ON THE FEE

It is readily verified that (6) is equivalent to
\[ r_u'x + (r_u - r_C)(1 - r_u) > 0. \]  
This expression will be used below.

Output. Differentiating (10) leads to
\[
\frac{(qV''(\bar{q}) + V'(\bar{q}))C(\bar{q}) - q[V'(\bar{q})]^2}{[C(\bar{q})]^2} \frac{d\bar{q}}{dL} = -r_u' \left( \frac{1}{L} \frac{d\bar{q}}{dL} - \frac{\bar{q}}{L^2} \right).
\]
Using
\[ V'(\bar{q})\bar{q} = (1 - r_u)C(\bar{q}), \]
we obtain
\[
(r_u - r_C)(1 - r_u)\frac{L}{\bar{q}} \cdot \frac{d\bar{q}}{dL} = -r_u' \left( \frac{d\bar{q}}{dL} - \frac{\bar{q}}{L} \right),
\]
which amounts to
\[
(r_u - r_C)(1 - r_u)\mathcal{E}_{\bar{q}/L} = -r_u' \frac{\bar{q}}{L}(\mathcal{E}_{\bar{q}/L} - 1).
\]
Thus, the elasticity of \( \bar{q} \) with respect to (w.r.t.) to \( L \) is equal to
\[
\mathcal{E}_{\bar{q}/L} = \frac{r_u'\bar{x}}{r_u'\bar{x} + (r_u - r_C)(1 - r_u)}.
\]
It follows from (6) that the denominator is positive. Consequently, a firm’s output increases (decreases) when the RLV is increasing (decreasing). Furthermore, the weak convexity of \( V \) implies that
\[ \mathcal{E}_{\bar{q}/L} < 1. \]
Consumption per capita. It is readily verified that the elasticity of $\bar{x}$ w.r.t. $L$ can be derived from $\mathcal{E}_{\bar{q}/L}$ as

$$
\mathcal{E}_{\bar{x}/L} = \mathcal{E}_{\bar{q}/L} - 1 = -\frac{(r_u - r_C)(1 - r_u)}{r'_u \bar{x} + (r_u - r_C)(1 - r_u)}.
$$

Thus, $\bar{x}$ decreases with $L$ when $r_u - r_C > 0$. Observe that this inequality holds when $V$ is convex or not too concave.

Markup. From the comparative statics above, it is straightforward that markups decrease (increase) with $L$ if and only if the RLV is increasing (decreasing).

Price. It follows from (9) that

$$
(A.3) \quad \frac{d \bar{p}}{dL} = \frac{V'(\bar{q})\bar{q} - C(\bar{q})}{\bar{q}^2} \cdot \frac{d\bar{q}}{dL}.
$$

Then, firms’ output and market price move in opposite directions with $L$:

$$
\frac{d \bar{p}}{dL} = -r'_u \frac{C(\bar{q})}{\bar{q}^2} \cdot \frac{d\bar{q}}{dL}.
$$

Number of varieties. The number of varieties $\tilde{N}$ is determined by labor market clearing:

$$
\tilde{N} C(\bar{q}) = L.
$$

Thus, the elasticity of $\tilde{N}$ w.r.t. $L$ is

$$
\mathcal{E}_{\tilde{N}/L} + \mathcal{E}_C \cdot \mathcal{E}_{\bar{q}/L} = 1,
$$

which amounts to

$$
\mathcal{E}_{\tilde{N}/L} = 1 - \mathcal{E}_C \cdot \frac{r'_u \bar{x}}{r'_u \bar{x} + (r_u - r_C)(1 - r_u)}.
$$

Again, the denominator of the second term is strictly positive by (A.1). Furthermore, at the equilibrium, it must be that $0 < \mathcal{E}_C(L\bar{x}) = 1 - r_u(\bar{x}) < 1$ and, thus, the sign of $\mathcal{E}_{\tilde{N}/L} - 1$ is determined by $r'_u$. Consequently, the elasticity of $\tilde{N}$ w.r.t. $L$ is smaller (larger) than 1 if the RLV is increasing (decreasing).

APPENDIX B: THE MULTISECTOR ECONOMY

Properties of the Expenditure Function in the Two-Sector Economy

The following two lemmas provide a rationale for the following assumptions made in Section 4.1:

$$
(B.1) \quad 0 \leq \frac{p}{E} \cdot \frac{\partial E}{\partial p} < 1, \quad \frac{N}{E} \cdot \frac{\partial E}{\partial N} < 1.
$$
Set
\[ D = U''_1 \cdot (v'_E)^2 - 2U''_{12} v'_E + U''_{22} + U'_1 v''_{EE}. \]

**Lemma 1:** If \( U''_{21} \geq 0 \), then the elasticity of \( E \) w.r.t. \( N \) is such that
\[ \frac{\partial E}{\partial N} \cdot \frac{N}{E} - 1 = -\frac{U''_{11} v'_E v + U''_{21} (v + v'E) - U''_{22} E}{D E} \leq 0. \]

**Lemma 2:** If \( U''_{21} \geq 0 \) and the inequality
\[ \frac{1 - r_u(x)}{E_u(x)} \leq \frac{U''_{21}(X, Y)X}{U'_1(X, Y)} - \frac{U''_{11}(X, Y)X}{U'_1(X, Y)} \]
hold at a symmetric outcome, then the elasticity of \( E \) w.r.t. \( p \) is such that
\[ -1 \leq \frac{\partial E}{\partial p} \cdot \frac{p}{E} - 1 = \frac{U'_1 v'_E + U''_{21} v'E - E U''_{22}}{D E} \leq 0. \]

**Remark:** Under \( u(0) = 0 \), the indirect utility function
\[ v(p, E, N) = Nu\left(\frac{E}{pN}\right) \]
is homogeneous of degree 0 w.r.t. \( (p, E) \) and of degree 1 w.r.t. \( (E, N) \). Therefore, \( v'_E \) and \( v'_p \) are homogeneous of degree \(-1\) w.r.t. \( (p, E) \) and of degree 0 w.r.t. \( (E, N) \). Finally, we have \( v''_{EE} < 0 \).

Let \( E(p, N) \) be the unique solution to the first-order condition for the upper-tier utility maximization,
\[ U'_1(v(p, E, N), 1 - E) v'_E(p, E, N) - U'_2(v(p, E, N), 1 - E) = 0, \]
where the second-order condition is given by
\[ D < 0. \]
Note that \( U(v(p, E, N), 1 - E) \) is concave w.r.t. \( E \) because \( U \) is concave, while the concavity of \( u \) implies that of \( v \).

**Proof of Lemma 1:** Differentiating (B.4) w.r.t. \( N \) and solving for \( \partial E/\partial N \), we get
\[ \frac{\partial E}{\partial N} = -\frac{U''_{11} v'_E v_N + U''_{12} v'_E}{D} + \frac{(U''_{11} v'_E - U''_{21}) v'_N + U'_1 v''_{EN}}{D}. \]
Consequently,
\[
\frac{\partial E}{\partial N} \cdot \frac{N}{E} - 1 = -N \frac{(U_{11}'' v_E' - U_{21}'') v_N' + U_1' v_{EN}''}{D E} - 1
\]
\[
= \left(-U_{11}'' [v_E' N v_N' + E(v_E')^2] + U_{21}'' (N v_N' + 2v_E')\right)
\]
\[
- U_1' (N v_{EN}' + E v_{EE}') - E U_{22}'
\]
\[
/ (D E).
\]

Applying the Euler theorem to \(v\) and \(v'\), we obtain the equalities
\[
-U_{11}'' [v_E' N v_N' + E(v_E')^2] = -U_{11}'' v_E' (N v_N' + E v_E') = -U_{11}'' v_E' v,
\]
\[
U_{21}'' (N v_N' + 2E v_E') = U_{21}'' (v + E v_E'),
\]
\[
-U_1' (N v_{EN}' + E v_{EE}') = 0.
\]

As a result, we have
\[
\frac{\partial E}{\partial N} \cdot \frac{N}{E} - 1 = \frac{-U_{11}'' v_E' v + U_{21}'' (v + E v_E') - E U_{22}''}{D E}.
\]

Since \(U_{21}'' \geq 0\), the numerator of this expression is positive. Since \(D < 0\), we have
\[
\frac{\partial E}{\partial N} \cdot \frac{N}{E} - 1 \leq 0.
\]

Q.E.D.

PROOF OF LEMMA 2: Differentiating (B.4) w.r.t. \(p\) and solving for \(\partial E/\partial p\), we get
\[
(B.5) \quad \frac{\partial E}{\partial p} = \frac{-U_{11}'' v_p' v_E' - U_1' v_{pE}' + U_{21}'' v_p'}{D},
\]
which implies
\[
\frac{\partial E}{\partial p} \cdot \frac{p}{E} - 1 = p \frac{-U_{11}'' v_p' v_E' - U_1' v_{pE}' + U_{21}'' v_p'}{D E} - 1
\]
\[
= \left(-U_{11}'' [p v_p' v_E' + E(v_E')^2] - U_1' (p v_{pE}' + E v_{EE}')
\]
\[
+ U_{21}'' (p v_p' + 2E v_E') - E U_{22}'
\]
\[
/ (D E).
\]
Applying the Euler theorem to $v$ and $v'$ yields

$$-U'_{12} \left[ p v'_p v'_E + E (v'_E)^2 \right] = -U''_{11} v'_E (p v'_p + E v'_E) = 0$$

and

$$-U'_1 (p v''_{Ep} + E v''_{EE}) = U'_1 v'_E > 0.$$ 

Therefore,

$$\frac{\partial E}{\partial p} \cdot p E \cdot -1 = \frac{U'_1 v'_E + U''_{22} E v'_E - E U''_{22}}{DE} \leq 0$$

since $U''_{22} \geq 0$. Consequently, the right inequality of (B.3) is proven.

To show that $\partial E/\partial p > 0$, we rewrite (B.4) as

$$\frac{\partial E}{\partial p} = \frac{v'_p}{D} \left( -U''_{11} v'_E - U''_{1} \frac{v''_{Ep}}{v'_p} + U''_{21} \right).$$

By definition of $v$, we have

$$v'_p = \frac{-Eu'}{p^2} < 0, \quad v'_E = \frac{u'}{p}, \quad v''_{Ep} = -\frac{u'}{p^2} - \frac{Eu''}{Np^3}.$$

Since $v'_p/D > 0$, the sign of $\partial E/\partial p$ is the same as that of the bracketed term of (B.5). Substituting these three expressions into (B.5) leads to

$$-U''_{11} v'_E - U'_1 \frac{v''_{Ep}}{v'_p} + U''_{21}$$

$$= -U''_{11} \frac{u'}{p} - U'_1 \frac{-u'}{p^2} - \frac{Eu''}{Np^3} + U''_{21}$$

$$= \frac{-U'_1}{E} \left[ \left( \frac{U''_{11} Nu}{U_1} - \frac{U''_{21} Nu}{U_2} \right) \frac{Eu'}{Np u} + 1 + \frac{Eu''}{Np u} \right].$$

Using $-U'_1/E < 0$ and $U'_1 v'_E(p, E, N) = pU'_2/u'$, it follows from (B.2) that

$$\left( \frac{U''_{11} Nu}{U_1} - \frac{U''_{21} Nu}{U_2} \right) \frac{Eu'}{Np u} + 1 + \frac{Eu''}{Np u} < 0 \quad \Rightarrow \quad \frac{\partial E}{\partial p} > 0,$$

which implies the left inequality of (B.3). \ Q.E.D.
The Impact of Market Size on the Mass of Firms in the Two-Sector Economy

We now show that the equilibrium mass of firms decreases with market size. Using the budget constraint and the zero-profit condition yields

\[ N[F + V(\bar{q}(L))] = LE(\bar{p}(L), N). \]

Rewriting this expression in elasticity terms w.r.t. \( L \), we get

\[ \mathcal{E}_N + \frac{\bar{q}V'(\bar{q})}{F + V(\bar{q})} \mathcal{E}_q = 1 + \frac{\partial E \bar{p}}{\partial \bar{p}} : \mathcal{E}_p + \frac{\partial E}{\partial N} \cdot \mathcal{E}_N, \]

which can be rewritten as

\[ \mathcal{E}_N \left(1 - \frac{\partial E N}{\partial N \mathcal{E}_E}\right) = 1 + \frac{\partial E \bar{p}}{\partial \bar{p} \mathcal{E}_p} - \frac{\bar{q}V'(\bar{q})}{F + V(\bar{q})} \mathcal{E}_q. \]

The expression (A.3) is equivalent to

\[ \mathcal{E}_p = -r_u \mathcal{E}_q. \]

Using (10) and (B.7), (B.6) implies

\[ \mathcal{E}_N \left(1 - \frac{\partial E N}{\partial N \mathcal{E}_E}\right) = 1 + \frac{\partial E \bar{p}}{\partial \bar{p} \mathcal{E}_p} - (1 - r_u) \mathcal{E}_q \]

\[ > 1 - \left(\frac{\partial E \bar{p}}{\partial \bar{p} \mathcal{E}_p} + \frac{1 - r_u}{r_u} \mathcal{E}_p\right)r_u = (1 - \frac{\partial E \bar{p}}{\partial \bar{p} \mathcal{E}_p})r_u, \]

where we have used (A.2) for the inequality. Since the elasticity of \( E \) w.r.t. \( p \) is smaller than 1 by assumption, the last term in the above expression is positive. Since the elasticity of \( E \) w.r.t. \( N \) in the first term is also smaller than 1, it must be that

\[ \mathcal{E}_N = \frac{dN}{dL} \cdot \frac{L}{N} > 0. \]

APPENDIX C: RELATIONSHIP BETWEEN THE TRANSLOG AND CARA MODELS

Under the translog, the profit is given by

\[ \pi(p_i; A_{trans}, L) - F = (p_i - c) \frac{L}{p_i} (A_{trans} - \beta \ln p_i) - F. \]
Differentiating this expression w.r.t. $p_i$ yields
\[
\frac{c}{p_i^3} (A_{\text{trans}} - \beta \ln p_i) - \beta \frac{p_i - c}{p_i^2} = 0.
\]
Solving for
\[
A_{\text{trans}} - \beta \ln p_i = \beta \frac{p_i - c}{c},
\]
plugging this expression into (C.1), and rearranging terms leads to the equilibrium condition
\[
\beta (p - c)^2 / (cp) = F/L.
\]
Applying the same argument to the CARA model yields the desired expression:
\[
\beta (p - c)^2 / p = F/L.
\]