SUPPLEMENT TO “CONTINUOUS IMPLEMENTATION”
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BELOW WE PROVIDE THE PROOF of Theorem 4, which is omitted in the main text.

PROOF OF THE “IF PART” OF THEOREM 4: Assume that \( f : \tilde{T} \rightarrow A \) is rationalizable implementable by a finite mechanism \( \mathcal{M} = (M, g) \), that is, that, for all \( \tilde{i} \in \tilde{T} \), \( m \in R(\tilde{i}|\mathcal{M}, \tilde{T}) \Rightarrow g(m) = f(\tilde{i}) \).

We first recall the following well known lemma.

**LEMMA 1—Dekel, Fudenberg, and Morris (2006):** Fix any model \( T = (T, \kappa) \) such that \( \tilde{T} \subset T \) and any finite mechanism \( \mathcal{M} \). (i) For any \( \tilde{i} \in \tilde{T} \) and any sequence \( \{t[n]\}_{n=0}^\infty \) in \( T \), if \( t[n] \rightarrow_p \tilde{i} \), then, for \( n \) large enough, we have \( R(t[n]|\mathcal{M}, T) \subset R(\tilde{i}|\mathcal{M}, T) \). (ii) For any type \( t \in T \), \( R(t|\mathcal{M}, T) \) is nonempty.

Now pick any model \( T = (T, \kappa) \) such that \( \tilde{T} \subset T \). We show that there exists an equilibrium that continuously implements \( f \) on \( \tilde{T} \). For each player \( i \) and each type \( \tilde{i} \in \tilde{T}_i \), fix some \( m_i(\tilde{i}) \in \tilde{R}_i(\tilde{i}|\mathcal{M}, \tilde{T}_i) \) and restrict the space of strategies of player \( i \) by assuming that \( \sigma_i(\tilde{i}) = m_i(\tilde{i}) \) for each type \( \tilde{i} \in \tilde{T}_i \). Because \( \mathcal{M} \) is finite and \( \tilde{T} \) is countable, standard arguments\(^1\) show that there exists a Bayes Nash equilibrium in \( U(\mathcal{M}, T) \). Let us first establish that \( \sigma \) is a Bayes Nash equilibrium in \( U(\mathcal{M}, T) \). It is clear by construction that, for each \( i \in \mathcal{I} \) and \( t_i \notin \tilde{T}_i \),

\[
m_i \in \text{Supp}(\sigma_i(t_i)) \Rightarrow m_i \in \tilde{B}R_i(\pi_i(\cdot|t_i, \sigma_{-i})|\mathcal{M}).
\]

Now fix a player \( i \in \mathcal{I} \) and a type \( \tilde{i} \in \tilde{T}_i \). Since \( \tilde{T} \subset T \) is a model (and hence, \( \kappa(\tilde{i}) \) takes its support in \( \Theta \times \tilde{T}_i \)), it is easily checked that, by construction of \( \sigma \), \( \pi_i(m_{-i}|\tilde{i}, \sigma_{-i}) > 0 \Rightarrow m_{-i} \in \tilde{R}_{-i}(\tilde{i}_{-i}|\mathcal{M}, \tilde{T}_i) \) for some \( \tilde{i}_{-i} \in \tilde{T}_{-i} \). Hence, by a well known argument, \( \tilde{B}R_i(\pi_i(\cdot|\tilde{i}, \sigma_{-i})|\mathcal{M}) \subset \tilde{R}_{i}(\tilde{i}|\mathcal{M}, \tilde{T}_i) \). Since \( g(R(\tilde{i}|\mathcal{M}, \tilde{T})) = \{f(\tilde{i})\} \), we have, for all \( \tilde{m}_i \in \tilde{R}_{i}(\tilde{i}|\mathcal{M}, \tilde{T}_i) \),

\[
\sum_{(\theta, m_{-i}) \in \Theta \times \tilde{M}_{-i}} \pi_i(\theta, m_{-i}|\tilde{i}, \sigma_{-i})[u_i(g(\tilde{m}_i, m_{-i}), \theta)]
\]

\[
= \sum_{\theta, \tilde{i}_{-i}} \kappa(\tilde{i})(\theta, \tilde{i}_{-i})u_i(f(\tilde{i}, \tilde{i}_{-i}), \theta),
\]

\(^1\)The existence of a Bayes Nash equilibrium can be proved using Kakutani–Fan–Glicksberg’s fixed point theorem. The space of strategy profiles is compact in the product topology. Using the fact that \( u_i : A \times \Theta \rightarrow \mathbb{R} \) is bounded, all the desired properties of the best-response correspondence (in particular upper hemicontinuity) can be established.

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and so $BR_i(\pi_i(\cdot|\bar{\nu}_i, \sigma_\pi)|\mathcal{M}) = R_i(\bar{\nu}_i|\mathcal{M}, \bar{T})$. Hence we must have $m_i(\bar{\nu}_i) = \sigma_i(\bar{\nu}_i) \in BR_i(\pi_i(\cdot|\bar{\nu}_i, \sigma_\pi)|\mathcal{M})$. Thus, $\sigma$ is a Bayes Nash equilibrium in $U(\mathcal{M}, T)$ and $\sigma_\pi$ is a pure Nash equilibrium in $U(\mathcal{M}, \bar{T})$. Now, pick any sequence $\{t[n]\}_{n=0}^\infty$ in $T$, such that $t[n] \to \bar{\nu}$. It is clear that, for each $n$: $\text{Supp}(\sigma(t[n])) \subset R(t[n] | \mathcal{M}, T)$. In addition, for $n$ large enough, we know by Lemma 1 that $R(t[n] | \mathcal{M}, T) \subset R(\bar{\nu} | \mathcal{M}, \bar{T})$. Then, for $n$ large enough, $\text{Supp}(\sigma(t[n])) \subset R(\bar{\nu} | \mathcal{M}, \bar{T})$ and so, $(g \circ \sigma)(t[n]) = f(\bar{\nu})$ as claimed. \hfill Q.E.D.

**Proof of the “Only if” Part of Theorem 4:** We show that a social choice function $f : \bar{T} \to A$ is continuously implementable by a countable mechanism $\mathcal{M}$ only if it is rationalizable implementable by some mechanism $\mathcal{M}' \subset \mathcal{M}$ (i.e., $M_i' \subset M_i$ for each $i$ and $g' = g|_{\mathcal{M}'}$).

Since $f$ is continuously implementable, there exists a mechanism $\mathcal{M} = (M, g)$ such that, for any model $T = (T, \kappa)$ satisfying $\bar{T} \subset T$, there is a Bayes Nash equilibrium $\sigma$ in the induced game $U(\mathcal{M}, T)$ where, for each $\bar{\nu} \in \bar{T}$, (i) $\sigma(\bar{\nu})$ is pure, and (ii) for any sequence $t[n] \to \bar{\nu}$ where, for each $n$: $t[n] \in T$, we have $(g \circ \sigma)(t[n]) \to f(\bar{\nu})$. We let $C$ be the set of pure Bayes Nash equilibria of $U(\mathcal{M}, \bar{T})$. Note that because $\bar{T}$ is finite and $M$ is countable, $C$ is countable. For each $\tilde{\sigma} \in C$, we build the set of message profiles $M(\tilde{\sigma})$ in the following way.

For each player $i$ and each positive integer $\ell$, we define inductively $M_i^\ell(\tilde{\sigma})$. First, we set $M_i^0(\tilde{\sigma}) = \tilde{\sigma}_i(\bar{T}_i)$. Then, for each $\ell \geq 1$,

$$M_i^{\ell+1}(\tilde{\sigma}) = BR_i(\Delta(\theta \times \{\tilde{\theta}^{\ell}\} \times M_i^\ell(\tilde{\sigma})) | \mathcal{M})$$

Recall that in the model $\tilde{T} = (\bar{T}, \tilde{\kappa})$, $\text{marg}_{\tilde{\theta}} \tilde{\kappa}(\bar{\nu}_i)(\tilde{\theta}^{\ell}) = 1$, for each $i \in I$ and $\bar{\nu}_i \in \bar{T}_i$. Since $\tilde{\sigma}$ is an equilibrium in $U(\mathcal{M}, \bar{T})$, $M_i^0(\tilde{\sigma}) = \tilde{\sigma}_i(\bar{T}_i) \subset BR_i(\Delta(\theta \times \{\tilde{\theta}^0\} \times M_i^0(\tilde{\sigma})) | \mathcal{M}) = M_i^1(\tilde{\sigma})$. Consequently, it is clear that, for each $\ell$, $M_i^\ell(\tilde{\sigma}) \subset M_i^{\ell+1}(\tilde{\sigma})$. Finally, set $M_i(\tilde{\sigma}) = \lim_{\ell \to +\infty} M_i^\ell(\tilde{\sigma}) = \bigcup_{\ell \in \mathbb{N}} M_i^\ell(\tilde{\sigma})$. In the sequel, for each $\tilde{\sigma} \in C$, we will note by $\mathcal{M}(\tilde{\sigma})$ the mechanism $(M(\tilde{\sigma}), g^{|_{M(\tilde{\sigma})}})$.

A first interesting property of the family of sets $\{M(\tilde{\sigma})\}_{\tilde{\sigma} \in C}$ is that there is a model $T$, satisfying $\bar{T} \subset T$, for which any equilibrium $\sigma$ in $U(\mathcal{M}, T)$ has full range in $M(\sigma|_T)$, that is, each message profile in $M(\sigma|_T)$ is played under $\sigma$ at some profile of types in the model $T$. More precisely, Proposition 1 is the first step of the proof of the only if part of Theorem 4.

**Proposition 1:** There exists a model $T = (T, \kappa)$ such that, for any $\tilde{\sigma} \in C$ and $m \in M(\tilde{\sigma})$, there exists $t[\tilde{\sigma}, m] \in T$ such that $\sigma(t[\tilde{\sigma}, m]) = m$ for any equilibrium $\sigma$ in $U(\mathcal{M}, T)$ such that $\sigma|_T = \tilde{\sigma}$.

\footnote{As already mentioned, the only if part of the theorem holds beyond finite mechanisms.}
CONTINUOUS IMPLEMENTATION

PROOF: We build the model $\mathcal{T} = (T, \kappa)$ as follows. For each equilibrium $\hat{\sigma} \in C$, player $i$, and integer $\ell$, we define inductively $t_i[\hat{\sigma}, \ell, m_i]$ for each $m_i \in M^\ell_i(\hat{\sigma})$ and set

$$
T_i = \bigcup_{\hat{\sigma} \in C} \bigcup_{\ell = 1}^{\infty} t_i[\hat{\sigma}, \ell, m_i] \cup \hat{T}_i.
$$

Note that $T_i$ is countable. In the sequel, we fix an arbitrary $\hat{\sigma} \in C$. This equilibrium $\hat{\sigma}$ is sometimes omitted in our notations.

For each $\ell \geq 1$ and $m_i \in M^\ell_i(\hat{\sigma})$, we know that there exists $\pi_i^\ell, m_i \in \Delta(\Theta \times \{\bar{\theta}^0\} \times M^{\ell-1}_{\bar{\sigma}}(\hat{\sigma}))$ such that $m_i \in BR_i(\pi_i^\ell, m_i | \mathcal{M})$. Thus we can build $\hat{\pi}_i^\ell, m_i \in \Delta(\Theta \times \bar{\Theta} \times M^{\ell-1}_{\bar{\sigma}}(\hat{\sigma}))$ such that

$$
marg_{\Theta \times M^{\ell-1}_{\bar{\sigma}}(\hat{\sigma})} \hat{\pi}_i^\ell, m_i = \marg_{\Theta \times M^{\ell-1}_{\bar{\sigma}}(\hat{\sigma})} \pi_i^\ell, m_i,
$$

while $\marg_{\Theta} \hat{\pi}_i^\ell, m_i = \delta_{\bar{\theta}^0}$. Note that $BR_i(\hat{\pi}_i^\ell, m_i | \mathcal{M}) = \{m_i\}$.

In the sequel, for each player $i$ and message $m_i \in M^0_i(\hat{\sigma})$, we pick one type denoted $t_i[\hat{\sigma}, 0, m_i]$ in $\hat{T}_i$ satisfying $\hat{\sigma}(t_i[\hat{\sigma}, 0, m_i]) = m_i$. This is well defined because, by construction, $M^0_i(\hat{\sigma}) = \bar{\sigma}(\hat{T}_i)$. Now, for each $\ell \geq 1$ and $m_i \in M^\ell_i(\hat{\sigma})$, we define inductively $t_i[\hat{\sigma}, \ell, m_i]$ by

$$
\kappa(t_i[\hat{\sigma}, \ell, m_i])[\theta, \bar{\theta}, t_{-i}] = \begin{cases} 
0, & \text{if } t_{-i} \neq t_{-i}[\hat{\sigma}, \ell - 1, m_{-i}] \\
\hat{\pi}_i^\ell, m_i(\theta, \bar{\theta}, m_{-i}), & \text{if } t_{-i} = t_{-i}[\hat{\sigma}, \ell - 1, m_{-i}]
\end{cases}
$$

for each $m_{-i} \in M^{\ell-1}_{\bar{\sigma}}(\hat{\sigma})$.

This probability measure is well defined since $\hat{\pi}_i^\ell, m_i(\Theta \times \bar{\Theta} \times M^{\ell-1}_{\bar{\sigma}}(\hat{\sigma})) = 1$.

To complete the proof, we show that, for any equilibrium $\sigma$ of $U(\mathcal{M}, \mathcal{T})$ such that $\sigma_{\hat{\sigma}} = \hat{\sigma}$, we have

(S1) $\sigma(t_i[\hat{\sigma}, \ell, m_i]) = m_i$

for each player $i$, integer $\ell$, and message $m_i \in M^\ell_i(\hat{\sigma})$. The proof proceeds by induction on $\ell$.

First note that, by construction of $t_i[\hat{\sigma}, 0, m_i]$, we must have, for any equilibrium $\sigma$ of $U(\mathcal{M}, \mathcal{T})$ such that $\sigma_{\hat{\sigma}} = \hat{\sigma}$,

$$
\sigma(t_i[\hat{\sigma}, 0, m_i]) = m_i,
$$

3Here again, we abuse notation and write $t_{-i}[\hat{\sigma}, 0, m_{-i}]$ for $(t_{-i}[\hat{\sigma}, 0, m_{-i}])_{j \neq i}$. Similarly, $t[\hat{\sigma}, 0, m]$ stands for $(t[i][\hat{\sigma}, 0, m_i])_{i \in \mathbb{Z}}$. Similar abuses will be used throughout this proof.
for each player \(i\) and message \(m_i \in M^{0}_i(\tilde{\sigma})\). Now, assume that Equation (S1) is satisfied at rank \(\ell - 1\) and let us prove that it is also satisfied at rank \(\ell\). Fix any \(m_i \in M^{\ell}_i(\tilde{\sigma})\) and any equilibrium \(\sigma\) of \(U(\mathcal{M}, T)\) such that \(\sigma_i^{\tilde{T}} = \tilde{\sigma}\). Note that \(\text{Supp}(\sigma(t_i[\tilde{\sigma}, \ell, m_i])) \subseteq BR_i(\pi_i | \mathcal{M})\), where \(\pi_i \in \Delta(\Theta \times \tilde{\Theta} \times M_{-i})\) is such that

\[
\pi_i(\theta, \tilde{\theta}, m_{-i}) = \sum_{t_{-i}} \kappa(t_i[\tilde{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}]\sigma_{-i}(m_{-i} | t_{-i}).
\]

In addition, by the inductive hypothesis and the fact that \(\sigma\) is an equilibrium of \(U(\mathcal{M}, T)\) satisfying \(\sigma_i^{\tilde{T}} = \tilde{\sigma}\), we have \(\sigma_{-i}(m_{-i} | t_{-i}[\tilde{\sigma}, \ell - 1, m_{-i}]) = 1\) for any \(m_{-i} \in M_{-i}^{\ell - 1}(\tilde{\sigma})\). Hence, by construction of \(\kappa(t_i[\tilde{\sigma}, \ell, m_i])\), we have

\[
\pi_i(\theta, \tilde{\theta}, m_{-i}) = \sum_{t_{-i}} \kappa(t_i[\tilde{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}]\sigma_{-i}(m_{-i} | t_{-i})
= \kappa(t_i[\tilde{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}[\tilde{\sigma}, \ell, m_{-i}]]
= \hat{\pi}_i^{\ell, m_i}(\theta, \tilde{\theta}, m_{-i}).
\]

We get that \(\text{Supp}(\sigma(t_i[\tilde{\sigma}, \ell, m_i])) \subseteq BR_i(\pi_i | \mathcal{M}) = BR_i(\hat{\pi}_i^{\ell, m_i} | \mathcal{M}) = \{m_i\}\) as claimed.

Q.E.D.

We now give a first insight on the second step of the proof. First notice that, by construction, each \(M(\tilde{\sigma})\) satisfies the following closure property: taking any belief \(\pi_i \in \Delta(\Theta \times \{\tilde{\theta}^0\} \times M_{-i}(\tilde{\sigma}))\) such that \(BR_i(\pi_i | \mathcal{M}) \neq \emptyset\), we must have \(BR_i(\pi_i | \mathcal{M}) \subseteq M(\tilde{\sigma})\) and hence, \(BR_i(\pi_i | \mathcal{M}) = BR_i(\pi_i | \mathcal{M}(\tilde{\sigma}))\).

Now pick a type \(\tilde{t}_i \in \tilde{T}\) and a message \(m_i \in R^1_i(\tilde{t}_i | M(\tilde{\sigma}), \tilde{T})\); it is possible to add a type \(t_i^{m_i}\) to the model \(\tilde{T}\) defined in Proposition 1 satisfying the following two properties.\(^4\) First, \(h_i^1(t_i^{m_i})\) is arbitrarily close to \(h_i^1(\tilde{t}_i)\); second, for any equilibrium \(\sigma\) with \(\sigma_i^{\tilde{T}} = \tilde{\sigma}\), \(\sigma_i(t_i^{m_i}) = m_i\). Indeed, by definition of \(R^1_i(\tilde{t}_i | M(\tilde{\sigma}), \tilde{T}_i)\), there exists a belief \(\pi_i^{m_i} \in \Delta(\Theta^* \times T_{-i} \times M_{-i}(\tilde{\sigma}))\), where \(\text{marg}_{\Theta^* \times \tilde{T}_{-i} \times \tilde{T}_i}(\pi_i^{m_i} | \mathcal{M}(\tilde{\sigma}))\). Using our assumption on cost of messages, we can slightly perturb \(\pi_i^{m_i}\) so that \(m_i\) becomes a unique best reply. So let us assume for simplicity that \(\{m_i\} = BR_i(\text{marg}_{\Theta^* \times \tilde{T}_{-i} \times \tilde{T}_i}(\pi_i^{m_i} | \mathcal{M}(\tilde{\sigma}))\). We can define the type \(t_i^{m_i}\) assigning probability \(\text{marg}_{\Theta^* \times \tilde{T}_{-i} \times \tilde{T}_i}(\pi_i^{m_i} | \mathcal{M}(\tilde{\sigma}))\) to \((\theta^*, t_{-i}, m_{-i})\), where \(t_{-i}[\tilde{\sigma}, m_{-i}]\) is defined as in Proposition 1 (i.e., \(t_{-i}[\tilde{\sigma}, m_{-i}]\) plays \(m_{-i}\) under any equilibrium \(\sigma\) in \(U(\mathcal{M}, T)\) such that \(\sigma_i^{\tilde{T}} = \tilde{\sigma}\)). Now pick any equilibrium \(\sigma\) in \(U(\mathcal{M}, T \cup \{t_i^{m_i}\})\)

\(^4\)In this section, for any mechanism \(\mathcal{M}\), we use the standard notation where \(R^1_i(\tilde{t}_i | \mathcal{M}, \tilde{T})\) stands for the \(\ell\)th round of elimination at type \(\tilde{t}_i\) of messages that are not best responses (see, for instance, Dekel, Fudenberg, and Morris (2007)). Recall that, for any \(\ell\) and \(\tilde{t}_i\), we have \(R_i(\tilde{t}_i | \mathcal{M}, \tilde{T}) \subseteq R^1_i(\tilde{t}_i | \mathcal{M}, \tilde{T})\) (for additional details on the relationship between \(R_i(\tilde{t}_i | \mathcal{M}, \tilde{T})\) and \(R^1_i(\tilde{t}_i | \mathcal{M}, \tilde{T})\) when the set of messages is countably infinite, see Lipman (1994)).
such that \( \sigma_{\tilde{t}} = \tilde{\sigma} \). By construction, \( \text{Supp}(\sigma_i(t_i^{m_i})) \subset BR_i(\text{marg}_{\Theta \times M-i} \pi_i^{m_i} | M) \) and so \( BR_i(\text{marg}_{\Theta \times M-i}(\tilde{\sigma}) \pi_i^{m_i} | M) \neq \emptyset \). By the closure property described above, \( BR_i(\text{marg}_{\Theta \times M-i}(\tilde{\sigma}) \pi_i^{m_i} | M) = BR_i(\text{marg}_{\Theta \times M-i}(\tilde{\sigma}) \pi_i^{m_i} | M(\tilde{\sigma})) \) and so we get that type \( t_i^{m_i} \) plays \( m_i \) under the equilibrium \( \sigma \) and satisfies the desired property. Using a similar reasoning, we show inductively the following “contagion” result.

**Proposition 2:** There exists a model \( \hat{T} = (\hat{T}, \hat{\kappa}) \) such that, for each equilibrium \( \tilde{\sigma} \in C \) and each player \( i \), the following statement holds: For all \( \tilde{i} \in \hat{T}_i \) and \( m_i \in R_i(\tilde{i}_i | M(\tilde{\sigma}), \hat{T}) \), there exists a sequence of types \( \{\tilde{i}_i[n]\}_{n=0}^{\infty} \) in \( \hat{T}_i \) such that (i) \( \tilde{i}_i[n] \to_{p} \tilde{i}_i \), and (ii) \( \sigma_i(\tilde{i}_i[n]) = m_i \) for each integer \( n \) and equilibrium \( \sigma \) of \( U(M, \hat{T}) \) satisfying \( \sigma_{\tilde{t}_i} = \tilde{\sigma} \).

**Proof:** We again define the set \( E \) by

\[
E := \bigcup_{q \in \mathbb{N}} \{ \frac{1}{q} \} \cup \{0\}.
\]

We build the model \( \hat{T} = (\hat{T}, \hat{\kappa}) \) as follows. For each \( \varepsilon \in E, \ell \in \mathbb{N}^*, \tilde{\sigma} \in C, \tilde{i}_i \in \hat{T}_i, \) and \( m_i \in R_i(\tilde{i}_i | M(\tilde{\sigma}), \hat{T}) \), we build inductively \( \tilde{i}_i[\varepsilon, \ell, \tilde{\sigma}, \tilde{i}_i, m_i] \) and set

\[
\hat{T}_i = \bigcup_{\varepsilon \in E} \bigcup_{\ell = 1}^{\infty} \bigcup_{\tilde{\sigma} \in C} \bigcup_{\tilde{i}_i \in \hat{T}_i} \bigcup_{m_i \in R_i(\tilde{i}_i | M(\tilde{\sigma}), \hat{T})} \tilde{i}_i[\varepsilon, \ell, \tilde{\sigma}, \tilde{i}_i, m_i] \cup T_i,
\]

where \( T_i \) is as defined in Proposition 1. Note that \( \hat{T}_i \) is countable. In the sequel, we fix an arbitrary \( \tilde{\sigma} \in C \). This equilibrium \( \tilde{\sigma} \) is sometimes omitted in our notations.

We know that, for each integer \( \ell \), player \( i \) of type \( \tilde{i}_i \in \hat{T}_i \), and message \( m_i \in R_i(\tilde{i}_i | M(\tilde{\sigma}), \hat{T}) \), there exists \( \pi_i^{\ell, m_i} \in \Delta(\Theta \times \hat{T}_{-i} \times M_{-i}(\tilde{\sigma})) \) such that

\[
\text{marg}_{\Theta \times \hat{T}_{-i}} \pi_i^{\ell, m_i} = \hat{\kappa}(\tilde{i}_i),
\]

\[
\text{marg}_{\hat{T}_{-i} \times M_{-i}(\tilde{\sigma})} \pi_i^{\ell, m_i}(\tilde{i}_{-i}, m_{-i}) > 0 \Rightarrow m_{-i} \in R_{-i}(\tilde{i}_{-i} | M(\tilde{\sigma}), \hat{T}),
\]

and

\[
m_i \in BR_i(\text{marg}_{\Theta \times M_{-i}(\tilde{\sigma})} \pi_i^{\ell, m_i} | M(\tilde{\sigma})).
\]

For ease of exposition, we sometimes consider \( \pi_i^{\ell, m_i} \) as a measure over \( \Theta \times \hat{T}_{-i} \times M_{-i}(\tilde{\sigma}) \) and sometimes as a measure over \( \Theta^* \times \hat{T}_{-i} \times M_{-i}(\tilde{\sigma}) \) assigning probability 1 to \( \{\tilde{\sigma}\} \). Similar abuses will be used throughout the proof.
First, we let \( \hat{i}[\epsilon, 1, \bar{\sigma}, \bar{i}, m_i] \) be such that \( \hat{k}(\hat{i}[\epsilon, 1, \bar{\sigma}, \bar{i}, m_i]) \) satisfies the two conditions

\[
\text{(S2)} \quad \text{marg}_{\Theta} \hat{k}(\hat{i}[\epsilon, 1, \bar{\sigma}, \bar{i}, m_i]) = \epsilon \delta_{\bar{\vartheta}m_i} + (1 - \epsilon) \delta_{\bar{\varrho}}
\]

and

\[
\text{(S3)} \quad \text{marg}_{\Theta \times \hat{T}_{-i}} \hat{k}(\hat{i}[\epsilon, 1, \bar{\sigma}, \bar{i}, m_i]) = \pi_{\hat{i}}^{\bar{\vartheta}, m_i} \circ (\tau_{-i}^{\vartheta, 1})^{-1},
\]

where \((\tau_{-i}^{\vartheta, 1})^{-1}\) stands for the preimage of the function \(\tau_{-i}^{\vartheta, 1} : \Theta \times \hat{T}_{-i} \times M_{-i} \to \Theta \times \hat{T}_{-i}, \) defined by \(\tau_{-i}^{\vartheta, 1}(\theta, \bar{i}_{-i}, m_{-i}) = (\theta, t_{-i}[\bar{\sigma}, m_{-i}]),\) and \(t_{-i}[\bar{\sigma}, m_{-i}] \in T_{-i}\) is the type profile defined in Proposition 1. Recall that \(\Theta\) are all endowed with the discrete topology, while \(\Theta\) and \(\bar{\vartheta}\) and \(\bar{\varrho}\)

First, we let \( \hat{i}[\epsilon, 1, \bar{\sigma}, \bar{i}, m_i] \) be such that \( \hat{k}(\hat{i}[\epsilon, 1, \bar{\sigma}, \bar{i}, m_i]) \) satisfies the two conditions

\[
\text{(S2)} \quad \text{marg}_{\Theta} \hat{k}(\hat{i}[\epsilon, 1, \bar{\sigma}, \bar{i}, m_i]) = \epsilon \delta_{\bar{\vartheta}m_i} + (1 - \epsilon) \delta_{\bar{\varrho}}
\]

and

\[
\text{(S3)} \quad \text{marg}_{\Theta \times \hat{T}_{-i}} \hat{k}(\hat{i}[\epsilon, 1, \bar{\sigma}, \bar{i}, m_i]) = \pi_{\hat{i}}^{\bar{\vartheta}, m_i} \circ (\tau_{-i}^{\vartheta, 1})^{-1},
\]

where \((\tau_{-i}^{\vartheta, 1})^{-1}\) stands for the preimage of the function \(\tau_{-i}^{\vartheta, 1} : \Theta \times \hat{T}_{-i} \times M_{-i} \to \Theta \times \hat{T}_{-i}, \) defined by \(\tau_{-i}^{\vartheta, 1}(\theta, \bar{i}_{-i}, m_{-i}) = (\theta, t_{-i}[\bar{\sigma}, m_{-i}]),\) and \(t_{-i}[\bar{\sigma}, m_{-i}] \in T_{-i}\) is the type profile defined in Proposition 1. Recall that \(\Theta\) are all endowed with the discrete topology, while \(\Theta\) and \(\bar{\vartheta}\) and \(\bar{\varrho}\)

\[
\text{CLAIM 1: For each } \hat{i}, \in \hat{T}_i \text{ and } m_i \in R(\hat{i}, \bar{\vartheta} \bar{\vartheta}_i, \bar{\vartheta}_i, \bar{\vartheta}_i, m_i) \to \varrho \hat{i}, \text{ as } \ell \to \infty \text{ for some mapping } \varrho \text{ taking values in } E \setminus \{0\}.
\]

**PROOF:** In the sequel, we will denote by \( \hat{h} \) the (continuous) mapping that projects \( \hat{T} \) into \( T^* \) and, in a similar way, by \( \hat{h} \) the (continuous) mapping from \( \hat{T} \) to \( T^* \).

For any \( \hat{i} \in \hat{T}_i \), since \(5\) for all \( \ell \geq 1 \) and all \( m_i \in R(\hat{i}, \bar{\vartheta} \bar{\vartheta}_i, \bar{\vartheta}_i, \bar{\vartheta}_i, m_i) \to \hat{i}[0, \ell, \bar{\vartheta}, \bar{i}, m_i] \) as \( \varrho \to 0 \), by Lemma 2 in the main text, for all \( \ell \geq 1 \), for all \( \ell' \geq 1 \), and all \( m_i \in R(\hat{i}, \bar{\vartheta} \bar{\vartheta}_i, \bar{\vartheta}_i, \bar{\vartheta}_i, m_i) \to \hat{i}[0, \ell', \bar{\vartheta}, \bar{i}, m_i] \) as \( \varrho \to 0 \).

\(5\) A type in \( \hat{T}_i \) is either in \( T_i \)—which is endowed with the discrete topology, say \( \tau_{T_i} \)—or in \( \hat{T}_i \setminus T_i \). Any point in \( \hat{T}_i \setminus T_i \) is identified with an element of the set \( E \times R \times C \times \hat{T}_i \times M_i \), where \( R, C, \hat{T}_i, M_i \) are all endowed with the discrete topology, while \( E \) is endowed with the usual topology on \( R \) induced on \( E \). Finally, \( E \times R \times C \times \hat{T}_i \times M_i \) is endowed with the product topology; call this topology \( \tau_{T_i \times T} \). The topology over \( \hat{T}_i \) is the coarsest topology that contains \( \tau_{T_i} \cup \tau_{T_i \times T} \). It can easily be checked that under such a topology, \( \hat{T} \) satisfies the conditions of Lemma 2 in the main text.
Let us now show that, for all \( \ell \geq 1 \) and \( \ell' \geq \ell \): \( \hat{h}^\ell_i([0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \hat{h}^{\ell'}_i(\bar{t}_i) \) for all \( \bar{t}_i \in \bar{T}_i \) and \( m_i \in R^\ell_i(\bar{t}_i | \mathcal{M}(\bar{\sigma}), \bar{T}) \). First notice that the first-order beliefs are equal, that is, for all \( \ell' \geq 1 \), \( \bar{t}_i \in \bar{T}_i \), and \( m_i \in R^\ell_i(\bar{t}_i | \mathcal{M}(\bar{\sigma}), \bar{T}) \),

\[
\hat{h}^\ell_i([0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \text{marg}_{\Theta} \kappa(\bar{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \text{marg}_{\Theta} \pi^{\ell, m_i}_{\bar{t}_i} \circ \left( \tau^{0, \ell'}_{\bar{t}_i} \right)^{-1} = \text{marg}_{\Theta} \pi^{\ell, m_i}_{\bar{t}_i} = \text{marg}_{\Theta} \kappa(\bar{t}_i) = \hat{h}^{\ell'}_i(\bar{t}_i),
\]

where the third and the fourth equalities are by definition of \( \tau^{0, \ell'}_{\bar{t}_i} \) and \( \pi^{\ell, m_i}_{\bar{t}_i} \), respectively. Now fix some \( \ell \geq 2 \) and let \( L \) be the set of all belief profiles of players other than \( i \) at order \( \ell - 1 \). Toward an induction, assume that, for all \( \ell' \geq \ell - 1 \): \( \hat{h}^{\ell'-1}_{j}([0, \ell', \bar{\sigma}, \bar{t}_i, m_j]) = \hat{h}^{\ell'}_{j-1}(\bar{t}_j) \) for each \( j, \bar{t}_j \in \bar{T}_j \) and \( m_j \in R^{\ell'}_{\bar{t}_j}(\bar{t}_j | \mathcal{M}(\bar{\sigma}), \bar{T}) \). Then for all \( \ell' \geq \ell \): \( \text{proj}_{\Theta \times L} \circ (\text{id}_\Theta \times \hat{h}_{\bar{t}_j}) \circ \tau^{0, \ell'}_{\bar{t}_j} = \text{proj}_{\Theta \times L} \circ (\text{id}_\Theta \times \hat{h}_{\bar{t}_j}) \circ \tau^{0, \ell'}_{\bar{t}_j} \), where \( \text{id}_\Theta \) (resp. \( \text{id}_{M_{\bar{t}_j}(\bar{\sigma})} \)) is the identity mapping from \( \Theta \) to \( \Theta \) (resp. from \( M_{\bar{t}_j}(\bar{\sigma}) \) to \( M_{\bar{t}_j}(\bar{\sigma}) \)), while \( \text{proj}_{\Theta \times L} \) (resp. \( \text{proj}_{\Theta \times L} \)) is the projection mapping from \( \Theta \times T^* \to \Theta \times L \) (resp. from \( \Theta \times T^* \times M_{\bar{t}_j}(\bar{\sigma}) \to \Theta \times L \); hence, for all \( \ell' \geq \ell \), \( \bar{t}_j \in \bar{T}_j \), and \( m_j \in R^{\ell'}_{\bar{t}_j}([0, \ell', \bar{\sigma}, \bar{t}_j, m_j]) \),

\[
\text{marg}_{\Theta \times L} \kappa(\bar{t}_j[0, \ell', \bar{\sigma}, \bar{t}_j, m_j]) = \left( \text{id}_\Theta \times \hat{h}_{\bar{t}_j} \right)^{-1} = \left( \text{id}_\Theta \times \hat{h}_{\bar{t}_j} \right)^{-1} = \left( \text{id}_\Theta \times \hat{h}_{\bar{t}_j} \right)^{-1} = \left( \text{id}_\Theta \times \hat{h}_{\bar{t}_j} \right)^{-1} = \left( \text{id}_\Theta \times \hat{h}_{\bar{t}_j} \right)^{-1}.
\]

Therefore,

\[
\hat{h}^{\ell'}_i([0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \delta_{\hat{h}^{\ell'-1}_i([0, \ell', \bar{\sigma}, \bar{t}_i, m_i])} \times \text{marg}_{\Theta \times L} \kappa(\bar{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) \circ (\text{id}_\Theta \times \hat{h}_{\bar{t}_i})^{-1} = \delta_{\hat{h}^{\ell'-1}_i(\bar{t}_i)} \times \text{marg}_{\Theta \times L} \kappa(\bar{t}_i) \circ (\text{id}_\Theta \times \hat{h}_{\bar{t}_i})^{-1} = \hat{h}^{\ell'}_i(\bar{t}_i),
\]

showing that \( \hat{h}^{\ell'}_i([0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \hat{h}^{\ell'}_i(\bar{t}_i) \). Thus, we have proved that, for all \( \ell \geq 1 \), all \( \ell' \geq \ell \): \( \hat{h}^{\ell'}_i([0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \hat{h}^{\ell'}_i(\bar{t}_i) \) for any \( \bar{t}_i \in \bar{T}_i \) and \( m_i \in R^\ell_i(\bar{t}_i | \mathcal{M}(\bar{\sigma}), \bar{T}) \), that is, \( \hat{h}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i] \to p \bar{t}_i \) as \( \ell' \to \infty \) for any \( \bar{t}_i \in \bar{T}_i \).
and $m_i \in R_i(\widehat{t}_i | M(\bar{\sigma}), \widehat{T})$. In addition, we know that, for all $\ell' \geq 1$ and all $m_i \in R_i(\widehat{t}_i | M(\bar{\sigma}), \widehat{T}) : \hat{t}_i(\ell, \bar{\sigma}, \widehat{t}_i, m_i) \rightarrow p \hat{t}_i(0, \ell, \bar{\sigma}, \widehat{t}_i, m_i)$ as $\epsilon \to 0$. Since $T^*$ is a metrizable space, $\hat{t}_i(\hat{\epsilon}(\ell'), \ell', \bar{\sigma}, \widehat{t}_i, m_i) \rightarrow p \hat{t}_i$ as $\ell' \to \infty$ for some function $\hat{\epsilon} : \mathbb{N}^* \to \mathcal{E} \setminus \{0\}$ satisfying $\lim_{\ell' \to \infty} \hat{\epsilon}(\ell') = 0$. Q.E.D.

CLAIM 2: For each $\epsilon \in \mathcal{E} \setminus \{0\}$, $\ell, \widehat{t}_i \in \widehat{T}_i$, and $m_i \in R_i(\widehat{t}_i | M(\bar{\sigma}), \widehat{T})$, we have $\sigma_i(\hat{t}_i(\epsilon, \ell, \bar{\sigma}, \widehat{t}_i, m_i)) = m_i$ for any equilibrium $\sigma$ of $U(M, \widehat{T})$ satisfying $\sigma_{\widehat{T}} = \bar{\sigma}$.

PROOF: Fix a type $\widehat{t}_i \in \widehat{T}_i$ and an equilibrium $\sigma$ of $U(M, \widehat{T})$ satisfying $\sigma_{\widehat{T}} = \bar{\sigma}$. We will show by induction on $\ell$ that, for all $\epsilon \in \mathcal{E} \setminus \{0\}$ and $\ell \geq 1$:

$\sigma_i(\hat{t}_i(\epsilon, \ell, \bar{\sigma}, \widehat{t}_i, m_i)) = m_i$ for all messages $m_i \in R_i(\widehat{t}_i | M(\bar{\sigma}), \widehat{T})$.

Recall that, by construction of $\hat{t}_i(\epsilon, 1, \bar{\sigma}, \widehat{t}_i, m_i)$ on $\Theta^* \times \widehat{T}_i \times M_{-i}$, for all $m_{-i}$ is played at each $\bar{\sigma}$, $t_{-i}[\bar{\sigma}, m_{-i}] = m_i$. First, fix $\epsilon \in \mathcal{E} \setminus \{0\}$ and $m_i \in R_i(\widehat{t}_i | M(\bar{\sigma}), \widehat{T})$ and let us prove that $\sigma_i(\hat{t}_i(\epsilon, 1, \bar{\sigma}, \widehat{t}_i, m_i)) = m_i$. For each $\hat{t}_i(\epsilon, 1, \bar{\sigma}, \widehat{t}_i, m_i)$, define the belief

$\pi^{\epsilon, 1}_i = \hat{\mathcal{K}}(\hat{t}_i(\epsilon, 1, \bar{\sigma}, \widehat{t}_i, m_i)) \circ \gamma^{-1} \in \Delta(\Theta^* \times \widehat{T}_i \times M_{-i}),$

where $\gamma : (\Theta^*, t_{-i}[\bar{\sigma}, m_{-i}]) \mapsto (\Theta, t_{-i}[\bar{\sigma}, m_{-i}])$. Note that by construction, $\pi^{\epsilon, 1}_i$ is the belief of type $\hat{t}_i(\epsilon, 1, \bar{\sigma}, \widehat{t}_i, m_i)$ on $\Theta^* \times \widehat{T}_i \times M_{-i}$ when he believes that $m_{-i}$ is played at each $\bar{\sigma}$, $t_{-i}[\bar{\sigma}, m_{-i}]$. Hence, for each $\epsilon \geq 0$, $\pi^{\epsilon, i}_i$ corresponds to beliefs of type $\hat{t}_i(\epsilon, 1, \bar{\sigma}, \widehat{t}_i, m_i)$ when the equilibrium $\sigma$ is played. Now, by Equations (S2) and (S3), the belief $\pi^{0, 1}_i$ of type $\hat{t}_i(0, 1, \bar{\sigma}, \widehat{t}_i, m_i)$ satisfies

$\text{marg}_{\Theta^* \times M_{-i}} \pi^{0, 1}_i = \text{marg}_{\Theta^* \times M_{-i}} \pi^{1, m_i}_i \circ (\tau^{0, 1}_i)^{-1} \circ (\gamma^\Theta)^{-1}$

$= \text{marg}_{\Theta^* \times M_{-i}} \pi^{1, m_i}_i,$

where $\gamma^\Theta : (\theta, t_{-i}[\bar{\sigma}, m_{-i}]) \mapsto (\theta, \bar{\theta}, t_{-i}[\bar{\sigma}, m_{-i}])$. Since $\text{Supp}(\sigma_i(\hat{t}_i(0, 1, \bar{\sigma}, \widehat{t}_i, m_i))) \subset BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi^{0, 1}_i | \mathcal{M})$, we have $BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi^{1, m_i}_i | \mathcal{M}) \neq \emptyset$. In addition, since $\text{marg}_{\Theta^* \times M_{-i}} \pi^{1, m_i}_i (\Theta \times \{\bar{\theta}\} \times M_{-i}(\bar{\sigma})) = 1$, by construction of $M_i(\bar{\sigma})$ we have $BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi^{1, m_i}_i | \mathcal{M}) \subset M_i(\bar{\sigma})$. Thus,

$BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi^{1, m_i}_i | \mathcal{M}(\bar{\sigma})) = BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi^{1, m_i}_i | \mathcal{M}(\bar{\sigma})).$

Recall that, by construction of $\pi^{1, m_i}_i$, $m_i \in BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi^{0, 1}_i | \mathcal{M}(\bar{\sigma}))$. Consequently,

$m_i \in BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi^{0, 1}_i | \mathcal{M}).$

In addition, we have

$\text{marg}_{\Theta^* \times M_{-i}} \pi^{\epsilon, 1}_i = \text{marg}_{\Theta^* \times M_{-i}} \pi^{0, 1}_i.$
Hence, for $\varepsilon \in \mathcal{E} \setminus \{0\}$, by construction of $\pi_{i}^{\varepsilon,1}, \{m_i\} = BR_i(\operatorname{arg\,max}_{\Theta^* \times M} \pi_{i}^{\varepsilon,1} | \mathcal{M})$ and $\sigma_i(\hat{t}_i[\varepsilon, 1, \tilde{\sigma}, \tilde{t}_i, m_i]) = m_i$.

Now, for each $\ell \geq 2$, proceed by induction and assume that $\sigma_{-i}(\hat{t}_{-i}[\varepsilon, \ell - 1, \tilde{\sigma}, \tilde{t}_{-i}, m_{-i}]) = m_{-i}$ for any $\tilde{t}_{-i} \in \tilde{T}_{-i}, m_{-i} \in R_{-i}^{\varepsilon,1}(\tilde{t}_{-i} | \mathcal{M}(\tilde{\sigma}), \tilde{T})$, and $\varepsilon \in \mathcal{E} \setminus \{0\}$. Fix $\varepsilon \in \mathcal{E} \setminus \{0\}$ and $m_i \in R_i(\tilde{t}_i | \mathcal{M}(\tilde{\sigma}), \tilde{T})$. For each $\hat{t}_i[\varepsilon, \ell, \tilde{\sigma}, \tilde{t}_i, m_i]$, define the belief

$$\pi_{i}^{\varepsilon,\ell} = \hat{k}(\hat{t}_i[\varepsilon, \ell, \tilde{\sigma}, \tilde{t}_i, m_i]) \circ \gamma_{\ell}^{-1} \in \Delta(\Theta^* \times \tilde{T}_{-i} \times M_{-i}),$$

where $\gamma_{\ell} : (\theta^*, \hat{t}_{-i}[\varepsilon, \ell - 1, \tilde{\sigma}, \tilde{t}_{-i}, m_{-i}]) \mapsto (\theta^*, \hat{t}_{-i}[\varepsilon, \ell - 1, \tilde{\sigma}, \tilde{t}_{-i}, m_{-i}, m_{-i})$.

Note that, by construction, $\pi_{i}^{\varepsilon,\ell}$ is the belief of type $\hat{t}_i[\varepsilon, \ell, \tilde{\sigma}, \tilde{t}_i, m_i]$ on $\Theta^* \times \tilde{T}_{-i} \times M_{-i}$ when he believes that $m_{-i}$ is played at each $(\theta^*, \hat{t}_{-i}[\varepsilon, \ell - 1, \tilde{\sigma}, \tilde{t}_{-i}, m_{-i}$). Hence, by the induction hypothesis, for each $\varepsilon \geq 0$, $\pi_{i}^{\varepsilon,\ell}$ corresponds to beliefs of type $\hat{t}_i[\varepsilon, \ell, \tilde{\sigma}, \tilde{t}_i, m_i]$ when the equilibrium $\sigma$ is played. The end of the proof mimics the case $\ell = 1$.

This completes the proof of Proposition 2. Q.E.D.

COMPLETION OF THE PROOF OF THE “ONLY IF PART” OF THEOREM 4: Pick $\hat{T} = (\hat{T}, \hat{k})$ as defined in Proposition 2. By definition of continuous implementation, there exists an equilibrium $\sigma$ in $U(M, \hat{T})$ that continuously implements $f$, and point (i) in this definition ensures that $\sigma|_{\hat{T}}$ is a pure equilibrium. Now pick any $\tilde{t} \in \tilde{T}$ and $m \in R(\tilde{t} | \mathcal{M}(\sigma|_{\hat{T}}), \hat{T})$; we show that $g_{\mathcal{M}(\sigma)}(m) = f(\tilde{t})$, proving that the mechanism $\mathcal{M}(\sigma)$ implements $f$ in rationalizable messages. Applying Proposition 2, we know that there exists a sequence of types $\{\hat{t}[n]\}_{n=0}^{\infty}$ in $\hat{T}$ such that (i) $\hat{t}[n] \rightarrow_{\rho} \tilde{t}$ and (ii) $\sigma(\hat{t}[n]) = m$ for all $n$. By (i) and the fact that $\sigma$ continuously implements $f$, we have $(g \circ \sigma)(\hat{t}[n]) \rightarrow f(\tilde{t})$, while by (ii), we have $(g \circ \sigma)(\hat{t}[n]) = g(m)$ for all $n$. Hence, we must have $g(m) = f(\tilde{t})$ and so $g_{\mathcal{M}(\sigma)}(m) = f(\tilde{t})$, as claimed. Q.E.D.

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