SUPPLEMENT TO “REPEATED GAMES WHERE THE PAYOFFS AND MONITORING STRUCTURE ARE UNKNOWN”  
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S.1. PROOF OF THEOREM 1

THEOREM 1: If a subset \( W \) of \( \mathbb{R}^I \times |\Omega| \) is bounded and ex post self-generating with respect to \( \delta \), then \( W \subseteq E(\delta) \).

PROOF: Let \( v \in W \). We will construct a PPXE that yields \( v \). Since \( v \in B(\delta, \text{or} W) \), there exist a profile \( \alpha \) and a function \( w: Y \rightarrow W \) such that \( (\alpha, v) \) is ex post enforced by \( w \). Set the action profile in period one to be \( s_{h0} = \alpha \) and for each \( h_1 \in Y \), set \( v_{h1} = w(h_1) \in W \). The play in later periods is determined recursively, using \( v_{ht} \) as a state variable. Specifically, for each \( t \geq 2 \) and for each \( h^{t-1} = (y^{\tau})_{\tau=1}^{t-1} \in H^{t-1} \), given a \( v_{ht-1} \in W \), let \( \alpha_{ht-1} \) and \( w_{ht-1}: Y \rightarrow W \) be such that \( (\alpha_{ht-1}, v_{ht-1}) \) is ex post enforced by \( w_{ht-1} \). Then set the action profile after history \( h^{t-1} \) to be \( s_{ht-1} = \alpha_{ht-1} \) and for each \( y^{t} \in Y \), set \( v_{ht} = (h^{t-1} / \text{or} y^{t}) = w_{ht-1}(y^{t}) \in W \).

Because \( W \) is bounded and \( \delta \in (0, 1) \), payoffs are continuous at infinity, so finite approximations show that the specified strategy profile \( s \in S \) generates \( v \) as an average payoff, and its continuation strategy \( s_{ht} \) yields \( v_{ht} \) for each \( h^t \in H^t \). Also, by construction, nobody wants to deviate at any moment of time, given any state \( \omega \in \Omega \). Because payoffs are continuous at infinity, the one-shot deviation principle applies, and we conclude that \( s \) is a PPXE, as desired. Q.E.D.

S.2. PROOF OF THEOREM 2

THEOREM 2: If a subset \( W \) of \( \mathbb{R}^I \times |\Omega| \) is compact, convex, and locally ex post generating, then there is \( \delta \in (0, 1) \) such that \( W \subseteq E(\delta) \) for all \( \delta \in (\delta, 1) \).

PROOF: Suppose that \( W \) is locally ex post generating. Since \( \{U_y\}_{y \in W} \) is an open cover of the compact set \( W \), there is a subcover \( \{U_{y_m}\}_m \) of \( W \). Let \( \delta = \max_m \delta_{y_m} \). Choose \( u \in W \) arbitrarily and let \( U_{y_m} \) be such that \( u \in U_{y_m} \). Since \( W \cap U_{y_m} \subseteq B(\delta_{y_m}, W) \), there exist \( \alpha_u \) and \( w_u: Y \rightarrow W \) such that \( (\alpha_u, u) \) is ex post enforced by \( w_u \) for \( \delta_{y_m} \). Given a \( \delta \in (\delta, 1) \), let

\[
    w(y) = \frac{\delta - \delta_u}{\delta(1 - \delta_u)} u + \frac{\delta_u(1 - \delta)}{\delta(1 - \delta_u)} w_u(y)
\]

for all \( y \in Y \). Then it is straightforward that \( (\alpha_u, u) \) is enforced by \( (w(y))_{y \in Y} \) for \( \delta \). Also, \( w(y) \in W \) for all \( y \in Y \), since \( u \) and \( w(y) \) are in \( W \) and \( W \) is convex. Therefore, \( u \in B(\delta, W) \), meaning that \( W \subseteq B(\delta, W) \) for all \( \delta \in (\delta, 1) \). (Recall

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that $u$ and $\delta$ are arbitrarily chosen from $W$ and $(\delta, 1)$. Then, from Theorem 1, $W \subseteq E(\delta)$ for $\delta \in (\delta, 1)$, as desired. 

**Q.E.D.**

**S.3. PROOF OF LEMMA 2**

**LEMMA 2:** For every $\delta \in (0, 1)$, $E(\delta) \subseteq E^*(\delta) \subseteq Q$, where $E^*(\delta)$ is the convex hull of $E(\delta)$.

**PROOF:** It is obvious that $E(\delta) \subseteq E^*(\delta)$. Suppose $E^*(\delta) \not\subseteq Q$. Then, since the score is a linear function, there is $v \in E(\delta)$ and $\lambda$ such that $\lambda \cdot v > k^*(\lambda)$. In particular, since $E(\delta)$ is compact, there exist $v^* \in E(\delta)$ and $\lambda$ such that $\lambda \cdot v^* > k^*(\lambda)$ and $\lambda \cdot v^* \geq \lambda \cdot \tilde{v}$ for all $\tilde{v} \in E^*(\delta)$. By definition, $v^*$ is enforced by $(w(y))_{y \in Y}$ such that $w(y) \in E(\delta) \subseteq E^*(\delta) \subseteq H(\lambda, \lambda \cdot v^*)$ for all $y \in Y$. But this implies that $k^*(\lambda)$ is not the maximum score for direction $\lambda$, a contradiction. **Q.E.D.**

**S.4. PROOF OF LEMMA 3**

**LEMMA 3:** For any smooth set $W$ in the interior of $Q$, there is $\tilde{\delta} \in (0, 1)$ such that $W \subseteq E(\delta)$ for $\delta \in (\tilde{\delta}, 1)$.

**PROOF:** Since $W$ is bounded, it suffices to show that it is also locally ex post generating, that is, for each $v \in W$, there exist $\delta_v \in (0, 1)$ and an open neighborhood $U_v$ of $v$ such that $W \cap U_v \subseteq B(\delta_v, W)$.

First, consider $v \in \text{bd } W$. Let $\lambda$ be normal to $W$ at $v$ and let $k = \lambda \cdot v$. Since $W \subseteq Q \subseteq H^*(\lambda)$, there exist $\alpha, \tilde{v}$, and $(\tilde{w}(y))_{y \in Y}$ such that $\lambda \cdot \tilde{v} > \lambda \cdot v = k$, $(\alpha, \tilde{v})$ is enforced using continuation payoffs $(\tilde{w}(y))_{y \in Y}$ for some $\tilde{\delta} \in (0, 1)$, and $\tilde{w}(y) \in H(\lambda, \lambda \cdot \tilde{v})$ for all $y \in Y$. For each $\delta \in (\tilde{\delta}, 1)$ and $y \in Y$, let

$$w(y, \delta) = \frac{\delta - \tilde{\delta}}{\delta(1 - \tilde{\delta})} v + \frac{\tilde{\delta}(1 - \delta)}{\delta(1 - \tilde{\delta})} \left( \tilde{w}(y) + \frac{v - \tilde{v}}{\tilde{\delta}} \right).$$

By construction, $(\alpha, v)$ is enforced by $(w(y, \delta))_{y \in Y}$ for $\delta$, and there is $\kappa > 0$ such that $|w(y, \delta) - v| < \kappa(1 - \delta)$. Also, since $\lambda \cdot \tilde{v} > \lambda \cdot v = k$ and $\tilde{w}(y) \in H(\lambda, \lambda \cdot \tilde{v})$ for all $y \in Y$, there is $\varepsilon > 0$ such that $\tilde{w}(y) - \frac{v - \tilde{v}}{\tilde{\delta}}$ is in $H(\lambda, k - \varepsilon)$ for all $y \in Y$, thereby

$$w(y, \delta) \in H\left( \lambda, k - \frac{\tilde{\delta}(1 - \delta)}{\delta(1 - \tilde{\delta})} \varepsilon \right)$$

for all $y \in Y$. Then, as in the proof of FL’s Theorem 3.1, it follows from the smoothness of $W$ that $w(y, \delta) \in \text{int } W$ for sufficiently large $\delta$, that is, $(\alpha, v)$ is enforced with respect to $\text{int } W$. To enforce $u$ in the neighborhood of $v$, use $\alpha$ and a translate of $(w(y, \delta))_{y \in Y}$.  

Next, consider \( v \in \text{int} W \). Choose \( \lambda \) arbitrarily, and let \( \alpha \) and \((w(y, \delta))_{y \in Y}\) be as in the above argument. By construction, \((\alpha, v)\) is enforced by \((w(y, \delta))_{y \in Y}\). Also, \( w(y, \delta) \in \text{int} W \) for sufficiently large \( \delta \), since \( |w(y, \delta) - v| < \kappa(1 - \delta) \) for some \( \kappa > 0 \) and \( v \in \text{int} W \). Thus, \((\alpha, v)\) is enforced with respect to \( \text{int} W \) when \( \delta \) is close to 1. To enforce \( u \) in the neighborhood of \( v \), use \( \alpha \) and a translate of \((w(y, \delta))_{y \in Y}\), as before. Q.E.D.

S.5. ALTERNATE PROOF OF LEMMA 6

LEMMA 6: Suppose that a profile \( \alpha \) has statewise full rank for \((i, \omega)\) and \((j, \tilde{\omega})\) satisfying \( \omega \neq \tilde{\omega} \), and that \( \alpha \) has individual full rank for all players and states. Then \( k^\ast(\alpha, \lambda) = \infty \) for direction \( \lambda \) such that \( \lambda_i^\omega \neq 0 \) and \( \lambda_j^{\tilde{\omega}} \neq 0 \).

PROOF: Let \((i, \omega)\) and \((j, \tilde{\omega})\) be such that \( \lambda_i^\omega \neq 0 \), \( \lambda_j^{\tilde{\omega}} \neq 0 \), and \( \tilde{\omega} \neq \omega \). Let \( \alpha \) be a profile that has statewise full rank for all \((i, \omega)\) and \((j, \tilde{\omega})\) satisfying \( \omega \neq \tilde{\omega} \).

First, we claim that for every \( K > 0 \), there exist \( z_i^\omega = (z_i^\omega(y))_{y \in Y} \) and \( z_j^{\tilde{\omega}} = (z_j^{\tilde{\omega}}(y))_{y \in Y} \) such that

\[
\pi^\omega(a_i, \alpha_{-i}) \cdot z_i^\omega = \frac{K}{\delta \lambda_i^\omega}
\]

for all \( a_i \in A_i \),

\[
\pi^\omega(a_j, \alpha_{-j}) \cdot z_j^{\tilde{\omega}} = 0
\]

for all \( a_j \in A_j \), and

\[
\lambda_i^\omega z_i^\omega(y) + \lambda_j^{\tilde{\omega}} z_j^{\tilde{\omega}}(y) = 0
\]

for all \( y \in Y \). To prove that this system of equations indeed has a solution, eliminate (S3) by solving for \( z_j^{\tilde{\omega}}(y) \). Then there remain \(|A_i| + |A_j|\) linear equations, and its coefficient matrix is \( \Pi_{(i, \omega)(j, \tilde{\omega})}(\alpha) \). Since statewise full rank implies that this coefficient matrix has rank \(|A_i| + |A_j|\), we can solve the system.

Next, for each \((l, \overline{\omega}) \in I \times \Omega\), we choose \((\tilde{w}_l^\omega(y))_{y \in Y}\) so that

\[
(1 - \delta)g_l^\overline{\omega}(a_l, \alpha_{-i}) + \delta \pi^\overline{\omega}(a_l, \alpha_{-i}) \cdot \tilde{w}_l^\overline{\omega} = 0
\]

for all \( a_l \in A_l \). Note that this system has a solution, since \( \alpha \) has individual full rank. Intuitively, continuation payoffs \( \tilde{w}_l^\overline{\omega} \) are chosen so that players are indifferent over all actions and their payoffs are zero.

Let \( K > \max_{y \in Y} \lambda \cdot \tilde{w}(y) \), and choose \((z_i^\omega(y))_{y \in Y}\) and \((z_j^{\tilde{\omega}}(y))_{y \in Y}\) to satisfy (S1)–(S3). Then let

\[
\tilde{w}_l^\overline{\omega}(y) = \begin{cases} 
\tilde{w}_l^\omega(y) + z_i^\omega(y), & \text{if } (l, \overline{\omega}) = (i, \omega), \\
\tilde{w}_l^{\tilde{\omega}}(y) + z_j^{\tilde{\omega}}(y), & \text{if } (l, \overline{\omega}) = (j, \tilde{\omega}), \\
\tilde{w}_l^\overline{\omega}(y), & \text{otherwise}
\end{cases}
\]
for each $y \in Y$. Also, let

$$v_l = \begin{cases} \frac{K}{\lambda^i_l}, & \text{if } (l, \omega) = (i, \omega), \\ 0, & \text{otherwise}. \end{cases}$$

We claim that this $(v, w)$ satisfies constraints (i) through (iii) in LP Average. It follows from (S4) that constraints (i) and (ii) are satisfied for all $(l, \omega) \in (I \times \Omega) \setminus \{(i, \omega), (j, \tilde{\omega})\}$. Also, using (S1) and (S4), we obtain

$$\left(1 - \delta\right) g_i^\omega(a_i, \alpha_{-i}) + \delta \pi^\omega(a_i, \alpha_{-i}) \cdot w_i^\omega = \left(1 - \delta\right) g_i^\omega(a_i, \alpha_{-i}) + \delta \pi^\omega(a_i, \alpha_{-i}) \cdot (\tilde{w}_i^\omega + z_i^\omega)$$

for all $a_i \in A_i$. This shows that $(v, w)$ satisfies constraints (i) and (ii) for $(i, \omega)$. Likewise, from (S2) and (S4), $(v, w)$ satisfies constraints (i) and (ii) for $(j, \tilde{\omega})$. Furthermore, using (S3) and $K > \max_{y \in Y} \lambda \cdot \tilde{w}(y)$,

$$\lambda \cdot w(y) = \lambda \cdot \tilde{w}(y) + \lambda^\omega_iz_i^\omega(y) + \lambda^\omega_jz_j^\omega(y) = \lambda \cdot \tilde{w}(y) < K = \lambda \cdot v$$

for all $y \in Y$, and hence constraint (iii) holds.

Therefore, $k^*(\alpha, \lambda) \geq \lambda \cdot v = K$. Since $K$ can be arbitrarily large, we conclude $k^*(\alpha, \lambda) = \infty$. \[Q.E.D.]\n
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