SUPPLEMENT TO “OPTIMAL INATTENTION TO THE STOCK MARKET WITH INFORMATION COSTS AND TRANSACTIONS COSTS”  
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BY ANDREW B. ABEL, JANICE C. EBERLY, AND STAVROS PANAGEAS  

APPENDIX A  

PROOF OF LEMMA 1: Since  
\[ e^{-\rho \tau_i} A(t_i, \tau_i) = \kappa b(\tau_i) \int_{t_i}^{t_i + \tau_i} c_i^{1-\alpha} e^{-\rho (t_i - t)} \, dt \]  
we have  
\[ \lim_{\tau_i \to 0} \tau_i b(\tau_i) = \lim_{\tau_i \to 0} \kappa b(\tau_i) \int_{t_i}^{t_i + \tau_i} c_i^{1-\alpha} e^{-\rho (t_i - t)} \, dt \]  

Equation (9a) states that the numerator on the right hand side of (A.1) has a positive finite limit as  \( \tau_i \to 0 \). The limit of the denominator is  
\[ \lim_{\tau_i \to 0} \kappa c_i^{1-\alpha} e^{-\rho (t_i - t)} \, dt = \kappa c_i^{1-\alpha} \]  
which is positive and finite since we are confining attention to cases with positive (and finite) consumption. Therefore, statement (ii) holds.  

Statement (iii) follows from the fact that  
\[ e^{-\rho \tau_i} A(t_i, \tau_i) = \kappa b(\tau_i) \int_{t_i}^{t_i + \tau_i} c_i^{1-\alpha} e^{-\rho (t_i - t)} \, dt \]  
and (9b) along with the assumptions that  \( \kappa > 0 \) and  \( c_i > 0 \).  

Equation (11) and  \( \kappa > 0 \) can be used to rewrite (9c) as  
\[ b(\tau_i) \int_{t_i}^{t_i + \tau_i} c_i^{1-\alpha} e^{-\rho (t_i - t)} \, dt + e^{-\rho \tau_i} b(\tau_i) \int_{t_i + 1}^{t_i + \tau_i + 1} c_i^{1-\alpha} e^{-\rho (t_i - t)} \, dt \]  

To see the implications of (A.2) for  \( b(\tau_i) \), we first state the following lemma.  

**LEMMA 3:** Suppose  \( q_1 b(z_1) + q_2 b(z_2) > (q_1 + q_2) b(z_1 + z_2) \) for all positive  \( q_i \) and  \( z_i, i = 1, 2 \), and that  \( b(z) > 0 \) for all  \( z > 0 \). Then  \( b(z) \) is nonincreasing.  

**PROOF:** The assumption that  \( q_1 b(z_1) + q_2 b(z_2) > (q_1 + q_2) b(z_1 + z_2) \) for all positive  \( q_i \) and  \( z_i, i = 1, 2 \), implies that  \( q_1 [b(z_1) - b(z_1 + z_2)] + q_2 [b(z_2) - \[31\text{Let } \gamma = \lim_{\tau \to 0} \tau b(\tau) = \lim_{\tau \to 0} \frac{\tau}{b(\tau)}, \text{ which, by L'Hôpital's rule (and assuming that the derivative of } 1/b(\tau) \text{ exists and is non-zero in a neighborhood of } \tau = 0) \text{ implies } \gamma = \lim_{\tau \to 0} \frac{\frac{\delta}{\delta \tau} [\frac{\tau}{b(\tau)}]}{\frac{\delta}{\delta \tau} [\frac{1}{b(\tau)}]} = -\gamma^{-1}. \text{ Then } \lim_{\tau \to 0} \frac{\tau b(\tau)}{b(\tau)^2} = \lim_{\tau \to 0} \frac{\tau}{b(\tau)} = [\lim_{\tau \to 0} \tau b(\tau)] [\lim_{\tau \to 0} \frac{\delta}{\delta \tau} [\frac{\tau}{b(\tau)}]] = \gamma (-\gamma^{-1}) = -1. \]
b(z_1 + z_2) > 0 for all positive q_i and z_i, i = 1, 2. Suppose that, contrary to what is to be proved, for some positive z_1 and z_2, b(z_1) < b(z_1 + z_2). Then for any q_1 > -q_2[b(z_2)-b(z_1+z_2)], q_1[b(z_1) - b(z_1 + z_2)] + q_2[b(z_2) - b(z_1 + z_2)] < 0, which is a contradiction. Therefore, b(z_1) ≥ b(z_1 + z_2) for any positive z_1 and z_2.

Q.E.D.

Applying Lemma 3 to (A.2) while setting z_1 = \tau_i, z_2 = \tau_{i+1}, q_1 = \int_{t_i}^{t_{i+1}} c_t^{1-\alpha} \times e^{-\rho(t-t_i)} dt, and q_2 = e^{-\rho t_i} \int_{t_{i+1}}^{t_i} c_t^{1-\alpha} e^{-\rho(t-t_i)} dt, implies that b(\tau) is non-increasing, which is statement (i) in Lemma 1.

Q.E.D.

PROOF OF PROPOSITION 1: We start by proving the following lemma.

LEMMA 4: Optimal behavior requires y^s y^b = 0. If the optimal asset transfer increases x, then y^s < 0. If the optimal transfer decreases x, then y^b > 0.

PROOF: To prove that y^s y^b = 0, suppose y^s y^b \neq 0, which implies that y^s < 0 and y^b > 0. Now consider reducing y^b by \varepsilon > 0 and increasing y^s by \varepsilon > 0, which will have no effect on the value of S relative to the original transfer, but will increase X by (\psi_s + \psi_b)\varepsilon > 0 relative to the original transfer by reducing the amount of proportional transactions cost incurred. Therefore, it could not have been optimal for y^s y^b \neq 0. Hence, y^s y^b = 0.

The value function V(X, S) is strictly increasing in X and S, so an optimal transfer will never decrease both X and S. Therefore, if the optimal transfer increases x \equiv \frac{X}{S}, then the optimal transfer cannot decrease X and must decrease S, which implies that y^b = 0 and y^s < 0. Similarly, if the optimal transfer decreases x \equiv \frac{X}{S}, then the optimal transfer cannot decrease S and must decrease X, which implies that y^s = 0 and y^b > 0.

Q.E.D.

Proof of statement (ii)(a). Suppose that x < \omega_1. The definition of \omega_1 in (25) implies that v(x) \neq \bar{v}(x). The optimal asset transfer will change the value of x to some value z for which v(z) = \bar{v}(z). The definition of \omega_1 implies that such a z cannot be less than \omega_1, so the optimal transfer increases x. Lemma 4 implies that y^s < 0.

Proof of statement (ii)(b). Suppose that on an observation date normalized to be t = 0, X_0 < \omega_1 S_0. Statement (ii)(a) implies that y^s < 0. Let (X^*, S^*) be the value of (X_{t+}, S_{t+}) resulting from the optimal value of y^s. Define P \equiv \{(X, S) : X = X^* + (1 - \psi_s)z and S = S^* - z for z \in (0, S^*)\}. Because (X^*, S^*) is the result of an optimal transfer of assets from the investment portfolio to the transactions account (and the fixed costs \theta_X X_0 and \theta_S S_0 have already been paid to reach (X^*, S^*)), there is no (X^**, S^**) \in P such that V(x^* S^*, S^*) \geq V(x^* S^*, S^*) and V(x^* S^*, S^*) > \bar{V}(x^* S^*, S^*). [If there
were such a \((X^{**}, S^{**})\), then either (a) \(V(x^{**}S^{**}, S^{**}) > V(x^{**}S^{*}, S^{*})\) or (b) \(V(x^{**}S^{**}, S^{**}) = V(x^{**}S^{*}, S^{*})\). If (a) holds, then \((X^{*}, S^{*})\) is not optimal. If (b) holds, then \(V(x^{*}S^{*}, S^{*}) > \tilde{V}(x^{*}S^{*}, S^{*})\) and hence it cannot be optimal to remain at \((X^{*}, S^{*})\). Now suppose that \(x^{*} < \pi_1\). Then consider \((X^{**}, S^{**}) \in \mathcal{P}\) for which \(x^{**} = \frac{x^{**}}{S^{**}}\) is between \(x^{*}\) and \(\pi_1\). The definition of \(\pi_1\) implies that \(V(x^{**}S^{**}, S^{**}) \geq V(x^{*}S^{*}, S^{*})\) and \(V(x^{**}S^{**}, S^{**}) > \tilde{V}(x^{*}S^{*}, S^{*})\), which contradicts the statement that there is no \((X^{**}, S^{**}) \in \mathcal{P}\) such that \(V(x^{**}S^{**}, S^{**}) \geq V(x^{*}S^{*}, S^{*})\) and \(V(x^{**}S^{**}, S^{**}) > \tilde{V}(x^{*}S^{*}, S^{*})\). Hence, \(x^{*} < \pi_1\) is not optimal.

**Proof of statement (ii)(c).** Consider the point \((X_0, S_0)\) with \(x_0 = \frac{x_0}{S_0} = \omega_1\) and define \(D\) as the set of \((X, S)\) for which \(x < \omega_1\) and from which the consumer can instantaneously move to \((X_0, S_0)\) by transferring assets from the investment portfolio to the transactions account. Specifically,

\[
(A.3) \quad D \equiv \{ (X, S) \text{ with } X < \omega_1 S : \exists y^1 < 0 \text{ for which } (1 - \theta_X)X - (1 - \psi_s)y^1 = X_0 \text{ and } (1 - \theta_S)S + y^1 = S_0 \}.
\]

Define \(F\) as the set of \((X, S)\) for which \(x \geq \omega_1\) and to which the consumer can instantaneously move from any point in \(D\) by transferring assets from the investment portfolio to the transactions account. Specifically,

\[
(A.4) \quad F \equiv \{ (X, S) \text{ with } X \geq \omega_1 S : \exists y^1 < 0 \text{ for which } X = X_0 - (1 - \psi_s)y^1 \text{ and } S = S_0 + y^1 \geq 0 \}.
\]

Consider two arbitrary points \((X_1, S_1)\) and \((X_2, S_2)\) in set \(D\). Since \(x_1 < \omega_1\) and \(x_2 < \omega_1\), the optimal value of \(y^1\) will be strictly negative starting from either point. Moreover, \(y^1\) must be large enough in absolute value so that the post-transfer value of \((X, S)\) satisfies \(x = x_1 + \frac{x_1}{S_1} \geq x_1\), because it is always optimal to transfer assets from the investment portfolio to the transactions account from any point in set \(D\). Therefore, the post-transfer value of \((X, S)\) will be an element of set \(F\). Thus, regardless of whether the consumer starts from point \((X_1, S_1)\) or \((X_2, S_2)\), the consumer’s choice of asset transfer can be described as choosing \((X^+, S^+) \in F\) to maximize the value function. Therefore, \(V(X_1, S_1) = V(X_2, S_2)\), so all of the points in set \(D\) lie on the same indifference curve of \(V(X, S)\). The slope of this indifference curve is \(\frac{dX}{dS} = \frac{dX}{dy^1} \frac{dy^1}{dS} = -(1 - \theta_S)^{\frac{1 - \theta_X}{1 - \theta_X} - \psi_s} \), which proves statement (ii)(c).

**Proof of statement (ii)(d).** We have shown that if \(x < \omega_1\), then \(m(x) = (1 - \psi_s)^{\frac{1 - \theta_X}{1 - \theta_X}}\). The expression for \(V(X_{tj}, S_{tj})\) in (21) can be used to rewrite the marginal rate of substitution, \(m(x_{tj}) \equiv \frac{V_5(X_{tj}, S_{tj})}{V_5(X_{tj}, S_{tj})} - x_{tj}\), as \(m(x_{tj}) = \frac{(1 - \alpha)v(x_{tj})}{\alpha v(x_{tj})} - x_{tj}\), so
that
\[ (1 - \alpha)v(x) - x = (1 - \psi_s) \frac{1 - \theta_s}{1 - \theta_X} \quad \text{for} \quad 0 \leq x < \omega_1, \]

which implies
\[ v(x) = \left[ \frac{(1 - \theta_X)x + (1 - \theta_s)(1 - \psi_s)}{(1 - \theta_X)\omega_1 + (1 - \theta_s)(1 - \psi_s)} \right]^{1-\alpha} v(\omega_1) \quad \text{for} \quad 0 \leq x \leq \omega_1. \]

**Proof of statement (i).** We start by proving the following lemma.

**LEMMA 5:** For sufficiently small \( \bar{x} > 0 \), \( \frac{1 - \alpha}{1 - \alpha} \tilde{v}(x) < \frac{1}{1 - \alpha} v(x) \) for all \( x \in (0, \bar{x}) \).

**PROOF:** Substitute the expression for \( U(C(t_j, \tau_j)) \) from (16) into the restricted value function in (23) to obtain
\[ \tilde{V}(X_{t_j}, S_{t_j}) \]
\[ = \max_{C(t_j, \tau_j), \phi_j, \tau_j} \left[ 1 - (1 - \alpha)\kappa b(\tau_j) \right] \frac{1}{1 - \alpha} \left[ h(\tau_j) \right]^{\alpha} \left[ C(t_j, \tau_j) \right]^{1-\alpha} \]
\[ + e^{-\rho \tau_j} E_{t_j} \{ V(e^{L\tau_j}(X_{t_j} - C(t_j, \tau_j)), R(t_j, \tau_j)S_{t_j}) \}. \]

Equation (***) in footnote 18 states that \( C(t_j, \tau_j) = h(\tau_j)c_{t_j}^{\alpha} \), so that
\[ \left[ 1 - (1 - \alpha)\kappa b(\tau_j) \right] \frac{1}{1 - \alpha} \left[ h(\tau_j) \right]^{\alpha} \left[ C(t_j, \tau_j) \right]^{1-\alpha} \]
\[ = \frac{1}{1 - \alpha} \left[ 1 - (1 - \alpha)\kappa b(\tau_j) \right] h(\tau_j) c_{t_j}^{1-\alpha}. \]

Substitute (A.8) into (A.7) to obtain
\[ \tilde{V}(X_{t_j}, S_{t_j}) = \max_{C(t_j, \tau_j), \phi_j, \tau_j} \frac{1}{1 - \alpha} \left[ 1 - (1 - \alpha)\kappa b(\tau_j) \right] h(\tau_j) c_{t_j}^{1-\alpha} \]
\[ + e^{-\rho \tau_j} E_{t_j} \{ V(e^{L\tau_j}(X_{t_j} - C(t_j, \tau_j)), R(t_j, \tau_j)S_{t_j}) \}. \]

Because the choice of \( C(t_j, \tau_j) \) must satisfy the constraint \( X_{t_j} - C(t_j, \tau_j) \geq 0 \), the partial derivative with respect to \( C(t_j, \tau_j) \) of the maximand on the right hand side of (A.7) must be nonnegative. Therefore, differentiation of this maximand with respect to \( C(t_j, \tau_j) \) yields
\[ \left[ 1 - (1 - \alpha)\kappa b(\tau_j) \right] \left[ h(\tau_j) \right]^{\alpha} \left[ C(t_j, \tau_j) \right]^{-\alpha} \]
\[ - e^{-(\rho - L)\tau_j} E_{t_j} \{ V_X(e^{L\tau_j}(X_{t_j} - C(t_j, \tau_j)), R(t_j, \tau_j)S_{t_j}) \} \geq 0. \]
Since $V_X( ) > 0$, $[h(\tau_j)]^\alpha [C(t_j, \tau_j)]^{-\alpha} > 0$, and $e^{-(\rho - r_L)\tau_j} > 0$, (A.10) implies that

(A.11) $1 - (1 - \alpha)\kappa b(\tau_j^*) > 0$,

where $\tau_j^*$ is the value of $\tau_j$ that maximizes the restricted value function. Equation (A.11) implies that we can confine attention to values of $\tau_j$ that are greater than $\tau \equiv \inf\{\tau > 0: \kappa (1 - \alpha) b(\tau) < 1\}$. If $\alpha > 1$, then $1 - \kappa (1 - \alpha) b(\tau_j) > 0$ for any positive value of $\tau_j$, so $\tau = 0$. However, if $\alpha < 1$, Lemma 1 implies $\tau > 0$.

Now we consider the cases in which $\alpha < 1$ and $\alpha > 1$ separately.

Case I: $\alpha < 1$. When $\alpha < 1$, $\tau^* > \tau > 0$. Since $C(t_j, \tau_j) = h(\tau_j) c_{t_j} + j$, then

(A.12) $c_{t_j} = \frac{C(t_j, \tau_j^*)}{h(\tau_j)} < \frac{X_{t_j}}{h(\tau)}$,

where the inequality follows from the constraint $C(t_j, \tau_j^*) \leq X_{t_j}$ and the facts that $h(\tau_j)$ is strictly increasing in $\tau_j$ and $\tau_j^* > \tau$. Equation (A.12) implies

(A.13) $\lim_{x_{t_j} \to 0} \tilde{V}(X_{t_j}, S_{t_j}) = \lim_{x_{t_j} \to 0} e^{-\rho \tau_j} E_{t_j} \{V(0, R(t_j, \tau_j^*) S_{t_j})\} = \lim_{x_{t_j} \to 0} e^{-\rho \tau_j} E_{t_j}\{[R(t_j, \tau_j^*)]^{1-\alpha}\} \frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v(0)$.

Use (B.9) and the fact that $\tau^* > \tau$ to obtain

(A.14) $\lim_{x_{t_j} \to 0} \tilde{V}(X_{t_j}, S_{t_j}) < \frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v(0) = V(0, S_{t_j})$.

Case II: $\alpha > 1$. We start by showing that optimal $y^i(t_j) < 0$, when $x_{t_j} = 0$. Suppose, contrary to what is to be proved, that it is optimal to set $y^i(t_j) = 0$ when $x_{t_j} = 0$, which implies that $c_t = 0$ for all $t \in [t_j, t_{j+1}]$ and $x_{t_{j+1}} = 0$. In turn, $x_{t_{j+1}} = 0$ implies $c_t = 0$ for all $t \in [t_{j+1}, t_{j+2}]$ and so on ad infinitum. Accordingly, $\frac{1}{1-\alpha} v(0) = -\infty$ when $\alpha > 1$. Clearly, $\frac{1}{1-\alpha} v(0)$ is smaller than the value associated with the policy of setting $y^i(t_j) = -(1 - \theta_S) S_{t_j}$, so that $X_{t_j}^+ = (1 - \psi_S)(1 - \theta_S) S_{t_j}$ and then consuming optimally from the transactions account over the infinite future, never incurring any information costs or transactions costs. As we show in (A.26), the value of such a policy is given by $\frac{1}{1-\alpha} X_{t_j}^{1-\alpha}$, which is finite. Accordingly, the policy of setting $y^i(t_j) = 0$ whenever $x_{t_j} = 0$ cannot be optimal.
We show next that \( \lim_{x_{tj} \to 0} \frac{1}{1-\alpha}v(x_{tj}) \geq \frac{1}{1-\alpha}v(0) \). Let \( x^*_{tj} \) denote the optimal value of \( x_{tj}^+ \) associated with the optimal transfer \( y^*(t_j) \) when \( x_{tj} = 0 \). Value matching implies that \( \frac{1}{1-\alpha}v(0)S^1_{tj} = \frac{1}{1-\alpha}v(x^*_{tj})S^1_{tj} \). Now we will compute the size of the transfer \( y^s \) that changes \( x_t \) from arbitrary \( x_{tj} \) at time \( t_j \) to \( x^*_{tj} \) at time \( t^+ \). When \( y^b = 0 \), (4) and (5) imply that

\[
(1 - \theta X)x_{tj} - (1 - \theta_S)x^*_{tj} = \frac{(1 - \theta X)x_{tj} - (1 - \theta_S)x^*_{tj}}{S_{tj}}.
\]

Solving for \( \frac{y^s}{S_{tj}} \) gives

\[
\frac{y^s}{S_{tj}} = \frac{(1 - \theta X)x_{tj} - (1 - \theta_S)x^*_{tj}}{x^*_{tj} + 1 - \psi_s}.
\]

Furthermore, when \( x_{tj} = 0 \), then

\[
\frac{S^1_{tj}}{S_{tj}} = (1 - \theta_S) + \frac{y^s}{S_{tj}} = (1 - \theta_S) - \frac{(1 - \theta_X)x^*_{tj}}{x^*_{tj} + 1 - \psi_s} = (1 - \theta_S)\frac{1 - \psi_s}{x^*_{tj} + 1 - \psi_s} \text{ and, accordingly,}
\]

\[
(A.15) \quad \frac{v(0)}{v(x^*_{tj})} = \left( 1 - \theta_S \right) \frac{1 - \psi_s}{x^*_{tj} + 1 - \psi_s} \right)^{1-\alpha}.
\]

Now take \( \epsilon > 0 \) and suppose that \( x_{tj} = \epsilon \). For sufficiently small \( \epsilon > 0 \), set

\[
\frac{y^s}{S_{tj}} = \frac{(1 - \theta X)\epsilon - (1 - \theta_S)x^*_{tj}}{x^*_{tj} + 1 - \psi_s} \text{, which will be negative as } \epsilon \text{ approaches } 0.\]

By construction, this feasible transfer implies that \( x^*_{tj} = x^*_{tj} \). Moreover, \( \frac{S^1_{tj}}{S_{tj}} = (1 - \theta_S) + \frac{(1 - \theta_X)\epsilon - (1 - \theta_S)x^*_{tj}}{x^*_{tj} + 1 - \psi_s} = (1 - \theta_S)\frac{1 - \psi_s}{x^*_{tj} + 1 - \psi_s} + \frac{\epsilon}{x^*_{tj} + 1 - \psi_s}. \) Accordingly,

\[
(A.16) \quad \frac{1}{1-\alpha}v(x^*_{tj}) \left[ (1 - \theta_S) \frac{1 - \psi_s}{x^*_{tj} + 1 - \psi_s} + (1 - \theta_X) \frac{\epsilon}{x^*_{tj} + 1 - \psi_s} \right]^{1-\alpha} \leq \frac{1}{1-\alpha}v(\epsilon).
\]
Using (A.15) to solve for $v(x_{t_j}^*)$, substituting the resulting expression inside (A.16), and taking limits on both sides of (A.16) as $\varepsilon = x_{t_j} \to 0$ implies
\[
\lim_{x_{t_j} \to 0} \frac{1}{1-\alpha} v(x_{t_j}) \geq \frac{1}{1-\alpha} v(0).
\]

Next we show that $\lim_{x_{t_j} \to 0} v(x_{t_j}) = v(0)$. The proof proceeds by contradiction. Indeed, suppose that $\lim_{x_{t_j} \to 0} \frac{1}{1-\alpha} v(x_{t_j}) > \frac{1}{1-\alpha} v(0)$. Then for any $t_j$, it cannot be optimal to set $C(t_j, \tau_j) = X_{t_j}$, so that $X_{t_j+1} = 0$. [To see why, suppose otherwise. If it were optimal to set $X_{t_j+1} = 0$, then consider the following deviation: Reduce $C(t_j, \tau_j)$ by an arbitrarily small $\varepsilon > 0$, so that $X_{t_j+1} = e^{\rho \tau_j} \varepsilon$. This deviation is feasible for sufficiently small $\varepsilon > 0$, because $C(t_j, \tau_j) = 0$ can never be optimal when $\alpha > 1$. The deviation changes the value of the program by $A(\varepsilon) = \left(1 - 1-\alpha \kappa b(\tau_j)\right) \times \left(U(C(t_j, \tau_j)) - \varepsilon \right) - U(C(t_j, \tau_j)) + e^{-\rho \tau_j} E_{t_j} \left(V(e^{\rho \tau_j} \varepsilon, S_{t_j+1}) - V(0, S_{t_j+1})\right)$.]

For given $X_{t_j}$ and $\tau_j$, $\lim_{\varepsilon \to 0} \left(1 - 1-\alpha \kappa b(\tau_j)\right) \times \left(U(C(t_j, \tau_j)) - \varepsilon \right) - U(C(t_j, \tau_j)) = 0$, so that $\lim_{\varepsilon \to 0} A(\varepsilon) = e^{-\rho \tau_j} \frac{1}{1-\alpha} \lim_{\varepsilon \to 0} E_{t_j} \left(S_{t_j+1}^{1-\alpha} \left(1-\varepsilon\right) - v(0)\right) \right)$. Since the function $\frac{1}{1-\alpha} v(x_{t_j})$ is increasing in $x_{t_j}$, and $\alpha > 1$, it follows that $v(x_{t_j}^{x_j})$ is increasing as $\varepsilon$ decreases to 0. Therefore, the monotone convergence theorem, along with the supposition that $\lim_{x_{t_j} \to 0} v(x_{t_j}) > \frac{1}{1-\alpha} v(0)$, implies that $\lim_{\varepsilon \to 0} A(\varepsilon) = e^{-\rho \tau_j} \frac{1}{1-\alpha} \lim_{\varepsilon \to 0} E_{t_j} \left(S_{t_j+1}^{1-\alpha} \left(1-\varepsilon\right) - v(0)\right) \right) > 0$. Accordingly, there always exists a small enough $\varepsilon > 0$, so that the deviation dominates the supposed optimal path.

Next we show that for any $\delta > 0$, there exists a $z \in (0, \delta)$ such that if $x_{t_j} = z$ on observation date $t_j$, then $y^*(t_j) < 0$. The proof proceeds by contradiction. Suppose otherwise, that is, suppose that there exists a $\delta > 0$, such that it is optimal to set $y^* = 0$ for all $x_{t_j} \in (0, \delta)$. Now fix $T > 0$, and take $x_{t_j} < \delta$. Let $\bar{t}_j+1$ denote the last observation date before $t_j + T$. We will show next that under this (counterfactual) supposition, the discounted sum of the observation costs $\sum_{t_j \in [t_j, \bar{t}_j+1]} e^{-\rho(t_j-t_j)} (1-\alpha) \kappa b(\tau_j) U(C(t_j, \tau_j))$ approaches infinity with probability approaching 1 as $x_{t_j} \to 0$.

To start, we note that because $\alpha > 1$, it must be the case that $c_{t_j}^* > 0$. (Otherwise utility would be negatively infinite between $t_j^+$ and $t_j^+ + \tau_j$, and that would make the value function unboundedly negative.) Since $C(X_{t_j}) = c_{t_j}^* h(\tau_j) < X_{t_j}$, this implies that $\lim_{x_{t_j} \to 0} h(\tau_j) = 0$ or, equivalently, $\lim_{x_{t_j} \to 0} \tau_j = 0$. Now note that $x_{t_j+1} < x_{t_j} \frac{e^{\rho \tau_j}}{R(\tau_j)}$, so that $\lim_{x_{t_j} \to 0} \Pr(x_{t_j+1} > \delta) = 0$.

More generally, for any $\varepsilon \in (0, \delta)$, as long as (i) $x_{t_j} < \varepsilon$ and (ii) $x_{t_j} \times \max_{k \in [t_j, \bar{t}_j]} \frac{e^{\rho \tau_j}}{R(\tau_j)} \frac{e^{\rho \tau_j}}{R(\tau_j)} < \varepsilon$, it follows that $\max_{k \in [t_j, \bar{t}_j+1]} x_{t_k} < x_{t_j} \times \max_{k \in [t_j, \bar{t}_j]} \frac{e^{\rho \tau_j}}{R(\tau_j)} \frac{e^{\rho \tau_j}}{R(\tau_j)} < \varepsilon$. Next we show that the probability that $\max_{k \in [t_j, \bar{t}_j+1]} x_{t_k} \leq \varepsilon$ approaches 1 as $x_{t_j}$ approaches 0. Indeed, since $x_{t_j} \times
Before proceeding, we make a few observations. We start by noting that

\[ (A.17) \quad \Pr\left( \max_{k \in [t_i, t_{j+1}]} x_k > \varepsilon \right) \]

\[ < \Pr \left( x_{t_i} \max_{k \in [t_i, t_j]} \prod_{t_i \in [t_i, t_k]} \frac{e^{\tau_i R(t_i, \tau_i)}}{R(t_i, \tau_i)} > \varepsilon \right) \]

\[ = \Pr \left( \max_{k \in [t_i, t_j]} \sum_{t_i \in [t_i, t_k]} (rL \tau_i - \log R(t_i, \tau_i)) > \log \varepsilon - \log x_{t_j} \right). \]

We next observe that \( g(0) = 0, g'(y) = \frac{\phi_i e^{\gamma}}{\phi_i e^{\gamma} + (1 - \phi_i)} \leq 1, \) and \( g''(y) = \frac{\phi_i e^{\gamma}(1 - \phi_i)}{[\phi_i e^{\gamma} + (1 - \phi_i)]^2} \geq 0. \) Therefore, if \( y > 0, \) then \( g(y) = g(0) + \int_0^y g'(y) dy \leq y. \) By a similar logic, if \( y < 0, \) then \( g(y) \geq y. \) Accordingly, \( y^2 \geq g^2(y) \) and also \( E(y^2) \geq E(g^2(y)). \) Finally, since \( g''(y) \geq 0, \) Jensen’s inequality implies that \( E(g(y)) \geq g(E(y)). \) Accordingly,

\[ (A.19) \quad E(z_{t_{i+1}}) = (rL - r_f)\tau_i - E\left(g(\sigma \Delta B_{t_{i+1}})\right) \leq (rL - r_f)\tau_i - g\left[E(\sigma \Delta B_{t_{i+1}})\right] \]

\[ = (rL - r_f)\tau_i < 0, \]

where the last equality in (A.19) follows from \( E(\Delta B_{t_{i+1}}) = 0 \) and \( g(0) = 0. \) Now let \( Z_{t_{i+1}} = \sum_{t_j \leq t_i \leq t_j} (z_{t_{i+1}} - E\left(z_{t_{i+1}}\right)). \) By construction, \( Z_{t_{i+1}} \) is a martingale, and Jensen’s inequality implies that \( |Z_{t_{i+1}}| \) is a nonnegative submartingale.\(^{32}\) Equation (A.19) implies that \( \max_{t_i \in [t_j, t_{j+1}]} \sum_{t_i \in [t_i, t_{j+1}]} z_{t_{i+1}} < \max_{t_i \in [t_j, t_{j+1}]} \sum_{t_i \in [t_i, t_{j+1}]} z_{t_{i+1}} \leq \]

\[^{32}E_{\eta} |Z_{t_{i+1}}| = E_{\eta} |Z_{\eta} + z_{t_{i+1}} - E_{\eta} (z_{t_{i+1}})| > |Z_{\eta} + E_{\eta} (z_{t_{i+1}} - z_{t_{i+1}})| = |Z_{\eta}|.\]
max_{t_k \in [t_j, \bar{t}_j]} |Z_{t_k+1}| and, therefore,

\begin{equation}
\Pr \left( \max_{t_k \in [t_j, \bar{t}_j]} \sum_{t_l \in [t_j, t_k]} z_{t_l+1} > \log \epsilon - \log x_{t_j} \right) < \Pr \left( \max_{t_k \in [t_j, \bar{t}_j]} |Z_{t_k+1}| > \log \epsilon - \log x_{t_j} \right)
\end{equation}

\begin{equation}
\leq \frac{E_{t_j}[Z_{t_{\bar{t}_j+1}}^2]}{(\log \epsilon - \log x_{t_j})^2},
\end{equation}

where the last inequality follows from Doob’s inequality for submartingales applied to the process $|Z_{t_{\bar{t}_j+1}}|$. Since $Z_{t_{\bar{t}_j+1}}$ is a martingale,

\begin{equation}
E_{t_j}[Z_{t_{\bar{t}_j+1}}^2] = E_{t_j} \left\{ \sum_{t_l \in [t_j, \bar{t}_j]} (z_{t_{l+1}} - E_{t_l}(z_{t_{l+1}}))^2 \right\}
\end{equation}

\begin{align*}
&= E_{t_j} \left\{ \sum_{t_l \in [t_j, \bar{t}_j]} E_{t_l} \left[ g(\sigma \Delta B_{t_{l+1}}) - E_{t_l} [g(\sigma \Delta B_{t_{l+1}})] \right]^2 \right\} \\
&= E_{t_j} \left\{ \sum_{t_l \in [t_j, \bar{t}_j]} E_{t_l} \left[ g(\sigma \Delta B_{t_{l+1}}) \right]^2 \right\} - \sum_{t_l \in [t_j, \bar{t}_j]} \left[ E_{t_l} g(\sigma \Delta B_{t_{l+1}}) \right]^2 \\
&\leq E_{t_j} \left\{ \sum_{t_l \in [t_j, \bar{t}_j]} (\sigma \Delta B_{t_{l+1}})^2 \right\} \\
&\leq \sigma^2 T,
\end{align*}

where the next to last inequality follows from $g^2(y) \leq y^2$ for any $y$. Equations (A.17), (A.18), (A.20), and (A.21) imply $\lim_{x_{t_{\bar{t}_j}} \to 0} \Pr(\max_{t_k \in [t_j, \bar{t}_j+1]} x_{t_k} > \epsilon) = 0$. Since $\epsilon$ is an arbitrary number in $(0, \delta)$, it can be chosen arbitrarily close to 0. In turn, this implies that for any $t_k \in [t_j, \bar{t}_j+1]$, $x_{t_k}$ approaches 0 with probability 1 as $x_{t_j}$ becomes arbitrarily small. Accordingly, the lengths $\tau_k$ of all the inattention intervals between $t_j$ and $\bar{t}_{j+1}$ approach 0 with probability approaching 1. Using this result together with (8) and assumption (9a) implies that the discounted sum of the observation costs $\sum_{t_k \in [t_j, \bar{t}_{j+1}]} e^{-\rho(t_{k+1} - t_j)}(1 - \epsilon)$ becomes arbitrarily small as $x_{t_j}$ approaches 0.
\(\alpha k b(\tau_k) U(C(t_k, \tau_k))\) approaches infinity with probability approaching 1. \(^{33}\)

Accordingly, there cannot exist a \(\delta > 0\), such that \(y^* = 0\) for all \(x_{t_j} < \delta\).

This finding implies that for any \(\delta > 0\) (however small), there exists a \(z \in (0, \delta)\) such that if \(x_{t_j} = z\) on observation date \(t_j\), then optimal \(y^*(t_j) < 0\). This finding implies that for any \(\delta > 0\) (however small), there exists a \(z \in (0, \delta)\) such that if \(x_{t_j} = z\) on observation date \(t_j\), then optimal \(y^*(t_j) < 0\). Now take some \(x(n) \in \mathcal{X}\), such that if \(x_{t_j} = z\) on observation date \(t_j\), then optimal \(y^*(t_j) < 0\). Accordingly, it is possible to find a set of positive values \(X = \{x(1), x(2), \ldots\}\) with the properties that (i) \(\inf_{x \in \mathcal{X}} x = 0\) and (ii) if \(x_{t_j} \in \mathcal{X}\) on observation date \(t_j\), then \(y^*(t_j) < 0\).

As we established at the beginning of the proof, it is always optimal to set \(y^* < 0\) whenever \(x_{t_j} = 0\) on an observation date. Let \(x^0\) denote the optimal post-transfer value of \(x_{t_j}\). Since \(\frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v(x_{t_j}) = \frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v(x_{t_j}^+), \frac{S_{t_j}^+}{S_{t_j}} = (1 - \theta_S) + \\frac{(1-\theta_X)x_{t_j} - (1-\theta_S)x^+_{t_j}}{x^+_{t_j} + 1 - \psi_s}, x_{t_j} = x(n), \) and \(x_{t_j} = x(n^+),\) we have that

\begin{equation}
\frac{v(x(n))}{v(x(n^+))} = \left(1 - \theta_S + \frac{(1-\theta_X)x(n) - (1-\theta_S)x(n^+)}{x(n^+) + 1 - \psi_s}\right)^{1-\alpha}.
\end{equation}

As we established at the beginning of the proof, it is always optimal to set \(y^* < 0\) whenever \(x_{t_j} = 0\) on an observation date. Let \(x_0^n\) denote the optimal post-transfer value of \(x_{t_j}\) when \(x_{t_j} = 0\). Since the consumer can choose any \(y^* < 0\), optimality of \(x_{t_j}^+\) requires that

\begin{equation}
\frac{1}{1-\alpha} v(0) = \frac{1}{1-\alpha} v(x_0^n) \left(1 - \theta_S \frac{1 - \psi_s}{x_0^n + 1 - \psi_s}\right)^{1-\alpha} \geq \frac{1}{1-\alpha} v(x) \left(1 - \theta_S \frac{1 - \psi_s}{x + 1 - \psi_s}\right)^{1-\alpha}
\end{equation}

for any \(x > 0\). However, dividing (A.15) by (A.22) implies that

\begin{equation}
\frac{1}{1-\alpha} \frac{v(0)}{v(x(n))} = \frac{1}{1-\alpha} \frac{v(x_0^n)}{v(x(n^+))} \left(1 - \theta_S + \frac{x(n)(1 - \theta_X) - (1 - \theta_S)x(n^+)}{x(n^+) + 1 - \psi_s}\right)^{1-\alpha}.
\end{equation}

\(^{33}\)We note that it would be impossible to set \(c_t\) arbitrarily close to infinity for almost all values between \(t_j\) and \(t_{j+1}\), since this would violate the constraint \(X_{t_j} > \int_{t_j}^{t_{j+1}} e^{-r_L(s-t)} c_t\ ds\).
Since \( \inf_{x \in X} x = 0 \), it is possible to take the limit as \( x(n) \to 0 \) on both sides of (A.24). Using the supposition that \( \lim_{x(n) \to 0} \frac{1}{1-\alpha} v(x(n)) > \frac{1}{1-\alpha} v(0) \) and noting that \( \alpha > 1 \) gives

\[
(A.25) \quad 1 < \lim_{x(n) \to 0} \frac{1}{1-\alpha} v(0) = \frac{1}{1-\alpha} v(x(n)) = \frac{1}{1-\alpha} v(x(n))^{1-\alpha}.
\]

The fact that \( \alpha > 1 \) along with (A.25) implies that \( \frac{1}{1-\alpha} v(x(n))((1 - \theta_1) + 1 - \psi_s) \), which contradicts (A.23). Accordingly, \( \lim_{x(n) \to 0} \frac{1}{1-\alpha} v(x_n) = \frac{1}{1-\alpha} v(0) \).

The continuity of the function \( v \) in a positive neighborhood of zero, together with the theorem of the maximum, implies the continuity of \( \tilde{v} \) in a positive neighborhood of zero. Moreover, noting that \( y^* < 0 \) when \( x_{ij} = 0 \) implies that

\[
1 = \frac{1}{1-\alpha} v(x_{ij}) < \frac{1}{1-\alpha} v(0).
\]

**Proof of \( \omega_1 > 0 \).** Since Lemma 5 implies that \( \lim_{x(n) \to 0} \frac{1}{1-\alpha} \tilde{v}(x_{ij}) < \frac{1}{1-\alpha} v(0) \), there exists \( \bar{X} > 0 \) such that \( \frac{1}{1-\alpha} \tilde{v}(x) < \frac{1}{1-\alpha} v(x) \) \( \forall x \in [0, \bar{X}] \). Therefore, \( \omega_1 > 0 \).

**Proof of \( \pi_2 \geq \pi_1 \).** To prove that \( \pi_2 \geq \pi_1 \), suppose the contrary, that is, \( \pi_2 > \pi_1 \), and consider three points \((X_A, S_A)\), \((X_B, S_B)\), and \((X_C, S_C)\), where \( X_A = \pi_1 S_A \), \((X_B, S_B) = (\pi_1 S_A - (1 - \psi_s) z^*)\), \((S_A + z^*)\) where \( z^* = \frac{\pi_1 - \pi_2}{\pi_1 + 1 - \psi_s} S_B \), which implies \( X_B = \pi_2 S_B \), and \((X_C, S_C) = (\pi_2 S_B + (1 + \psi_b) z^**)\), where \( z^** = \frac{\pi_1 - \pi_2}{\pi_1 + 1 - \psi_s} S_B \), which implies \( X_C = \pi_1 S_A \). The definition of \( \pi_1 \) implies that \( V(X_A, S_A) \geq V(X_B, S_B) \) and the definition of \( \pi_2 \) implies that \( V(X_B, S_B) \geq V(X_C, S_C) \) so that \( V(X_A, S_A) \geq V(X_C, S_C) \). But \( S_C = S_B + z^** = S_B - \frac{\pi_1 - \pi_2}{\pi_1 + 1 - \psi_s} S_B = \frac{\pi_1 + 1 - \psi_s}{\pi_1 + 1 - \psi_s} S_B = \frac{\psi_s + \psi_b}{\pi_1 + 1 - \psi_s} S_B = \frac{\pi_1 - \pi_2}{\pi_1 + 1 - \psi_s} S_A = \frac{\psi_s + \psi_b}{\pi_1 + 1 - \psi_s} S_A \), which implies \( \pi_2 S_B > S_A \) since \( \psi_s + \psi_b > 0 \). Therefore, since \( X_C = \pi_1 S_A \), we have \( V(X_C, S_C) > V(X_A, S_A) \), which contradicts the earlier statement that \( V(X_A, S_A) \geq V(X_C, S_C) \).

**Proof of \( \omega_1 \leq \pi_1 \).** We prove this statement using a geometric argument to show that \( \omega_1 > \pi_1 \) leads to a contradiction. We consider three cases: \( \theta_S < \theta_X \), \( \theta_S > \theta_X \), and \( \theta_S = \theta_X \).

Suppose that \( \omega_1 > \pi_1 \) and consider the case in which \( \theta_S < \theta_X \), so that in Figure 2(a), the line through points \( B, C, \) and \( E \), which has slope \(-1 - \psi_s \), is steeper than the line through points \( C \) and \( D \), which has slope \(-1 - \psi_s \). Statement (ii)(c) of Proposition 1 implies that for values of \( x = \frac{X}{S} \) less than \( \omega_1 \), indifference curves of the value function are straight lines with slope \(-1 - \psi_s \). Therefore, \( V(B) = V(C) = V(E) \), where the notation \( V(j) \) indicates the value
FIGURE 2.—Proof of $\omega_1 \leq \pi_1$.

of the value function evaluated at point $j$. The definition of $\pi_1$ implies that $V(C) \geq V(D)$. Therefore, $V(E) \geq V(D)$, which contradicts strict monotonicity of the value function since both $X$ and $S$ are larger at point $D$ than at point $E$. Therefore, $\omega_1 \leq \pi_1$ if $\theta_S < \theta_X$.

Suppose that $\omega_1 > \pi_1$, and consider the case in which $\theta_S > \theta_X$, so that in Figure 2(b) the line through points $D$ and $E$, which has slope $-(1 - \psi_s) \frac{1 - \theta_S}{1 - \theta_X}$, is less steep than the line through points $C$ and $E$, which has slope $-(1 - \psi_s)$. Statement (ii)(c) of Proposition 1 implies that the line from point $D$ through point $E$ is an indifference curve and all points on this indifference curve are preferred to all points below and to the left of the indifference curve for which $x < \omega_1$. In particular, point $E$ is preferred to all points below point $E$ along the line through points $E$ and $C$. Since the value of $x$ at point $E$ is higher than $\pi_1$, the fact that the value function evaluated at point $E$ is greater than the value function, and hence greater than the restricted value function, evaluated at all
points below point $E$ with slope $-(1 - \psi_s)$ contradicts the definition of $\pi_1$. Therefore, $\omega_1 \leq \pi_1$ if $\theta_S > \theta_X$.

Suppose that $\omega_1 > \pi_1$ and consider the case in which $\theta_S = \theta_X$, so that in Figure 2(c), the slope of the line through points $C$ and $E$ is $-(1 - \psi_s)\frac{1 - \theta_S}{1 - \theta_X}$. Statement (ii)(c) of Proposition 1 implies that for values of $x \equiv \frac{X}{\bar{X}} < \omega_1$, indifference curves of the value function are straight lines with slope $-(1 - \psi_s)\frac{1 - \theta_S}{1 - \theta_X}$ so points $E$ and $C$ are on the same indifference curve. Indeed, point $E$ yields the same value of the value function as all points below point $E$ on the line through points $E$ and $C$. That is, for any point $J$ below point $E$ along the line through points $E$ and $C$ with $X \geq 0$, $V(E) = V(J)$. Since $x < \omega_1$ at point $J$, the definition of $\omega_1$ implies that $V(J) > \tilde{V}(J)$. Therefore, $V(E) = V(J) > \tilde{V}(J)$. Since $x > \pi_1$ at point $E$, the facts that for arbitrary point $J$ we have $V(E) = V(J)$ and $V(E) > \tilde{V}(J)$ contradict the definition of $\pi_1$. Therefore, $\omega_1 \leq \pi_1$ if $\theta_S = \theta_X$.

Putting together the cases in which $\theta_S < \theta_X$, $\theta_S > \theta_X$, and $\theta_S = \theta_X$, we have proved that $\omega_1 = \pi_1$.

To prove $\omega_2 \geq \pi_2$, use a set of arguments similar to the proof that $\omega_1 \leq \pi_1$.

{f Proof of $\omega_2 < \infty$.} We prove that $\omega_2$ is finite by showing that if the investment portfolio has zero value on an observation date, the consumer will use some of the liquid assets in the transactions account to buy assets for the investment portfolio. We use proof by contradiction. That is, suppose that time $0$ is an observation date, and that at this observation date, the transactions account has a balance $X_0 > 0$ and the investment portfolio has a zero balance so that $S_0 = 0$ and $x_0$ is infinite. Suppose that whenever the investment portfolio has zero value on an observation date, the consumer does not transfer any assets to the investment portfolio. Then the consumer will simply consume from the transactions account over the infinite future, never incurring any information costs or transactions costs. In this case, with the values of the variables denoted with asterisks, $c_{0+}^* = \frac{X_0}{h(\infty)} = \chi X_0$ and $c_t^* = \exp(-\frac{\rho + \lambda}{\alpha} t) c_{0+}^* = \chi X_t^*$, so $X_t^* = \exp(-\frac{\rho + \lambda}{\alpha} t) X_0$. Equation (16) implies that lifetime utility is

\begin{align}
(\text{A.26}) \quad U^* &= \frac{1}{1 - \alpha} [h(\infty)]^a X_0^{1-a} = \frac{1}{1 - \alpha} \chi^{-a} X_0^{1-a}.
\end{align}

Now consider an alternative feasible path that sets $c_t = c_t^*$ for $0 < t \leq T$ and at time $0^+$ transfers to the investment portfolio any liquid assets in the transactions account that will not be needed to finance consumption until time $T$. Under this alternative policy, the present value of consumption up to date $T$ is $h(T)c_{0+}^* = h(T)\chi X_0$, so

\begin{align}
(\text{A.27}) \quad X_{0+} &= h(T)\chi X_0.
\end{align}
The consumer uses \((1 - \theta_X - \chi h(T))X_0\) liquid assets to purchase assets in the investment portfolio. After paying the transactions cost,

\[
S_{0^+} = \frac{1 - \theta_X - \chi h(T)}{1 + \psi_b} X_0.
\]  
(A.28)

Suppose that the consumer invests the investment portfolio entirely in the riskless bond. At time \(T\), the transactions account has a zero balance, and the investment portfolio is worth \(S_T = \exp(r_f T) \frac{1 - \theta_X - \chi h(T)}{1 + \psi_b} X_0\). The consumer transfers the entire investment portfolio to the transactions account, so that after paying the transactions costs, the balance in the transactions account is

\[
X_{T^+} = (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} \exp(r_f T) \left[1 - \theta_X - \chi h(T)\right] X_0.
\]  
(A.29)

Define \(P \equiv \frac{X_{T^+}}{X_T^*}\) as the ratio of the transactions account balance at time \(T^+\) under this alternative policy to the transactions account balance under the initial policy. Use (A.29) and \(X_T^* = \exp\left(-\frac{\rho - r_L}{\alpha} T\right) X_0\), along with \(\chi \equiv \frac{\rho - (1 - \alpha)r_L}{\alpha}\), to obtain

\[
P \equiv \frac{X_{T^+}}{X_T^*} = (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} F(T),
\]  
(A.30)

where

\[
F(T) \equiv \exp\left[(r_f - r_L) T\right]\left[1 - \theta_X \exp(\chi T)\right].
\]  
(A.31)

Equation (A.30) and \(X_T^* = \exp\left(-\frac{\rho - r_L}{\alpha} T\right) X_0\) imply

\[
X_{T^+} = (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} F(T) \exp\left(-\frac{\rho - r_L}{\alpha} T\right) X_0.
\]  
(A.32)

Now choose \(T\) to maximize \(F(T)\). Differentiate \(F(T)\) and set the derivative equal to zero to obtain

\[
\exp(-\chi \hat{T}) = \left(1 + \frac{\chi}{r_f - r_L}\right) \theta_X < 1,
\]  
(A.33)
where \( \hat{T} \) is the optimal value of \( T \), and the inequality follows from the assumption that \( \theta_X < \overline{\theta_X} \) and the fact that \( \frac{x}{r_f - r_L} > 0 \). Use (A.33) to evaluate \( F(\hat{T}) \) to obtain

\[
F(\hat{T}) = \left(1 + \frac{x}{r_f - r_L}\right)^{-\left(\frac{r_f - r_L}{x}\right)} \frac{x}{r_f - r_L} \overline{\theta_X}^{\frac{(r_f - r_L)}{x}}.
\]

Use (A.33) and the definition of \( h(T) \) to obtain

\[
\chi h(\hat{T}) = 1 - \left(1 + \frac{x}{r_f - r_L}\right) \theta_X.
\]

The present value of lifetime utility under the alternative plan is

\[
U = \left[1 - (1 - \alpha)\kappa b(\hat{T})\right] \frac{1}{1 - \alpha} \left[h(\hat{T})\right]^\alpha [X_{0^+}]^{1 - \alpha} + \exp(-\rho \hat{T}) \frac{1}{1 - \alpha} \left[h(\infty)\right]^\alpha [X_{\hat{T}^+}]^{1 - \alpha}.
\]

Substitute (A.27) and (A.32) into (A.36), and use the fact that \( h(\infty) = \frac{1}{\chi} \) to obtain

\[
U = \left[1 - (1 - \alpha)\kappa b(\hat{T})\right] \frac{1}{1 - \alpha} h(\hat{T}) [\chi X_0]^{1 - \alpha}
\]

\[
+ \exp(-\rho \hat{T}) \frac{1}{1 - \alpha} \chi^{-\alpha}
\]

\[
\times \left[(1 - \theta_S) \frac{1}{1 + \psi_b} F(\hat{T}) \exp\left(-\frac{\rho - r_L}{\alpha} \hat{T}\right) X_0\right]^{1 - \alpha}.
\]

Now divide the utility under the alternative plan in (A.37) by the utility under the initial plan in (A.26), and use the definition of \( \chi \) and the fact that \( \chi h(T) = 1 - \exp(-\chi T) \) to obtain

\[
\frac{U}{U^*} = \left[1 - (1 - \alpha)\kappa b(\hat{T})\right] \frac{1}{1 - \alpha} \left[1 - \exp(-\chi \hat{T})\right]
\]

\[
+ \exp(-\chi \hat{T}) \left[(1 - \theta_S) \frac{1}{1 + \psi_b} F(\hat{T})\right]^{1 - \alpha}.
\]

\[34\text{From (27), } \overline{\theta_X} \equiv [(1 - \theta_S) \frac{1 - \phi_b}{1 + \phi_b} \frac{x}{r_f - r_L + \chi}]^{\frac{r_f - r_L}{x}}, \text{ which implies } (1 + \frac{x}{r_f - r_L}) \theta_X = [(1 - \theta_S) \frac{1 - \phi_b}{1 + \phi_b} \frac{x}{r_f - r_L + \chi}]^{\frac{r_f - r_L}{x}} < 1 \text{ because } (1 - \theta_S) \frac{1 - \phi_b}{1 + \phi_b} < 1, \frac{x}{r_f - r_L} > 0, \text{ and hence } \frac{x}{r_f - r_L + \chi} < 1.\]
Then rearrange to obtain

$$(A.39) \quad \frac{U}{U^*} = 1 + \left( (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} F(\hat{T}) \right)^{1-\alpha} \left[ 1 + (1 - \alpha) \kappa b(\hat{T})(\exp(\chi \hat{T}) - 1) \right] \exp(-\chi \hat{T}).$$

If $\alpha < 1$, utility under the alternative plan, $U$, will exceed $U^*$ if $\frac{U}{U^*} > 1$; if $\alpha > 1$, utility under the alternative plan, $U$, will exceed $U^*$ if $\frac{U}{U^*} < 1$. A sufficient condition for $U$ to exceed $U^*$, regardless of whether $\alpha$ is less than or greater than 1, is

$$(A.40) \quad \left[ (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} \right] F(\hat{T}) > \left[ 1 + (1 - \alpha) \kappa b(\hat{T})(\exp(\chi \hat{T}) - 1) \right]^{1/(1-\alpha)}.$$

Multiply both sides of $(A.34)$ by $\left[ (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} \right]$ to obtain

$$(A.41) \quad \left[ (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} \right] F(\hat{T}) = \left[ (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} r_f - r_L + \chi \right] \left( \frac{r_f - r_L}{r_f - r_L + \chi} \right)^{(r_f - r_L)/\chi} \theta_X^{-(r_f - r_L)/\chi}.$$

Use the definition of $\bar{\theta}_X$ in $(27)$ and the assumption that $\theta_X < \bar{\theta}_X$ to write $(A.41)$ as

$$(A.42) \quad \left[ (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} \right] F(\hat{T}) = \left( \frac{\theta_X}{\bar{\theta}_X} \right)^{-(r_f - r_L)/\chi} > 1.$$

Substitute $(A.42)$ into $(A.40)$ to obtain the sufficient condition for $U$ to exceed $U^*$:

$$(A.43) \quad \left( \frac{\theta_X}{\bar{\theta}_X} \right)^{-(r_f - r_L)/\chi} > \left[ 1 + (1 - \alpha) \kappa b(\hat{T})(\exp(\chi \hat{T}) - 1) \right]^{1/(1-\alpha)}.$$

$^{35}$If $\alpha > 1$, then $\kappa$ must be less than $\hat{\kappa} \equiv \frac{1}{\alpha - 1} \frac{1}{b(\hat{T})(\exp(\chi \hat{T}) - 1)}$ so that the right hand side of $(A.40)$ is defined. Since we assume that $\kappa < \bar{\kappa}$ in $(28)$ and $\hat{\kappa} = 1 - \left( \frac{\theta_X}{\bar{\theta}_X} \right)^{-(r_f - r_L)/\chi} [1/(1-\alpha)]^{-1} \bar{\kappa} > \bar{\kappa}$, we have $\kappa < \hat{\kappa}$. 
Regardless of whether $\alpha$ is larger or smaller than 1, the condition in (A.43) is satisfied if $\theta_X < \overline{\theta_X}$ and $\kappa < \overline{\kappa}$, where

$$
\overline{\kappa} \equiv \left( \frac{\theta_X}{\overline{\theta_X}} \right)^{-(r_f-r_L)/\chi(1-\alpha)} - 1
$$

(A.44)

Since $\theta_X < \overline{\theta_X}$ and $\kappa < \overline{\kappa}$, the original plan, in which the consumer does not buy any assets in the investment portfolio, is not optimal.

The proof of statement (i) is now complete.

Proof of statement (iii). The proof of statement (iii) follows the proof of statement (ii).

The proof of Proposition 1 is now complete. Q.E.D.

To prepare for the proof of Proposition 2, we state and prove the following lemma.

**Lemma 6:** If $C(t_j, \tau_j) \leq X_{t_j}$, then, for sufficiently small $\theta_S \geq 0$, $y^*(t_j) = 0$.

**Proof:** Consider some path for $c_t$, $X_t$, $S_t$, $y^*(t)$, and $y^b(t)$, $t \in [t_j, t_{j+1}]$, and let $c^0_t$, $X^0_t$, $S^0_t$, $y^{t,0}(t)$, and $y^{b,0}(t)$ denote the values of these variables along this path. Suppose that $C(t_j, \tau_j) \leq X^0_{t_j}$ (and contrary to what is to be proved) that $y^{t,0}(t_j) < 0$, so that Lemma 4 implies that $y^{b,0}(t_j) = 0$. Consider a deviation from $y^{t,0}(t_j) < 0$ that reduces $-y^*(t_j)$ to zero so that $X_{t_j}^+$ changes by $y^{t,0}(t_j) - \psi_s$ and $S_{t_j}^+$ increases by $-y^{b,0}(t_j) + \theta_S S_{t_j}^0$.

Since under the deviation, $X_{t_j}^+ = X_{t_j} = X^0_{t_j} \geq C(t_j, \tau_j)$, it is feasible to maintain $c_t = c^0_t$ for $t_j \leq t \leq t_{j+1}$, and we suppose that the consumer does so. Also suppose that the consumer invests the additional assets in the investment portfolio in the riskless bond, which pays a rate of return $r_f$. Thus, at the next observation date $t_{j+1}$, the transactions account will have changed by $\Delta X = [y^{t,0}(t_j) - \psi_s] + \theta_X X^0_{t_j} e^{r_f \tau_j}$ and the investment portfolio will have increased by $\Delta S = [-y^{b,0}(t_j) + \theta_S S^0_{t_j}] e^{r_f \tau_j} > 0$, relative to the original path. The deviation at time $t_{j+1}$ depends on the direction of the transfer along the original path at time $t_{j+1}$.

(i) If $y^{t,0}(t_{j+1}) < 0$, increase $-y^*(t_{j+1})$ by $(1 - \theta_S) \Delta S$, which makes the value of the investment portfolio under the deviation equal to the value under the original path. Compared to the original path, the transactions account at time $t_{j+1}^+$ changes by $\xi = (1 - \theta_X) \Delta X + (1 - \psi_s) (1 - \theta_S) \Delta S$. Using the definitions of $\Delta S$ and $\Delta X$ implies

$$
\xi = [y^{t,0}(t_j) - \psi_s] [(1 - \theta_S) e^{r_f \tau_j} - (1 - \theta_X) e^{r_f \tau_j}] + (1 - \theta_X) \theta_X X^0_{t_j} e^{r_f \tau_j} + (1 - \psi_s) (1 - \theta_S) \theta_S S^0_{t_j} e^{r_f \tau_j},
$$
which in turn implies that \( \lim_{\theta_S \to 0} \xi = [-y^{s,0}(t_j)](1 - \psi_s)[e^{\ell_j} - (1 - \theta_X)Ye^{\ell_j}] + (1 - \theta_X)\theta_X Y_0 e^{\ell_j} > 0. \)

(ii) If the consumer would not have transferred assets in either direction between the investment portfolio and the transactions account at time \( t_{j+1} \), then \( \omega_1 \leq x_{t_{j+1}}^0 \leq \omega_2 \). We begin by showing that \( \frac{S_{t_{j+1}}^0}{S_{t_j}^0} = \frac{x_{t_{j+1}}^0}{x_{t_j}^0} \left( \frac{x_{t_{j+1}}^0}{x_{t_j}^0} \right) = \frac{1}{x_{t_j}^0} \left( \frac{x_{t_{j+1}}^0}{x_{t_j}^0} \right) x_{t_j}^0 \) is bounded above by a quantity that is finite and \( F_t \)-measurable. First, the fact that \( \omega_1 \leq x_{t_{j+1}}^0 \leq \omega_2 \) implies that \( \frac{1}{x_{t_{j+1}}^0} \leq \frac{1}{x_{t_j}^0} \), which is finite since \( \omega_1 > 0 \). Second, \( X_{t_{j+1}}^0 \equiv [(1 - \theta_X)X_{t_j}^0 - (1 - \psi_s)\theta_s y^{s,0}(t_j)]e^{\ell_j} - C(t_j, \tau_j)Y_{t_{j+1}} \) so that \( \frac{x_{t_{j+1}}^0}{x_{t_j}^0} = [(1 - \theta_X) - (1 - \psi_s)\theta_s y^{s,0}(t_j)]e^{\ell_j} - C(t_j, \tau_j)Y_{t_{j+1}} \), which is finite and \( F_t \)-measurable. Third, since \(-y^{s,0}(t_j) > 0\), we know that \( S_{t_j}^0 \geq \frac{1}{1 - \psi_s} [-y^{s,0}(t_j)] > 0 \), which implies that \( x_{t_j}^0 \equiv \frac{x_{t_{j+1}}^0}{S_{t_j}^0} \) is finite; it is also \( F_t \)-measurable. Therefore, \( \frac{S_{t_{j+1}}^0}{S_{t_j}^0} = \frac{1}{x_{t_j}^0} \left( \frac{x_{t_{j+1}}^0}{x_{t_j}^0} \right) x_{t_j}^0 \) is bounded above by \( \frac{1}{x_{t_j}^0} \left( \frac{x_{t_{j+1}}^0}{x_{t_j}^0} \right) x_{t_j}^0 \), which is the product of three quantities that are finite and \( F_t \)-measurable.

For sufficiently small \( \theta_S \geq 0 \), the alternative path sets \( y^s(t_{j+1}) \) equal to \(-y^{s,0}(t_j)\Delta^S + \theta_X S_{t_{j+1}}^0 = -S_{t_j}^0 \left( (1 - \theta_S)[-\frac{y^{s,0}(t_j)}{S_{t_j}^0}]e^{\ell_j} + \theta_S [(1 - \theta_S)Ye^{\ell_j} - \frac{S_{t_{j+1}}^0}{S_{t_j}^0}] \right) \), which is negative because \(-\frac{y^{s,0}(t_j)}{S_{t_j}^0} > 0 \) and \( \frac{S_{t_{j+1}}^0}{S_{t_j}^0} \) is bounded above by an \( F_t \)-measurable quantity. With \( y^s(t_{j+1}) = -(1 - \theta_S)\Delta^S + \theta_X S_{t_{j+1}}^0 \), the value of the investment portfolio on the alternative path equals the value on the hypothesized optimal path. Compared to the hypothesized optimal path, the transactions account at time \( t_{j+1} \) changes by \( \xi_2 \equiv (1 - \theta_X)\Delta^X - \theta_X X_{t_{j+1}}^0 - (1 - \psi_s)\theta_s y^{s,0}(t_j)\Delta^S - \theta_X S_{t_{j+1}}^0 \). Use the definitions of \( \Delta^X \) and \( \Delta^S \) to obtain \( \xi_2 = (1 - \theta_X)\left( (1 - \theta_S)[-\frac{y^{s,0}(t_j)}{S_{t_j}^0}]e^{\ell_j} - (1 - \theta_X)Ye^{\ell_j} \right) + \theta_X \left( (1 - \theta_X)X_{t_j}^0 e^{\ell_j} - X_{t_{j+1}}^0 \right) + (1 - \psi_s)\left( (1 - \theta_S)\theta_S X_{t_j}^0 e^{\ell_j} - (1 - \theta_S)\theta_S S_{t_{j+1}}^0 \right). \)

Now use the fact that \( X_{t_{j+1}}^0 = [(1 - \theta_X)X_{t_j}^0 - (1 - \psi_s)\theta_s y^{s,0}(t_j)]e^{\ell_j} - C(t_j, \tau_j) \times e^{\ell_j} \) to obtain \( (1 - \theta_X)X_{t_j}^0 e^{\ell_j} - X_{t_{j+1}}^0 = (1 - \theta_S)\theta_S y^{s,0}(t_j) e^{\ell_j} + C(t_j, \tau_j) e^{\ell_j} \), substitute this expression into the expression for \( \xi_2 \), and factor out \( S_{t_j}^0 \) to obtain

\[
\xi_2 = S_{t_j}^0 \left\{ (1 - \psi_s) \left[ -\frac{y^{s,0}(t_j)}{S_{t_j}^0} \right] \left( (1 - \theta_S)Ye^{\ell_j} - e^{\ell_j} \right) + \theta_X \frac{C(t_j, \tau_j)}{S_{t_j}^0} e^{\ell_j} + (1 - \psi_s)(1 - \theta_S)\theta_S e^{\ell_j} - (1 - \theta_S)\theta_S S_{t_{j+1}}^0 \right\}.
\]
Since \( \frac{S^0_{t+j}}{S^j_t} \) is bounded above by a quantity that is \( F_j \)-measurable and finite, \[
\lim_{\theta_S \to 0} \xi_2 = S^0_t \left( (1 - \psi_s) - \frac{-\psi_s(t_j)}{S^j_t} \left[ e^{v_j(t_j)} - e^{v_L(t_j)} \right] + \theta_x C(t_j) e^{v_L(t_j)} \right) > 0.
\]

(iii) If \( y^b,0(t_{j+1}) > 0 \), the deviation depends on whether \( (1 - \theta_S)\Delta^S \) is larger or smaller than \( y^b,0(t_{j+1}) \). (a) If \( (1 - \theta_S)\Delta^S > y^b,0(t_{j+1}) \), set \( y^*(t_{j+1}) = -(1 - \theta_S)\Delta^S + y^b,0(t_{j+1}) < 0 \) and set \( y^b(t_{j+1}) = 0 \) so that the value of the investment portfolio at time \( t_{j+1}^+ \) is the same for the deviation and for the original path. Compared to the original path, the transactions account at time \( t_{j+1}^+ \) changes by \( \xi_3 \equiv (1 - \theta_S)\Delta^X + (1 - \psi_s)[(1 - \theta_S)\Delta^S - y^b,0(t_{j+1})] + (1 + \psi_b)(y^b,0(t_{j+1}) = (1 - \theta_S)\Delta^X + (1 - \psi_s)[(1 - \theta_S)\Delta^S + (\psi_s + \psi_b)y^b,0(t_{j+1})]. \) Using the definitions of \( \Delta^X \) and \( \Delta^S \), rewrite \( \xi_3 \) as \( \xi_3 = (1 - \psi_s)[-y^0,0(t_j)][(1 - \theta_S)\Delta^L(t_j) - (1 - \theta_X)\Delta^L(t_j)] + (1 - \theta_X)\theta_x X^0_{t_j} e^{v_L(t_j)} + (1 - \theta_S)\theta_S S^0_{t_j} e^{v_L(t_j)} + (\psi_s + \psi_b)y^b,0(t_{j+1}) \). Therefore,
\[
\lim_{\theta_S \to 0} \xi_3 = (1 - \psi_s)[-y^0,0(t_j)][(1 - \theta_S)\Delta^L(t_j) - (1 - \theta_X)\Delta^L(t_j)] + (1 - \theta_S)\theta_x X^0_{t_j} e^{v_L(t_j)} + (1 - \theta_S)\theta_S S^0_{t_j} e^{v_L(t_j)} + (\psi_s + \psi_b)y^b,0(t_{j+1}) > 0.
\]

(b) If \( (1 - \theta_S)\Delta^S < y^b,0(t_{j+1}) \), set \( y^b(t_{j+1}) = y^b,0(t_{j+1}) - (1 - \theta_S)\Delta^S \) and \( y^*(t_{j+1}) = y^b(t_{j+1}) = 0 \) so that the value of the investment portfolio at time \( t_{j+1}^+ \) is the same for the deviation and for the original path. Compared to the original path, the transactions account at time \( t_{j+1}^+ \) changes by \( \xi_4 \equiv (1 - \theta_S)\Delta^X + (1 + \psi_b)(1 - \theta_S)\Delta^S \). Using the definitions of \( \Delta^X \) and \( \Delta^S \), rewrite \( \xi_4 \) as \( \xi_4 = [y^0,0(t_j)][(1 + \psi_b)(1 - \theta_S)\Delta^L(t_j) - (1 - \theta_X)\Delta^L(t_j)] + (1 - \theta_X)\theta_x X^0_{t_j} e^{v_L(t_j)} + (1 + \psi_b)(1 - \theta_S)\theta_S S^0_{t_j} e^{v_L(t_j)} \). Therefore, \( \lim_{\theta_S \to 0} \xi_4 = [y^0,0(t_j)][(1 + \psi_b)\Delta^L(t_j)] + (1 - \theta_X)\theta_x X^0_{t_j} e^{v_L(t_j)} > 0 \).

(c) If \( (1 - \theta_S)\Delta^S = y^b,0(t_{j+1}) \), set \( y^b(t_{j+1}) = y^*(t_{j+1}) = 0 \). Compared to the original path, the deviation increases \( S^0_{t+j} \) by \( \Delta^S + \theta_S S^0_{t+j} - y^b,0(t_{j+1}) = \theta_S S^0_{t+j} + \theta_S \Delta^S = \theta_S S^j_t > 0 \). Compared to the original path, the transactions account at time \( t_{j+1}^+ \) changes by \( \xi_5 \equiv \Delta^X + \theta_x X^0_{t_{j+1}} + (1 + \psi_b)y^b,0(t_{j+1}) = \Delta^X + \theta_x X^0_{t_{j+1}} + (1 + \psi_b)(1 - \theta_S)\Delta^S \). Using the definitions of \( \Delta^X \) and \( \Delta^S \), rewrite \( \xi_5 \) as \( \xi_5 = -y^0,0(t_j)][(1 + \psi_b)(1 - \theta_S)\Delta^L(t_j) - (1 - \theta_X)\Delta^L(t_j)] + \theta_x X^0_{t_{j+1}} e^{v_L(t_j)} + \theta_x X^0_{t_{j+1}} + (1 + \psi_b)(1 - \theta_S)\theta_S S^0_{t_{j+1}} e^{v_L(t_j)} \). Therefore, \( \lim_{\theta_S \to 0} \xi_5 = -y^0,0(t_j)][(1 + \psi_b)\Delta^L(t_j)] + \theta_x X^0_{t_{j+1}} e^{v_L(t_j)} + (1 - \theta_X)\theta_x X^0_{t_{j+1}} > 0 \).

To summarize, we have shown that along all possible branches, the deviation leads to an unchanged or increased value of \( S_{t+j}^j \) and an increased value of \( X^0_{t_{j+1}} \) (because \( \xi_i, i = 1, 2, 3, 4, 5 \), have positive limits for \( \theta_S \) approaching 0) for sufficiently small \( \theta_S \geq 0 \). Therefore, the hypothesized optimal path could not have been optimal. Therefore, the optimal value of \( y^*(t_j) = 0 \). Q.E.D.
PROOF OF PROPOSITION 2: Consider some path for \( c_t, X_t, S_t, y^v(t), \) and \( y^b(t), t \in [t_j, t_{j+1}] \), and let \( c^0_t, X^0_t, S^0_t, y^{v,0}(t), \) and \( y^{b,0}(t) \) denote the values of these variables along this path. Suppose that \( x_{ij} < \omega_1 \) and (contrary to what is to be proved) \( X^0_{ij+1} > 0 \). Since \( \kappa > 0 \), the consumer will not continuously observe the value of the investment portfolio. That is, \( \tau_j > 0 \).

\( \tau_j \) differs between the investment portfolio and the transactions account at time \( t_j \) compared to the original path. Compared to the original path, this deviation will change the consumer sets \( x_{ij} < \omega_1 \) at time \( t_j \). The deviation from the original path at time \( t_j \) will have a zero balance at time \( t_j \).

First, consider the case in which \( y^{v,0}(t_j) < 0 \). Since \( X^0_{t_j} < \omega \), the consumer reduces \( x_{ij} < \omega_1 \) and invests this amount in the riskless bond in the investment portfolio. With this deviation, the value of the investment portfolio at time \( t_j \) will exceed its value under the original policy by \( \left( C(t_j, \omega_1) - C(t_j, \omega) \right) \), which is positive for sufficiently small \( \omega_1 \). Therefore, the deviation dominates the original path in this case when \( \omega_1 \) is sufficiently small.

Consider a deviation in which the consumer reduces \( y^v(t_j) \) by \( \frac{X^0_{ij+1} - C(t_j, \omega_1)}{1 - \psi_S} \) and invests this amount in the riskless bond in the investment portfolio. With this deviation, the value of the investment portfolio at time \( t_{j+1} \) will exceed its value under the original policy by \( \frac{X^0_{ij+1} e^{(r_f - r_L) \tau_j}}{1 - \psi_S} \) and the transactions account will have a zero balance at time \( t_{j+1} \).

The deviation from the original path at time \( t_{j+1} \) depends on whether, and in which direction, the consumer would transfer assets between the transactions account and the investment portfolio under the original path at that time. First, consider the case in which \( y^{v,0}(t_{j+1}) < 0 \) so that the consumer transfers assets from the investment portfolio to the transactions account at time \( t_{j+1} \). In this case, the consumer can increase \( -y^v(t_{j+1}) \) by \( (1 - \theta_S) X^0_{ij+1} e^{(r_f - r_L) \tau_j} \), which leaves the value of the investment portfolio at time \( t_{j+1} \) equal to its value on the original path. Compared to the original path, this deviation will change the balance in the transactions account at time \( t_{j+1} \) by \( -((1 - \theta_X) X^0_{ij+1} - (1 - \theta_S) X^0_{ij+1} e^{(r_f - r_L) \tau_j}) = [(1 - \theta_S) e^{r_f - r_L \tau_j} - (1 - \theta_X)] X^0_{ij+1} \), which is positive for sufficiently small \( \theta_S \geq 0 \). Therefore, the deviation dominates the original path in this case when \( \theta_S \geq 0 \) is sufficiently small.

Second, consider the case in which the consumer would not make any transfers between the investment portfolio and the transactions account at time \( t_{j+1} \) under the original policy. Since the consumer does not make any transfers at time \( t_{j+1} \), if the original path were optimal, Proposition 1 implies that

\[ 0 < \omega_1 \leq \frac{X^0_{ij+1}}{S^0_{ij+1}} \leq \omega_2, \]

which implies that \( S^0_{ij+1} \leq \frac{X^0_{ij+1}}{\omega_1} \). In this case, under the deviation, the consumer sets \( -y^v(t_{j+1}) = (1 - \theta_S) X^0_{ij+1} e^{(r_f - r_L) \tau_j} - \theta_S S^0_{ij+1} \). There-
fore, 

\[-y^s(t_{j+1}) \geq \left[\frac{1 - \theta_s}{1 - \theta_s} e^{(r_f - r_L)t_j} - \frac{\theta_s}{\omega_1}\right]X^0_{t_{j+1}},\]

which is positive for sufficiently small \(\theta_s \geq 0\). (Proposition 1 states that \(\omega_1 > 0\) for all admissible values of \(\theta_s \geq 0\), including \(\theta_s = 0\), so that \(\lim_{\theta_s \to 0} \frac{\theta_s}{\omega_1} = 0\).) With this transfer, the value of assets in the investment portfolio at time \(t^+_{j+1}\) will be the same under the deviation as under the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time \(t^+_{j+1}\) by 

\[-X^0_{t_{j+1}} -(1 - \psi_s)y^s(t_{j+1}) = -X^0_{t_{j+1}} + (1 - \theta_S)X^0_{t_{j+1}} e^{(r_f - r_L)t_j} - (1 - \psi_s)\theta_S S^0_{t_{j+1}} = [(1 - \theta_S) e^{(r_f - r_L)t_j} - 1]X^0_{t_{j+1}} - (1 - \psi_s)\theta_S S^0_{t_{j+1}} \geq ((1 - \theta_S) e^{(r_f - r_L)t_j} - 1 - (1 - \psi_s)\theta_S)X^0_{t_{j+1}},\]

which is positive for sufficiently small \(\theta_S \geq 0\). Therefore, the deviation dominates the original path in this case when \(\theta_S \geq 0\) is sufficiently small.

Third, consider the case in which \(y^{b,0}(t_{j+1}) > 0\) so that the consumer transfers assets from the transactions account to the investment portfolio at time \(t_{j+1}\). If \(y^{b,0}(t_{j+1}) > (1 - \theta_S)X^0_{t_{j+1}} e^{(r_f - r_L)t_j}\), the deviation reduces \(y^b(t_{j+1})\) by 

\[(1 - \theta_S)X^0_{t_{j+1}} e^{(r_f - r_L)t_j}\]

and sets \(y^s(t_{j+1}) = 0\), which will leave the value of the investment portfolio at time \(t^+_{j+1}\) under the deviation equal to its value on the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time \(t^+_{j+1}\) by 

\[-(1 - \psi_b)(1 - \theta_S)X^0_{t_{j+1}} e^{(r_f - r_L)t_j} = [(1 - \theta_S)1+\psi_b e^{(r_f - r_L)t_j} - (1 - \theta_X)]X^0_{t_{j+1}},\]

which is positive for sufficiently small \(\theta_S \geq 0\). Therefore, the deviation dominates the original path in this case when \(\theta_S \geq 0\) is sufficiently small. If \(y^{b,0}(t_{j+1}) < (1 - \theta_S)X^0_{t_{j+1}} e^{(r_f - r_L)t_j}\), the deviation sets \(y^b(t_{j+1}) = 0\) and sets \(-y^s(t_{j+1}) = (1 - \theta_S)X^0_{t_{j+1}} e^{(r_f - r_L)t_j} - y^{b,0}(t_{j+1}) > 0\), which will leave the value of the investment portfolio at time \(t^+_{j+1}\) under the deviation equal to its value on the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time \(t^+_{j+1}\) by 

\[-(1 - \theta_X)X^0_{t_{j+1}} + (1 + \psi_b) y^{b,0}(t_{j+1}) + (1 - \psi_b)(1 - \theta_S)1+\psi_b e^{(r_f - r_L)t_j} - y^{b,0}(t_{j+1}) = [(1 - \theta_S) e^{(r_f - r_L)t_j} - (1 - \theta_X)]X^0_{t_{j+1}} + (\psi_b + \psi_s) y^{b,0}(t_{j+1}),\]

which is positive for sufficiently small \(\theta_S \geq 0\). Therefore, the deviation dominates the original path in this case when \(\theta_S\) is sufficiently small. Finally, if \(y^{b,0}(t_{j+1}) = (1 - \theta_S)X^0_{t_{j+1}} e^{(r_f - r_L)t_j}\), the deviation sets \(y^s(t_{j+1}) = y^b(t_{j+1}) = 0\). Compared to the original path, the deviation changes \(S^0_{t_{j+1}}\) by 

\[X^0_{t_{j+1}} e^{(r_f - r_L)t_j} + \theta_S S^0_{t_{j+1}} - y^{b,0}(t_{j+1}) = X^0_{t_{j+1}} e^{(r_f - r_L)t_j} + \theta_S S^0_{t_{j+1}} - (1 - \theta_S)X^0_{t_{j+1}} e^{(r_f - r_L)t_j} = \theta_S S^0_{t_{j+1}} + \theta_S X^0_{t_{j+1}} e^{(r_f - r_L)t_j} > 0.\]

Compared to the original
path, the deviation changes $X_{j+1}^+$ by $-X^0_{j+1} + \theta_X X^0_{j+1} + (1 + \psi_b)(1 + \theta_S) e^{(\gamma - r_L)\tau_j} = -(1 - \theta_X) X^0_{j+1} + (1 + \psi_b) e^{(\gamma - r_L)\tau_j} = [(1 - \theta_S) + \phi_b e^{(\gamma - r_L)\tau_j} - (1 - \theta_X)] X^0_{j+1}$, which is positive for sufficiently small $\theta_S > 0$.

We have shown that the deviation path dominates the original path; hence it cannot be optimal for $X_{j+1}$ to be positive. Since the optimal value of $X_{j+1} = 0$, we have $x_{j+1} = 0 < \omega_1$, which implies $x_{j+2} = 0$ and so on, ad infinitum. *Q.E.D.*

**Proof of Lemma 2:** Lemma 11 states that the optimal value of $\phi_j$ is positive. Since $\tau_j > 0$ as a consequence of the information cost, there exists some $\delta > 0$ such that between any two consecutive observation dates, $t_j$ and $t_{j+1} = t_j + \tau_j$, $Pr\{e^{-r_L}\tau_j R(t_j, \tau_j) > \omega_1\} \geq \delta$. Therefore, since $x_{j+1} = x^0_{j+1} e^{(\gamma - r_L)\tau_j} \leq \frac{\omega_2}{e^{r_L} R(t_j, \tau_j)}$ (where the final inequality follows from Corollary 1), then $Pr\{x_{j+1} < \omega_1\} \geq \delta$. Let $t_k \geq t_j$ be the first observation date at which $x_k < \omega_1$. Then by Williams (1991, p. 233), $Pr\{t_k < \infty\} = 1$ and $E\{t_k\} < \infty$. *Q.E.D.*

**Proof of Proposition 3:** Lemma 2 states that eventually, with probability 1, $x_{t_j} < \omega_1$ on an observation date. Proposition 2 implies that when this event occurs, $x_{j+1} = 0$ on the next observation date and on all subsequent observation dates, provided that $\theta_S \geq 0$ is sufficiently small. Since the optimal value of $\tau_j$ is simply a function of $x_{t_j}$, $\tau_j$ will be constant when $x_{t_j}$ becomes constant. *Q.E.D.*

**Proposition 5:** Let $T^s(t_j, t) \equiv \int_{t_j}^t dY^s(t)$ denote the cumulative transfer process from the investment portfolio to the transactions account from time $t_j$ to time $t \in [t_j, t_{j+1}]$, and let $T^b(t_j, t) \equiv \int_{t_j}^t dY^b(t)$ denote the cumulative transfer process from the transactions account to the investment portfolio from time $t_j$ to time $t \in [t_j, t_{j+1}]$. We define automatic transfers as $F_t$-measurable functions $T^s(t_j, t)$ and $T^b(t_j, t)$ that satisfy three requirements: (i) $T^s(t_j, t)$ is nonincreasing in $t$, (ii) $T^b(t_j, t)$ is nondecreasing in $t$, and (iii) given $T^s(t_j, t)$ and $T^b(t_j, t)$, along with the $F_t$-measurable path of consumption from $t_j$ to $t_{j+1}$, $X_t \geq 0$ and $S_t \geq 0$ for any path of $P_t$. If the consumer can utilize automatic transfers and $\theta_X = \theta_S = 0$, then the stochastic process for $x_{t_j}$ is eventually, with probability 1, absorbed at zero and the time between consecutive observations is constant.

To prepare for the proof of Proposition 5, we first introduce some notation and then prove three ancillary lemmas.

Define $F^s(t, z; r)$ to be the (negative of the) future value, as of time $z$, of transfers from the investment portfolio to the transactions account from time $t$ until, but not including, time $z$. The future value is computed using the discount rate $r$. Formally, $F^s(t, z; r) \equiv \lim_{\delta \to z} \int_t^Z e^{(x-v)} dY^s(v)$, where
\(dY_s(v) \leq 0\) denotes the increments of the cumulative transfer from the investment portfolio to the transactions account (so that \(F^s(t, z; 0) = T^s(t, z)\)). We use the notation \(F^s(t, \tilde{t}^+; r)\) to capture potential lump-sum transfers at time \(t\) \((F^s(t, \tilde{t}^+; r) = \lim_{\tilde{t} \to t} F^s(\tilde{t}, z; r)\), which equals \(y^s(t)\) using the notation in the baseline version of the model with transfers confined to observation dates). Similarly, \(F^b(t, z, r)\) is the future value, as of time \(z\), of transfers from the transactions account to the investment portfolio from time \(t\) until, but not including, time \(z\) (so that \(F^b(t, z; 0) = T^b(t, z)\)). The notation \(F^b(t, \tilde{t}^+; r)\) captures lump-sum transfers from the transactions account to the investment account at time \(t\). Finally, \(FV(t, z) = \int_{\tilde{t}}^{\tilde{t}^+} c_v e^{\rho(z-v)} \text{d}v\) is the future value, as of time \(z\), of consumption from time \(t\) to \(z\), compounded at the rate \(\rho\).

We next prove the three ancillary lemmas.

**Lemma 7: Along an optimal path that includes the possibility of automatic transfers, if \(\theta_X = \theta_S = 0\) and if \(X_t > 0\) for all \(t \in [t_j, t_{j+1}]\), then \(F^s(t_j, t_{j+1}, r_L) = 0\).**

**Proof:** Assume otherwise, that is, suppose that for an optimal path, \(X^0 > 0\) for all \(t \in [t_j, t_{j+1}]\) and yet \(F^{x,0}(t_j, t_{j+1}, r_L) < 0\). Now consider the following deviation: Do not transfer any assets from the investment portfolio to the transactions account until the next observation time, \(t_{j+1}\), or until the transactions account under this deviation reaches a nonpositive balance, whichever comes first. Formally, denote this time as \(t^* = \min\{t_{j+1}, \inf\{t: \tilde{X}_t \leq 0\}\}\), where \(\tilde{X}_t\) is the balance in the transactions account under this deviation. We next argue that \(t^* \neq t_j\) and hence that \(t^* > t_j\). We proceed by contradiction. Suppose, contrary to what is to be proved, that \(t^* = t_j\), so that \(0 \geq \tilde{X}_j\). Since \((i)\) \(\tilde{X}_j = X_j - (1 - \psi_s)\tilde{F}^s(t_j, t_j^+; r_L) - (1 + \psi_b)F^{b,0}(t_j, t_j^+; r_L)\), \((ii)\) \(X_{t_j} > 0\), and \((iii)\) \(\tilde{F}^s(t_j, t_j^+; r_L)\) cannot be positive under any circumstance, then \(\tilde{X}_j\) can be nonpositive only if \(F^{b,0}(t_j, t_j^+; r_L) > 0\). But if the original path is optimal, then \(F^{b,0}(t_j, t_j^+; r_L) > 0\) and Lemma 4 imply that \(F^{x,0}(t_j, t_j^+; r_L) = 0\). Since \(X^0_{t_j} = X_j - (1 - \psi_s)F^{x,0}(t_j, t_j^+; r_L) - (1 + \psi_b)F^{b,0}(t_j, t_j^+; r_L)\), the fact that \(F^{x,0}(t_j, t_j^+; r_L) = 0\) implies that \(0 < X^0_{t_j} = X_j - (1 + \psi_b)F^{b,0}(t_j, t_j^+; r_L) \leq \tilde{X}_j\), which contradicts \(0 \geq \tilde{X}_j\) above. Therefore, \(t^* > t_j\).

Also, by construction, \(t^* \leq t_{j+1}\) and \(F^{x,0}(t_j, t^*, r_L) < 0\).

To complete the construction of the deviation, suppose that between \(t_j\) and \(t^*\) the consumer...
invests the funds she would have transferred into the transactions account in riskless bonds in the investment portfolio. At time \( t^* \), the consumer sets \( \tilde{F}^b(t^*, t^{++}, r_L) = F^{x,0}(t^*, t^{++}, r_L) + F^{x,0}(t^*, t^*, r_f) < 0 \). From \( t^{++} \) to \( t_{j+1} \), the consumer simply follows the same transfer and consumption policies she would have followed under the original path.

Under this deviation, the consumption process does not change between \( t_j \) and \( t^* \) or between \( t^{++} \) and \( t_{j+1} \), so that consumption is unchanged in \([t_j, t_{j+1}]\). Moreover, at time \( t^{++} \), the investment portfolio has the same value as under the original path, and since the consumer follows the same transfer policies from \( t^{++} \) onward, the investment portfolio at \( t_{j+1} \) is the same under the deviation as under the original path. The transactions account changes by \((1 - \psi_s)[F^{x,0}(t_j, t^*, r_L) - F^{x,0}(t_j, t^*, r_f)] > 0 \) at \( t^{++} \). Since the consumer follows the same transfer policies from \( t^{++} \) onward, the deviation increases the transactions account at time \( t_{j+1} \) relative to the original path by \((1 - \psi_s)e^{\theta(t_{j+1} - t^*)} \times [F^{x,0}(t_j, t^*, r_L) - F^{x,0}(t_j, t^*, r_f)] > 0 \). Hence, the original path could not have been optimal.

**LEMMA 8:** Along an optimal path that includes the possibility of automatic transfers, let \( \tilde{t} = \inf\{t \geq t_j : X_t = 0\} \). If \( \theta_X = \theta_S = 0 \), then \( X_t = 0 \) for all \( t \geq \tilde{t} \).

**PROOF:** Suppose that there are no transactions costs \((\psi_s = \psi_b = 0)\). In that case, the consumer can move freely and instantaneously between the investment portfolio and the transactions account. The allocation between the investment portfolio and the transactions account is part of an asset allocation problem with three assets: risky equity, riskless bonds paying \( r_f \), and riskless liquid assets paying \( r_L < r_f \). In the absence of the requirement \( X_t \geq 0 \), there would be an arbitrage opportunity that would send the holding of riskless bonds in the investment portfolio to infinity and the holding of the liquid assets in the transactions account to minus infinity. Given the requirement \( X_t \geq 0 \) and the ability to undertake costless transfers between \( X_t \) and \( S_t \), the consumer would immediately set \( X_t = 0 \), and then would keep \( X_t \) at zero forever by setting \( F^b(t, \infty) = 0 \) and \( \int_t^{\infty} dT^+ = -\int_t^\infty c_ds \) so that \( F^v(t, z, r_L) = -\text{FVC}(t, z) \) for any \( z \geq t \); in words, the consumer would transfer infinitesimal amounts from \( S_t \) to \( X_t \) as needed to finance instantaneous consumption. Any allocation to riskless bonds would take place exclusively inside the investment portfolio and on observation dates, the consumer would simply adjust the consumption rate.

Now introduce transactions costs so that \( \psi_s + \psi_b > 0 \). We will prove that, also in this case, it is optimal to keep \( X_t = 0 \) for \( t \geq \tilde{t} \). Let \( c_t^{x+}, X_t^{x+}, \) and \( S_t^{x+} \) denote values of \( c_t, X_t, \) and \( S_t \) along an optimal path for \( \psi_s + \psi_b > 0 \) and \( t \geq \tilde{t} \). Now consider the case with \( \psi_s = \psi_b = 0 \), and let \( c_t^*, \text{FVC}^*(\cdot), F^{x,*}(\cdot) \), and

\[ \text{But then } F^{b,0}(t^*, t^{++}, r_L) = 0 \text{ so } X_{t^{++}} = X_{t^*} - (1 - \psi_s)\tilde{F}^b(t^*, t^{++}; r_L) - (1 + \psi_b)\tilde{F}^b(t^*, t^{++}; r_f) = X_{t^*} - (1 - \psi_s)\tilde{F}^b(t^*, t^{++}; r_L) - (1 + \psi_b)\tilde{F}^{b,0}(t^*, t^{++}; r_L) \geq \tilde{X}_{t^*} - X_{t^*} > 0 \text{. So under the deviation, } X_t \text{ is positive both at time } t^* \text{ and at time } t^{++}, \text{ which contradicts the definition of } t^*. \]
\( F^{bs}(\cdot) \) denote the values of \( c_t, \) FVC(\( \cdot \)), \( F^f(\cdot) \), and \( F^b(\cdot) \) in this case. In this case, setting \( c_t^* = \frac{1}{1-\theta_s}c_t^{**} \) is feasible. To see this, simply set \( c_t^* = \frac{1}{1-\theta_s}c_t^{**} \), and keep the observation dates, the allocations within the investment portfolio, and the transfers between the investment portfolio and the transactions account unchanged. Clearly the path of \( S_t \) does not change, so to show feasibility, it suffices to show that the path of \( X_t^* \) is nonnegative. To that end, note that for arbitrary \( \psi_s \) and \( \psi_b \), and any feasible consumption and transfer policies, the dynamics of \( X_t \) for \( t \geq \bar{t} \) are characterized by

\[
(A.45) \quad X_t = -\text{FVC}(\bar{t}, t) - (1 - \psi_s)F^s(\bar{t}, t; r_L) - (1 + \psi_b)F^b(\bar{t}, t; r_L).
\]

For the optimal path associated with \( \psi_s + \psi_b > 0 \), we have

\[
(A.46) \quad X_t^{**} = -\text{FVC}^{**}(\bar{t}, t) - (1 - \psi_s)F^{**s}(\bar{t}, t; r_L) - (1 + \psi_b)F^{**b}(\bar{t}, t; r_L).
\]

For the alternative path, which has \( \psi_s = \psi_b = 0 \), we have \( F^s(\bar{t}, t) = \frac{1}{1-\theta_s}\text{FVC}^s(\bar{t}, t), \quad F^{**s}(\bar{t}, t; r_L) = F^{**s}(\bar{t}, t; r_L), \) and \( F^{bs}(\bar{t}, t; r_L) = F^{bs}(\bar{t}, t; r_L), \) which implies

\[
(A.47) \quad X_t^* = -\frac{1}{1-\psi_s}F^{**s}(\bar{t}, t) - F^{**s}(\bar{t}, t; r_L) - F^{bs}(\bar{t}, t; r_L).
\]

Dividing (A.46) by \( 1 - \psi_s \), recognizing that \( \frac{1+\psi_b}{1-\psi_s} > 1 \) when \( \psi_s + \psi_b > 0 \), and then using (A.47) yields

\[
(A.48) \quad \frac{1}{1-\psi_s}X_t^{**} = -\frac{1}{1-\psi_s}F^{**s}(\bar{t}, t) - F^{**s}(\bar{t}, t; r_L) - \frac{1+\psi_b}{1-\psi_s}F^{bs}(\bar{t}, t; r_L)
\]

\[
(A.49) \quad \leq -\frac{1}{1-\psi_s}\text{FVC}^{**}(\bar{t}, t) - F^{**s}(\bar{t}, t; r_L) - F^{**bs}(\bar{t}, t; r_L)
\]

\[
(A.50) \quad = X_t^*.
\]

Since the original path was feasible with \( X_t^{**} \geq 0 \), (A.48) implies that \( X_t^* \geq \frac{1}{1-\theta_s}X_t^{**} \geq 0 \) for all \( t \). Therefore, it is feasible to set \( c_t^* = \frac{1}{1-\theta_s}c_t^{**} \) when \( \psi_s = \psi_b = 0 \). Accordingly, letting \( V_t^{\psi_s,\psi_b}(\cdot) \) denote the time-\( t \) value function of the consumer when the transactions costs parameters are \( \psi_s \) and \( \psi_b \), we obtain \( \frac{1}{1-\theta_s}V_t^{\psi_s,\psi_b}(\cdot) \leq V_t^{(0,0)}(\cdot) \) or, equivalently, \( V_t^{\psi_s,\psi_b}(\cdot) \leq (1 - \psi_s)^{1-\alpha}V_t^{(0,0)}(\cdot) \). In words, \( (1 - \psi_s)^{1-\alpha}V_t^{(0,0)}(\cdot) \) provides an upper bound to \( V_t^{\psi_s,\psi_b}(\cdot) \). Next observe that when \( \psi_s + \psi_b > 0 \), the policy that sets \( c_t^{**} = (1 - \psi_s)c_t^{**}, \quad F^{**s}(\bar{t}, t; r_L) = 0, \) and \( F^{**s}(\bar{t}, t; r_L) = F^{**s}(\bar{t}, t; r_L) = -\text{FVC}^{**}(t, t_1) \) for all \( t \geq \bar{t} \) keeps \( X_t = 0 \) for all \( t \geq \bar{t} \), is feasible, and delivers welfare equal to \( (1 - \psi_s)^{1-\alpha}V_t^{(0,0)}(\cdot) \). That is, for
\( \psi_t + \psi_h > 0, \) this policy attains the upper bound \((1 - \psi_t)^{1-a} V_t^{(0,0)}\) and hence is optimal. \( \quad Q.E.D. \)

**Lemma 9:** Along an optimal path that includes the possibility of automatic transfers, if \( \theta_X = \theta_S = 0 \) and if \( F^i(t_j, t_{j+1}; r_L) < 0 \), then optimal \( X_{t_{j+1}} = 0. \)

**Proof:** Lemma 7 implies that if \( F^i(t_j, t_{j+1}; r_L) < 0 \), then \( t_j \equiv \inf \{ t \geq t_j: X_t = 0 \} < t_{j+1} \). Then Lemma 8 implies that \( X_t = 0 \) for all \( t \geq t_i \), so that in particular, \( X_{t_{j+1}} = 0. \)

**Proof of Proposition 5:** The arguments of Lemma 2, appropriately adjusted for automatic transfers, imply that if, along an optimal path, \( x_{t_j} \) becomes smaller than some number \( \Omega_i \) on some observation date \( t_j \), then \( C(t_j, t_{j+1}) > X_{t_j} \), which requires \( F^i(t_j, t_{j+1}; r_L) < 0 \). Accordingly Lemma 9 implies \( X_{t_{j+1}} = 0 \), which implies \( x_t = 0 \) for all \( t \geq t_{j+1} \) (by Lemma 8) so that, in particular, \( x_{t_{j+k}} = 0 \) for all \( k \geq 1 \).

Next we argue that eventually, with probability 1, there will exist some \( k \geq 1 \), such that \( x_{t_{j+k}} \leq \Omega_1 \). We start by observing that in the presence of automatic transfers, \( X_{t_{j+1}} \) is \( F_{t_j} \)-measurable.\(^{37}\) Lemmas 7 and 8 imply that as long as \( X_{t_{j+1}} > 0 \), it follows that \( F^i(t_j, t_{j+1}; r_L) = 0 \), which, together with the fact that consumption and transfers from the transactions account to the investment account are both nonnegative, implies that \( X_{t_{j+1}} \leq e^{\ell_j X_{t_j}} \), and \( S_{t_{j+1}} \geq S_t R(t, \tau_j) \). Accordingly, \( x_{t_{j+1}} = \frac{x_{t_{j+1}}}{S_{t_{j+1}}} \leq \frac{e^{\ell_j X_{t_j}}}{S_j R(t, \tau_j)} = x_t e^{\ell_j R(t, \tau_j)} \). Taking logs gives \( \log x_{t_{j+1}} \leq \log x_t + r_L \tau_j - \log R(t, \tau_j) \). Taking expectations as of time \( t_j \) gives \( E_t \log x_{t_{j+1}} \leq \log x_t + r_L \tau_j - E_t \log R(t, \tau_j) \). We next observe that \( -E_t \log R(t, \tau_j) \leq \max_{\phi_j \in [0, 1]} \{-E_t \log R(t, \tau_j)\} = -r_t \tau_j \).\(^{38}\) Accordingly, \( \log x_t \) is bounded above by a random walk with drift \( r_L - r_f \), which is strictly negative. Since a random walk with negative drift eventually, with probability 1, becomes smaller than any finite number (and in particular \( \log \Omega_1 \)) with probability 1, there will exist a \( k \), such that \( x_{t_{j+k}} \leq \Omega_1 \). Therefore, as discussed above, \( x_{t_{j+k+n}} = 0 \) for all \( n \geq 1 \).

Since the optimal value of \( \tau_j \) is simply a function of \( x_{t_j} \) and since \( x_{t_j} \) eventually, with probability 1, becomes constant (namely, zero), the inattention intervals \( \tau_j \) will eventually become constant with probability 1. \( \quad Q.E.D. \)

\(^{37}\)Since any transfers from the investment portfolio must be \( F_{t_j} \)-measurable, and feasible, these transfers will not be financed from the risky holdings in the investment portfolio.

\(^{38}\)Note that \( -E_t \log R(t, \tau_j) \) is a convex function of \( \phi_j \), since \( \frac{\{E_t \{-E_t \log R(t, \tau_j)}\}}{\phi_j} = E_t \frac{1}{R(t, \tau_j)} \left[ \frac{\phi_j}{P_j} - e^{-r_f \tau_j} \right] > 0 \) Hence the maximum value of \( -E_t \log R(t, \tau_j) \) for \( \phi_j \in [0, 1] \) is attained either when \( \phi_j = 0 \), or when \( \phi_j = 1 \). When \( \phi_j = 0 \), \( -E_t \log R(t, \tau_j) = -r_f \tau_j \), whereas when \( \phi_j = 1 \), \( -E_t \log R(t, \tau_j) = -(\mu - \frac{\sigma^2}{2}) \tau_j \). Given the maintained assumption \( (\mu - \frac{\sigma^2}{2}) > r_f \), it follows that \( \max_{\phi_j \in [0, 1]} \{-E_t \log R(t, \tau_j)\} = -r_f \tau_j \).
The following lemma proves that although $x_i$ is eventually absorbed at zero with probability 1, this absorption need not occur immediately.

**Lemma 10:** Suppose that we allow automatic transfers, $\theta_X = \theta_S = 0$, and $x_i$ is sufficiently large. Then optimal $X_{j+} > 0$ so that $x_i$ is not immediately absorbed at zero.

**Proof:** Let $X_j^0$ be the value of $X_j$ along the hypothesized optimal path, and suppose, contrary to what is to be proved, that $X_j^0 = 0$, which implies that $F^b(t_j, t_j^+, r_L) = \frac{x_j^0}{1 + \psi_b}$ and $F^s(t_j, t, r_L) = -\frac{FVC(t_j, t)}{1 - \psi_s}$ for $t > t_j$. Define $\tau^*$ such that $\frac{1 - \psi_s}{1 + \psi_b} e^{(t_j - L) \tau^*} = 1$ and note that for $0 \leq \tau^{**} < \tau^*$, any dollar transferred from the transactions account to the investment portfolio at time $t_j$, invested in the riskless bond, and then transferred back to the transactions account at time $t_j + \tau^*$ will be worth less at time $t_j + \tau^*$ than a dollar simply left in the transactions account from $t_j$ to $t_j + \tau^*$. Now let $\tau^{**}$ be a positive number less than $\min\{t_{j+} - t_j, \tau^*\}$ that is small enough that $e^{-rL\tau^{**}} FVC(t_j, t_j + \tau^{**}) < X_j^0$. Consider an alternative path that sets $F^b(t_j, t_j^+, r_L) = \frac{x_j^0 - e^{-rL\tau^{**}} FVC(t_j, t_j^+, \tau^{**})}{1 + \psi_b} > 0$ and does not change any other transfers from the transactions account to the investment portfolio so that $F^b(t_j, t, 0) = F^b(t_j, t, 0) - \frac{e^{-rL\tau^{**}} FVC(t_j, t_j^+, \tau^{**})}{1 + \psi_b}$ for $t > t_j^+$. In addition, the alternative path sets $F^s(t_j, t_j + \tau^{**}, r_L) = 0$ and then maintains $F^s(t_j + \tau^{**}, r_L) = F^s(t_j + \tau^{**}, t, r_L)$ for all $t \in (t_j + \tau^{**}, t_{j+1})$. Suppose that any changes in the size of the investment portfolio affect only the amount invested in riskless bonds. Relative to the originally hypothesized optimal path, the alternative path changes $S_j + \tau^{**}$ by $\Delta S \equiv -e^{rL\tau^{**}} e^{-rL\tau^{**}} FVC(t_j, t_j^+, \tau^{**}) - F^s(t_j, t_j + \tau^{**}, r_L)$, where the first term reflects the reduction in $S_j + \tau^{**}$ arising from the reduced transfer into the investment portfolio at time $t_j$ and the second term reflects the fact that the consumer does not need to transfer assets from the investment portfolio to the transactions account to finance the original path of consumption until $t_j + \tau^{**}$. Relative to the originally hypothesized optimal path, the alternative path changes $X_{j+} + \tau^{**}$ by $\Delta X \equiv (1 + \psi_b) \frac{e^{-rL\tau^{**}} FVC(t_j, t_j^+, \tau^{**})}{1 + \psi_b} + (1 - \psi_s) F^s(t_j, t_j + \tau^{**}, r_L)$, where the first term reflects the increase in $X_{j+} + \tau^{**}$ that arises from the reduction in the transfer out of the transactions account at time $t_j$ and the second term reflects the reduction in transfers into the transactions account between $t_j$ and $t_j + \tau^{**}$. Use the fact that $-F^s(t_j, t_j + \tau^{**}, r_L) \equiv -\frac{FVC(t_j, t_j^+, \tau^{**})}{1 - \psi_s}$ to obtain $\Delta^S \geq -e^{rL\tau^{**}} e^{-rL\tau^{**}} FVC(t_j, t_j^+, \tau^{**}) - \frac{FVC(t_j, t_j^+, \tau^{**})}{1 + \psi_b} = [1 - \frac{1 - \psi_s}{1 + \psi_b} e^{rL\tau^{**}} + 1] - \frac{FVC(t_j, t_j^+, \tau^{**})}{1 + \psi_b} > 0$ since $\tau^{**} > \tau^*$. Observe that $\Delta^X \equiv FVC(t_j, t_j + \tau^{**}) + (1 - \psi_s) F^s(t_j, t_j + \tau^{**}, r_L) = 0$. Since $\Delta^S > 0$ and $\Delta^X = 0$, the original path could not be optimal. Therefore, optimal $X_{j+} > 0$. Q.E.D.
PROPOSITION 6: Define $V(0, S_j; \psi_s)$ as the value function for a given value of the transactions cost parameter $\psi_s$ on observation date $t_j$ when $(X_{t_j}, S_{t_j}) = (0, S_j)$, and define $\pi_1(\psi_s)$ as the optimal return value of $x^*_{t_j}$ for $x_{t_j} < \omega_1$. Suppose that $\theta_s$ is sufficiently small that for any admissible value of $\psi_s$, if $x_{t_j} < \omega_1$ on observation date $t_j$, then on all subsequent observation dates $x_{t_{j+1}} = 0$. Then the following statements hold:

(i) $V(0, S_j; \psi_s) = (1 - \psi_s)^{1+\alpha}V(0, S_j; 0)$.

(ii) The optimal observation dates $t_k = t_j + (k - j)\tau^*$ for $k \geq j$ are invariant to $\psi_s$.

(iii) $\pi_1(\psi_s) = (1 - \psi_s)\pi_1(0)$.

PROOF: Suppose that $\psi_s = 0$ and let $\{S^*_t\}_{t=0}^{\infty}$ be the path of $S_t$ under the optimal policy starting from observation date $t_j$ when the consumer observes $X_{t_j} = 0$ and $S_{t_j} = S^*_j$. Let $\tau^*$ be the constant optimal interval of time between consecutive observations so that observation date $t_k = t_j + (k - j)\tau^*$ for $k \geq j$. For any observation date $t_k \geq t_j$, the transactions account balance will be $X_{t_k} = 0$, and immediately after each observation date, the transactions account balance will be $X^*_{t_k} = X^*_k = \pi_1(0)S^*_k$. Since $0 = X^*_{t_{j+1}} = e^\tau^* (X^*_{t_j} - C(t_k, \tau^*))$, we have $C(t_k, \tau^*) = X^*_{t_k}$.

Now let $\psi_s$ take an arbitrary admissible value and suppose that the consumer continues to observe the value of the investment portfolio on dates $t_k = t_j + (k - j)\tau^*$ for $k \geq j$ and maintains the same path of $S_t$, that is, that $S_t = S^*_t$ for $t \geq t_j$. Since the consumer will make the same transfers out of the investment portfolio as in the initial case with $\psi_s = 0$, a feasible path of the transactions account balance immediately after each observation date would be $X^*_{t_k} = (1 - \psi_s)X^*_k$, which supports a feasible path of consumption $C(t_k, \tau^*) = (1 - \psi_s)X^*_k$. Therefore, $V(0, S_j; \psi_s) \geq (1 - \psi_s)^{1+\alpha}V(0, S_j; 0)$.

A similar argument, starting with an arbitrary admissible value of $\psi_s$ less than 1, implies $V(0, S_j; 0) \geq (\frac{1}{1 - \psi_s})^{1+\alpha}V(0, S_j; \psi_s)$. Therefore, $V(0, S_j; \psi_s) \geq (1 - \psi_s)^{1+\alpha}V(0, S_j; 0) \geq V(0, S_j; \psi_s)$, which implies $\pi_1(0, S_j; \psi_s) = (1 - \psi_s)^{1+\alpha}V(0, S_j; 0)$ (statement (i)). We showed that by maintaining the same observation dates when $\psi_s$ is positive as when $\psi_s = 0$ allows a path of consumption that achieves $V(0, S_j; \psi_s) \geq (1 - \psi_s)^{1+\alpha}V(0, S_j; 0) = V(0, S_j; \psi_s)$. Similarly, by maintaining the same observation dates when $\psi_s = 0$ as when $\psi_s$ is positive allows a path of consumption that achieves $V(0, S_j; 0) \geq (\frac{1}{1 - \psi_s})^{1+\alpha}V(0, S_j; \psi_s) = V(0, S_j; 0)$. Therefore, we have proven statement (ii).

For any observation date $t_k \geq t_j$, $x^*_k = \pi_1(\psi_s)$. Therefore, $\pi_1(\psi_s) = \frac{X^*_k}{S^*_k} = \frac{(1 - \psi_s)X^*_k}{S^*_k} = (1 - \psi_s)\pi_1(0)$, which proves statement (iii).

Q.E.D.
PROOF OF PROPOSITION 4: At each observation date $t_j$ the consumer chooses the share $\phi_j$ of the investment portfolio to allocate to equity to maximize $E_t(V(X_{t_{j+1}}, S_{t_{j+1}}))$ subject to the constraints $0 \leq \phi_j \leq 1$. Using (2) and (3), we can write the Lagrangian for this constrained maximization as

$$L_j = E_t\left\{ V(X_{t_{j+1}}, \phi_j\frac{P_{t_{j+1}}}{P_{t_j}} S_{t_{j+1}} + (1 - \phi_j) e^{-\tau_{t_{j+1}}} S_{t_{j+1}}^+ \right\}$$

$$+ \delta_j S_{t_{j+1}} \phi_j + \nu_j S_{t_{j+1}}^+ (1 - \phi_j),$$

where $\delta_j S_{t_{j+1}}^+ \geq 0$ is the Lagrange multiplier on the constraint $\phi_j \geq 0$ and $\nu_j S_{t_{j+1}}^+ \geq 0$ is the Lagrange multiplier on the constraint $\phi_j \leq 1$. Differentiating the Lagrangian in (A.51) with respect to $\phi_j$, setting the derivative equal to zero, and then dividing both sides by $S_{t_{j+1}}$ yields

$$E_t\left\{ V(X_{t_{j+1}}, \phi_j)\frac{P_{t_{j+1}}}{P_{t_j}} - e^{-\tau_{t_{j+1}}} \right\} = \nu_j - \delta_j.$$  \hspace{1cm} (A.52)

Next, we prove the following lemma.

**LEMMA 11:** We have $\phi_j > 0$ and $\delta_j = 0$.

**PROOF:** We proceed by contradiction. Suppose that $\phi_j = 0$, which implies that $\nu_j = 0$ and that $S_{t_{j+1}}^+$ is known at time $t_j$. Therefore, (A.52) can be written as $V_S(X_{t_{j+1}}, S_{t_{j+1}}) E_t\left\{ \frac{P_{t_{j+1}}}{P_{t_j}} - e^{-\tau_{t_{j+1}}} \right\} = -\delta_j \leq 0$, which is a contradiction because $V_S(X_{t_{j+1}}, S_{t_{j+1}}) > 0$ and, by assumption, the expected equity premium, $E_t\left\{ \frac{P_{t_{j+1}}}{P_{t_j}} - e^{-\tau_{t_{j+1}}} \right\}$, is positive. Therefore, $\phi_j$ must be positive, which implies $\delta_j = 0$. \hspace{1cm} Q.E.D.

To replace the marginal valuation of the investment portfolio $V_S(X_{t_{j+1}}, S_{t_{j+1}})$ by a function of the marginal utility of consumption, first use the definition of the marginal rate of substitution $m(x_{t_{j+1}})$ to obtain

$$V_S(X_{t_{j+1}}, S_{t_{j+1}}) = m(x_{t_{j+1}}) V_X(X_{t_{j+1}}, S_{t_{j+1}}).$$  \hspace{1cm} (A.53)

Then use the envelope theorem to obtain

$$V_X(X_{t_{j+1}}, S_{t_{j+1}}) = \left[ 1 - (1 - a) \alpha b(\tau_{j+1}) \right] U'(C(t_{j+1}, \tau_{j+1})),$$

which implies that $V_X(X_{t_{j+1}}, S_{t_{j+1}})$, the increase in expected lifetime utility made possible by a $1$ increase in $X_{t_{j+1}}$, equals the increase in utility that would
accompany an increase of $1 - \left( 1_{(\phi_j > 0)} + 1_{(\phi_j < 0)} \right) \theta_X \text{ dollars in } C(t_{j+1}, \tau_{j+1})$. That is, if the consumer transfers assets between the investment portfolio and the transactions account at time $t_{j+1}$, a $1$ increase in $X_{t_{j+1}}$ would allow $C(t_{j+1}, \tau_{j+1})$ to increase by $1 - \theta_X$ dollars; otherwise, $C(t_{j+1}, \tau_{j+1})$ can increase by $1$. Differentiate (16) with respect to $C(t_j, \tau_j)$ and use (**) in footnote 18 to obtain

(A.55) \[ U'(C(t_j, \tau_j)) = c^{-\alpha} \]

Substitute (A.54) into (A.53) and use (A.55) to obtain

(A.56) \[ V_s(X_{t_{j+1}}, S_{t_{j+1}}) = m(x_{t_{j+1}}) \left[ 1 - \left( 1_{(\phi_j > 0)} + 1_{(\phi_j < 0)} \right) \theta_X \right] \times \left( 1 - (1 - \alpha) \kappa b(\tau_{j+1}) \right) c_{t_{j+1}}^{-\alpha} \]

Substituting the right hand side of (A.56) for $V_s(X_{t_{j+1}}, S_{t_{j+1}})$ in (A.52) and using Lemma 11 to set $\delta_j = 0$ yields

(A.57) \[ E_t \left\{ m(x_{t_j}) \left[ 1 - \left( 1_{(\phi_j > 0)} + 1_{(\phi_j < 0)} \right) \theta_X \right] \times \left( 1 - (1 - \alpha) \kappa b(\tau_{j+1}) \right) c_{t_{j+1}}^{-\alpha} \left( \frac{P_{t_{j+1}}}{P_{t_j}} - e^{\varepsilon t_j} \right) \right\} = \nu_j \]

In standard models without information costs and transfer costs, and without the constraints $0 \leq \phi_j \leq 1$, the corresponding Euler equation, which is widely used in financial economics, is

(A.58) \[ E_t \left\{ c_s^{-\alpha} \left( \frac{P_s}{P_t} - e^{\varepsilon (s-t)} \right) \right\} = 0 \quad \text{for} \quad s > t. \]

In general, the Euler equation in the presence of information costs and transactions costs in (A.57) differs from the standard Euler equation in (A.58) in five ways: (i) the Euler equation in (A.57) contains the Lagrange multiplier on the constraint $\phi_j \leq 1$, but this Lagrange multiplier does not appear in the standard Euler equation; (ii) the Euler equation in (A.57) contains the marginal rate of substitution $m(x_{t_{j+1}})$, which is a random variable, but this marginal rate of substitution is absent (or implicitly equal to a constant) in the standard Euler equation; (iii) the Euler equation in (A.57) contains the term $1 - \left( 1_{(\phi_j > 0)} + 1_{(\phi_j < 0)} \right) \theta_X$, which reflects the additional fixed transfer cost associated with having an additional dollar in the transactions account; (iv) the

39 If assets could be transferred without any resource costs (i.e., if $\theta_X = \theta_s = \psi_s = \psi_b = 0$), then $m(x_{t_j}) = 1$ at all observation dates and, hence, can be eliminated from (A.57).
Euler equation in (A.57) contains the term \(1 - (1 - \alpha)\kappa b(\tau_{j+1})\), which reflects the utility cost of the next observation; and (v) in the presence of information costs, the Euler equation holds only for rates of return between observation dates, whereas the Euler equation in the standard case holds for rates of return between any arbitrary pair of dates because all dates are observation dates in the standard case. We show that in the long run, in an interesting special case, the first four of these differences disappear. Before showing this result, we prove the following lemma.

**Lemma 12:** Suppose that \(\theta_S\) is sufficiently small in the sense described in the proof of Proposition 2. If \(x_{ij} \leq \omega_1\), then (i) \(\phi_j < 1\) if \(\alpha > \frac{\mu - r_f}{\sigma^2}\) and (ii) \(\phi_j = 1\) if \(\alpha \leq \frac{\mu - r_f}{\sigma^2}\).

**Proof:** Proposition 2 implies that if \(x_{ij} \leq \pi_1\), then \(x_{ij+1} = 0\). The optimal value of \(\phi_j\), \(0 \leq \phi_j \leq 1\), maximizes \(E_{ij}\{V(X_{ij+1}, S_{ij+1})\} = \frac{1}{1-\alpha}E_{ij}\{S_{ij+1}^{1-\alpha}u(0)\}\), which is equivalent to maximizing \(\varphi(\phi_j; \alpha) \equiv \frac{1}{1-\alpha}E_{ij}\{(\phi_j \frac{P_{ij+\tau_j}}{P_{ij}} + (1 - \phi_j) \times e^{r\tau_j})^{1-\alpha}\}\). Define \(\alpha^*\) such that \(\arg\max_{\phi_j} \varphi(\phi_j; \alpha^*) = 1\) and note that \(\varphi'(1; \alpha^*) = 0\).

Differentiating the definition of \(\varphi(\phi_j; \alpha)\) with respect to \(\phi_j\) and setting \(\phi_j = 1\) yields

\[
\varphi'(1; \alpha) = E_{ij}\left\{ \left( \frac{P_{ij+\tau_j}}{P_{ij}} \right)^{1-\alpha} \right\} - e^{r\tau_j}E_{ij}\left\{ \left( \frac{P_{ij+\tau_j}}{P_{ij}} \right)^{-\alpha} \right\}.
\]

Use the fact that \(\frac{P_{ij+\tau_j}}{P_{ij}}\) is log normal to obtain

\[
\varphi'(1; \alpha) = \exp\left[ (1 - \alpha)\left( \mu - \frac{1}{2}\alpha\sigma^2 \right)\tau_j \right] - e^{r\tau_j}\exp\left[ -\alpha\left( \mu + \frac{1}{2}(-\alpha - 1)\sigma^2 \right)\tau_j \right] - \exp\left[ -\alpha\left( \mu + \frac{1}{2}(-\alpha - 1)\sigma^2 \right)\tau_j \right] - \exp\left[ \alpha\sigma^2\tau_j \right].
\]

Further rearrangement yields

\[
\varphi'(1; \alpha) = \exp\left[ \left( -\alpha\mu + r_f - \frac{1}{2}\alpha(1 - \alpha)\sigma^2 \right)\tau_j \right] \times \left[ \exp((\mu - r_f)\tau_j) - \exp(\alpha\sigma^2\tau_j) \right],
\]

which implies that

\[
\varphi'(1; \alpha) \leq 0 \quad \text{as} \quad \alpha \geq \alpha^* \equiv (\mu - r_f)/\sigma^2.
\]
Differentiate $\varphi(\phi_j; \alpha)$ twice with respect to $\phi_j$ to obtain

$$\varphi''(\phi_j; \alpha) = -\alpha E_t \left\{ \left( \frac{P_{t_j+\tau_j}}{P_{t_j}} + (1 - \phi_j) e^{\rho(t_j)} \right)^{-\alpha} \left( \frac{P_{t_j+\tau_j}}{P_{t_j}} - e^{\rho(t_j)} \right)^2 \right\} < 0,$$

which implies that $\varphi(\phi_j; \alpha)$ is concave. If $\alpha > \alpha^*$, then $\varphi'(1; \alpha) < 0$, so the concavity of $\varphi(\phi_j; \alpha)$ implies that the optimal value of $\phi_j$ is less than 1 and the Lagrange multiplier on the constraint $\phi_j \leq 1$ is $\nu_j = 0$. If $\alpha \leq \alpha^*$, then $\varphi'(1; \alpha) \geq 0$, so the concavity of $\varphi(\phi_j; \alpha)$ implies that the optimal value of $\phi_j$ equals 1. If $\alpha < \alpha^*$, the Lagrange multiplier on the constraint $\phi_j \leq 1$ is $\nu_j > 0$. 

Q.E.D.

APPENDIX B: BASIC PROPERTIES OF THE OPTIMIZATION PROBLEM AND THE VALUE FUNCTION

The goal of this section is to establish some basic properties of the optimization problem that we consider in the paper. Specifically, we show that the value function $V(X_{t_j}, S_{t_j})$ is finite, homogeneous of degree $1 - \alpha$, continuous, and satisfies the Bellman equation (20). Moreover, we show that there exist policies that attain the supremum on the right hand side of (20) and that these policies are optimal. The main result is formulated in Proposition 7. In preparation for Proposition 7, we state and prove four lemmas.

**Lemma 13:** Let $a_j \equiv \{C(t_j, \tau_j), y^{b}(t_j), y^{s}(t_j), \phi_j, \tau_j\}$ denote a strategy that is feasible given $X_{t_j}$ and $S_{t_j}$, and let

$$U(a_j=1, \ldots, \infty) \equiv E_t \left\{ \int_{t_j}^{\infty} \frac{1}{1 - \alpha} c_t^{1-\alpha} e^{-\rho(t-t_j)} \, dt - \sum_{i=j}^{\infty} A(t_i, \tau_i) e^{-\rho(t_i+\tau_i-t_j)} \right\}$$
denote the expected payoff from following the policy \( a_j = 1, \ldots, \infty \). Furthermore, let 
\[ V(X_{t_j}, S_{t_j}) = \sup_{a_j = 1, \ldots, \infty} U(\alpha_j = 1, \ldots, \infty) \]
denote the value function of the problem. Then \( V(X_{t_j}, S_{t_j}) \) satisfies (20).

PROOF: First, we observe that \(-\infty < V(X_{t_j}, S_{t_j}) < \infty\). To see this, we note first that there exist policies that are feasible and lead to a finite \( U(\alpha_j = 1, \ldots, \infty) \). For instance, setting \( \tau_1 = \infty, y_s(t_j) = -[1 - \theta_s]S_{t_j} \), and \( C(t_j, \infty) = X_{t_j}^+ = (1 - \theta_X)X_{t_j}^+ + (1 - \psi_s)(1 - \theta_s)S_{t_j} \) implies the discounted utility 
\[ \frac{1}{1 - \alpha} \chi^{-\alpha} X_{t_j}^{1 - \alpha}, \]
where \( \chi \equiv \frac{\psi - \theta}{\alpha} > 0 \). Accordingly, the value function is bounded below. Furthermore, the value function is bounded above by the value function that can be attained if we remove all transactions and observations costs \( (\kappa = \theta_s = \theta_X = \psi_s = \psi_b = 0) \). But by removing all these frictions, the problem becomes identical to the standard, continuous-time Merton (1971) problem with portfolio weights restricted to \( \phi \in [0, 1] \). Since that problem has a finite value function (as long as (7) holds), we conclude that the value function is bounded above and, hence, is finite.40

From this point on, the proof mimics closely the proof given in Stokey and Lucas (1989, Theorem 4.2). We observe that the definition of \( V(X_{t_j}, S_{t_j}) \) implies that

\[
V(X_{t_j}, S_{t_j}) \geq U(\alpha_j = 1, \ldots, \infty) \quad \text{for all } a_j = 1, \ldots, \infty
\]

and that for (arbitrarily small) \( \delta > 0 \), there exists some strategy \( a_j = 1, \ldots, \infty \) such that

\[
U(\alpha_j = 1, \ldots, \infty) \geq V(X_{t_j}, S_{t_j}) - \delta.
\]

(If (B.2) were not true, then we must have
\[
V(X_{t_j}, S_{t_j}) = \sup_{\alpha_j = 1, \ldots, \infty} U(\alpha_j = 1, \ldots, \infty) < V(X_{t_j}, S_{t_j}) - \delta,
\]
which is absurd.) Now, take \( \varepsilon > 0 \). To show that \( V(X_{t_j}, S_{t_j}) \) satisfies (20), we will show that

\[
V(X_{t_j}, S_{t_j}) \geq \left[ 1 - (1 - \alpha)\kappa b(\tau_j) \right] \times U(C(t_j, \tau_j))
\]

\[ + e^{-\rho \tau_j}E_{t_j} \left\{ V(e^{\rho \tau_j}(X_{t_j}^+ - C(t_j, \tau_j)), R(t_j, \tau_j)S_{t_j}^+) \right\} \]

40For a detailed analysis of the infinite-horizon version of Merton’s problem and the condition for its value function to be finite, see, for example, the monograph Karatzas and Shreve (1998, p. 149).
for all $a_j$ and that there exists $a_j$ such that for any (arbitrarily small) $\epsilon > 0$,

\[
V(X_{t_j}, S_{t_j}) \leq \left[ 1 - (1 - \alpha)\kappa b(\tau_j) \right] \times U(C(t_j, \tau_j)) + e^{-\rho t_j} E_t \{ V(e^{\rho t_j}(X_{t_j}^+ - C(t_j, \tau_j)), R(t_j, \tau_j)S_{t_j}^+) \} + \epsilon.
\]

To show (B.3), note that by (B.2) there exists a policy sequence $a' = a'_{t_j,t_j+1,\ldots}$ such that $U(\alpha'_{t_j+1,t_j+2,\ldots}) \geq V(X_{t_j+1}, S_{t_j+1}) - \frac{\epsilon}{2}$. Moreover, using the definition of $U(C(t_j, \tau_j))$ implies the existence of a policy $\alpha'_{t_j,t_j+1} \epsilon_{t_j,t_j+1}$ such that $\frac{1}{1-\alpha} [1 - (1 - \alpha)\kappa b(\tau_j)] \times \int_{t_j}^{t_j+\tau_j} \epsilon_{t_j,t_j+1}^{t_j+\tau_j}(c_j')^{1-\alpha} e^{-\rho(t-t_j)} dt \geq [1 - (1 - \alpha)\kappa b(\tau_j)] \times U(C(t_j, \tau_j)) - \frac{\epsilon}{2}$. Accordingly,

\[
V(X_{t_j}, S_{t_j}) \geq U(a'_{t_j,t_j+1,\ldots})
\]

\[
= \frac{1}{1 - \alpha} \left\{ [1 - (1 - \alpha)\kappa b(\tau_j)] \int_{t_j}^{t_j+\tau_j} (c_j')^{1-\alpha} e^{-\rho(t-t_j)} dt \right\} + e^{-\rho t_j} E_t U(\alpha'_{t_j+1,t_j+2,\ldots})
\]

\[
\geq [1 - (1 - \alpha)\kappa b(\tau_j)] \times U(C(t_j, \tau_j)) - \frac{\epsilon}{2}
\]

\[
+ e^{-\rho t_j} E_t V(e^{\rho t_j}(X_{t_j}^+ - C(t_j, \tau_j)), R(t_j, \tau_j)S_{t_j}^+) - e^{-\rho t_j} \frac{\epsilon}{2}
\]

\[
\geq [1 - (1 - \alpha)\kappa b(\tau_j)] \times U(C(t_j, \tau_j))
\]

\[
+ e^{-\rho t_j} E_t V(e^{\rho t_j}(X_{t_j}^+ - C(t_j, \tau_j)), R(t_j, \tau_j)S_{t_j}^+) - \epsilon.
\]

Since $\epsilon > 0$ was arbitrary, we obtain (B.3). To show (B.4), choose $\epsilon > 0$ and take a policy $a'_{t_j,t_j+1,\ldots}$ such that $V(X_{t_j}, S_{t_j}) \leq U(a'_{t_j,t_j+1,\ldots}) + \epsilon$. Accordingly

\[
V(X_{t_j}, S_{t_j}) \leq U(\alpha'_{t_j,t_j+1,\ldots}) + \epsilon
\]

\[
= \frac{1}{1 - \alpha} \left\{ [1 - (1 - \alpha)\kappa b(\tau_j)] \int_{t_j}^{t_j+\tau_j} (c_j')^{1-\alpha} e^{-\rho(t-t_j)} dt \right\} + e^{-\rho t_j} E_t U(\alpha'_{t_j+1,t_j+2,\ldots}) + \epsilon
\]

\[
\leq [1 - (1 - \alpha)\kappa b(\tau_j)] \times U(C(t_j, \tau_j))
\]

\[
+ e^{-\rho t_j} E_t V(e^{\rho t_j}(X_{t_j}^+ - C(t_j, \tau_j)), R(t_j, \tau_j)S_{t_j}^+) + \epsilon.
\]

\[Q.E.D.\]
Lemma 13 shows that the value function satisfies (20). The next lemma shows that the reverse conclusion holds, subject to two additional conditions.

**Lemma 14:** If $\hat{V}(X_{t_j}, S_{t_j})$ satisfies (20) with the supremum on the right hand side of (20) attained for some policy, and if $\lim_{t_k \to \infty} e^{-\rho t_k} E_{t_j} \hat{V}(X_{k}, S_{k}) = 0$ for all $(X_{t_j}, S_{t_j}) \in \mathbb{R}^2_+$ and for all feasible $a_{j=1, \ldots, \infty}$, then $\hat{V} = V$ and the policy that attains the supremum on the right hand side of (20) is an optimal policy for the intertemporal optimization problem.

**Proof:** The proof closely follows Stokey and Lucas (1989), so we give a brief sketch of some minor adaptations that are required so as to deal with the specifics of our setup. Iterating on (20) implies that if we adopt any feasible policy tuple $a_j$, we obtain

\[
\hat{V}(X_{t_j}, S_{t_j}) \geq E_{t_j} \sum_{i=j}^k \left[1 - (1 - \alpha)\kappa b(\tau_i)\right] e^{-\rho(t_i - t_j)} U(C(t_i, \tau_i))
\]

\[+ e^{-\rho(t_{k+1} - t_j)} E_{t_j} \hat{V}(X_{k+1}, S_{k+1}) \quad \text{for any } k \geq j.
\]

Now if the feasible policy $a_j$ involves a finite number of observations (so that $\tau_{k+1} = \infty$), then (B.6) shows that $\hat{V}(X_{t_j}, S_{t_j})$ is an upper bound to the payoff of $a_j$ since $\hat{V}(X_{k+1}, S_{k+1}) \geq \hat{V}(X_{k+1}, S_{k+1}; \tau_j = \infty) \geq \mathcal{U}(a_{k+1})$, where $\hat{V}(X_{k+1}, S_{k+1}; \tau_j = \infty)$ is the maximized value of the right hand side of (20) restricted by $\tau_{k+1} = \infty$ and $\mathcal{U}(a_{k+1})$ denotes the payoff from following the strategy $a_{k+1}$ for $t_{k+1}$ onward. If the feasible policy involves an infinite number of observations, then taking $t_{k+1} \to \infty$ and using $\lim_{t_{k+1} \to \infty} e^{-\rho t_{k+1}} E_{t_j} \hat{V}(X_{k+1}, S_{k+1}) = 0$, we once again conclude that $\hat{V}(X_{t_j}, S_{t_j})$ provides an upper bound to the payoff from following $a_{j=1, \ldots, \infty}$. Furthermore, the inequality in (B.6) becomes an equality for the policy that attains the maximum on the right hand side of (20). Accordingly, that policy is optimal and $\hat{V}(X_{t_j}, S_{t_j})$ is the value function.

Q.E.D.

The next lemma shows that the value function is homogeneous of degree $1 - \alpha$.

**Lemma 15:** Letting $x_t \equiv \frac{X_t}{S_t}$, the value function satisfies (21).

**Proof:** Consider an optimal policy $a_{j=1, \ldots, \infty}$ associated with the initial state variables $(X_{t_j}, S_{t_j})$. Now suppose that we consider the initial state variables $(X_{t_j}, 1)$ and, additionally, we assume that $x_{t_j} = x_{t_j}^B$. Construct a policy $a_{j=1, \ldots, \infty}$ as follows. For all $j = 1, \ldots, \infty$, let $\tau_j = \tau_j^A$ and $\phi_j = \phi_j^A$, and also let
\[ C^B(t_j, \tau^B_j) = \frac{1}{S^B_{t_j}} C^A(t_j, \tau^A_j), \quad y^{b,B}(t_j) = \frac{1}{S^B_{t_j}} y^{b,A}(t_j), \quad \text{and} \quad y^{s,B}(t_j) = \frac{1}{S^B_{t_j}} y^{s,A}(t_j). \]

Using (4), (5), and (19), and the fact that policy \( A \) is feasible, it is straightforward to verify that policy \( B \) is feasible and implies a consumption process that is equal to \( c^B_t = c^A_t S^B_{t_j} \) for all \( t \in (t_j, t_{j+1}] \), all \( t_j \), and all realizations of uncertainty.

Accordingly, \( V(X^B_{t_j}, 1) = V(x^B_{t_j}, 1) = V(x^A_{t_j}, 1) \geq \frac{1}{(S^A_{t_j})^{1-\alpha}} V(X^A_{t_j}, S^A_{t_j}) \).

Similarly, consider a path that is optimal for \( (X^B_{t_j}, 1) \). Repeating the same arguments as above, the policy defined by \( \tau^A_j = \tau^B_j, \phi^A_j = \phi^B_j, \ C^A(t_j, \tau^A_j) = S^A_{t_j} C^B(t_j, \tau^B_j), \ y^{b,A}(t_j) = S^A_{t_j} y^{b,B}(t_j), \quad \text{and} \quad y^{s,A}(t_j) = S^A_{t_j} y^{s,B}(t_j) \) is feasible starting from \( (X^A_{t_j}, S^A_{t_j}) \), assuming always that \( x^A_{t_j} = x^B_{t_j} \). Moreover, this policy implies that \( c^A_t = S^A_{t_j} c^B_t \) for all \( t \in (t_j, t_{j+1}] \), all \( t_j \), and all realizations of uncertainty.

Accordingly, \( V(X^A_{t_j}, S^A_{t_j}) \geq (S^A_{t_j})^{1-\alpha} V(X^B_{t_j}, 1) = (S^A_{t_j})^{1-\alpha} V(x^A_{t_j}, 1) \).

Now letting \( v(x^A_{t_j}) \equiv (1 - \alpha) V(x^A_{t_j}, 1) \) and using \( \frac{v(x^A_{t_j})}{1-\alpha} = V(x^A_{t_j}, 1) \geq \frac{1}{(S^A_{t_j})^{1-\alpha}} \times V(X^A_{t_j}, S^A_{t_j}) \) together with \( \frac{v(x^A_{t_j})}{1-\alpha} (S^A_{t_j})^{1-\alpha} = V(x^A_{t_j}, 1) (S^A_{t_j})^{1-\alpha} \leq V(X^A_{t_j}, S^A_{t_j}) \) yields (21).

**Q.E.D.**

In preparation for the main proposition, we also introduce the norm

\[
\|f\| \equiv \max_{X_t, S_t \in R^2_+ \text{ s.t. } X_t + S_t = 1} |f(X_t, S_t)|.
\]

We let \( B \) denote the set of functions that map \( R^2_+ \rightarrow R_+ \) if \( \alpha < 1 \) (respectively, \( R^2_+ \rightarrow R_- \) if \( \alpha > 1 \)) that are homogeneous of degree \( 1 - \alpha \) and bounded in the norm defined in (B.7). Similarly, we let \( H \) denote the set of functions that belong in \( B \) and additionally are continuous. Finally, define the operator \( T \) applied to function \( f \) as

\[
Tf \equiv \sup_{C(t_j, \tau_j), y^{b}(t_j), y^{s}(t_j), \phi_j, \tau_j} \left\{ \left[ 1 - (1 - \alpha) \kappa b(\tau_j) \right] U(C(t_j, \tau_j)) \right\} + e^{-\rho \tau_j} E_i \left[ f(X_{t_{j+1}}, S_{t_{j+1}}) \right].
\]

The next proposition contains our main result.

**PROPOSITION 7:** The operator \( Tf \) maps \( H \) into \( H \) and has a fixed point in \( H \), which is the value function \( V \). Moreover, for \( f = V \), there exist policies that attain the optimum on the right hand side of (B.8) and these policies are optimal.

**PROOF:** First, we prove that \( Tf \) maps \( H \) into \( H \). Using the definition of \( U(C(t_j, \tau_j)) \) and inspection of (B.8), it is immediate that if \( f \) is homogeneous of
degree $1 - \alpha$ (so that it can be expressed as $\frac{1}{1-\alpha} (X_t + S_t)^{1-\alpha} \tilde{f}(\tilde{x}_t)$), then so is $Tf$. Next we observe that if $f$ is bounded in the norm (B.7), so is $Tf$. In the case $\alpha > 1$, the result is immediate, since $Tf$ is bounded above by zero and below by the feasible policy that sets $\tau_1 = \infty$, $y_s(t_j) = -[1 - \theta_S S_t]$ and $C(t_j, \infty) = X_t^+$, which implies the discounted utility $U \equiv \frac{1}{1-\alpha} (X_t + S_t)^{1-\alpha} \chi^{-(1-\alpha)}[(1 - \theta_X) \tilde{x}_t + (1 - \psi_S)(1 - \theta_S)(1 - \tilde{x}_t)]^{1-\alpha}$, where $\chi \equiv \frac{\rho - (1-\alpha) r_L}{\alpha} > 0$. Clearly, this feasible policy is bounded in the norm (B.7), and thus $Tf$ is bounded in the norm (B.7). In the case where $\alpha < 1$, we note that $U$ still provides a lower bound.

To derive an upper bound, let $l(\tau_j) \equiv \left[1 - (1 - \alpha) \kappa b(\tau_j)\right] \times [h(\tau_j)]^a$, define $G \equiv \sup_{\tau_j > 0} l(\tau_j)$, and observe that the assumptions of Lemma 1 imply that $G$ is finite. In turn, this implies that $[1 - (1 - \alpha) \kappa b(\tau_j)] \times U(C(t_j, \tau_j)) = [1 - (1 - \alpha) \kappa b(\tau_j)] \times \left[1 - \frac{1}{1-\alpha} h(\tau_j)\right]^{1-\alpha} \leq \frac{1}{1-\alpha} G[X_t + S_t]^{1-\alpha}$ is bounded in the norm (B.7). Moreover, $\|e^{-\rho \tau_j} E_{t_j} f(X_{t_{j+1}}, S_{t_{j+1}})\| \leq \|f(X_t, S_t)\|$ is bounded in the norm (B.7), since $f$ is bounded in the norm (B.7). Accordingly, $Tf$ is bounded in the norm (B.7) for both $\alpha < 1$ and $\alpha > 1$. Finally, $Tf$ maps continuous functions to continuous functions. (To see this, note that the right hand side of (B.8) can be expressed as the maximum of three functions, namely the maximal value conditional on $y^b > 0$, conditional on $y^s < 0$, and conditional on $y^b = y^s = 0$. Each of these functions is continuous by a version of the theorem of the maximum (see in particular Alvarez and Stokey (1998)\textsuperscript{42}, and hence so is the maximum of the three functions.) We conclude that $Tf$ maps $\mathcal{H}$ into $\mathcal{H}$.

To show that $Tf$ has a fixed point in $\mathcal{H}$, we adapt the arguments in Alvarez and Stokey (1998). Specifically, we distinguish two cases, depending on whether $\alpha < 1$ or $\alpha > 1$. The case $\alpha < 1$ allows a relatively straightforward proof based on a contraction mapping argument. The case $\alpha > 1$ requires a different set of arguments. It is useful to note that the proof that we develop for the case $\alpha > 1$ would provide an alternate proof (with obvious modifications) for the case $\alpha < 1$, but the reverse is not true.

We start with the case $\alpha < 1$. For this case, we start by proving the following implication of assumption (7).

**Lemma 16:** For all $\tau_j > 0$ and all $\phi_j \in [0, 1]$, assumption (7) implies that

$$e^{-\rho \tau_j} E_{t_j} \left\{ \left[ R(t_j, \tau_j) \right]^{1-\alpha} \right\} < 1.$$  

\textsuperscript{41}To see this, note that $l(\tau_j)$ is continuous, $\lim_{\tau_j \to 0} l(\tau_j) \leq 0$, and $\lim_{\tau_j \to \infty} l(\tau_j) = \frac{1}{\rho} < \infty$.

\textsuperscript{42}Notice in particular that $Tf$ is always bounded below by $U$ for any $f \in \mathcal{H}$. 

PROOF: To simplify notation, we fix some \( t_j \) and \( t_j + 1 \), and for any \( t \in [t_j, t_{j+1}] \), we let \( R_t \equiv R(t_j, t - t_j) = \phi_j \frac{P_t}{P_{t_j}} + (1 - \phi_j) e^{r(t - t_j)}. \) Applying Itô’s lemma gives

\[
\begin{align*}
    dR_t &= \phi_j \frac{P_t}{P_{t_j}} \mu dt + (1 - \phi_j) e^{r(t - t_j)} r_f dt + \phi_j \frac{P_t}{P_{t_j}} \sigma dz_t. 
\end{align*}
\]

Dividing both sides of (B.10) by \( R_t \) and letting \( \pi_t \equiv \frac{\phi_j(P_t/P_{t_j})}{\phi_j(P_t/P_{t_j}) + (1 - \phi_j)e^{r(t - t_j)}} \) gives

\[
\begin{align*}
    \frac{dR_t}{R_t} &= \pi_t \mu dt + (1 - \pi_t) r_f dt + \pi_t \sigma dz_t. 
\end{align*}
\]

The unique solution of the linear stochastic differential equation (B.11) for \( t \in [t_j, t_{j+1}] \) subject to the initial condition \( R_t \equiv R(t_j, 0) = 1 \) is given by

\[
R_t = e^{\int_{t_j}^t \left[ \pi_t \mu - \frac{1}{2} \pi_t^2 \sigma^2 + (1 - \pi_t) r_f \right] dt + \int_{t_j}^t \pi_t \sigma dz_t}. 
\]

Using (B.12) and recalling that \( R_{t_{j+1}} = R(t_j, t_{j+1} - t_j) = R(t_j, \tau_j) \), we obtain

\[
\begin{align*}
    e^{-\rho \tau_j} E \left\{ \left[ R(t_j, \tau_j) \right]^{1-\alpha} \right\} &= E \left\{ e^{-\rho \tau_j + (1 - \alpha) \int_{t_j}^{t_{j+1}} (\pi_t \mu - (1/2) \pi_t^2 \sigma^2 + (1 - \pi_t) r_f) dt + \int_{t_j}^{t_{j+1}} \pi_t \sigma dz_t} \right\} \\
    &\leq \max_{\pi_t} E \left\{ e^{-(\rho \tau_j + (1 - \alpha)) \int_{t_j}^{t_{j+1}} (\pi_t \mu - (1/2) \pi_t^2 \sigma^2 + (1 - \pi_t) r_f) dt + \int_{t_j}^{t_{j+1}} \pi_t \sigma dz_t} \right\}.
\end{align*}
\]

In light of (B.11) and (B.12), the maximization problem in (B.13) is identical to the Merton-type problem of maximizing \( \max_{\pi_t} e^{-\rho \tau_j} E \{ R_{t_{j+1}}^{1-\alpha} \} \) subject to the constant-investment-opportunity-set dynamics (B.11), which has the well known constant rebalancing solution \( \pi_t = \pi = \frac{\mu - r_f}{\sigma \alpha^2} \). Substituting this solution into (B.13), letting

\[
\nu \equiv (1 - \alpha) \left[ r_f + \frac{1}{2\alpha} \left( \frac{\mu - r_f}{\sigma} \right)^2 \right],
\]

and utilizing properties of the log-normal distribution gives

\[
\begin{align*}
    \max_{\pi_t \in [0, 1]} E \left\{ e^{-(\rho \tau_j + (1 - \alpha)) \int_{t_j}^{t_{j+1}} (\pi_t \mu - (1/2) \pi_t^2 \sigma^2 + (1 - \pi_t) r_f) dt + \int_{t_j}^{t_{j+1}} \pi_t \sigma dz_t} \right\} &= e^{\nu - \rho} r_f < 1.
\end{align*}
\]

Combining (B.14) with (B.13) and noting that \( \phi_j, \tau_j \) are arbitrary implies (B.9). \( \Box \)
We next define $\tilde{x}_t \equiv \frac{x_t}{x_{t+1}}$ = $\frac{x_t}{x_{t+1}}$ and observe that $x_t = \frac{\tilde{x}_t}{1-\tilde{x}_t}$. Because of (21), we obtain that

$$V(X_t, S_t) = \frac{1}{1 - \alpha} (X_t + S_t)^{1-\alpha} \left( \frac{S_t}{X_t + S_t} \right)^{1-\alpha} v(x_t)$$

$$= \frac{1}{1 - \alpha} (X_t + S_t)^{1-\alpha} (1 - \tilde{x}_t)^{1-\alpha} v\left( \frac{\tilde{x}_t}{1 - \tilde{x}_t} \right)$$

$$= \frac{1}{1 - \alpha} (X_t + S_t)^{1-\alpha} v^*(\tilde{x}_t),$$

where $v^*(\tilde{x}_t) \equiv (1 - \tilde{x}_t)^{1-\alpha} v\left( \frac{\tilde{x}_t}{1 - \tilde{x}_t} \right)$. In that case, we obtain that (B.16). To see why (B.16) holds, note that for any $T(\varepsilon)$ fine

Next, we show that

$$E_t \left\{ e^{-\rho\tau_j} \left( \frac{X_{t+1} + S_{t+1}}{X_t + S_t} \right)^{1-\alpha} \right\}$$

$$= E_t \left\{ e^{-\rho\tau_j} \left( e^{t\tau_j} (X_{t+j} - C(t_j, \tau_j)) + R(t_j, \tau_j) S_{t+j} \right)^{1-\alpha} \right\}$$

$$\leq \left( \frac{X_{t+j} + S_{t+j}}{X_t + S_t} \right)^{1-\alpha} e^{-\rho\tau_j} E_t \left\{ \left( e^{t\tau_j} \tilde{x}_{t+j} + R(t_j, \tau_j) (1 - \tilde{x}_{t+j}) \right)^{1-\alpha} \right\},$$

where the inequality follows from $C(t_j, \tau_j) \geq 0$ and the definition of $\tilde{x}_{t+j}$. Next we show that

$$e^{-\rho\tau_j} E_t \left\{ \left( e^{t\tau_j} \tilde{x}_{t+j} + R(t_j, \tau_j) (1 - \tilde{x}_{t+j}) \right)^{1-\alpha} \right\}$$

$$\leq \max_{\phi_j \in [0,1]} e^{-\rho\tau_j} E_t \{ R(t_j, \tau_j)^{1-\alpha} \}.$$

To see why (B.16) holds, note that for any $\phi_j \in [0, 1]$ and $\tilde{x}_{t+j} \in [0, 1]$, we obtain

$$e^{t\tau_j} \tilde{x}_{t+j} + R(t_j, \tau_j; \phi_j) \times (1 - \tilde{x}_{t+j}) = e^{t\tau_j} \tilde{x}_{t+j} + (\phi_j \frac{P_{t+j} + \tau_j}{P_{t+j}} + (1 - \phi_j)) \times (1 - \tilde{x}_{t+j}) \leq$$

$$\phi_j (1 - \tilde{x}_{t+j}) \frac{P_{t+j}}{P_{t+j}} + ((1 - \phi_j)(1 - \tilde{x}_{t+j}) + \tilde{x}_{t+j}) e^{t\tau_j} = \phi_j (1 - \tilde{x}_{t+j}) \frac{P_{t+j}}{P_{t+j}} + (1 - \phi_j)(1 - \tilde{x}_{t+j}) e^{t\tau_j} = R(t_j, \tau_j; \phi_j (1 - \tilde{x}_{t+j})).$$

Therefore, $E_t \{ e^{t\tau_j} \tilde{x}_{t+j} + R(t_j, \tau_j; \phi_j) (1 - \tilde{x}_{t+j}) \}^{1-\alpha} \leq E_t \{ R(t_j, \tau_j; \phi_j (1 - \tilde{x}_{t+j})) \}^{1-\alpha} \leq \max_{\phi_j \in [0,1]} E_t \{ R(t_j, \tau_j)^{1-\alpha} \}$.

Using (B.16) inside (B.15), noting that $\frac{X_{t+j} + S_{t+j}}{X_t + S_t} \leq 1$, and using (B.9), implies that $E_t \{ e^{-\rho\tau_j} \left( \frac{X_{t+1} + S_{t+1}}{X_t + S_t} \right)^{1-\alpha} \} < 1$. Suppose next that we choose some (arbitrarily small) $\varepsilon > 0$ and we confine attention to choices $\tau_j \geq \varepsilon$. (Also define $T(\varepsilon)$ to equal $T$ subject to $\tau_j \geq \varepsilon$.) Then we obtain that there exists
\( \beta < 1 \) such that \( e^{-\rho \tau_j} E_t j R(t_j, \tau_j)^{1-a} \leq \beta. \) Therefore, for any constant \( \eta \) and any function \( f \in \mathcal{H}, \) the operator \( T^{(\varepsilon)} \) satisfies the “discounting” property \( T^{(\varepsilon)}(f + \eta(X_t + S_t))^{1-a} \leq T^{(\varepsilon)} f + \eta \beta(X_t + S_t)^{1-a}. \) Furthermore, the operator \( T^{(\varepsilon)} \) satisfies the monotonicity property \( f \geq g \Rightarrow T^{(\varepsilon)} f \leq T^{(\varepsilon)} g. \) Accordingly, the operator \( T^{(\varepsilon)} \) is a contraction by Lemma 1 in Alvarez and Stokey (1998) (Boyd’s lemma) and possesses a unique fixed point \( V^{(\varepsilon)}. \) Since this fixed point is in \( \mathcal{H} \) (so that, in particular, \( TV^{(\varepsilon)} \) is bounded below and above in the norm (B.7) and continuous), it implies that the supremum on the right hand side of (B.8) is attained. Furthermore, the fixed point \( V^{(\varepsilon)} \) in \( \mathcal{H} \) and hence in \( \mathcal{B}. \) But note that for any function \( f \in \mathcal{B}, \) we obtain

\[
0 \leq \lim_{k \to \infty} e^{-\rho \varepsilon_k} E_t j f(X_t k + S_t k) = \lim_{k \to \infty} e^{-\rho \varepsilon_k} E_t j (X_t k + S_t k)^{1-a} \frac{f(X_t k, S_t k)}{(X_t k + S_t k)^{1-a}} \leq (X_t j + S_t j)^{1-a} f(X_t j, S_t j) = 0.
\]

Accordingly, by Lemma 14, \( V^{(\varepsilon)} \) is the value function subject to the additional constraint \( \tau j \geq \varepsilon. \)

Next consider a sequence of \( \varepsilon_k > 0 \) such that \( \lim_{k \to \infty} \varepsilon_k = 0. \) The associated sequence \( V^{(\varepsilon_k)}(X_t j, S_t j) \) is a nondecreasing sequence of functions, which is bounded above by \( V^{(0)}(X_t j, S_t j) \). Hence this sequence of functions converges pointwise to a limit \( \overline{V} = \lim_{\varepsilon_k \to 0} V^{(\varepsilon_k)}. \) The completeness of \( \mathcal{H} \) implies that \( \overline{V} \in \mathcal{H}. \) Moreover, \( \overline{V} = \lim_{\varepsilon_k \to 0} V^{(\varepsilon_k)} = \lim_{\varepsilon_k \to 0} T(\varepsilon_k)^{1/\varepsilon} V^{(\varepsilon_k)} = \overline{T} \overline{V}, \) where the last equality follows upon applying the theorem of the maximum to

\[
\lim_{\varepsilon_k \to 0} \sup_{C(t_j, \tau j), \gamma^h(t_j), \gamma^z(t_j), \phi_j, \tau j \geq \varepsilon_k} [1 - (1 - \alpha)\kappa b(\tau j)] U(C(t_j, \tau j)) + e^{-\rho \tau_j} E_t j \{V^{(\varepsilon_k)}(X_{t j+1}, S_{t j+1})\}
\]

and observing that the monotone convergence theorem implies that \( \lim_{\varepsilon_k \to 0} E_t j \{V^{(\varepsilon_k)}(X_{t j+1}, S_{t j+1})\} = E_t j \overline{V}(X_{t j+1}, S_{t j+1}) \). Accordingly, \( \overline{V} \in \mathcal{H} \) is a fixed point of (B.8). And since \( \overline{V} \in \mathcal{H} \) (so that, in particular, it is continuous and bounded in the norm (B.7)), it satisfies the rest of the requirements\(^{44}\) of

\(^{43}\) The fact that there exists such \( \beta \) follows from the fact that \( e^{-\rho \tau_j} E_t j R(t_j, \tau_j)^{1-a} \leq \sup_{\tau j \geq 0, \phi_j \in [0,1]} e^{-\rho \tau j} E_t j R(t_j, \tau j)^{1-a} = \max_{t j \geq 0, \phi_j \in [0,1]} e^{-\rho \tau j} E_t j R(t_j, \tau j)^{1-a}. \) To see why the supremum is attained, we note that a continuous function on a closed set attains its maximum, so that on any set \([\varepsilon, \overline{\varepsilon}],[\overline{\varepsilon} \geq \varepsilon > 0, \phi_j \in [0,1]) e^{-\rho \tau j} E_t j R(t_j, \tau j)^{1-a} \) attains a maximum. Moreover, since \( \lim_{\tau j \to \infty} e^{-\rho \tau_j} E_t j R(t_j, \tau_j)^{1-a} = 0, \) there exists \( \overline{\varepsilon} \geq \overline{\varepsilon} \) such that \( e^{-\rho \tau_j} E_t j R(t_j, \tau_j)^{1-a} \geq e^{-\rho \tau_j} E_t j R(t_j, \tau_j)^{1-a} \) for all \( \tau j \geq \overline{\varepsilon} \). Accordingly, we can confine attention to closed sets of \( \tau. \) Finally, by (B.9), \( \max_{t j \geq 0, \phi_j \in [0,1]} e^{-\rho \tau j} E_t j R(t_j, \tau_j)^{1-a} = 1. \)

\(^{44}\) We note that even if we remove the requirement that \( \tau j \geq \varepsilon, \) it is still the case that \( \lim_{k \to \infty} e^{-\rho \varepsilon_k} E_t j \left(\frac{X_t k + S_t k}{X_t k + S_t k}\right)^{1-a} = 0. \) Indeed, the proof of Lemma 16 implies that

\[
\lim_{k \to \infty} E_t j \prod_{k=1, \ldots, \infty} e^{-\rho \varepsilon_k} (\phi_k e^{-\tau j} + (1 - \phi_j) e^{-\rho \varepsilon_k})\left(\frac{X_t k + S_t k}{X_t k + S_t k}\right)^{1-a} \leq \lim_{k \to \infty} e^{-\rho (\varepsilon_k - \varepsilon)} = 0.
\]
Lemma 14 and, hence, is the value function. Moreover, the policies that attain the maximum on the right hand side of (B.8) are optimal.

We next consider the case $\alpha > 1$. In this case a contraction mapping argument does not necessarily apply (see, e.g., Alvarez and Stokey (1998)) and hence we need to take a more direct approach. The value function $V$ satisfies (B.8) by Lemma 13. Hence, it is a fixed point of (B.8). So it suffices to show that $V \in \mathcal{H}$. By Lemma 14, $V$ is homogeneous of degree $1 - \alpha$. Also by the arguments given as part of the proof of Lemma 13, $V$ is bounded above by zero and below by the (homogeneous of degree $1 - \alpha$) function

$$\frac{1}{1-\alpha}(X_{t_j} + S_{t_j})^{1-\alpha} \chi^{-a}[(1 - \theta_X)\tilde{x}_{t_j} + (1 - \psi_S)(1 - \theta_S)(1 - \tilde{x}_{t_j})]^{1-\alpha},$$

which corresponds to the feasible policy $\tau_1 = \infty$, $y_s(t_j) = -[1 - \theta_S]S_{t_j}$, and $C(t_j, \infty) = X_{t_j}$. Accordingly, $V \in \mathcal{B}$.

We next show that $V$ is continuous. To that end we start by introducing some notation. Let $W_{t_j} \equiv (X_{t_j}, S_{t_j})$ denote a two-dimensional vector with $X_{t_j}$ and $S_{t_j}$ its two elements. We also let $\|W_{t_j}\|_d \equiv \max(X_{t_j}, S_{t_j})$ and let $a_j = \{C(t_j, \tau_j), y^*(t_j), y^-(t_j), \phi_j, \tau_j\}$ denote some optimal policies starting from $W_{t_j}$.

We next show that for any $\eta > 0$, there exists $\Delta > 0$, such that $\|W_{t_j} - \hat{W}_{t_j}\|_d < \delta$ implies $V(W_{t_j}) < V(\hat{W}_{t_j}) - \eta$. To see this, fix $\eta > 0$. We next show that it is possible to find such $\epsilon > 0$, $\delta > 0$, $\hat{y}^a \leq 0$, $\hat{y}^b \geq 0$ so that

$$\left|1 - (1 - \alpha)\kappa b(\tau_j)\right| U(C(t_j, \tau_j)) - U(\tilde{C}(t_j, \tau_j)) < \frac{\eta}{2} \quad \text{for all } \|W_{t_j} - \tilde{W}_{t_j}\|_d < \delta,$$

where $\tilde{C}(t_j, \tau_j) = \tilde{X}_{t_j}^\top_+ - e^{-\rho_\tau_j}(1 - \epsilon)X_{t_j+1},$

$$\tilde{S}_{t_j+1} \geq (1 - \epsilon)S_{t_j+1},$$

and

$$\left|1 - (1 - \epsilon)^{1-\alpha}\sum_{i=j+1, \ldots, \infty} \left[1 - (1 - \alpha)\kappa b(\tau_i)\right] e^{-\rho(t_i - t_j)} U(C(t_i, \tau_i))\right| < \frac{\eta}{2}. $$

To show that it is possible to find such $\epsilon$, $\delta$, $\hat{y}^a \leq 0$, $\hat{y}^b \geq 0$, we start by observing that it is clearly possible to find sufficiently small $\epsilon > 0$ that satisfies (B.19). To show the existence of $\delta > 0$, $\hat{y}^a \leq 0$, $\hat{y}^b \geq 0$ satisfying (B.17) and (B.18), we distinguish three cases, namely (i) $y^a = y^b = 0$, (ii) $y^a < 0$, and (iii) $y^b > 0$. (Because of Lemma 4, it is never optimal to set $y^a < 0$ and simultaneously $y^b > 0$.) If $y^a = 0$ and $y^b = 0$, then setting $\hat{y}^a = 0$, $\hat{y}^b = 0$ implies

$$\text{Hence, using (B.15) and (B.16), it follows that } \lim_{t_k \to \infty} e^{-\rho t_k} E_t \left(\frac{X_{t_k} + S_{t_k}}{X_{t_j} + S_{t_j}}\right)^{1-\alpha} = 0.$$
that \( \hat{X}_{ij} = \hat{X}_{ij}, \hat{S}_{ij} = \hat{S}_{ij} \). In turn, for any \( \varepsilon > 0 \), there exists sufficiently small \( \tilde{\delta}(\varepsilon) > 0 \) so that for all \( \hat{S}_{ij} \) with \( |\hat{S}_{ij} - S_{ij}| < \tilde{\delta}(\varepsilon) \), condition (B.18) holds, since \( \hat{S}_{ij} - S_{ij} = \hat{S}_{ij} - S_{ij} \). Moreover, for a sufficiently small \( \varepsilon > 0 \), there exists \( \tilde{\delta}(\varepsilon) \) so that condition (B.17) holds for \( |\hat{X}_{ij} - X_{ij}| < \tilde{\delta}(\varepsilon) \). Accordingly, there exists sufficiently small \( \varepsilon \) and sufficiently small \( \delta(\varepsilon) < \min(\tilde{\delta}(\varepsilon), \tilde{\delta}(\varepsilon)) \) such that conditions (B.17), (B.18), and (B.19) hold. Next suppose that \( y^b > 0 \), then set \( \hat{y}^b = y^b - \varepsilon S_{ij} - (1 - \theta_S)[S_j - \hat{S}_{ij}] \). Once again observe that for (sufficiently small) \( \varepsilon > 0 \) and \( \tilde{\delta}(\varepsilon) \), we obtain that \( -\hat{y}^b > 0 \) as long as \( |\hat{S}_{ij} - S_{ij}| < \tilde{\delta}(\varepsilon) \). By construction, this choice of \( -\hat{y}^b \) satisfies constraint (B.18) with equality. Furthermore, for this choice of \( -\hat{y}^b \) and sufficiently small \( \varepsilon > 0 \), there exists \( \tilde{\delta}(\varepsilon) \) such that for all \( \|W_{ij} - \hat{W}_{ij}\| < \tilde{\delta}(\varepsilon) \), condition (B.17) also holds. Finally, if \( \hat{y}^b > 0 \), then set \( \tilde{\delta}(\varepsilon) \) such that conditions (B.17), (B.18), and (B.19) hold. Next suppose that \( \varepsilon > 0 \), and choose \( \tilde{\delta}(\varepsilon) \) such that for all \( \|W_{ij} - \hat{W}_{ij}\| < \tilde{\delta}(\varepsilon) \), condition (B.17) holds.

From this point onward, the proof follows from Alvarez and Stokey (1998, p. 177) and we repeat their argument for completeness. Specifically choose \( \varepsilon > 0, \delta > 0, \tilde{\delta}(\varepsilon) \leq 0, \tilde{\delta} > 0 \) so that conditions (B.17), (B.18), and (B.19) hold, and take some \( \hat{W}_{ij} \) with \( \|W_{ij} - \hat{W}_{ij}\| < \delta \). Consider the following policy \( a_{j=1, \ldots, \infty} \) with initial conditions \( \hat{W}_{ij} \). Set \( \hat{\tau}_j = \tau_j, \hat{\phi}_j = \phi_j \) for all \( j \geq 1 \). Furthermore, at \( t_i \), choose \( \hat{y}^a < 0, \tilde{\delta}(\varepsilon) \leq 0 \) consistent with (B.17), (B.18), and (B.19), and set \( \hat{C}(t_i, \tau_i) = \hat{X}_{ij} - \varepsilon^{-\kappa b(\tau_i)}(1 - \varepsilon)X_{ij+1} \). From \( t_i+1 \) onward, set \( \hat{y}^a = (1 - \varepsilon)y^a, \hat{y}^b = (1 - \varepsilon)y^b \), and \( \hat{C}(t_i+k, \tau_{i+k}) = (1 - \varepsilon)C(t_i+k, \tau_{i+k}) \) for all \( k \geq 1 \). By construction, this policy satisfies \( \hat{W}_{i+1} \geq (1 - \varepsilon)W_{i+1} \) and, hence, it is feasible. Furthermore, we obtain

\[
\begin{align*}
|V(W_{ij}) - U(\hat{W}_{ij}; a_{j=1, \ldots, \infty})| & \\
& \leq [1 - (1 - \alpha)\kappa b(\tau_i)] \times |U(C(t_i, \tau_i)) - U(\hat{C}(t_i, \tau_i))| \\
& \quad + |1 - (1 - \varepsilon)^{1 - \sigma}| \\
& \quad \times E_i \sum_{i=j+1, \ldots, \infty} [1 - (1 - \alpha)\kappa b(\tau_i)]e^{-\rho(t_i-t_j)}|U(C(t_i, \tau_i))| \\
& < \eta.
\end{align*}
\]

Accordingly, \( V(\hat{W}_{ij}) \geq U(\hat{W}_{ij}; a_{j=1, \ldots, \infty}) \geq V(W_{ij}) - \eta \).
By similar arguments, letting \( \hat{a}_j = \{\hat{C}(t_j, \tau_j), \hat{y}^p(t_j), \hat{y}^s(t_j), \hat{\phi}_j, \hat{\tau}_j\} \) denote an optimal policy starting from \( \hat{W}_{t_j} \), and reversing the roles of \( (W_{t_j}, a_j) \) and \( (\hat{W}_{t_j}, \hat{a}_j) \) in (B.17)–(B.19) implies the existence of small enough \( \delta_1 > 0 \), such that for all \( \hat{W}_{t_j} \) with \( \|W_{t_j} - \hat{W}_{t_j}\|_d < \delta_1 \), we also obtain \( V(W_{t_j}) \geq V(\hat{W}_{t_j}) - \eta \). We conclude that there exists small enough \( \Delta = \min\{\delta, \delta_1\} \) such that for all \( \|W_{t_j} - \hat{W}_{t_j}\|_d < \Delta \), we obtain \( |V(W_{t_j}) - V(\hat{W}_{t_j})| < \eta \), proving the continuity of \( V \).

Since, as we showed above, \( V \in \mathcal{H} \) and there always exists a choice (namely \( \tau_1 = \infty, y_j(t_j) = -[1 - \theta_S]S_j \)) that provides a lower bound to the expression inside curly brackets in (B.8), it follows that the supremum in (B.8) is attained when \( f = V \). Furthermore, Theorem 4.5 in Stokey and Lucas (1989) implies that the policy that maximizes the right hand side of (B.8) for \( f = V \) is optimal. Q.E.D.

**Remark 1:** We note that since the value function is unique, an implication of Lemma 14 and Proposition 7 is that \( V \) is the unique fixed point of \( T \) in \( \mathcal{H} \) satisfying the condition \( \lim_{t_k \to \infty} e^{-\rho t_k} E_t V(X_{t_k}, S_{t_k}) = 0 \).

**References**


The Wharton School, University of Pennsylvania, 3620 Locust Walk, Philadelphia, PA 19104, U.S.A. and National Bureau of Economic Research; abel@wharton.upenn.edu,

Kellogg School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208, U.S.A. and National Bureau of Economic Research; eberly@kellogg.northwestern.edu,

and

Booth School of Business, University of Chicago, 5807 S. Woodlawn Avenue, Chicago, IL 60637, U.S.A. and National Bureau of Economic Research; stavros.panageas@chicagobooth.edu.

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