SUPPLEMENT TO “BELIEF-FREE EQUILIBRIA IN GAMES WITH INCOMPLETE INFORMATION”

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This supplement contains some omitted details on the existence of belief-free equilibria for two families of games that are studied in the literature on reputation. Namely, for the specific cases considered in footnotes 15 and 19, we claim that it is possible to find games arbitrarily close to the respective original reputation games such that \( V^* \) has nonempty interior. In this supplementary material, we explain this in greater detail.

Consider first a one-sided incomplete information game \( \Gamma \) with known-own payoffs, where player 2’s payoff matrix is \( u_2 \), while player 1’s payoff is \( u_1 \) in state \( j = 1 \) and \(-u_2\) in state \( j = 2 \). In footnote 15 of the paper, we claim that there exists a game \( \hat{\Gamma} \) arbitrarily close to \( \Gamma \) for which the set of belief-free equilibria is nonempty.

Let us start with a two-player full information game where \( u_1 \) and \( u_2 \) are players’ payoff matrices, and assume that the set of individually rational payoffs of this game has nonempty interior (otherwise the question of reputation is trivial). Consider a complete information zero-sum two-player game of this type. Because player \( i \) is not using \( s_i^* \) in state \( j = 1 \), that is, \( u_i(s_i^*, s_{-i}) = v_i \), we claim that there exists a feasible payoff \( \Gamma' \) arbitrarily close to \( \Gamma \) and whose set of individually rational payoffs has nonempty interior. To this purpose we perturb payoffs in such a way that \((s_1^*, s_2^*)\) remains an equilibrium of \( \Gamma' \), but there exists a feasible payoff Pareto dominating \((s_1^*, s_2^*)\).

1. \( s_i^* \) is not completely mixed, for some \( i = 1, 2 \):

   (a) \( M_i > v_i \): Let \( s_i' \) denote some action assigned zero probability by \( s_i^* \), and increase \( u_{-i}(s_i', s_{-i}) \) by \( \epsilon > 0 \) for all \( s_{-i} \). Call \( u' \) the new payoff matrix. Since player \( i \) is not using \( s_i^* \), \( s_i' \) remains a best reply to \( s_i^* \), and since \( i \)'s payoff matrix has not changed, \( s_i' \) remains a best reply to \( s_i^* \). So \( s_i^* \) remains an equilibrium. Because player \( i \) does not use \( s_i' \), it means that \( u_i(s_i', s_{-i}^*) \leq v_i \) and so \( u_{-i}(s_i', s_{-i}^*) + \epsilon > v_{-i} \), while also \( u_i(s_i', s_{-i}^*) + u_{-i}(s_i', s_{-i}^*) + \epsilon > 0 \) (since the game is zero sum), that is, \( u_i(a^i) = M_i \), there exists a mixture \( \lambda a^i + (1 - \lambda)(s_i', s_{-i}^*) \) that strictly improves upon the Nash equilibrium \((s_1^*, s_2^*)\).

   (b) \( M_i = v_i \): This means that player \( i \) is getting his maximal payoff from playing \( s_i^* \) independently of player \(-i\)'s action, so that any strategy profile \((s_i', s_{-i})\), \( s_{-i} \in A_{-i} \), is a saddle point. So pick one action \( s_{-i} \) and consider the game in which \( u_i(s_i', s_{-i}) = u_i(s_i, s_{-i}) + \epsilon \) for all \( s_i \in A_i \) and some \( \epsilon > 0 \), and

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all other payoffs remain unchanged. Clearly, \((s^*_i, s_{-i})\) is an equilibrium point of the game \(u'\), and in this equilibrium point, player \(i\) receives \(v_i + \epsilon\). We then proceed as in the previous case: There exists a mixture \(\lambda a_i^{-1} + (1 - \lambda)(s^*_i, s_{-i})\) that strictly improves upon the Nash equilibrium \((s^*_i, s^*_2)\).

2. Both \(s^*_i\) and \(s^*_2\) are completely mixed: \(u(a^1) \in \mathbb{R}^2\) and \(u(a^2) \in \mathbb{R}^2\) are the two extremes of the set of feasible payoffs that is a segment with slope \(-45^\circ\), while \(u(s^*)\) is somewhere on the interior of this segment. Let \(a^1 = (a_1, a_2)\) and \(a^2 = (a'_1, a'_2)\). Both \(a_i\) and \(a'_i\) are in the support of \(s^*_i\) for \(i = 1, 2\), and let \(\alpha_i, \alpha'_i\) denote the probabilities of those actions given \(s^*_i\). Consider the payoffs \(u'\) such that \(u'(a_1, a_2) = u(a_1, a_2) + (\epsilon/\alpha_2, \epsilon/\alpha_1)\), \(u'(a_1, a'_2) = u(a_1, a'_2) + (-\epsilon/\alpha'_2, -\epsilon/\alpha_1)\), \(u'(a'_1, a_2) = u(a'_1, a_2) + (-\epsilon/\alpha_2, -\epsilon/\alpha'_1)\), and \(u'(a'_1, a'_2) = u(a'_1, a'_2) + (\epsilon/\alpha'_2, \epsilon/\alpha'_1)\) (all other entries are left unchanged). By construction, \(s^*\) remains an equilibrium in game \(u'\), leading to payoffs \(v_i = v\) and \(v_2 = -v\). Now the action profiles \((a_1, a_2)\) and \((a'_1, a'_2)\) provide two points that are above the \(-45^\circ\) line, namely \((M_1 + \epsilon, -M_1 + \epsilon)\) and \((-M_2 + \epsilon, M_2 + \epsilon)\), respectively. Hence there exists a convex combination of \(a^1\) and \(a^2\) that is a Pareto improvement with respect to the Nash equilibrium \((s^*_i, s^*_2)\).

Let \(\tilde{\Gamma}\) be the one-sided incomplete information game with known-own payoffs where player 2’s payoff matrix is \(u'_{2}\), while player 1’s payoff is \(u_{1}\) in state \(j = 1\) and \(u'_{1}\) in state \(j = 2\). Here \(u'_{1}\) and \(u'_{2}\) are obtained as described above and are such that \(u'_{1}\) and \(u'_{2}\) are arbitrarily close to \(-u_{2}\) and \(u_{2}\), respectively.

The purpose is to show that the set of belief-free equilibria in \(\tilde{\Gamma}\) is nonempty. Consider the following construction.

Let \(\alpha^*\) be the occupation measure generated by strategy profile \((s^*_1, s^*_2)\). Let \(A^{IR}_2\) be the set of occupation measures leading to payoffs that are individually rational for player 2. This set has nonempty interior and includes \(\alpha^*\). Let \(\alpha^{1,j}\) be the \(\alpha \in A^{IR}_2\) preferred by player 1 in state \(j = 1, 2\). The payoffs originated by \(\alpha^{1,1}\) and \(\alpha^{1,2}\) are incentive compatible for player 1 (and generically strictly incentive compatible provided \(|A| > 3\)). We shall show that \(\alpha^{1,1}\) and \(\alpha^{1,2}\) generate strictly individually rational payoffs for player 1, that is to say, player 2 has a strategy \(\hat{s}_2\) that punishes player 1 in the two states. Let \(B^1\) be player 1’s best reply correspondence in state \(j\). Note first that payoffs that strictly Pareto dominate \((u'_1(s^*_1, s'_2), u'_2(s^*_1, s'_2))\) exist by construction of \(\Gamma'\) and are reachable with occupation measures that are in \(A^{IR}_2\). Thus, \(u'_1(\alpha^{1,2}) > u'_1(s^*_1, s'_2)\). Also, since \(s^*\) minimaxes player 2’s payoff, it results in \(u'_2(B^1(s^*_2), s'_2) \geq u'_2(s^*_1, s'_2)\) and hence \(u'_1(\alpha^{1,1}) \geq u'_1(B^1(s^*_2), s'_2)\), as \((B^1(s^*_2), s'_2)\) is in \(A^{IR}_2\). Let \(\epsilon := (u'_1(\alpha^{1,2}) - u'_1(s^*_1, s'_2))/2 > 0\). Then there are strategies \(s_2\) close to \(s^*_2\) such that \(u'_1(B^2(s_2), s_2) < u'_1(s^*_1, s'_2) + \epsilon < u'_1(\alpha^{1,2})\). Thus, we can define player 2’s punishment strategy, \(\hat{s}_2\), as the \(s_2\) that solves

\[
\inf_{s_2} u_1(B^1(s_2), s_2)
\]

\(\epsilon\) Recall that the set of individually rational payoffs in the initial game has nonempty interior.
\[ u'_1(B^2(s_2), s_2) < u'_1(s^*_2, s^*_2) + \varepsilon. \]

Noting that \( u_1(B^1(\hat{s}_2), \hat{s}_2) \leq u_1(B^1(s^*_2), s^*_2) \leq u_1(\alpha^{1.1}) \) and considering that the set of individually rational payoff has nonempty interior in the full information game corresponding to state \( j = 1 \), it follows that generically \( u_1(B^2(\hat{s}_2), \hat{s}_2) < u_1(\alpha^{1.1}). \) Finally note that \( \alpha^{1,1} \) and \( \alpha^{1,2} \) are individually rational for player 2 as they are in \( A^{\text{IR}_2}. \) Strict individual rationality can be obtained by slightly perturbing \( \alpha^{1,1} \) and \( \alpha^{1,2} \) if necessary. This can be done without violating player 1 individual rationality and incentive compatible constraint since these constraints are strictly satisfied at \( \alpha^{1,1} \) and \( \alpha^{1,2}. \)

A similar, but simpler, construction works for the case of dominant action games (footnote 19 in the paper): Pick, for instance, the commitment type’s payoff (for whom the payoff from the dominant action is only “nearly” independent of his opponent’s action) to be such that the ranking over player 2’s pure actions (given his own dominant action) is the same for both types. Then, since the minmax action of player 2 is independent of player 1’s type, we can find a distribution over action profiles that is both weakly incentive compatible and strictly individual rational for both players (take a “pooling” distribution in which player 1 plays his dominant action and player 2 does not play for sure his minmax action)\(^2\); since there are two states and four action profiles, there will also be strictly individually rational, strictly incentive compatible distributions.

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\(^2\)More precisely, this works if, as in the example in the paper, the dominant action for player 1 is not the action that minmaxes player 2; otherwise, player 1 must also play another action with small probability.