SUPPLEMENT TO "ASYMPTOTICS FOR STATISTICAL TREATMENT RULES"

(*Econometrica*, Vol. 77, No. 5, September 2009, 1683–1701)

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This supplement contains the proofs for "Asymptotics for Statistical Treatment Rules."

APPENDIX 1: PROOFS FOR THE PARAMETRIC CASE

We give some assumptions and lemmas that can be used to extend the results to other loss functions. Let $\mathcal{D}_\infty$ denote the set of all randomized treatment rules in the $N(h, I_0^{-1})$ experiment.

ASSUMPTION 1: *Given a loss function L, there exists $L_\infty(\delta, h)$ such that for some sequence r_n,*

$$\lim_{n\to\infty} r_n\left[L\left(1, \theta_0 + \frac{h}{\sqrt{n}}\right) - L\left(0, \theta_0 + \frac{h}{\sqrt{n}}\right)\right] = L_\infty(1, h) - L_\infty(0, h)$$

and

$$\lim_{n\to\infty} r_n L\left[\left(0, \theta_0 + \frac{h}{\sqrt{n}}\right)\right] = L_\infty(0, h)$$

for almost every h (with respect to Lebesgue measure on $\mathbb{R}^k$).

ASSUMPTION 2: *Assume that loss L_∞ in the limit experiment depends on h only through $\dot{g}'h$. That is, there exists L_g such that $L_\infty(a, h) = L_g(a, \dot{g}'h)$ for $a \in \{0, 1\}$.*

LEMMA 1: *Suppose the conditions of Theorem 3.2 are satisfied by a sequence of treatment assignment rules $\delta_n \in \mathcal{D}$. Let loss L satisfy Assumption 1.*

(i) *Then,*

$$\liminf_{n\to\infty} r_n B_n(\delta_n, \Pi) \geq \pi(\theta_0) \inf_{\delta\in\mathcal{D}_\infty} B_\infty(\delta).$$

(ii) *Moreover, if $\delta_n^* \in \mathcal{D}$ is matched by $\delta^* \in \mathcal{D}_\infty$ in the sense of Proposition 3.1 and if δ^* is the flat-prior Bayes rule in the limit experiment, then δ_n^* is the asymptotically optimal rule for Bayes risk:*

$$\lim_{n\to\infty} r_n B_n(\delta_n^*, \Pi) = \pi(\theta_0) B_\infty(\delta^*) = \pi(\theta_0) \inf_{\delta\in\mathcal{D}_\infty} B_\infty(\delta).$$

 DOI: 10.3982/ECTA6630

PROOF: For the sequence $\{\delta_n\}$, let $\bar{\delta}$ be the matching treatment assignment rule in the limit experiment as given by Proposition 3.1. Then

$$\begin{aligned}
&\liminf_{n\longrightarrow\infty} r_n B_n(\delta_n, \pi) \\
&\quad \geq \int \liminf_{n\longrightarrow\infty} \left\{ \int r_n L\left(\delta_n(z), \theta_0 + \frac{h}{\sqrt{n}}\right) dP^n_{\theta_0+h/\sqrt{n}}(z) \right. \\
&\qquad \left. \times\, \pi\left(\theta_0 + \frac{h}{\sqrt{n}}\right)\right\} dh \\
&\quad = \int \liminf_{n\to\infty} \left\{ \left(E_{\theta_0+h/\sqrt{n}}[\delta_n(Z_n)] \right.\right. \\
&\qquad \times r_n \left[L\left(1, \theta_0 + \frac{h}{\sqrt{n}}\right) - L\left(0, \theta_0 + \frac{h}{\sqrt{n}}\right)\right] \\
&\qquad \left.\left. + r_n L\left(0, \theta_0 + \frac{h}{\sqrt{n}}\right)\right)\pi\left(\theta_0 + \frac{h}{\sqrt{n}}\right)\right\} dh \\
&\quad = \pi(\theta_0) \int \big(E_h[\bar{\delta}(\Delta)][L_\infty(1,h) - L_\infty(0,h)] + L_\infty(0,h)\big)\, dh \\
&\quad = \pi(\theta_0) B_\infty(\bar{\delta}) \\
&\quad \geq \pi(\theta_0) \inf_{\delta\in\mathcal{D}_\infty} B_\infty(\delta),
\end{aligned}$$

where the first inequality follows by Fatou's lemma and the second equality follows by Proposition 3.1, Assumption 1, and the continuity of π. An analogous argument yields $\limsup_{n\to\infty} r_n B_n(\delta_n, \Pi) \leq \pi(\theta_0) B_\infty(\bar{\delta})$. Applying these conclusions to δ_n^* and δ^* proves (ii). *Q.E.D.*

LEMMA 2: *Suppose that loss L_∞ satisfies Assumption 2. Let*

$$\delta^*(\Delta) = \mathbf{1}\big\{E_s[L_g(1,s)] \leq E_s[L_g(0,s)]\big\},$$

where $s \sim N(\dot{g}'\Delta, \sigma_g^2)$. Then δ^* is the flat-prior Bayes rule in the limit experiment:

$$B_\infty(\delta^*) = \inf_{\delta\in\mathcal{D}_\infty} B_\infty(\delta).$$

PROOF: By Fubini's theorem, we can rewrite the flat-prior Bayes risk as

$$\begin{aligned}
B_\infty(\delta) = \iint & L_\infty(\delta(\Delta), h)(2\pi)^{-k/2}|I_0|^{-1/2} \\
& \times \exp(-(\Delta - h)' I_\theta(\Delta - h)/2)\, dh\, d\Delta.
\end{aligned}$$

As usual, the Bayes optimality problem is equivalent to minimizing posterior expected loss for each observable Δ. The posterior expected loss for the rule δ in the Gaussian limit experiment, at a fixed Δ, is

$$\begin{aligned}
&\int L_\infty(\delta(\Delta), h)\, dN(\Delta, I_\theta^{-1})(h) \\
&\quad = \int L_g(0, s)\, dN(\dot{g}'\Delta, \sigma_g^2)(s) \\
&\qquad + \delta(\Delta) \int [L_g(1, s) - L_g(0, s)]\, dN(\dot{g}'\Delta, \sigma_g^2)(s).
\end{aligned}$$

The optimal Bayes rule then is determined by the last term and the statement of the lemma follows. *Q.E.D.*

LEMMA 3: *Suppose the conditions of Theorem* 3.2 *are satisfied. Then, for every* $h \in \mathbb{R}^k$,

$$\sqrt{n}\frac{g(x, \hat{\theta}_n)}{\hat{\sigma}_g} \overset{h}{\rightsquigarrow} N(\dot{g}'h, 1).$$

PROOF: By differentiability in quadratic mean, the sequence of experiments is locally asymptotically normal. For all sequences $h_n \to h$ in $\mathbb{R}^k$,

$$\log \frac{dP^n_{\theta_0 + h_n/\sqrt{n}}}{dP^n_{\theta_0}} = h'S_n - \frac{1}{2}h'I_0 h + o_{P_{\theta_0}}(1),$$

where $S_n \overset{\theta_0}{\rightsquigarrow} N(0, I_0)$. Since $\hat{\theta}_n$ is best regular, Lemma 8.14 of Van der Vaart (1998) implies $\sqrt{n}(\hat{\theta}_n - \theta_0) = I_0^{-1}S_n + o_{P^n_{\theta_0}}(1)$ under θ_0. By Slutsky's theorem and the delta method, under θ_0,

$$\begin{aligned}
&\left(\sqrt{n}\frac{g(x, \hat{\theta})}{\hat{\sigma}_g}, \log \frac{dP^n_{\theta_0 + h/\sqrt{n}}}{dP^n_{\theta_0}}\right) \\
&\quad = \left(\frac{1}{\sigma_g}\dot{g}'I_0^{-1}S_n, h'S_n - \frac{1}{2}h'I_0 h\right) + o_{P^n_{\theta_0}}(1) \\
&\quad \rightsquigarrow N\left(\begin{pmatrix} 0 \\ -\frac{1}{2}h'I_0 h \end{pmatrix}, \begin{pmatrix} 1 & \frac{\dot{g}'h}{\sigma_g} \\ \frac{\dot{g}'h}{\sigma_g} & h'I_0 h \end{pmatrix}\right).
\end{aligned}$$

Then by Le Cam's third lemma, the conclusion of the lemma follows. *Q.E.D.*

PROOF OF THEOREM 3.2: Since losses L^H and L^T satisfy Assumption 1, and L^H_∞ and L^T_∞ satisfy Assumption 2, Lemma 2 establishes Bayes rules in the limit experiment and Lemma 3 can be used to show that these rules are the matching rules for the sequences of rules in the statement of the theorem. Lemma 1 then states that these sequences of rules are asymptotically Bayes optimal as desired.

Suppose $s \sim N(\dot{g}'\Delta, \sigma_g^2)$. Then $E[L^H_g(1, s)] = K\Phi(-\dot{g}'\Delta/\sigma_g)$ and $E[L^H_g(0, s)] = \Phi(\dot{g}'\Delta/\sigma_g)$. By Lemma 2, the flat-prior Bayes rule in the limit experiment for L^H_∞ is $\delta^{H,B}(\Delta) = \mathbf{1}\{\frac{\dot{g}'\Delta}{\sigma_g} \geq c^{H,B}\}$. By Lemma 3, $\lim_{n\to\infty} E_{\theta_0+h/\sqrt{n}}[\delta^{H,B}_n(Z_n)] = \Pr_h(\dot{g}'\Delta/\sigma_g \geq c^{H,B}) = E_h[\delta^{H,B}(\Delta)]$.

Similarly, for loss L^T, $E[L^T_g(1, s)] = -K[\dot{g}'\Delta\Phi(-\dot{g}'\Delta/\sigma_g) - \sigma_g\phi(\dot{g}'\Delta/\sigma_g)]$ and $E[L^T_g(0, s)] = \dot{g}'\Delta\Phi(\dot{g}'\Delta/\sigma_g) + \sigma_g\phi(\dot{g}'\Delta/\sigma_g)$. Differentiation shows that $E[L^T_g(0, s)] - E[L^T_g(1, s)]$ is monotonically increasing in $(\dot{g}'\Delta)$, and the optimal decision rule will be determined by the cutoff $c^{T,B}$. By Lemmas 2 and 3, $\delta^{T,B}_n$ is matched by the flat-prior Bayes rule in the limit experiment. Lemma 1 then yields asymptotic optimality of $\delta^{H,B}_n$ and $\delta^{T,B}_n$. *Q.E.D.*

PROOF OF COROLLARY 3.3: From Lemma 1(i), $\liminf_{n\to\infty} B^j_n(\delta^{j,\text{Bayes}}_n, \Pi) \geq \pi(\theta_0)\inf_{\delta\in\mathcal{D}_\infty} B_\infty(\delta)$. Also, by definition, $B^j_n(\delta^{j,\text{Bayes}}_n, \Pi) \leq B^j_n(\delta^{j,B}_n, \Pi)$ for every n, so $\liminf_{n\to\infty} B^j_n(\delta^{j,\text{Bayes}}_n, \Pi) \leq \liminf_{n\to\infty} B^j_n(\delta^{j,B}_n, \Pi) = \pi(\theta_0) \times \inf_{\delta\in\mathcal{D}_\infty} B^j_\infty(\delta)$. *Q.E.D.*

PROOF OF THEOREM 3.4: Part (i) would follow from a multivariate extension of Karlin and Rubin (1956, Theorem 1). A direct proof follows.

Note that if $\dot{g}'h_0 \neq 0$, then for $\tilde{h}_0 = h_1(-\dot{g}'h_0, h_0)$, $\dot{g}'\tilde{h}_0 = 0$. Since $\{h_1(b, h_0) : b \in \mathbb{R}\} = \{h_1(b, \tilde{h}_0) : b \in \mathbb{R}\}$, we may assume without loss of generality that, in fact, $\dot{g}'h_0 = 0$.

Recall that $\dot{g}'\Delta \sim N(0, \dot{g}'I_0^{-1}\dot{g})$ under h_0, so $E_{h_0}[\delta_c(\Delta)] = 1 - \Phi(c/\sqrt{\dot{g}'I_0^{-1}\dot{g}})$. Let $\tilde{\delta}$ be an arbitrary treatment assignment rule. We can choose c to satisfy $E_{h_0}[\delta_c(\Delta)] = E_{h_0}[\tilde{\delta}(\Delta)]$.

Now, following the method in the proof of Van der Vaart (1998, Proposition 15.2), take some $b > 0$ and consider the test $H_0 : h = h_0$ against $H_1 : h = h_1(b, h_0)$ based on $\Delta \overset{h}{\sim} N(h, I_0^{-1})$. Note that $\dot{g}'h_1 = b > 0$. The likelihood ratio (LR) is:

$$\text{LR} = \frac{dN(h_1, I_0^{-1})}{dN(h_0, I_0^{-1})} = \exp\left(\frac{b}{\dot{g}'I_0^{-1}\dot{g}}\dot{g}'\Delta - \frac{b^2}{2\dot{g}'I_0^{-1}\dot{g}}\right).$$

By the Neyman–Pearson lemma, a most powerful test is based on rejecting for large values of $\dot{g}'\Delta$. Since the test δ_c has been defined to have the same size as $\tilde{\delta}$, $E_{h_1(b,h_0)}[\delta_c(\Delta)] \geq E_{h_1(b,h_0)}[\tilde{\delta}(\Delta)]$. Moreover, this inequality similarly holds for all $b \geq 0$. Similarly, for $b < 0$, $1 - \delta_c = 1(\dot{g}'\Delta \leq c)$ is most powerful, leading to $E_{h_1(b,h_0)}[\delta_c(\Delta)] \le E_{h_1(b,h_0)}[\tilde{\delta}(\Delta)]$ for all $b \le 0$. Since $R(\tilde{\delta}, h) - R(\delta_c, h) = [L(1,h) - L(0,h)]\{E_h[\delta_1(\Delta)] - E_h[\delta_2(\Delta)]\}$, we can conclude that $R(\tilde{\delta}, h) \ge R(\delta_c, h)$ for all $h \in \{h_1(b, h_0) : b \in \mathbb{R}\}$.

For part (ii), let $R^* = \inf_{\delta \in \mathcal{D}_\infty} \sup_h R(\delta, h)$ and let δ^* be such that $\sup_h R(\delta^*, h) = R^*$. By part (i), there exists c^* such that $R(\delta^*, h_1(b, 0)) \ge R(\delta_{c^*}, h_1(b, 0))$ for all b. Note that $\dot{g}'\Delta \sim N(b, \dot{g}' I_0^{-1} \dot{g})$ under $h = h_1(b, h_0)$. Hence $E_{h_1(b,h_0)}[\delta_{c^*}(\Delta)] = E_{h_1(b,0)}[\delta_{c^*}(\Delta)]$ for all h_0 with $\dot{g}' h_0 = 0$. Also, loss can be rewritten $L_g(a, \dot{g}'h) = L(a, h)$ for $a \in \{0, 1\}$. Recalling that $b = \dot{g}' h_1(b, h_0) = \dot{g}' h_1(b, 0)$, we have

$$\begin{aligned} R(\delta_{c^*}, h_1(b, h_0)) &= L_g(0, b) + E_{h_1(b,0)}[\delta_{c^*}(\Delta)][L_g(1, b) - L_g(0, b)] \\ &= R(\delta_{c^*}, h_1(b, 0)). \end{aligned}$$

Then,

$$\begin{aligned} R^* &\ge \sup_b R(\delta^*, h_1(b, 0)) \ge \sup_b R(\delta_{c^*}, h_1(b, 0)) \\ &= \sup_{h_0} \sup_b R(\delta_{c^*}, h_1(b, h_0)) = \sup_h R(\delta_{c^*}, h) \ge R^*, \end{aligned}$$

so δ_{c^*} attains the bound and must be a solution to $\inf_c \sup_b R(\delta_c, h_1(b, 0))$. *Q.E.D.*

LEMMA 4: *Suppose the conditions of Proposition* 3.1 *are satisfied and* $\delta_n \in \mathcal{D}$ *is a sequence of treatment assignment rules with matching rule* δ *in the limit experiment as given by Proposition* 3.1. *Let* J *be a finite subset of* $\mathbb{R}^k$. *If*

$$\lim_{n\to\infty} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right) = R_\infty(\delta, h) \quad \left[\liminf_{n\to\infty} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right) \ge R_\infty(\delta, h)\right] \tag{A.1}$$

for $h \in J$, *then*

$$\liminf_{n\to\infty} \sup_{h\in J} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right) = [\ge] \sup_{h\in J} R_\infty(\delta, h).$$

Furthermore, if (A.1) *holds for all* $h \in \mathbb{R}^k$, *then*

$$\sup_J \liminf_{n\to\infty} \sup_{h\in J} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right) = [\ge] \sup_h R_\infty(\delta, h), \tag{A.2}$$

where the outer supremum is taken over all finite subsets of $\mathbb{R}^k$.

PROOF: Fix a finite subset J. Then

$$\begin{aligned}\liminf_{n\to\infty}\sup_{h\in J} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right) &\geq \sup_{h\in J}\liminf_{n\to\infty} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right)\\ &= \sup_{h\in J}\lim_{n\to\infty} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right)\\ &= \sup_{h\in J} R_\infty(\delta, h)\end{aligned}$$

$$\left[\liminf_{n\to\infty}\sup_{h\in J} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right) \geq \sup_{h\in J}\liminf_{n\to\infty} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right)\right.$$
$$\left.\geq \sup_{h\in J} R_\infty(\delta, h)\right].$$

The bracketed inequality in (A.2) follows trivially from the above bracketed expression. Now we show that the equality in (A.2) holds. By the supposition of the lemma, take $\varepsilon > 0$ and any $h \in J$. Then there exists N_h such that for $n \geq N_h$, $r_n R(\delta_n, \theta_0 + h/\sqrt{n}) \leq R_\infty(\delta, h) + \varepsilon \leq \sup_{h'\in J} R_\infty(\delta, h') + \varepsilon$. Let $N = \max_{h\in J} N_h$. Then, for $n \geq N$,

$$\sup_{h\in J} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right) \leq \sup_{h'\in J} R_\infty(\delta, h') + \varepsilon$$

and

$$\liminf_{n\to\infty}\sup_{h\in J} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right) \leq \sup_{h\in J} R_\infty(\delta, h) + \varepsilon.$$

Since this holds for any $\varepsilon > 0$, we have $\liminf_{n\to\infty}\sup_{h\in J} r_n R(\delta_n, \theta_0 + h/\sqrt{n}) = \sup_{h\in J} R_\infty(\delta, h)$, so

$$\begin{aligned}\sup_J \liminf_{n\to\infty}\sup_{h\in J} r_n R\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right) &= \sup_J \sup_{h\in J} R_\infty(\delta, h)\\ &= \sup_h R_\infty(\delta, h).\end{aligned}$$

Q.E.D.

Let $\bar{\delta}_c(\Delta) = \mathbf{1}\{\dot{g}'\Delta/\sigma_g \geq c\}$.

LEMMA 5: *The solutions to*

$$\inf_c \sup_h R^j_\infty(\bar{\delta}_c, h)$$

for $j = H, T$ *are the constants* $c^{j,M}$.

PROOF: Let $b = \dot{g}'h/\sigma_g$. Then $R^H_\infty(\bar{\delta}_c, h) = R^H_g(\bar{\delta}_c, \sigma_g b) = \Phi(c-b)\mathbf{1}(b > 0) + K(1-\Phi(c-b))\mathbf{1}(b \leq 0)$, so $\sup_h R^H_\infty(\bar{\delta}_c, h) = \max\{\Phi(c), K(1-\Phi(c))\}$. The terms $\Phi(c)$ and $K(1-\Phi(c))$ are strictly increasing and decreasing in c, and cross at a unique point, which must minimize maximum risk. For loss L^H, the crossing point is the solution given in the conclusion of the lemma.

For loss L^T, we can treat $\dot{g}'\Delta$ as the scalar observable; the cutoff value given in the lemma was derived in Tetenov (2007). We note here that the solution is well behaved. Let $r^+(c, b) = b\Phi(c-b)$ and $r^-(c, b) = -Kb\Phi(b-c)$. Then $\sup_b R^T_g(\bar{\delta}_c, \sigma_g b) = \sigma_g \cdot \max\{\sup_{b>0} r^+(c, b), \sup_{b\leq 0} r^-(c, b)\}$. From the first and second derivatives of r^+ in b (for any fixed finite value of c), it is straightforward to show that r^+ is single-peaked with a unique, finite global maximum on $[0, \infty)$. The same conclusion is true of r^- on $(-\infty, 0]$. Also, $\sup_{b>0} r^+(c, b)$ is strictly increasing in c and $\sup_{b\leq 0} r^-(c, b)$ is strictly decreasing, and they cross at the unique value $c^{T,M}$, which must minimize the maximum risk. *Q.E.D.*

PROOF OF THEOREM 3.5: From Theorem 3.4(ii), it suffices to look at cutoff rules along a "slice" $h_1(b, 0)$ to obtain the minmax rule in the limit experiment. Note that the classes of rules $\{\delta_c\}$ and $\{\bar{\delta}_c\}$ are equivalent, and so it suffices to consider rules of the form $\bar{\delta}_c$. Lemma 5 provides the minmax rules for losses L^H and L^T in the limit experiment.

Given a sequence of rules δ_n and a matching rule $\bar{\delta}$ in the limit experiment, $\lim_{n\to\infty} r_n R^T(\delta_n, \theta_0 + h/\sqrt{n}) = R^T_\infty(\bar{\delta}, h)$. Also, by Lemma 3, $\delta^{T,M}_n$ is matched in the limit experiment by $\delta^{T,M}$. Lemma 4 states that the risk bound in the limit experiment is the asymptotic risk bound and that it is attained by $\delta^{T,M}_n$.

For loss L^H, for h such that $\dot{g}'h \neq 0$, $\lim_{n\to\infty} r_n R^H(\delta_n, \theta_0 + h/\sqrt{n}) = R^H_\infty(\bar{\delta}, h)$. For h such that $\dot{g}'h = 0$, $R^H_\infty(\bar{\delta}, h) = 0$, so for all h, $\liminf_{n\to\infty} r_n \times R^H(\delta_n, \theta_0 + h/\sqrt{n}) \geq R^H_\infty(\bar{\delta}, h)$. Hence, by Lemma 4,

$$\sup_J \liminf_{n\longrightarrow\infty} \sup_{h\in J} \sqrt{n} R^H\left(\delta_n, \theta_0 + \frac{h}{\sqrt{n}}\right) \geq \sup_h R^H_\infty(\bar{\delta}, h) \geq \inf_{\delta\in\mathcal{D}_\infty} \sup_h R^H_\infty(\delta, h) = \sup_h R^H_\infty(\delta^{H,M}, h),$$

where the outer supremum is taken over all finite subsets of $\mathbb{R}^k$. Also, by Lemma 3, $\delta^{H,M}_n$ is matched in the limit experiment by $\delta^{H,M}$.

Let $\tilde{J}$ be any finite subset such that $\dot{g}'h \neq 0$ for $h \in \tilde{J}$. By Lemma 4,

$$\sup_{\tilde{J}} \liminf_{n\longrightarrow\infty} \sup_{h\in\tilde{J}} R^H\left(\delta^{H,M}_n, \theta_0 + \frac{h}{\sqrt{n}}\right) = \sup_{\tilde{J}} \sup_{h\in\tilde{J}} R^H_\infty(\delta^{H,M}, h) \leq \sup_h R^H_\infty(\delta^{H,M}, h) = \inf_{\delta\in\mathcal{D}_\infty} \sup_h R^H_\infty(\delta, h). \tag{A.3}$$

Next, take $\varepsilon > 0$. We will show that

$$\sup_J \liminf_{n\longrightarrow\infty} \sup_{h\in J} R^H\left(\delta_n^{H,M}, \theta_0 + \frac{h}{\sqrt{n}}\right) \le \sup_{\tilde{J}} \liminf_{n\longrightarrow\infty} \sup_{h\in\tilde{J}} R^H\left(\delta_n^{H,M}, \theta_0 + \frac{h}{\sqrt{n}}\right) + \varepsilon. \tag{A.4}$$

Take a finite subset $J \subset \mathbb{R}^k$ such that for exactly one element $h_0 \in J$, $\dot{g}'h_0 = 0$. For $\tau > 0$, define $h' = h_0 + \tau\dot{g}$ and $h'' = h_0 - \tau\dot{g}$. Note that $\dot{g}'h' > 0$, and $\dot{g}'h'' < 0$, and by continuity of $E_h(\delta^*)$ in h we can choose τ small enough that $|E_{h_0}(\delta^*) - E_{h'}(\delta^*)| < \varepsilon$ and $|E_{h_0}(\delta^*) - E_{h''}(\delta^*)| < \varepsilon/K$. Take $\tilde{J} = (J\backslash h_0) \cup \{h', h''\}$. Then

$$\begin{aligned}
&\liminf_{n\longrightarrow\infty} \sup_{h\in J} R^H\left(\delta_n^{H,M}, \theta_0 + \frac{h}{\sqrt{n}}\right) \\
&\quad \le \liminf_{n\longrightarrow\infty}\left[\max\left\{\sup_{h\in J\backslash h_0} R^H\left(\delta_n^{H,M}, \theta_0 + \frac{h}{\sqrt{n}}\right),\right.\right. \\
&\qquad \left.\left.\left(1 - E_{h_0}(\delta_n^{H,M})\right), KE_{h_0}(\delta_n^{H,M})\right\}\right] \\
&\quad = \max\left\{\sup_{h\in J\backslash h_0} R_\infty^H(\delta^{H,M}, h), \left(1 - E_{h_0}(\delta^{H,M})\right), KE_{h_0}(\delta^{H,M})\right\} \\
&\quad \le \max\left\{\sup_{h\in J\backslash h_0} R_\infty^H(\delta^{H,M}, h), \left(1 - E_{h'}(\delta^{H,M})\right), KE_{h''}(\delta^{H,M})\right\} + \varepsilon \\
&\quad = \sup_{h\in\tilde{J}} R_\infty^H(\delta^{H,M}, h) + \varepsilon = \liminf_{n\longrightarrow\infty} \sup_{h\in\tilde{J}} R^H\left(\delta_n^{H,M}, \theta_0 + \frac{h}{\sqrt{n}}\right) + \varepsilon,
\end{aligned}$$

where the first equality follows by the same argument for the proof of the first conclusion of Lemma 4. This argument clearly generalizes to finite subsets J with more than one element h with $\dot{g}'h = 0$. Hence Equation (A.4) holds. Together with Equation (A.3) and the fact that ε was arbitrary, it follows that $\delta_n^{H,M}$ attains the risk bound. *Q.E.D.*

APPENDIX 2: PROOFS FOR SEMIPARAMETRIC CASE

Most of the proofs follow by obvious modification of the proofs for the parametric results in Appendix 1. Nontrivial modifications are noted below.

PROOF OF PROPOSITION 4.1: Let $\Delta = (\Delta_1, \Delta_2, \ldots)$. The limit experiment and the asymptotic representation theorem (Theorem 3.1) given in Van der

Vaart (1991) yield a randomized statistic $\tilde{\delta}(\Delta, U)$ that matches the limit distribution of δ_n, where U is uniform on [0, 1] independent of Δ. (This is a "doubly randomized" treatment assignment rule.) The desired rule comes from setting $\delta(\Delta) = \int_0^1 \tilde{\delta}(\Delta, u)\,du$. *Q.E.D.*

PROOF OF THEOREM 4.2: The analog to Lemma 2 follows by the assumed product measure form of the prior Π. An analog to Lemma 3 follows below. If Assumption 1 is modified to require the analogous conditions hold for almost every h with respect to $\lambda \times \rho$, then the analog to Lemma 1 holds with $B_\infty(\cdot, \Pi)$ replacing $\pi(\theta_0)B_\infty(\cdot)$. For loss L^T, the conditions of Assumption 1 hold for all h, so the modification entails no additional complications. For loss L^H, the conditions of Assumption 1 hold for all $(h_2, h_3, \dots)$ and almost every h_1 (with respect to the Lebesgue measure λ). Hence, Lemma 1 also extends to L^H in the semiparametric case and the conclusions of the theorem follow. *Q.E.D.*

LEMMA 3′: *Suppose the conditions of Theorem* 4.2 *are satisfied*. *Then*, *for every* h,

$$\frac{\sqrt{n}\hat{g}_n}{\hat{\sigma}_g} \overset{h}{\rightsquigarrow} N(\langle \dot{g}, h\rangle, 1).$$

PROOF: We revert to treating h and $\dot{g}$ as functions from $\mathcal{Z}$ to $\mathbb{R}$. Equation (4.1) implies that

$$\ln \prod_{i=1}^{n} \frac{dP_{1/\sqrt{n},h}}{dP_0}(Z_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h(Z_i) - \frac{1}{2}\|h\|^2 + o_{P_0}(1),$$

where $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} h(Z_i) \overset{P_0}{\rightsquigarrow} N(0, \|h\|^2)$. By Van der Vaart (1998, Lemma 25.23), $\sqrt{n}\hat{g}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \dot{g}(Z_i) + o_{P_0}(1)$. By Slutsky's lemma and the delta method,

$$\begin{aligned}
&\left(\frac{\sqrt{n}\hat{g}_n}{\hat{\sigma}_g}, \ln \prod_{i=1}^{n} \frac{dP_{1/\sqrt{n},h}}{dP_0}(Z_i) \right) \\
&\quad = \left(\frac{\sum_{i=1}^{n} \dot{g}(Z_i)}{\sigma_g \sqrt{n}}, \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h(Z_i) - \frac{1}{2}\|h\|^2 \right) + o_{P_0}(1) \\
&\quad \overset{P_0}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ -\frac{1}{2}\|h\|^2 \end{pmatrix}, \begin{bmatrix} 1 & \frac{\langle \dot{g}, h\rangle}{\sigma_g} \\ \frac{\langle \dot{g}, h\rangle}{\sigma_g} & \|h\|^2 \end{bmatrix} \right).
\end{aligned}$$

The conclusion then follows by applying Le Cam's third lemma. *Q.E.D.*

PROOF OF THEOREM 4.4: Note that analogs of Lemmas 4 and 5 follow for the semiparametric case with trivial modification of their proofs. The proof of Theorem 4.4 follows from the proof of Theorem 3.5 after letting $h_0 = (0, h_2, h_3, \ldots)$, $h' = (\tau, h_2, h_3, \ldots)$, and $h'' = (-\tau, h_2, h_3, \ldots)$. *Q.E.D.*

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Manuscript received August, 2006; final revision received August, 2008.