SUPPLEMENT TO "A DOUBLE-TRACK ADJUSTMENT PROCESS FOR DISCRETE MARKETS WITH SUBSTITUTES AND COMPLEMENTS"

(*Econometrica*, Vol. 77, No. 3, May, 2009, 933–952)

BY NING SUN AND ZAIFU YANG

PROOF OF LEMMA 1: Suppose that p^* is an equilibrium price vector. Then we know from Gul and Stacchetti (1999, Lemma 6) that for any efficient allocation π^*, (p^*, π^*) constitutes an equilibrium. Clearly, $\sum_{i\in I} u^i(\pi^*(i)) = R(N)$ the market value of the objects. Furthermore, we have $\mathcal{L}(p^*) = \sum_{i\in I} V^i(p^*) + \sum_{\beta_h\in N} p^*_h = \sum_{i\in I}(u^i(\pi^*(i)) - \sum_{\beta_h\in\pi^*(i)} p^*_h) + \sum_{\beta_h\in N} p^*_h = R(N)$. Note that for any $p \in \mathbb{R}^n$ and $i \in I$, $V^i(p) \geq u^i(\pi^*(i)) - \sum_{\beta_h\in\pi^*(i)} p_h$. Thus for any $p \in \mathbb{R}^n$, we have

$$\mathcal{L}(p) = \sum_{i\in I} V^i(p) + \sum_{\beta_h\in N} p_h \geq \sum_{i\in I} u^i(\pi^*(i)) = R(N) = \mathcal{L}(p^*).$$

Hence, $\mathcal{L}(p^*) = \min_{p\in\mathbb{R}^n} \mathcal{L}(p)$, that is, p^* is a minimizer of the function $\mathcal{L}$ with $\mathcal{L}(p^*) = R(N)$.

Suppose that $\hat{p}$ is a minimizer of $\mathcal{L}$ with its value $\mathcal{L}(\hat{p}) = R(N)$. Let ρ be any efficient allocation of the model. We will show that $(\hat{p}, \rho)$ is an equilibrium. Clearly, we have $V^i(\hat{p}) \geq u^i(\rho(i)) - \sum_{\beta_h\in\rho(i)} \hat{p}_h$ for every $i \in I$. We need to show that $V^i(\hat{p}) = u^i(\rho(i)) - \sum_{\beta_h\in\rho(i)} \hat{p}_h$ for every $i \in I$. Suppose to the contrary that $V^j(\hat{p}) > u^j(\rho(j)) - \sum_{\beta_h\in\rho(j)} \hat{p}_h$ for some bidder j. Adding the previous inequalities over all bidders leads to $\mathcal{L}(\hat{p}) > R(N)$. This contradicts the hypothesis that $\hat{p}$ is a minimizer of $\mathcal{L}$ with $\mathcal{L}(\hat{p}) = R(N)$. Thus $(\hat{p}, \rho)$ must be an equilibrium. *Q.E.D.*

PROOF OF LEMMA 3: Let $p = p(t) \in \mathbb{Z}^n$. Observe that by Theorem 3(i), $\mathcal{L}(p + \delta)$ as a function of δ is continuous, generalized submodular, and convex on the set $\square$. So minimum is attained and its minimizers form a nonempty generalized lattice. Analogous to the proof of Theorem 3(ii), one can further show that the set of minimizers is integrally convex and, consequently, both its smallest and largest elements are integer vectors. *Q.E.D.*

PROOF OF LEMMA 5: Sufficiency is obvious. Let us prove necessity. First, recall that Lemma 1 of Sun and Yang (2006) says a value function $u^i : 2^N \to \mathbb{R}$ satisfies the gross substitutes and complements (GSC) condition if and only if for any $p \in \mathbb{R}^n$, any $\beta_k \in S_j$ for $j = 1$ or 2, any $\delta \geq 0$, and any $A \in D^i(p)$, there exists $B \in D^i(p - \delta e(k))$ such that $[A^c \cap S_j] \setminus \{\beta_k\} \subseteq B^c$ and $[A \cap S^c_j] \subseteq B$.

For any $p \in \mathbb{R}^n$ and any $A \in D^i(p)$, we consider the following three basic cases; the other cases can be proved in an analogously recursive way.

 DOI: 10.3982/ECTA6514

Case (i), $\tilde{p} = p + \delta_k e(k) + \delta_{k'} e(k')$, where the two different objects β_k and $\beta_{k'}$ are both in S_j and $\delta_k > 0$, $\delta_{k'} > 0$. By the definition of the GSC condition, there exists $B' \in D^i(p + \delta_k e(k))$ such that $[A \cap S_j] \setminus \{\beta_k\} \subseteq B'$ and $[A^c \cap S_j^c] \subseteq B'^c$. Since $\tilde{p} = (p + \delta_k e(k)) + \delta_{k'} e(k')$, for $B' \in D^i(p + \delta_k e(k))$, there is $B \in D^i(\tilde{p})$ such that $[B' \cap S_j] \setminus \{\beta_{k'}\} \subseteq B$ and $[B'^c \cap S_j^c] \subseteq B^c$. Thus we have $[A \cap S_j] \setminus \{\beta_k, \beta_{k'}\} \subseteq B$ and $[A^c \cap S_j^c] \subseteq B^c$, namely,

$$\{\beta_x \mid \beta_x \in A \cap S_j \text{ and } \tilde{p}_x = p_x\} \subseteq B \quad \text{and}$$
$$\{\beta_y \mid \beta_y \in A^c \cap S_j^c \text{ and } \tilde{p}_y = p_y\} \subseteq B^c.$$

Case (ii), $\tilde{p} = p - \delta_l e(l) - \delta_{l'} e(l')$, where the two different objects β_l and $\beta_{l'}$ are both in S_j^c and $\delta_l > 0$, $\delta_{l'} > 0$. It follows from the above equivalent formulation of the GSC condition that there exists $B' \in D^i(p - \delta_l e(l))$ such that $[A^c \cap S_j^c] \setminus \{\beta_l\} \subseteq B'^c$ and $[A \cap S_j] \subseteq B'$. Since $\tilde{p} = (p - \delta_l e(l)) - \delta_{l'} e(l')$, for $B' \in D^i(p - \delta_l e(l))$ there is $B \in D^i(\tilde{p})$ such that $[B'^c \cap S_j^c] \setminus \{\beta_{l'}\} \subseteq B^c$ and $[B' \cap S_j] \subseteq B$. Thus we obtain that $[A^c \cap S_j^c] \setminus \{\beta_l, \beta_{l'}\} \subseteq B^c$ and $[A \cap S_j] \subseteq B$, namely,

$$\{\beta_x \mid \beta_x \in A \cap S_j \text{ and } \tilde{p}_x = p_x\} \subseteq B \quad \text{and}$$
$$\{\beta_y \mid \beta_y \in A^c \cap S_j^c \text{ and } \tilde{p}_y = p_y\} \subseteq B^c.$$

Case (iii), $\tilde{p} = p + \delta_k e(k) - \delta_l e(l)$, where $\beta_k \in S_j$, $\beta_l \in S_j^c$, and $\delta_k > 0$, $\delta_l > 0$. By the definition of the GSC condition, there exists $B' \in D^i(p + \delta_k e(k))$ such that $[A \cap S_j] \setminus \{\beta_k\} \subseteq B'$ and $[A^c \cap S_j^c] \subseteq B'^c$. Note that $\tilde{p} = (p + \delta_k e(k)) - \delta_l e(l)$. Then it follows from the above equivalent formulation of the GSC condition that for $B' \in D^i(p + \delta_k e(k))$ there is $B \in D^i(\tilde{p})$ such that $[B'^c \cap S_j^c] \setminus \{\beta_l\} \subseteq B^c$ and $[B' \cap S_j] \subseteq B$. So we have $[A^c \cap S_j^c] \setminus \{\beta_l\} \subseteq B^c$ and $[A \cap S_j] \setminus \{\beta_k\} \subseteq B$, namely,

$$\{\beta_x \mid \beta_x \in A \cap S_j \text{ and } \tilde{p}_x = p_x\} \subseteq B \quad \text{and}$$
$$\{\beta_y \mid \beta_y \in A^c \cap S_j^c \text{ and } \tilde{p}_y = p_y\} \subseteq B^c. \qquad \textit{Q.E.D.}$$

PROOF OF LEMMA 6: Necessity is obvious. Let us prove sufficiency. Pick up any $p \in \mathbb{R}^n$ and fix any $A \notin D^i(p)$, that is, any A for which $V^i(p) > v^i(A, p)$. By continuity of $V^i(\cdot)$ and $v^i(A, \cdot)$, there exists $\varepsilon > 0$ such that $V^i(q) > v^i(A, q)$, where $q = p + \varepsilon e(A^c) - \varepsilon e(A)$. Then there exists B ($\neq A$) such that it satisfies (i) or (ii) of Definition 3 and $v^i(B, q) \geq v^i(A, q)$, which implies

$$\begin{aligned} v^i(B, p) - v^i(A, p) &= v^i(B, q) - v^i(A, q) + [\sharp(A \setminus B) + \sharp(B \setminus A)]\varepsilon \\ &> v^i(B, q) - v^i(A, q) \geq 0. \end{aligned} \qquad \textit{Q.E.D.}$$

To prove Theorem 2, we need to introduce an equivalent characterization of the generalized submodular function. This characterization can be easily used to verify whether a function is a generalized submodular function or not. Property (i) is new but (ii) is a familiar property of the submodular function.

LEMMA 7: *A function f is a generalized submodular function if and only if the following statements hold*:

(i) *For any $x \in \mathbb{R}^n$, any $\beta_k \in S_j$, any $\beta_l \in S_j^c$, any $\delta_k > 0$, and $\delta_l > 0$,*

$$f(x + \delta_k e(k) - \delta_l e(l)) - f(x - \delta_l e(l)) \leq f(x + \delta_k e(k)) - f(x).$$

(ii) *For any $x \in \mathbb{R}^n$, any distinct $\beta_k, \beta_l \in S_j$, any $\delta_k > 0$, and $\delta_l > 0$,*

$$f(x + \delta_k e(k) + \delta_l e(l)) - f(x + \delta_l e(l)) \leq f(x + \delta_k e(k)) - f(x).$$

PROOF: Suppose that f is a generalized submodular function. In the case of (i), let $p = x + \delta_k e(k)$ and $q = x - \delta_l e(l)$. Then $p \wedge_g q = x$ and $p \vee_g q = x + \delta_k e(k) - \delta_l e(l)$. Clearly the part (i) conclusion holds. It is also easy to check the case of (ii).

Suppose that both (i) and (ii) hold. Take any $p, q \in \mathbb{R}^n$. With respect to S_1 and S_2, let

$$\begin{aligned}
J_{S_1} &= \{j \mid p_j > q_j \text{ and } \beta_j \in S_1\},\\
K_{S_1} &= \{k \mid p_k < q_k \text{ and } \beta_k \in S_1\},\\
J_{S_2} &= \{j \mid p_j > q_j \text{ and } \beta_j \in S_2\},\\
K_{S_2} &= \{k \mid p_k < q_k \text{ and } \beta_k \in S_2\}.
\end{aligned}$$

We consider the most general case, namely, all the above four sets are nonempty. So there exists a nonnegative vector $\delta = (\delta_1, \dots, \delta_n) \gg 0$, such that $p_j = q_j + \delta_j$ for all $j \in J_{S_1} \cup J_{S_2}$ and $p_j = q_j - \delta_j$ for all $j \in K_{S_1} \cup K_{S_2}$. Let $J_{S_1} = \{h_1, \dots, h_s\}$, $K_{S_1} = \{i_1, \dots, i_t\}$, $J_{S_2} = \{j_1, \dots, j_u\}$, and $K_{S_2} = \{k_1, \dots, k_v\}$. Then we have

$$\begin{aligned}
&f(p) - f(p \wedge_g q)\\
&\quad = f(p) - f\left(p - \sum_{l=1}^{s} \delta_{h_l} e(h_l) + \sum_{l=1}^{v} \delta_{k_l} e(k_l)\right)\\
&\quad = \sum_{l=1}^{s} \left[f\left(p - \sum_{r=1}^{l-1} \delta_{h_r} e(h_r)\right) - f\left(p - \sum_{r=1}^{l} \delta_{h_r} e(h_r)\right)\right]\\
&\qquad + \sum_{l=1}^{v} \left[f\left(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l-1} \delta_{k_r} e(k_r)\right)\right.\\
&\qquad\qquad \left. - f\left(p - \sum_{r=1}^{s} \delta_{h_r} e(h_r) + \sum_{r=1}^{l} \delta_{k_r} e(k_r)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{l=1}^{s}\Bigg[f\Bigg(p-\sum_{r=1}^{l-1}\delta_{h_r}e(h_r)+\delta_{i_1}e(i_1)\Bigg)\\
&\qquad -f\Bigg(p-\sum_{r=1}^{l}\delta_{h_r}e(h_r)+\delta_{i_1}e(i_1)\Bigg)\Bigg]\\
&\quad+\sum_{l=1}^{v}\Bigg[f\Bigg(p-\sum_{r=1}^{s}\delta_{h_r}e(h_r)+\sum_{r=1}^{l-1}\delta_{k_r}e(k_r)+\delta_{i_1}e(i_1)\Bigg)\\
&\qquad\quad -f\Bigg(p-\sum_{r=1}^{s}\delta_{h_r}e(h_r)+\sum_{r=1}^{l}\delta_{k_r}e(k_r)+\delta_{i_1}e(i_1)\Bigg)\Bigg]\\
&\vdots\\
&\geq \sum_{l=1}^{s}\Bigg[f\Bigg(p-\sum_{r=1}^{l-1}\delta_{h_r}e(h_r)+\sum_{r=1}^{t}\delta_{i_r}e(i_r)\Bigg)\\
&\qquad -f\Bigg(p-\sum_{r=1}^{l}\delta_{h_r}e(h_r)+\sum_{r=1}^{t}\delta_{i_r}e(i_r)\Bigg)\Bigg]\\
&\quad+\sum_{l=1}^{v}\Bigg[f\Bigg(p-\sum_{r=1}^{s}\delta_{h_r}e(h_r)+\sum_{r=1}^{l-1}\delta_{k_r}e(k_r)+\sum_{r=1}^{t}\delta_{i_r}e(i_r)\Bigg)\\
&\qquad\quad -f\Bigg(p-\sum_{r=1}^{s}\delta_{h_r}e(h_r)+\sum_{r=1}^{l}\delta_{k_r}e(k_r)+\sum_{r=1}^{t}\delta_{i_r}e(i_r)\Bigg)\Bigg]\\
&\geq \sum_{l=1}^{s}\Bigg[f\Bigg(p-\sum_{r=1}^{l-1}\delta_{h_r}e(h_r)+\sum_{r=1}^{t}\delta_{i_r}e(i_r)-\delta_{j_1}e(j_1)\Bigg)\\
&\qquad -f\Bigg(p-\sum_{r=1}^{l}\delta_{h_r}e(h_r)+\sum_{r=1}^{t}\delta_{i_r}e(i_r)-\delta_{j_1}e(j_1)\Bigg)\Bigg]\\
&\quad+\sum_{l=1}^{v}\Bigg[f\Bigg(p-\sum_{r=1}^{s}\delta_{h_r}e(h_r)+\sum_{r=1}^{l-1}\delta_{k_r}e(k_r)\\
&\qquad\qquad +\sum_{r=1}^{t}\delta_{i_r}e(i_r)-\delta_{j_1}e(j_1)\Bigg)\\
&\qquad\quad -f\Bigg(p-\sum_{r=1}^{s}\delta_{h_r}e(h_r)+\sum_{r=1}^{l}\delta_{k_r}e(k_r)
\end{aligned}
$$

$$+\sum_{r=1}^{t}\delta_{i_r}e(i_r)-\delta_{j_1}e(j_1)\Bigg)\Bigg]$$

$$\vdots$$

$$\begin{aligned}
&\geq \sum_{l=1}^{s}\Bigg[f\Bigg(p-\sum_{r=1}^{l-1}\delta_{h_r}e(h_r)+\sum_{r=1}^{t}\delta_{i_r}e(i_r)-\sum_{r=1}^{u}\delta_{j_r}e(j_r)\Bigg)\\
&\qquad -f\Bigg(p-\sum_{r=1}^{l}\delta_{h_r}e(h_r)+\sum_{r=1}^{t}\delta_{i_r}e(i_r)-\sum_{r=1}^{u}\delta_{j_r}e(j_r)\Bigg)\Bigg]\\
&\quad +\sum_{l=1}^{v}\Bigg[f\Bigg(p-\sum_{r=1}^{s}\delta_{h_r}e(h_r)+\sum_{r=1}^{l-1}\delta_{k_r}e(k_r)\\
&\qquad\qquad +\sum_{r=1}^{t}\delta_{i_r}e(i_r)-\sum_{r=1}^{u}\delta_{j_r}e(j_r)\Bigg)\\
&\qquad -f\Bigg(p-\sum_{r=1}^{s}\delta_{h_r}e(h_r)+\sum_{r=1}^{l}\delta_{k_r}e(k_r)\\
&\qquad\qquad +\sum_{r=1}^{t}\delta_{i_r}e(i_r)-\sum_{r=1}^{u}\delta_{j_r}e(j_r)\Bigg)\Bigg]\\
&=\sum_{l=1}^{s}\Bigg[f\Bigg(p\vee_g q-\sum_{r=1}^{l-1}\delta_{h_r}e(h_r)\Bigg)\\
&\qquad -f\Bigg(p\vee_g q-\sum_{r=1}^{l}\delta_{h_r}e(h_r)\Bigg)\Bigg]\\
&\quad +\sum_{l=1}^{v}\Bigg[f\Bigg(p\vee_g q-\sum_{r=1}^{s}\delta_{h_r}e(h_r)+\sum_{r=1}^{l-1}\delta_{k_r}e(k_r)\Bigg)\\
&\qquad -f\Bigg(p\vee_g q-\sum_{r=1}^{s}\delta_{h_r}e(h_r)+\sum_{r=1}^{l}\delta_{k_r}e(k_r)\Bigg)\Bigg]\\
&=f(p\vee_g q)-f(q).
\end{aligned}$$

Therefore we have $f(p\wedge_g q)+f(p\vee_g q)\leq f(p)+f(q)$. In the above derivation, the first two inequalities follow from case (ii) and the last two follow from case (i). *Q.E.D.*

PROOF OF THEOREM 2—Necessity: Choose any two distinct items β_k, $\beta_l \in N$, any $p \in \mathbb{R}^n$, any $\delta_k > 0$, and any $\delta_l > 0$. If $V^i(p) - V^i(p + \delta_k e(k)) = 0$, the monotonicity of $V^i(\cdot)$ implies that $V^i(p + \delta_l e(l) + \delta_k e(k)) - V^i(p + \delta_l e(l)) \leq 0 = V^i(p + \delta_k e(k)) - V^i(p)$ and $V^i(p - \delta_l e(l) + \delta_k e(k)) - V^i(p - \delta_l e(l)) \leq 0 = V^i(p + \delta_k e(k)) - V^i(p)$. We can now assume that $V^i(p) - V^i(p + \delta_k e(k)) = \varepsilon_k > 0$. Then it follows that $0 < \varepsilon_k \leq \delta_k$, $V^i(p + \varepsilon_k e(k)) = V^i(p + \delta_k e(k)) = V^i(p) - \varepsilon_k$, and there is a bundle $A \in D^i(p)$ and a bundle $B \in D^i(p + \varepsilon_k e(k))$ (for example, $B = A$) with $\beta_k \in A \cap B$. We need to consider the following two situations.

Case (i)—β_l and β_k are in the same set S_j: With regard to $A \in D^i(p)$ and $B \in D^i(p + \varepsilon_k e(k))$, it follows from the GSC condition and $\beta_k \in A \cap B$ that there are two bundles $C \in D^i(p + \delta_l e(l))$ with $\beta_k \in C$ and $D \in D^i(p + \delta_l e(l) + \varepsilon_k e(k))$ with $\beta_k \in D$. As a result, we have

$$\begin{aligned}
&V^i(p + \delta_l e(l) + \delta_k e(k)) - V^i(p + \delta_l e(l)) \\
&\quad \leq V^i(p + \delta_l e(l) + \varepsilon_k e(k)) - V^i(p + \delta_l e(l)) \\
&\quad = v^i(D, p + \delta_l e(l) + \varepsilon_k e(k)) - V^i(p + \delta_l e(l)) \\
&\quad = v^i(D, p + \delta_l e(l)) - \varepsilon_k - V^i(p + \delta_l e(l)) \\
&\quad \leq -\varepsilon_k = V^i(p + \delta_k e(k)) - V^i(p).
\end{aligned}$$

Case (ii)—β_l and β_k are not in the same set S_j: With regard to $A \in D^i(p)$ and $B \in D^i(p + \varepsilon_k e(k))$, it follows from the GSC condition, Lemma 5, and $\beta_k \in A \cap B$ that there are two bundles $C \in D^i(p - \delta_l e(l))$ with $\beta_k \in C$ and $D \in D^i(p - \delta_l e(l) + \varepsilon_k e(k))$ with $\beta_k \in D$, which leads to

$$\begin{aligned}
&V^i(p - \delta_l e(l) + \delta_k e(k)) - V^i(p - \delta_l e(l)) \\
&\quad \leq V^i(p - \delta_l e(l) + \varepsilon_k e(k)) - V^i(p - \delta_l e(l)) \\
&\quad = v^i(D, p - \delta_l e(l) + \varepsilon_k e(k)) - V^i(p - \delta_l e(l)) \\
&\quad = v^i(D, p - \delta_l e(l)) - \varepsilon_k - V^i(p - \delta_l e(l)) \\
&\quad \leq -\varepsilon_k = V^i(p + \delta_k e(k)) - V^i(p).
\end{aligned}$$

In summary, we see through Lemma 7 that V^i is a generalized submodular function.

Sufficiency: Suppose to the contrary that there are some $p \in \mathbb{R}^n$, $\beta_k \in S_j$, $\delta_k > 0$, and $A \in D^i(p)$ such that for every $B \in D^i(p + \delta_k e(k))$ we have $[A \cap S_j] \setminus \{\beta_k\} \not\subseteq B$ or $A^c \cap S_j^c \not\subseteq B^c$. Let $\varepsilon_k = V^i(p) - V^i(p + \delta_k e(k))$. Clearly, $0 \leq \varepsilon_k \leq \delta_k$, $V^i(p + \varepsilon_k e(k)) = V^i(p + \delta_k e(k))$, and $A \in D^i(p + \varepsilon_k e(k))$. Since $A \notin D^i(p + \delta_k e(k))$, it holds that $D^i(p + \varepsilon_k e(k)) \neq D^i(p + \delta_k e(k))$ and $\varepsilon_k < \delta_k$. Let $q = p + \varepsilon_k e(k)$ and $\theta_k = \delta_k - \varepsilon_k > 0$. Then $V^i(q) = V^i(q + \theta_k e(k))$. Observe that $A \in D^i(q)$ and $B \notin D^i(q + \theta_k e(k))$ for every bundle B satisfying $[A \cap S_j] \setminus \{\beta_k\} \subseteq B$ and $A^c \cap S_j^c \subseteq B^c$. This means that $V^i(q + \theta_k e(k)) >$

$v^i(B, q+\theta_k e(k))$ for every bundle B satisfying $[A\cap S_j]\setminus\{\beta_k\}\subseteq B$ and $A^c\cap S_j^c\subseteq B^c$. Furthermore, the continuity of $V^i(\cdot)$ and $v^i(B,\cdot)$ implies that there exists a sufficiently small positive number θ so that

$$V^i\big(q+\theta_k e(k)-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c)\big)$$
$$> v^i\big(B, q+\theta_k e(k)-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c)\big)$$

for every bundle B satisfying $[A\cap S_j]\setminus\{\beta_k\}\subseteq B$ and $A^c\cap S_j^c\subseteq B^c$. This means that if $B\in D^i(q+\theta_k e(k)-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c))$, then $[A\cap S_j]\setminus\{\beta_k\}\not\subseteq B$ or $A^c\cap S_j^c\not\subseteq B^c$. Then choosing a bundle $B\in D^i(q+\theta_k e(k)-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c))$ yields

$$\begin{aligned}
&V^i\big(q+\theta_k e(k)-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c)\big)\\
&\quad= v^i\big(B, q+\theta_k e(k)-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c)\big)\\
&\quad= v^i(B, q+\theta_k e(k)+\bar{p})\\
&\quad= v^i(B, q+\theta_k e(k))-\sum_{\beta_k\in B}\bar{p}_k\\
&\quad= v^i(B, q+\theta_k e(k))+\sharp\big(B\cap([A\cap S_j]\setminus\{\beta_k\})\big)\theta\\
&\qquad -\sharp(B\cap(A^c\cap S_j^c))\theta\\
&\quad< v^i(B, q+\theta_k e(k))+\sharp([A\cap S_j]\setminus\{\beta_k\})\theta\\
&\quad\le V^i(q+\theta_k e(k))+\sharp([A\cap S_j]\setminus\{\beta_k\})\theta,
\end{aligned}$$

where $\bar{p}=-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c)$. Therefore we have

$$\begin{aligned}
&V^i\big(q-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c)\big)\\
&\quad\ge v^i\big(A, q-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c)\big)\\
&\quad= v^i(A,q)+\sharp([A\cap S_j]\setminus\{\beta_k\})\theta\\
&\quad= V^i(q)+\sharp([A\cap S_j]\setminus\{\beta_k\})\theta\\
&\quad= V^i(q+\theta_k e(k))+\sharp([A\cap S_j]\setminus\{\beta_k\})\theta\\
&\quad> V^i\big(q+\theta_k e(k)-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c)\big).
\end{aligned}$$

Let $x=q$ and $y=q+\theta_k e(k)-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c)$. Then the above inequality leads to

$$\begin{aligned}
&V^i(x\wedge_g y)+V^i(x\vee_g y)\\
&\quad= V^i\big(q-\theta e([A\cap S_j]\setminus\{\beta_k\})+\theta e(A^c\cap S_j^c)\big)+V^i(q+\theta_k e(k))
\end{aligned}$$

$$> V^i(q) + V^i\big(q + \theta_k e(k) - \theta e([A \cap S_j] \setminus \{\beta_k\}) + \theta e(A^c \cap S_j^c)\big)$$
$$= V^i(x) + V^i(y),$$

contradicting the hypothesis that V^i is a generalized submodular function. *Q.E.D.*

To expedite reading, we also give a proof for the following statement in the third paragraph of the proof of Theorem 5 in the Appendix of the paper. The argument here is the same as in the second paragraph of the proof for the Step 2 case of the global dynamic double-track (GDDT) procedure.

STATEMENT: *By the symmetry between Step* 2 *and Step* 3*, similarly we can also show that* $\mathcal{L}(p \wedge_g p(t^*)) \geq \mathcal{L}(p(t^*))$ *for all* $p \in \mathbb{R}^n$.

PROOF: To prove the statement, we first show that $\mathcal{L}(p) \geq \mathcal{L}(p(t^*))$ for all $p \leq_g p(t^*)$. Suppose to the contrary that there exists some $p \leq_g p(t^*)$ such that $\mathcal{L}(p) < \mathcal{L}(p(t^*))$. By the convexity of $\mathcal{L}(\cdot)$ via Theorem 3(i), there is a strict convex combination p' of p and $p(t^*)$ such that $p' \in \{p(t^*)\} - \square$ and $\mathcal{L}(p') < \mathcal{L}(p(t^*))$. Because of the symmetry between Step 2 and Step 3, Lemma 3 (where $\square$ is replaced by $\square^* = -\square$) and Step 3 of the GDDT procedure imply that $\mathcal{L}(p(t^*) + \delta(t^*)) = \min_{\delta \in \square^*} \mathcal{L}(p(t^*) + \delta) = \min_{\delta \in \Delta^*} \mathcal{L}(p(t^*) + \delta) \leq \mathcal{L}(p') < \mathcal{L}(p(t^*))$ and so $\delta(t^*) \neq 0$, contradicting the fact that the GDDT procedure stops in Step 3 with $\delta(t^*) = 0$. So we have $\mathcal{L}(p) \geq \mathcal{L}(p(t^*))$ for all $p \leq_g p(t^*)$. Because $p \wedge_g p(t^*) \leq_g p(t^*)$ for all $p \in \mathbb{R}^n$, it follows that $\mathcal{L}(p \wedge_g p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. *Q.E.D.*

REFERENCES

GUL, F., AND E. STACCHETTI (1999): "Walrasian Equilibrium With Gross Substitutes," *Journal of Economic Theory*, 87, 95–124.

SUN, N., AND Z. YANG (2006): "Equilibria and Indivisibilities: Gross Substitutes and Complements," *Econometrica*, 74, 1385–1402.

School of Economics, Shanghai University of Finance and Economics, Shanghai, China; nsun@mail.shufe.edu.cn

and

Faculty of Business Administration, Yokohama National University, Yokohama, Japan; yang@ynu.ac.jp.

Manuscript received June, 2006; final revision received October, 2008.