SUPPLEMENT TO “TESTING MODELS WITH multiple EQUILIBRIA BY QUANTILE METHODS”
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This supplement contains the proofs of Proposition 1 and Lemma 1 that were stated in the paper.

PROOF OF PROPOSITION 1: For any \((y, x) \in \mathbb{R} \times \mathcal{X}\), \(F_{Y|X=x}(y) = \int_{-\infty}^{+\infty} P_{X=x}(y)f_{U|X=x}(u)\,du\) with \(P_{X=x}(y) = \sum_{i=1}^{n_x} \pi_{ix}\,\mathbb{1}(\xi_{ixu} \leq y)\), where \(\mathbb{1}\) denotes the standard indicator function: For any event \(A\) in \(B\), where \(B\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}\), \(\mathbb{1}(A) = 1\) if \(A\) is true and \(= 0\) otherwise. Combining all of the above, we get

\[
F_{Y|X=x}(y) = \sum_{i=1}^{n_x} \pi_{ix} \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leq y) f_{U|X=x}(u)\,du.
\]

For any \(x \in \mathcal{X}\) and any \(1 \leq i \leq n_x\), let \(F_{Y|X=x}(y) = \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leq y) f_{U|X=x}(u)\,du\) for all \(y \in \mathbb{R}\). Then \(F_{Y|X=x}(y) : \mathbb{R} \to \mathbb{R}\) is right-continuous, \(\lim_{y \to -\infty} F_{Y|X=x}(y) = 0\), \(\lim_{y \to +\infty} F_{Y|X=x}(y) = 1\), and \(F_{Y|X=x}\) is nondecreasing in \(y\). Hence, \(F_{Y|X=x}\)'s are distribution functions and the conditional distribution of the dependent variable can be written as in Proposition 1. Moreover, for any \((y, x) \in \mathbb{R} \times \mathcal{X}\), we have

\[
F_{Y|X=x}(y) - F_{Y|X=x}(y) = \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leq y < \xi_{jxu}) f_{U|X=x}(u)\,du \geq 0
\]

whenever \(\xi_{jxu} \geq \xi_{ixu}\), that is, \(F_{Y|X=x}(y) \leq F_{Y|X=x}(y)\) whenever \(j \geq i\). So, \(F_{Y|X=x}\) first-order stochastically dominates \(F_{Y|X=x}\) for any \(j \geq i\). Q.E.D.

PROOF OF LEMMA 1: Fix \((y_0, x) \in \mathbb{R} \times \mathcal{X}\): continuity and limit conditions on \(r(y, x)\) in S1 then ensure that the envelope \(r^e(y, x)\) is well defined on \([y_0, +\infty)\). Now consider \(y \geq y_0\). That \(\mathbb{1}(\xi_{n_xu} \leq y) = \mathbb{1}(u \leq r^e(y, x))\) follows from showing that \(r^e(\xi_{n_xu}, x) = r(\xi_{n_xu}, x)\), as \(r^e\) is nonincreasing and \(\xi_{n_xu}\) is the largest equilibrium. We proceed in two steps. First, we show that for all \(y > \xi_{n_xu}\), we have \(r(\xi_{n_xu}, x) > r(y, x)\). If that were not the case, then there would exist a \(y' > \xi_{n_xu}\) such that \(r(\xi_{n_xu}, x) \leq r(y', x)\). But this is incompatible with \(\xi_{n_xu}\) being the largest equilibrium: we would have \(u \leq r(y', x)\), so given the limit condition S1(ii) on \(r\) at \(+\infty\), there would be an equilibrium larger than \(\xi_{n_xu}\). Second, we show that \(r^e(\xi_{n_xu}, x) = r(\xi_{n_xu}, x)\). By definition of \(r^e\), we have \(r^e(\xi_{n_xu}, x) \geq r(\xi_{n_xu}, x)\), so we need to rule out that strict inequality holds. We again reason by contradiction: assume that \(r^e(\xi_{n_xu}, x) > r(\xi_{n_xu}, x)\). From the first step, we know that \(r(\xi_{n_xu}, x) > r(y, x)\) for all \(y > \xi_{n_xu}\). Then consider the function which coincides with \(r^e(y, x)\) for \(y < \xi_{n_xu}\) and with \(\min(r^e(y, x), r(y, x))\) for \(y \geq \xi_{n_xu}\). This function is nonincreasing, larger than \(r\), and smaller than \(r^e\) at \(\xi_{n_xu}\), which is impossible by the definition of \(r^e\). Q.E.D.

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