SUPPLEMENT TO “MARKOV PERFECT INDUSTRY DYNAMICS WITH MANY FIRMS”—TECHNICAL APPENDIX
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A. PROOFS AND MATHEMATICAL ARGUMENTS FOR SECTION 4: LONG-RUN BEHAVIOR AND THE INVARIANT INDUSTRY DISTRIBUTION

LEMMA A.3: Let Assumptions 3.2 and 3.3 hold. Assume that firms follow a common oblivious strategy \( \mu \in \tilde{\mathcal{M}} \), the expected entry rate is \( \lambda \in \tilde{\Lambda} \), and the expected time that each firm spends in the industry is finite. Let \( \{Z_x : x \in \mathbb{N}\} \) be a sequence of independent Poisson random variables with means \( \{\tilde{s}_{\mu, \lambda}(x) : x \in \mathbb{N}\} \), and let \( Z \) be a Poisson random variable with mean \( \sum_{x \in \mathbb{N}} \tilde{s}_{\mu, \lambda}(x) \). Then:
(a) \( \{s_t : t \geq 0\} \) is an irreducible, aperiodic, and positive recurrent Markov process;
(b) the invariant distribution of \( s_t \) is a product form of Poisson random variables;
(c) for all \( x \), \( s_t(x) \Rightarrow Z_x \);
(d) \( n_t \Rightarrow Z \).

PROOF: If every firm uses a strategy \( \mu \in \tilde{\mathcal{M}} \) and entry is according to an entry rate function \( \lambda \in \tilde{\Lambda} \), then \( A = \{s_t : t \geq 0\} \) is clearly an irreducible Markov process. All states reach the state \( \emptyset = \{0, 0, \ldots\} \) with positive probability and all states can be reached from \( \emptyset \) as well. Moreover, state \( \emptyset \) is aperiodic; hence, \( A \) is aperiodic. Finally, \( A \) is positive recurrent because the expected time to come back from state \( \emptyset \) to itself is finite (Kleinrock (1975)).

Now, let us write

\[
S_t(x) = \sum_{\tau=0}^{t} \sum_{i=1}^{W_\tau} 1_{[X_{i,t-\tau} = x]},
\]

where \( W_\tau \) are i.i.d. Poisson random variables with mean \( \lambda \), the first sum is taken over all periods previous to (and including) \( t \), the second sum is taken over the firms that entered the industry in each period, and for each \( \tau \), \( X_{i,t-\tau} \) are random variables that represent the state of firm \( i \) after \( t - \tau \) periods inside the industry when using strategy \( \mu \). Since firms use oblivious strategy \( \mu \in \tilde{\mathcal{M}} \) and shocks are idiosyncratic, their state evolutions are independent, so \( 1_{[X_{i,t-\tau} = x]} \) are i.i.d. across \( i \). It follows that \( \sum_{i=1}^{W_\tau} 1_{[X_{i,t-\tau} = x]} \) is a filtered Poisson random variable, so it is a Poisson random variable. Thus \( S_t(x) \), as a sum of independent Poisson random variables, is also Poisson. Given that the expected time a firm spends inside the industry is finite, using characteristic functions it is straightforward to show that \( S_t(x) \Rightarrow Z_x \forall x \in \mathbb{N} \). To show that \( \{Z_x : x \in \mathbb{N}\} \) is a

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sequence of independent random variables, note that by the filtering property of Poisson random variables, for all \( t \), \( \{ s_t(x) : x \in \mathbb{N} \} \) is a sequence of independent random variables (Durrett (1996)). By summing over \( x \in \mathbb{N} \), we can show that \( n_t \Rightarrow Z \).

**Q.E.D.**

**LEMMA A.4:** Let Assumptions 3.2 and 3.3 hold. Assume that firms follow a common oblivious strategy \( \mu \in \tilde{\mathcal{M}} \), the expected entry rate is \( \lambda \in \tilde{\Lambda} \), and the expected time that each firm spends in the industry is finite. Let \( \{ Y_n : n \in \mathbb{N} \} \) be a sequence of integer-valued i.i.d. random variables, each distributed according to \( \tilde{s}_{\mu, \lambda}(\cdot) / \sum_{x \in \mathbb{N}} \tilde{s}_{\mu, \lambda}(x) \). Then, for all \( n \in \mathbb{N} \),

\[
(x(1)_t, \ldots, x(n)_t | n_t = n) \Rightarrow (Y_1, \ldots, Y_n).
\]

**PROOF:** The proof relies on a well known result for Poisson processes; conditional on \( n \) arrivals on an interval \([0, T]\), the unordered arrival times have the same distribution as \( n \) i.i.d. uniform random variables in \([0, T]\).

Let us condition on \( n_t = n \). \( \{ x(j)_t : j = 1, \ldots, n \} \) are the random variables that represent the state of each of the \( n \) firms in the industry when they are sampled randomly. The expected time a firm spends inside the industry is finite, so the time a firm spends inside the industry is finite with probability 1. A firm can increase its quality level by at most \( w \) states each period. Therefore, for all \( \varepsilon > 0 \), there exists a state \( z \), such that, for all \( j \in \{1, \ldots, n\} \) and for all \( t \), \( \mathcal{P}[x(j)_t > z] < \frac{\varepsilon}{n} \). Hence, \( \mathcal{P}[\bigcup_{j=1}^n \{ x(j)_t > z \} | n_t = n] < \varepsilon \), for all \( t \), so the sequence of random vectors \( \{ (x(1)_t, \ldots, x(n)_t) | n_t = n \} : t \geq 0 \) is tight. By Theorem 9.1 in Durrett (1996) and tightness, to prove the lemma it is enough to show that for all \( n \), for all \( (z_1, \ldots, z_n) \),

\[
\lim_{t \to \infty} \mathcal{P}[x(j)_t = z_j, j = 1, \ldots, n | n_t = n] = \prod_{j=1}^n p(z_j),
\]

where \( p(\cdot) \) is the probability mass function (pmf) \( \tilde{s}_{\mu, \lambda}(\cdot) / \sum_{x \in \mathbb{N}} \tilde{s}_{\mu, \lambda}(x) \). Let \( \tilde{T}_j \) be the entry time period for firm \( j \) and let \( T_j = t - \tilde{T}_j \) be its age. Then we can write

(S.2) \[
\mathcal{P}[x(j)_t = z_j, j = 1, \ldots, n | n_t = n] = \sum_{0 \leq t_1 < \infty, \ldots, 0 \leq t_n < \infty} \mathcal{P}[x(j)_t = z_j, j = 1, \ldots, n | T_1 = t_1, \ldots, T_n = t_n, n_t = n] \times \mathcal{P}[T_1 = t_1, \ldots, T_n = t_n | n_t = n] = \prod_{0 \leq t_1 < \infty, \ldots, j=1}^n \mathcal{P}[x(j)_t = z_j | T_j = t_j] \]
\[ \times \mathcal{P}[T_1 = t_1, \ldots, T_n = t_n|n_t = n]. \]

The last equation follows because the evolution of firms is independent across firms. Note that if any \( t_j \) has a value greater than \( t \), then \( \mathcal{P}[T_1 = t_1, \ldots, T_n = t_n|n_t = n] = 0 \). We can write

\[ (S.3) \quad \mathcal{P}[x^{(j)} = z_j|T_j = t_j] = \frac{\mathcal{P}[x^{(j)} = z_j, T_j = t_j]}{\mathcal{P}[T_j = t_j]} \]

\[ = \frac{\mathcal{P}[T_j = t_j, X_{j,t_j} = z_j]}{\mathcal{P}[T_j = t_j]} \]

\[ = \frac{\mathcal{P}[T_j = t_j] \mathcal{P}[X_{j,t_j} = z_j]}{\mathcal{P}[T_j = t_j]} \]

\[ = \mathcal{P}[X_{j,t_j} = z_j], \]

where \( X_{j,t_j} \) is a random variable that represents a firm’s state after \( t_j \) periods, conditional on having stayed in the industry. Note that for all \( k \), \( \{X_{j,k} : j \geq 1 \} \) are i.i.d. The second to last equation follows because the evolution of a firm is independent of its entry time.

Now we show that

\[ \lim_{t \to \infty} \mathcal{P}[T_1 = t_1, \ldots, T_n = t_n|n_t = n] = \prod_{j=1}^{n} u[t_j] \]

for some pmf \( u \). We derive this equation by invoking the relationship between \( n_t \) and a Poisson process.

Similarly to equation (S.1), we can write

\[ n_t = \sum_{\tau=0}^{t} \sum_{i=1}^{W_{\tau}} A_{i,t-\tau}, \]

where \( A_{i,t-\tau} \) are i.i.d. Bernoulli random variables that equal one if the firm is still in the industry after \( t-\tau \) periods when using strategy \( \mu \) and zero otherwise. Since \( A_{i,t-\tau} \) are i.i.d., \( n_t = \sum_{i=1}^{W_{\tau}} A_{i,t-\tau} \) is a filtered Poisson random variable and is therefore Poisson. Let us denote its mean by \( \alpha_{t-\tau} \). It follows that \( n_t \) is a sum of independent Poisson random variables, so it is Poisson with mean \( \sum_{\tau=0}^{t} \alpha_{t-\tau} \).

Consider \( \{N(t) : t \geq 0\} \), a homogeneous Poisson process on the real line with rate \( 1 \). Note that \( N(t) \) and \( n_t \) are equivalent in the sense that we can construct \( n_t \) using the process \( \{N(s) : 0 \leq s \leq \sum_{\tau=0}^{t} \alpha_{t-\tau} \} \). For each \( 0 \leq \tau \leq t \), with some abuse of notation, let \( N(\alpha_{t-\tau-1}, \alpha_{t-\tau} + \alpha_{t-\tau}) \) be the total number of events of the Poisson process in the interval \([\alpha_{t-\tau-1}, \alpha_{t-\tau} + \alpha_{t-\tau}]\) where \( \alpha_{t-1} = 0 \). Then
we can construct $n_t = \sum_{\tau=0}^{t} n_{t,\tau}$ by defining $n_{t,\tau} = N(\alpha_{t,\tau-1}, \alpha_{t,\tau-1} + \alpha_{t,\tau})$ for all $\tau$.

Now, conditional on the event $N(\sum_{\tau=0}^{t} \alpha_{t,\tau}) = n$, the unordered arrival times of $N(t)$ have the same distribution as $n$ i.i.d. uniform random variables in $[0, \sum_{\tau=0}^{t} \alpha_{t,\tau}]$ (Durrett (1996)). By the equivalence argument described above, conditional on $n_t = n$, the unordered arrival times of the $n$ firms are i.i.d. discrete random variables with pmf:

$$v_t(\tau) = \frac{\alpha_{t,\tau}}{\sum_{j=0}^{\tau} \alpha_{t,j}}, \quad 0 \leq \tau \leq t.$$  

Recall that $\alpha_{t,\tau}$ is the expected number of firms that entered at time $\tau$ and are still inside the industry at time $t$. Since the entry rate is oblivious, all firms use the same oblivious strategy and shocks are idiosyncratic, $\alpha_{t,\tau} = \tilde{\alpha}_{t-\tau}$, where $\tilde{\alpha}_{t-\tau}$ is the expected number of firms that entered the industry at time $s$, for any $s$, and are still inside the industry at time $s + t - \tau$. This suggests making a change of variable and defining

$$u_t(k) = \frac{\tilde{\alpha}_k}{\sum_{j=0}^{\infty} \tilde{\alpha}_j}, \quad 0 \leq k \leq t.$$  

$u_t(k)$ is the probability a random sampled firm from the industry at time $t$ entered $k$ periods ago, conditional on $n_t = n$. Taking the limit as $t$ tends to infinity, we get that

$$\lim_{t \to \infty} u_t(k) = u(k) = \frac{\tilde{\alpha}_k}{\sum_{j=0}^{\infty} \tilde{\alpha}_j}, \quad 0 \leq k < \infty,$$

provided that $\lim_{t \to \infty} E[n_t] = \sum_{j=0}^{\infty} \tilde{\alpha}_j < \infty$, which is true because the expected time that each firm spends in the industry is finite. $u(k)$ is the probability a random sampled firm, while the industry state is distributed according to its invariant distribution, entered $k$ periods before the sampling period. Therefore,

$$\lim_{t \to \infty} P[T_1 = t_1, \ldots, T_n = t_n | n_t = n] = \prod_{j=1}^{n} u(t_j).$$

Replacing the previous equation together with equation (S.3) into equation (S.2) we obtain

$$\lim_{t \to \infty} P[x_{(j)t} = z_j, j = 1, \ldots, n | n_t = n] = \prod_{j=1}^{n} \sum_{0 \leq t < \infty} P[X_{j,t} = z_j] u(t),$$

where the interchange between the infinite sum and the limit follows by the dominated convergence theorem. The sum yields the pmf $p(\cdot)$. The previous equation proves that, for all $n \in \mathbb{N}$, $(x_{(1)t}, \ldots, x_{(n)t}) | n_t = n \Rightarrow (Y_1, \ldots, Y_n)$,
where \( Y_1, \ldots, Y_n \) are i.i.d. random variables with pmf \( p(\cdot) \) which does not depend on \( n \).

To finish, consider a very large time period. Formally, suppose that \( s_0 \) is sampled from the invariant distribution of \( \{s_t: t \geq 0\} \) (which is well defined by Lemma A.3). In this case, \( s_t \) is a stationary process; \( s_t \) is distributed according to the invariant distribution for all \( t \geq 0 \):

\[
\tilde{s}_{\mu, \lambda}(x) = E[s_t(x)] = E \left[ \sum_{j=1}^{n_t} 1_{\{x(j)_t = x\}} \right].
\]

Conditioning on \( n_t \) and considering that we already proved that \( \{x(j)_t: j = 1, \ldots, n\} \) are i.i.d. with pmf \( p(\cdot) \), we conclude that

\[
p(\cdot) = \frac{\tilde{s}_{\mu, \lambda}(\cdot)}{\sum_{x \in \mathbb{N}} \tilde{s}_{\mu, \lambda}(x)}.
\]

Q.E.D.

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