Online Appendix to "Signaling under Double-Crossing Preferences"

(For online publication only)

F. Other Variants of Double-Crossing Preferences

In the main text we use Assumptions 2 and 3 to specify double-crossing preferences. We adopt these two assumptions because they are economically relevant for many applications, as shown by our examples. Nevertheless we do not rule out the possibility that other variants of double-crossing preferences may also be relevant in some contexts. We provide a brief discussion of these variations below.

Consider the following assumptions.

Assumption 2′. *For any $\theta' > \theta''$, there exists a continuous function $D(\cdot;\theta',\theta'') : [\underline{\theta},\overline{\theta}] \to \mathbb{R}_+$ such that*

(a) if $a < a_0 \leq D(t_0;\theta',\theta'')$, then

$$u(a,t,\theta') \leq u(a_0,t_0,\theta') \implies u(a,t,\theta'') < u(a_0,t_0,\theta'');$$

(b) if $a > a_0 \geq D(t_0;\theta',\theta'')$, then

$$u(a,t,\theta') \leq u(a_0,t_0,\theta') \implies u(a,t,\theta'') < u(a_0,t_0,\theta'').$$

Assumption 3′. *For any t, $D(t;\theta',\theta'')$ is continuous and strictly increases in θ' and in θ''.*

Assumption 2′ (A2′) states that the reverse single-crossing property holds to the left of the dividing line (i.e., $(a,t) \in RSC(\theta)$ if $a < D(t;\theta,\theta)$) while the single-crossing property holds to the right of it (i.e., $(a,t) \in SC(\theta)$ if $a > D(t;\theta,\theta)$). This is the opposite of Assumption 2 (A2). Similarly, Assumption 3′ (A3′) is the opposite of Assumption 3 (A3). Any combination of (A2) or (A2′) with (A3) or (A3′) would lead to a different specification of double-crossing preferences; therefore, there are four possible specifications.

For each of the four specifications, there is always one domain that is "well-behaved" and another domain that is "ill-behaved." It is easy to check which domain is ill-behaved.

Under (A3), the dividing line decreases in type, which makes the RSC-domain ill-behaved regardless of whether (A2) or (A2′) holds. Under (A3′), the dividing line increases in type such that the SC-domain is ill-behaved. In the main text, we define θ_* as the largest type such that $(S(\theta), T(\theta))$ is not in the RSC-domain (i.e., $S(\theta) \leq D(T(\theta); \theta, \theta)$) for all $\theta \leq \theta_*$. We now extend this definition and let

$$\theta_* := \sup\{\theta' : (S(\theta), T(\theta)) \text{ is not in the ill-behaved domain for all } \theta \leq \theta'\}.$$

Similarly, let

$$\theta_{**} := \inf\{\theta' : (S(\theta), T(\theta)) \text{ is not in the ill-behaved domain for all } \theta \geq \theta'\}.$$

The choice of (A2) or (A2′) determines the direction in which we impose the pairwise-matching condition. Under (A2), the SC-domain is to the left of the dividing line, and higher types have more convex indifference curves; therefore, the pairwise-matching condition is imposed for higher types. In this case, θ_* is applicable. Under (A2′), the RSC-domain is to the left of the dividing line, and lower types have more convex indifference curves; therefore, the pairwise-matching condition is imposed for lower types. In this case, θ_{**} is applicable.

Our model assumes (A2) and (A3). In this specification, incentive compatibility is potentially an issue for allocations in the RSC-domain. We use the pairwise-matching condition to ensure global incentive compatibility via Proposition 3. Note that an extended version of the pairwise-matching condition can now be stated as follows.

Definition F.1. *An allocation satisfies condition (P) if for any $\theta' > \theta_*$, there exists $\theta'' < \theta_*$ such that $(S(\theta''), T(\theta'')) = (S(\theta'), T(\theta'))$ and $m(S(\theta''), T(\theta''), \theta'') = m(S(\theta'), T(\theta'), \theta')$.*

Proposition 3 shows that under (A2) and (A3), an allocation that satisfies local IC for $\theta < \theta_*$ and the pairwise-matching condition is incentive compatible. With the extended definition of (P), this conclusion can also be applied to the case under (A2) and (A3′).

Proposition F.1. *Under Assumption (A2), an allocation that satisfies local IC and condition (P) is incentive compatible.*

Proof. The case under (A3) is already discussed in Proposition 3, so we focus on (A3′). Because $S(\theta) \geq D(T(\theta); \theta, \theta)$ for all $\theta \in [\underline{\theta}, \theta_*]$, local IC implies that $S(\cdot)$ is weakly decreasing on this interval. Here, we only show that incentive compatibility holds for any pair of types on this interval. Once this is established, the remainder of the proof immediately follows from the proof of Proposition 3.

Consider types $\theta_1 < \theta_2 \leq \theta_*$. By (A3′),

$$S(\theta_2) \geq D(T(\theta_2); \theta_2, \theta_2) > D(T(\theta_2); \theta_2, \theta_1). \tag{1}$$

(A2) requires that $\phi^*(\cdot; \theta_2)$ cannot cross $D(\cdot; \theta_2, \theta_1)$ to the right of $S(\theta_2)$, such that the single-crossing property holds along this indifference curve:

$$a > D(\phi^*(a; \theta_2); \theta_2, \theta_1) \quad \text{for } a \geq S(\theta_2). \tag{2}$$

At any point on $\phi^*(a; \theta_2)$ for $a \geq S(\theta_2)$, any lower-type $\theta_1 < \theta_2$ always has a lower marginal rate of substitution than type θ_2.

We argue that any locally IC allocation must stay below $\phi^*(a; \theta_2)$ for $a \in [S(\theta_2), S(\theta_1)]$. Suppose the opposite is true, and let $T(\theta'') \geq \phi^*(S(\theta''); \theta_2)$ with $T(\theta) < \phi^*(S(\theta); \theta_2)$ for all $\theta \in (\theta'', \theta_2]$. Local IC then implies that $\phi^*(\cdot; \theta'')$ reaches $\phi^*(\cdot; \theta_2)$ from below at some $a'' \in (S(\theta_2), S(\theta'')]$. However, this is a contradiction because by (2), type θ'' must have a lower marginal rate of substitution at any point on $\phi^*(\cdot; \theta_2)$. This shows that $T(\theta_1) < \phi^*(S(\theta_1); \theta_2)$, and thus, type θ_2 has no incentive to mimic type θ_1. Similarly, any locally IC allocation must stay below $\phi^*(a; \theta_1)$ for $a \in [S(\theta_2), S(\theta_1)]$. Suppose the opposite is true, and let $\phi^*(S(\theta'); \theta_1) \geq T(\theta')$ with $T(\theta') < \phi^*(S(\theta'); \theta_1)$ for all $\theta \in [\theta_1, \theta')$. Local IC implies that $\phi^*(\cdot; \theta_1)$ reaches $\phi^*(\cdot; \theta')$ from above at some $a' \in [S(\theta'), S(\theta_1))$. However, this is a contradiction because by (2), type θ_1 must have a lower marginal rate of substitution at any point on $\phi^*(\cdot; \theta')$. This shows that $T(\theta_2) \leq \phi^*(S(\theta_2); \theta_1)$, and so type θ_1 has no incentive to mimic type θ_2. ■

Observe that for condition (P) to hold under (A2) and (A3′), θ_* must be strictly greater than $\underline{\theta}$, meaning that $(S(\underline{\theta}), T(\underline{\theta})) \in RSC(\underline{\theta})$. In signaling models, if we assume $(0, \underline{\theta}) \in SC(\underline{\theta})$, then whenever the lowest type separates, condition (P) would have no bite under (A2) and (A3′), and global incentive compatibility would become an issue. However, for general mechanism design models, condition (P) may help play a role in ensuring incentive compatibility.

When we maintain (A2′) instead of (A2), higher types have "less convex" indifference curves than lower types on the dividing line. Condition (P) needs to be modified accordingly to ensure incentive compatibility. Recall that we define that $\theta_{**} := \inf\{\theta' :$ $(S(\theta), T(\theta))$ is not in the ill-behaved domain for all $\theta > \theta'\}$.

Definition F.2. *An allocation satisfies condition (P′) if for any $\theta' < \theta_{**}$, there exists $\theta'' > \theta_{**}$ such that $(S(\theta''), T(\theta'')) = (S(\theta'), T(\theta'))$ and $m(S(\theta''), T(\theta''), \theta'') = m(S(\theta'), T(\theta'), \theta')$.*

We can now state an analogous result that applies when (A2′) holds. Because the argument leading to the following result is very similar to that leading to Propositions 3 and F.1, we only provide a brief proof here.

Proposition F.2. *Under Assumption (A2′), an allocation that satisfies local IC and condition (P′) is incentive compatible.*

Proof. Suppose (A3) holds. Local IC then implies global incentive compatibility for allocations in the SC-domain (i.e., for any pair of types above θ_{**}). The proof of this claim follows that for Proposition 3. This result, together with the modified pairwise matching condition (P′), ensures that any type $\theta' \geq \theta_{**}$ has no incentive to mimic any other type. Moreover, under (A2′), for type $\theta'' < \theta_{**}$, condition (P′) implies that we can find $\theta' > \theta_{**}$ that is "matched to" type θ'', with the property that the indifference curve of the lower type θ'' is "more convex" than that of type θ'. Since incentive compatibility holds for type θ', the greater convexity of the indifference curve for type θ'' implies that incentive compatibility also holds for type θ''.

Now suppose (A3′) holds. Local IC then implies global incentive compatibility for allocations in the RSC-domain (i.e., for any pair of types above θ_{**}). The proof of this claim follows that for Proposition F.1. This result, together with the modified pairwise matching condition (P′), ensures that any type $\theta' \geq \theta_{**}$ has no incentive to mimic any other type. Moreover, under (A2′), for type $\theta'' < \theta_{**}$, condition (P′) implies that we can find $\theta' > \theta_{**}$ that is "matched to" type θ'', with the property that the indifference curve of the lower type θ'' is "more convex" than that of type θ'. Since incentive compatibility holds for type θ', the greater convexity of the indifference curve for type θ'' implies that incentive compatibility also holds for type θ''. ■