Online Appendices

(not intended for publication)

APPENDIX B. A MAXMIN-NEGISHI METHOD

We describe an alternative recursive planning program with a maxmin-type social welfare function. This recursive approach allows us to implement efficient contracts exactly even in non-convex economies. Beyond computational (dis)advantages, the method clarifies that an optimal policy can always be expressed in terms of the evolution of welfare *shares*, as opposed to welfare *weights*, along with Markov states, that is, as transitions on the minimally extended state space $S \times \Theta$. This Markov property of the optimal policy only requires that *ex-ante* efficient contracts remain *ex-post* efficient as time and uncertainty unfold. This is certainly the case when the economy satisfies the additional Assumption 4.2. Furthermore, in a convex economy, efficient contracts admit an ergodic probability measure on the *minimal* state space $S \times \Theta$.[12]

Given welfare weights θ in Θ, the planner's objective $\Phi : \Theta \times V \to \mathbb{R}$ is given as

$$\Phi (\theta, v) = \max \left\{ \lambda \in \mathbb{R}^+ : \lambda \theta \leq v \right\},$$

where we take the lower bound on utility values to be $\underline{v} = 0$ for notational convenience. Here, as in our previous analysis, Θ represents the canonical simplex in $\mathbb{R}^I$, but welfare weights are more properly interpreted as welfare shares. We can equivalently express the planner's objective as a maxmin social welfare function,

$$\Phi (\theta, v) = \min \left\{ \ldots, \frac{v^i}{\theta^i}, \ldots \right\}.$$

Maxmin-type social welfare functions support weakly Pareto efficient distributions of utility values even under non-convexity. The advantage of this welfare evaluation, relative to the more traditional weighted sum of utilities, is illustrated by Figure 7.

We modify the Negishi operator consistently, though maintaining the same notation for parsimony. Feasible sets for utility values are now given by

$$\mathcal{U}_t (J_t) = \{ v_t \in \mathcal{V}_t : 0 \leq \Phi (\theta_t, v_t) \leq J_t (\theta_t) \text{ for every } \theta_t \in \Theta \}.$$

The recursive decomposition can so be expressed as

$$(TJ)_t (\theta_t) = \sup_{(z_t, v_{t+1}) \in \mathcal{G}_t} \Phi (\theta_t, W_t (z_t, v_{t+1}))$$

subject to

$$v_{t+1} \in \mathcal{U}_{t+1} (J_{t+1}).$$

The first constraint accounts for feasibility, whereas the second constraint reflects consistency of promised utility values over time.

[12]The absence of such a simple Markon representation for the optimal policy is the major concern in Cole and Kubler [9]. We also notice that Lucas and Stokey [22, Theorem 3]'s statement about recursive optimal policy is slightly deceptive: it does not establish that any plan generated by the optimal policy is a feasible allocation.

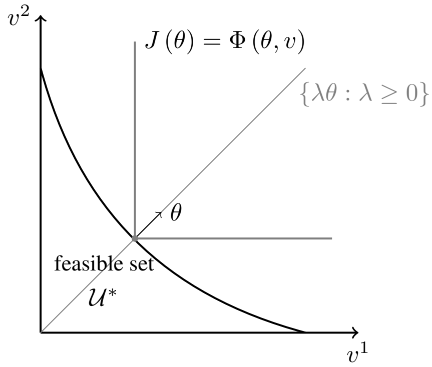

FIGURE 7. Maxmin-type social welfare function

The maxmin-Negishi value of contracts is compared with the actual maxmin-value of contracts, that is,

$$J_t^*(\theta_t) = \sup_{v_t \in \mathcal{U}_t^*} \Phi(\theta_t, v_t).$$

Not surprisingly, this approach permits the exact determination of the (weakly) efficient frontier. This is due to the fact that any allocation on a non-convex Pareto frontier can be supported by a positive sublinear (as opposed to linear) functional.

Proposition B.1 (Fixed points)**.** *Maxmin-Negishi operator $T : \mathcal{J} \to \mathcal{J}$ admits a* least *fixed point $\underline{J}$ in $\mathcal{J}$ and a* greatest *fixed point $\bar{J}$ in $\mathcal{J}$. In addition, $J^* = \bar{J}$, where J^* in $\mathcal{J}$ is the actual maxmin-value of contracts.*

Proof. We argue exactly as in the proof of Proposition 4.1. Let $J_t^-(\theta_t) = \Phi(\theta_t, U_t(z^0))$ and $J_t^+(\theta_t) = \Phi(\theta_t, \bar{v}_t)$, where contract z^0 in $\mathcal{Z}$ is given in Assumption 3.7 and the bounded processes $\bar{v}_t$ is defined in Proposition 3.1. The interval $[J^-, J^+] \subset \mathcal{J}$ is invariant for the maxmin-Negishi operator and is a complete lattice. Therefore, the first claim is a direct application of Tarski's Fixed Point Theorem [2, Theorem 1.11].

As for the second claim, consider the following recursive decomposition of the true value of contracts:

$$J_t^{**}(\theta_t) = \sup_{(z_t, v_{t+1}) \in \mathcal{G}_t} \Phi(\theta_t, W_t(z_t, v_{t+1}))$$

subject to

$$v_{t+1} \in \mathcal{U}_{t+1}^*,$$

where $\mathcal{U}_t^*$ denotes the utility possibilities set, that is, the set of utility values attainable by means of contracts which are feasible beginning from period t in $\mathbb{T}$. We so show that (the closure of) $\mathcal{U}_t^*$ coincides with (the closure of) $\mathcal{U}_t(J_t^*)$. This delivers $(TJ^*) = J^{**} \geq J^*$ and, thus, $\bar{J} \geq J^*$.

It is clear that $\mathcal{U}_t^* \subset \mathcal{U}_t(J_t^*)$, because J^* in $\mathcal{J}$ gives the maximum maxmin-value over feasible contracts. To the purpose of contradiction, at some contingency, assume that $\hat{v}_t$ lies in $\mathcal{U}_t(J_t^*)$, whereas it is not in the closure of $\mathcal{U}_t^*$. Choose $\hat{\lambda}$ in $\mathbb{R}^+$ such that $\hat{\lambda}\hat{\theta}_t = \hat{v}_t$ for some welfare weights $\hat{\theta}_t$ in Θ and, at no loss of generality, suppose that $\hat{\lambda} = 1$. As $\hat{v}_t$ is not in the closure of $\mathcal{U}_t^*$, there exists a sufficiently small ϵ in $\mathbb{R}^{++}$ such that v_t is not in $\mathcal{U}_t^*$ whenever $(1-\epsilon)\hat{v}_t \leq v_t$. Therefore,

$$J_t^*\left(\hat{\theta}_t\right) = \sup_{v_t \in \mathcal{U}_t^*} \Phi\left(\hat{\theta}_t, v_t\right) \leq 1 - \epsilon < \Phi\left(\hat{\theta}_t, \hat{v}_t\right) \leq J_t^*\left(\hat{\theta}_t\right),$$

thus revealing a contradiction.

Arguing as in the proof of Proposition 4.2, a similar argument also shows that (the closure of) $\bar{\mathcal{U}}_t$ coincides with (the closure of) $\mathcal{U}_t\left(\bar{J}_t\right)$, and we can proceed as in that proof to establish the coincidence $J^* = \bar{J}$. □

We complete our short exploration of the maxmin-Negishi method with a proof of existence of an ergodic distribution on the minimal state space $S \times \Theta$. In other terms, we show that this space exhausts all long-term dynamical properties of efficient contracts. The advantage of the maxmin-type social welfare function is that utility profiles on the efficient frontier are univocally supported by welfare shares θ in Θ. It follows that, subject to *ex post* efficiency, efficient contracts are governed by a Markov correspondence $F : S \times \Theta \twoheadrightarrow \Delta(S \times \Theta)$. Indeed, given a current state (s, θ) in $S \times \Theta$, the recursive optimal plan determines continuation utility values v' in V, contingent on next period state s' in S. As efficient contracts remain on the Pareto frontier when time evolves (by Assumption 4.2), contingent continuation utility values are supported by unique welfare shares θ' in Θ. Hence, the state in the next period can be unambiguously identified with some (s', θ') in $S \times \Theta$. Convexity (Assumption 4.1) guarantees that the Markov correspondence is convex-valued, so that a well-established theorem on the existence of ergodic measures can be applied (see Aliprantis and Border [2, Theorem 19.31]).

Proposition B.2 (Ergodic measure)**.** *Under additional Assumptions 4.1-4.2, efficient contracts are fully described by a Markov correspondence $F : S \times \Theta \twoheadrightarrow \Delta(S \times \Theta)$ admitting an ergodic probability measure.*

Proof. In the maximin-Negishi program, an optimal policy correspondence is described as $\gamma_t : \Theta_t \twoheadrightarrow \mathcal{Z}_t \times \Theta_{t+1}$. Indeed, an optimal plan is of the form (z_t, v_{t+1}) in $\mathcal{G}_t$ and, under Assumption 4.2, v_{t+1} in $\mathcal{V}_{t+1}$ achieves the maxmin social value for welfare weights θ_{t+1} in Θ_{t+1} given by

$$\theta_{t+1} = \frac{v_{t+1}}{\sum_{i \in I} v_{t+1}^i}.$$

Hence, the continuation utility values can be identified with those welfare weights θ_{t+1} in Θ_{t+1}. Under Assumptions 4.1-4.2 the correspondence $\gamma_t : \Theta_t \twoheadrightarrow \mathcal{Z}_t \times \Theta_{t+1}$ is upper hemicontinuous with nonempty convex values. Indeed, supposing v_{t+1}^0 and v_{t+1}^1 in $\mathcal{V}_{t+1}$ are both optimal, for all α_0 and α_1 in $\mathbb{R}^{++}$, we have that the convex combination is also optimal, where

$$v_{t+1} = \frac{\alpha_0}{\alpha_0 + \alpha_1} v_{t+1}^0 + \frac{\alpha_1}{\alpha_0 + \alpha_1} v_{t+1}^1,$$

Considering weights

$$\alpha_0 = (1-\lambda) \frac{1}{\sum_{i \in I} v_{t+1}^{i,0}} \text{ and } \alpha_1 = \lambda \frac{1}{\sum_{i \in I} v_{t+1}^{i,1}},$$

we obtain

$$\theta_{t+1} = (\alpha_0 + \alpha_1) v_{t+1} = (1-\lambda) \frac{v_{t+1}^0}{\sum_{i \in I} v_{t+1}^{i,0}} + \lambda \frac{v_{t+1}^1}{\sum_{i \in I} v_{t+1}^{i,1}} = (1-\lambda) \theta_{t+1}^0 + \lambda \theta_{t+1}^1.$$

We conclude that efficient contracts are governed by an upper hemicontinuous Markov correspondence $F : S \times \Theta \twoheadrightarrow \Delta (S \times \Theta)$ with nonempty convex compact values. To prove existence of an ergodic measure, we apply Aliprantis and Border [2, Theorem 19.31]. □

APPENDIX C. HISTORY DEPENDENCE

C.1. **Fundamentals.** We describe an economy in which a principal insures a risk-averse agent experiencing privately observed preference shocks. The unobservable preference shock s in the finite space S is governed by Markov transition $\pi : S \to \Delta (S)$. Consumption z in Z, a transfer from the principal to the agent, is restricted to a compact interval $[0, \eta] \subset \mathbb{R}^+$. Per-period utility of the agent is $u : Z \times S \to \mathbb{R}^+$, whereas the cost of the principal is $c : Z \to \mathbb{R}^-$, both subject to canonical assumptions. To describe the recursive contract, we adopt a more traditional notation.

Let $\mathcal{S}$ be the space of all partial histories of shocks and, given history s^t in $\mathcal{S}$, let $\mathcal{S}(s^t)$ be the space of all continuation histories (beginning from the next period). Given a contingent plan for consumption, the overall utility of the agent is

$$U(z)\left(s^t, \hat{s}_t\right) = \sum_{s^{t+j} \in \mathcal{S}(s^t)} \delta^j \pi\left(s^{t+j} | \hat{s}_t\right) u\left(z\left(s^{t+j}\right), s_{t+j}\right).$$

We assume that type declaration is truthful in all continuations, whereas the agent has initially declared type s_t in S when in state $\hat{s}_t$ in S. The principal utility (*i.e.*, the negative of the cost) is

$$U^0(z)\left(s^t, \hat{s}_t\right) = -\sum_{s^{t+j} \in \mathcal{S}(s^t)} \delta^j \pi\left(s^{t+j} | \hat{s}_t\right) c\left(z\left(s^{t+j}\right)\right).$$

Finally, we impose the incentive compatibility constraint, enforcing truthful revelation of private information. This takes the form

$$\begin{aligned} u\left(z\left(s^{t+1}\right), s_{t+1}\right) + \delta U\left(z\right)\left(s^{t+1}, s_{t+1}\right) \quad &\geq \\ u\left(z\left(s^{t}, \hat{s}_{t+1}\right), s_{t+1}\right) + \delta U\left(z\right)\left(\left(s^{t}, \hat{s}_{t+1}\right), s_{t+1}\right). \end{aligned}$$

It is a well-known property that preventing a single misreport of type is sufficient to implement truthful revelation over the entire infinite horizon.

C.2. **Efficiency.** The classical formulation features cost-minimization, subject to incentive compatibility, given a sustainable utility level for the truthful agent and for any untruthful agent. Though the agent reports the true type, an untruthful version of the agent serves as a counterfactual. We argue that efficient contracts can be equivalently represented as efficient utility profiles on the utility possibilities frontier, so setting the stage for the application of the Negishi method.

Fix an initial state s_0 in S, and assume initial truthful revelation, that is, $s_0 = \hat{s}_0$. A contract z in $\mathcal{Z}$ is feasible if it satisfies incentive compatibility at all histories s^{t+1} in $\mathcal{S}\left(s^0\right)$. A feasible contract z in $\mathcal{Z}$ is *efficient* it there exists no other feasible contract $\hat{z}$ in $\mathcal{Z}$, such that

$$U_0\left(\hat{z}\right)\left(s^0, s_0\right) \geq U_0\left(z\right)\left(s^0, s_0\right)$$

and, for every $\hat{s}_0$ in S,

$$U\left(\hat{z}\right)\left(s^0, \hat{s}_0\right) \geq U\left(z\right)\left(s^0, \hat{s}_0\right),$$

with at least one strict inequality.

Claim C.1 (Efficiency)**.** *A feasible contract z in $\mathcal{Z}$ is efficient only if it is cost-minimizing subject to incentive compatibility at every history s^{t+1} in $\mathcal{S}\left(s^0\right)$ and subject to the promise-keeping constraints, for every $\hat{s}_0$ in S,*

$$U\left(\hat{z}\right)\left(s^0, \hat{s}_0\right) \geq U\left(z\right)\left(s^0, \hat{s}_0\right).$$

Proof. Otherwise, for some feasible contract $\hat{z}$ in $\mathcal{Z}$, $U_0\left(\hat{z}\right)\left(s^0, s_0\right) > U_0\left(z\right)\left(s^0, s_0\right)$, so violating efficiency. □

Endowed with this simple characterization, we can develop the application of the Negishi method for the determination of efficient contracts. The advantage upon the more traditional approach is that the state space for the recursive program is exogenously given: it consists of (normalized) welfare weights, one for the principal, one for the truthful agent and one for each counterfactual untruthful agent.

C.3. **Recursive decomposition.** Let $v\left(s, \hat{s}\right)$ in $\mathbb{R}^+$ be the overall utility of an agent of type $\hat{s}$ in S having declared type s in S. The utility of the agent satisfies the recursive condition

(U) $$v(s,\hat{s}) = \sum_{s' \in S} \pi(s'|\hat{s})\left(u(z(s'), s') + \delta v(s', s')\right).$$

Similarly, the utility of the principal satisfies the recursive condition

(P) $$v_0(s,\hat{s}) = \sum_{s' \in S} \pi(s'|\hat{s})\left(-c(z(s')) + \delta v_0(s', s')\right).$$

Finally, the incentive compatibility constraint is

(IC) $$u(z(s'), s') + \delta v(s', s') \geq u(z(\hat{s}'), s') + \delta v(\hat{s}', s').$$

Let Θ be the simplex in $\mathbb{R} \times \mathbb{R}^S$. Welfare weights θ in Θ refer to the principal, θ_0, and to each agent conditional on (possible unfaithful) type declaration $\hat{s}$ in S, $\theta(\hat{s})$. Given a truthful state s in S, the objective of the Negishi planner is to maximize the weighted surplus,

$$J(\theta)(s) = \theta_0 v_0(s,s) + \sum_{\hat{s} \in S} \theta(\hat{s})\, v(\hat{s}, s).$$

Constraints are given by (U), (P) and (IC). Continuation values are chosen subject to the consistency constraint, for every state s' in S,

$$\sup_{\theta' \in \Theta} \theta'_0 v_0(s', s') + \sum_{\hat{s}' \in S} \theta'(\hat{s}')\, v(\hat{s}', s') - J(\theta')(s') \leq 0.$$

This ensures that values are in the convex envelope of the utility possibilities frontier.