Supplemental material for 'The converse envelope theorem'

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Supplemental appendix to the application (§4)

K The failure of the standard implementability argument

When the agent's preferences have the quasi-linear form $f(y,p,t) = h(y,t) - p$, a standard argument establishes the implementability of increasing allocations without resort to the converse envelope theorem. I first outline the argument, then show how it fails absent quasi-linearity, necessitating my alternative approach based on the converse envelope theorem.

Fix an increasing allocation $Y : [0,1] \to \mathcal{Y}$. Choose a P so that (Y,P) satisfies the envelope formula.[49] We then have for any $r,t \in [0,1]$ that

$$\begin{aligned} f(Y(t),P(t),t) - f(Y(r),P(r),t) \\ &= [V_{Y,P}(t) - V_{Y,P}(r)] - [f(Y(r),P(r),t) - f(Y(r),P(r),r)] \\ &= \int_r^t [f_3(Y(s),P(s),s) - f_3(Y(r),P(r),s)]\,\mathrm{d}s \end{aligned}$$

by the envelope formula and Lebesgue's fundamental theorem of calculus.

For quasi-linear preferences, $f_3(y,p,s)$ does not vary with p, and f is single-crossing iff $y \mapsto f_3(y,0,s)$ is increasing for every $s \in [0,1]$.[50] Since Y is also increasing, this implies that the above integrand is non-negative, which (since $r,t \in [0,1]$ were arbitrary) shows that (Y,P) is incentive-compatible.

These properties of quasi-linearity are very special, however. In general, single-crossing has nothing directly to say about the type derivative f_3, and so cannot be used to sign the integrand. The standard argument thus fails.

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[49]In the quasi-linear case, such a P is given explicitly by $P(t) \coloneqq h(Y(t),t) - \int_0^t h_2(Y(s),s)\mathrm{d}s$, obviating the need to invoke the existence lemma in appendix H.1.

[50]This is easily shown, and does not depend on exactly how 'single-crossing' is formalised.

The argument may of course be salvaged by replacing single-crossing with the brute assumption that the integrand is non-negative. But this assumption lacks a choice interpretation, being a restriction on the *type* derivative f_3 of the utility representation f. A theorem with such a hypothesis would have no economic meaning. (By contrast, single-crossing has a straightforward choice interpretation, described in the text.)

L Some regular outcome spaces (§4.2)

Proposition 4. The following partially ordered sets are regular:

(a) $\mathbf{R}^n$ equipped with the usual (product) order: $(y_1, \ldots, y_n) \lesssim (y_1', \ldots, y_n')$ iff $y_i \leq y_i'$ for every $i \in \{1, \ldots, n\}$.

(b) The space ℓ^1 of summable sequences equipped with the product order: $(y_i)_{i \in \mathbf{N}} \lesssim (y_i')_{i \in \mathbf{N}}$ iff $y_i \leq y_i'$ for every $i \in \mathbf{N}$.

(c) For any measure space $(\Omega, \mathcal{F}, \mu)$, the space $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ of (equivalence classes of μ-a.e. equal) μ-integrable functions $\Omega \to \mathbf{R}$, equipped with the partial order $\lesssim$ defined by $y \lesssim y'$ iff $y \leq y'$ μ-a.e.

(Special case: for any probability space, the space of finite-expectation random variables, ordered by 'a.s. smaller'.)

(d) For any finite set Ω and probability $\mu_0 \in \Delta(\Omega)$, the space of mean-μ_0 Borel probability measures on $\Delta(\Omega)$, equipped with the Blackwell informativeness order defined in §4.4.[51]

(e) The open intervals of $(0, 1)$ (including $\varnothing$), ordered by set inclusion $\subseteq$.

We will use the following sufficient condition for chain-separability.

Lemma 8. If there is a strictly increasing function $\mathcal{Y} \to \mathbf{R}$, then $\mathcal{Y}$ is chain-separable.

(The converse is false: there are chain-separable spaces that admit no strictly increasing real-valued function.)

Proof. Suppose that $\phi : \mathcal{Y} \to \mathbf{R}$ is a strictly increasing function, and let $Y \subseteq \mathcal{Y}$ be a chain; we will show that Y has a countable order-dense subset. By inspection, the restriction $\phi|_Y$ of ϕ to Y is an order-embedding of Y into $\mathbf{R}$; thus Y is order-isomorphic to a subset of $\mathbf{R}$ (namely $\phi(Y)$). The order-isomorphs of subsets of $\mathbf{R}$ are precisely those chains that have a countable order-dense subsets (see e.g. Theorem 24 in Birkhoff (1967, p. 200)); thus Y has a countable order-dense subset. ■

[51] A proof that this is a partial order (in particular, anti-symmetric) may be found in Müller (1997, Theorem 5.2).

Proof of Proposition 4(a)–(c). $\mathbf{R}^n$ is exactly $\mathcal{L}^1\left(\{1,\dots,n\}, 2^{\{1,\dots,n\}}, c\right)$ where c is the counting measure; similarly, ℓ^1 is $\mathcal{L}^1\left(\mathbf{N}, 2^{\mathbf{N}}, c\right)$. It therefore suffices to establish (c).

So fix a measure space $(\Omega, \mathcal{F}, \mu)$, and let $\mathcal{Y} \coloneqq \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ be ordered by 'μ-a.e. smaller'. $\mathcal{Y}$ is order-dense-in-itself since if $y \leq y''$ μ-a.e. and $y \neq y''$ on a set of positive μ-measure, then $y' \coloneqq (y + y'')/2$ lives in $\mathcal{Y}$ and satisfies $y \leq y' \leq y''$ μ-a.e. and $y \neq y' \neq y''$ on a set of positive μ-measure.

For countable-chain completeness, take any countable chain $Y \subseteq \mathcal{Y}$, and suppose that it has a lower bound $y \in \mathcal{Y}$; we will show that Y has an infimum. (The argument for upper bounds is symmetric.) Define $y_\star : \Omega \to \mathbf{R}$ by $y_\star(\omega) \coloneqq \inf_{y \in Y} y(\omega)$ for each $\omega \in \Omega$; it is well-defined (i.e. it maps into $\mathbf{R}$, with the possible exception of a μ-null set) since Y has a lower bound. Clearly $y' \leq y_\star \leq y''$ μ-a.e. for any lower bound y' of Y and any $y'' \in Y$, so it remains only to show that $y_\star$ lives in $\mathcal{Y}$, meaning that it is measurable and that its integral is finite. Measurability obtains since Y is countable (e.g. Proposition 2.7 in Folland (1999)). As for the integral, since $y \leq y_\star \leq y_0$ μ-a.e. and y and y_0 are integrable (live in $\mathcal{Y}$), we have

$$-\infty < \int_\Omega y \mathrm{d}\mu \leq \int_\Omega y_\star \mathrm{d}\mu \leq \int_\Omega y_0 \mathrm{d}\mu < +\infty.$$

For chain-separability, define $\phi : \mathcal{Y} \to \mathbf{R}$ by $\phi(y) \coloneqq \int_\Omega y \mathrm{d}\mu$ for each $y \in \mathcal{Y}$. ϕ is strictly increasing: if $y \leq y'$ μ-a.e. and $y \neq y'$ on a set of positive μ-measure, then $\phi(y) < \phi(y')$. Chain-separability follows by Lemma 8. ■

Proof of Proposition 4(d). Fix a finite set Ω and a probability $\mu_0 \in \Delta(\Omega)$, and let $\mathcal{Y}$ be the space of Borel probability measures with mean μ_0, equipped with the Blackwell informativeness order $\lesssim$. $\mathcal{Y}$ is order-dense-in-itself because if $y, y'' \in \mathcal{Y}$ satisfy $\int_{\Delta(\Omega)} v \mathrm{d}y \leq \int_{\Delta(\Omega)} v \mathrm{d}y''$ for every continuous and convex $v : \Delta(\Omega) \to \mathbf{R}$, with the inequality strict for some $v = \widehat{v}$, then $y' \coloneqq (y + y'')/2$ also lives in $\mathcal{Y}$ and satisfies $\int_{\Delta(\Omega)} v \mathrm{d}y \leq \int_{\Delta(\Omega)} v \mathrm{d}y' \leq \int_{\Delta(\Omega)} v \mathrm{d}y''$ for every continuous and convex $v : \Delta(\Omega) \to \mathbf{R}$, with both inequalities strict for $v = \widehat{v}$.

For countable chain-completeness, let $Y \subseteq \mathcal{Y}$ be a countable chain with an upper bound in $\mathcal{Y}$; we will show that it has a supremum. (The argument for infima is analogous.) This is trivial if Y has a maximum element, so suppose not. Then there is a strictly increasing sequence $(y_n)_{n \in \mathbf{N}}$ in Y that has no upper bound in Y. This sequence is trivially tight since $\Delta(\Omega)$ is a compact metric space, so has a weakly convergent subsequence $(y_{n_k})_{k \in \mathbf{N}}$ by Prokhorov's theorem;[52] call the limit $y^\star$. Then by the monotone convergence theorem for real numbers and the definition of weak convergence, we have for every for every continuous (hence bounded) and

[52]E.g. Theorem 5.1 in Billingsley (1999).

convex $v : \Delta(\Omega) \to \mathbf{R}$ that

$$\sup_{y\in Y} \int_{\Delta(\Omega)} v\mathrm{d}y = \lim_{k\to\infty} \int_{\Delta(\Omega)} v\mathrm{d}y_{n_k} = \int_{\Delta(\Omega)} v\mathrm{d}y^\star,$$

which is to say that $y^\star$ is the supremum of Y.

For chain-separability, it suffices by Lemma 8 to identify a strictly increasing function $\mathcal{Y} \to \mathbf{R}$. Let v be any strictly convex function $\Delta(\Omega) \to \mathbf{R}$,[53] and define $\phi : \mathcal{Y} \to \mathbf{R}$ by $\phi(y) \coloneqq \int_{\Delta(\Omega)} v\mathrm{d}y$. Take $y < y'$ in $\mathcal{Y}$; we must show that $\phi(y) < \phi(y')$. By a standard embedding theorem (e.g. Theorem 7.A.1 in Shaked and Shanthikumar (2007)), there exists a probability space on which there are random vectors X, X' with respective laws y, y' such that $\mathbf{E}(X'|X) = X$ a.s. and $X \neq X'$ with positive probability. Thus

$$\phi(y') = \mathbf{E}(v(X')) = \mathbf{E}(\mathbf{E}[v(X')|X]) > \mathbf{E}(v(\mathbf{E}[X'|X])) = \mathbf{E}(v(X)) = \phi(y)$$

by Jensen's inequality. ■

Proof of Proposition 4(e). Write $\mathcal{Y}$ for the open intervals of $(0, 1)$. $\mathcal{Y}$ is order-dense-in-itself since if $(a, b) \subsetneq (a'', b'')$ then $(a', b') \coloneqq ([a + a'']/2, [b + b'']/2)$ is an open interval (lives in $\mathcal{Y}$) and satisfies $(a, b) \subsetneq (a', b') \subsetneq (a'', b'')$.

For countable chain-completeness, we must show that every countable chain has an infimum and supremum. So take a countable chain $Y \subseteq \mathcal{Y}$, define $y^\star \coloneqq \bigcup_{y\in Y} y$, and let $y_\star$ be the interior of $\bigcap_{y\in Y} y$. Both are open intervals, so live in $\mathcal{Y}$. Clearly $y \subseteq y^\star \subseteq y^+$ for any $y \in Y$ and any set y^+ containing every member of Y, so $y^\star$ is the supremum of Y. Similarly $y_\star \subseteq \bigcap_{y'\in Y} y' \subseteq y$ for any $y \in Y$, and $y_- \subseteq y_\star$ for any *open* set y_- contained in every member of Y since $y_\star$ is by definition the $\subseteq$-largest open set contained in $\bigcap_{y\in Y} y$.

For chain-separability, define $\phi : \mathcal{Y} \to \mathbf{R}$ by $\phi((a, b)) \coloneqq b - a$. It is clearly strictly increasing, giving us chain-separability by Lemma 8. ■

M Proof of the approximation lemma (appendix H.2)

Let $Y : [0, 1] \to \mathcal{Y}$ be increasing. Then $Y([0, 1])$ is a chain. The result is trivial if $Y([0, 1])$ is a singleton, so suppose not.

We will first show (steps 1–3) that $Y([0, 1])$ may be embedded in a chain $\mathcal{C} \subseteq \mathcal{Y}$ with $\inf \mathcal{C} = Y(0)$ and $\sup \mathcal{C} = Y(1)$ that is order-dense-in-itself, order-complete and order-separable. We will then argue (step 4) that this chain $\mathcal{C}$ is order-isomorphic and homeomorphic to the unit interval, allowing us to treat Y as a function $[0, 1] \to [0, 1]$.

[53] E.g. the $\mathcal{L}^2$ norm $\|\cdot\|_2$, which is strictly convex on $\Delta(\Omega)$ by Minkowski's inequality.

Step 1: construction of $\mathcal{C}$. Write $\lesssim$ for the partial order on $\mathcal{Y}$. Define $\mathcal{Y}'$ to be the set of all outcomes $y' \in \mathcal{Y}$ that are $\lesssim$-comparable to every $y \in Y([0,1])$ and that satisfy $Y(0) \lesssim y' \lesssim Y(1)$.

We claim that $\mathcal{Y}'$ is order-dense-in-itself. Suppose to the contrary that there are $y < y''$ in $\mathcal{Y}'$ for which no $y' \in \mathcal{Y}'$ satisfies $y < y' < y''$. Observe that by definition of $\mathcal{Y}'$, any $x \in Y([0,1])$ must be comparable to both y and y'', so that

$$\{x \in Y([0,1]) : x \lesssim y \text{ or } y'' \lesssim x\} = Y([0,1]).$$

Since it is order-dense-in-itself, the grand space $\mathcal{Y}$ does contain an outcome y' such that $y < y' < y''$. Since $\lesssim$ is transitive (being a partial order), it follows that y' is comparable to every element of

$$\{x \in \mathcal{Y} : x \lesssim y \text{ or } y'' \lesssim x\} \supseteq \{x \in Y([0,1]) : x \lesssim y \text{ or } y'' \lesssim x\} = Y([0,1]).$$

But then y' lies in $\mathcal{Y}'$ by definition of the latter—a contradiction.

Clearly $Y(1)$ is an upper bound of any chain in $\mathcal{Y}'$. It follows by the Hausdorff maximality principle (which is equivalent to the Axiom of Choice) that there is a chain $\mathcal{C} \subseteq \mathcal{Y}'$ that is maximal with respect to set inclusion. (That is, $\mathcal{C} \cup \{y\}$ fails to be a chain for every $y \in \mathcal{Y}' \setminus \mathcal{C}$.)

Step 2: easy properties of $\mathcal{C}$. By definition of $\mathcal{Y}'$, any maximal chain in $\mathcal{Y}'$ (in particular, $\mathcal{C}$) contains $Y([0,1])$ and has infimum $Y(0)$ and supremum $Y(1)$.

To see that $\mathcal{C}$ is order-dense-in-itself, assume toward a contradiction that there are $c < c''$ for which no $c' \in \mathcal{C}$ satisfies $c < c' < c''$, so that (since $\mathcal{C}$ is a chain)

$$\{c' \in \mathcal{C} : c' \lesssim c\} \cup \{c' \in \mathcal{C} : c'' \lesssim c'\} = \mathcal{C}.$$

Because $\mathcal{Y}'$ is order-dense-in-itself, there is a $y' \in \mathcal{Y}' \setminus \mathcal{C}$ with $c < y' < c''$. It follows by transitivity of $\lesssim$ that y' is comparable to every element of

$$\{c' \in \mathcal{C} : c' \lesssim c\} \cup \{c' \in \mathcal{C} : c'' \lesssim c'\} = \mathcal{C}.$$

But then $\mathcal{C} \cup \{y'\}$ is a chain in $\mathcal{Y}'$, contradicting the maximality of $\mathcal{C}$.

To establish that $\mathcal{C}$ is order-separable, we must find a countable order-dense subset of $\mathcal{C}$. Because the grand space $\mathcal{Y}$ is chain-separable, it contains a countable set $\mathcal{K}$ that is order-dense in $\mathcal{C}$. Since $\mathcal{C}$ is a chain contained in

$$\{y \in \mathcal{Y} : Y(0) \lesssim y \lesssim Y(1)\},$$

we may assume without loss of generality that every $k \in \mathcal{K}$ satisfies $Y(0) \lesssim k \lesssim Y(1)$

and is comparable to every element of $\mathcal{C}$. It follows that $\mathcal{K}$ is contained in $\mathcal{Y}'$ (by definition of the latter). We claim that $\mathcal{K}$ is contained in $\mathcal{C}$. Suppose to the contrary that there is a $k \in \mathcal{K}$ that does not lie in $\mathcal{C}$; then $\mathcal{C} \cup \{k\}$ is a chain in $\mathcal{Y}'$, which is absurd since $\mathcal{C}$ is maximal.

Step 3: order-completeness of $\mathcal{C}$*.* Since every subset of $\mathcal{C}$ has a lower and an upper bound (viz. $Y(0)$ and $Y(1)$, respectively), what must be shown is that every subset of the chain $\mathcal{C}$ has an infimum and a supremum in $\mathcal{C}$. To that end, take any subset $\mathcal{C}'$ of $\mathcal{C}$, necessarily a chain.

We will first (step 3(a)) show that if $\inf \mathcal{C}'$ exists in $\mathcal{Y}$, then it must lie in $\mathcal{C}$. We will then (step 3(b)) construct a countable chain $\mathcal{C}''' \subseteq \mathcal{C}'$, for which $\inf \mathcal{C}'''$ exists in $\mathcal{Y}$ by countable-chain completeness of $\mathcal{Y}$, and show that it is also the infimum in $\mathcal{Y}$ of $\mathcal{C}'$. We omit the analogous arguments for $\sup \mathcal{C}'$.

Step 3(a): $\inf \mathcal{C}' \in \mathcal{C}$ *if the former exists in* $\mathcal{Y}$*.* Suppose that $\inf \mathcal{C}'$ exists in $\mathcal{Y}$. We claim that it lies in $\mathcal{Y}'$, meaning that $Y(0) \lesssim \inf \mathcal{C}' \lesssim Y(1)$ and that $\inf \mathcal{C}'$ is comparable to every $y \in Y([0,1])$. The former condition is clearly satisfied. For the latter, since $\inf \mathcal{C}'$ is a lower bound of $\mathcal{C}'$, transitivity of $\lesssim$ ensures that it is comparable to every $y \in Y([0,1])$ such that $c' \lesssim y$ for some $c' \in \mathcal{C}'$. To see that $\inf \mathcal{C}'$ is also comparable to every $y \in Y([0,1])$ with $y < c'$ for every $c' \in \mathcal{C}'$, note that any such y is a lower bound of $\mathcal{C}'$. Since $\inf \mathcal{C}'$ is the *greatest* lower bound, we must have $y \lesssim \inf \mathcal{C}'$, showing that $\inf \mathcal{C}'$ is comparable to y.

Now to show that $\inf \mathcal{C}'$ lies in $\mathcal{C}$, decompose the chain $\mathcal{C}$ as

$$\begin{aligned}\mathcal{C} &= \{c \in \mathcal{C} : c \lesssim c' \text{ for every } c' \in \mathcal{C}'\} \cup \{c \in \mathcal{C} : c' < c \text{ for some } c' \in \mathcal{C}'\} \\ &= \{c \in \mathcal{C} : c \lesssim \inf \mathcal{C}'\} \cup \{c \in \mathcal{C} : \inf \mathcal{C}' < c\}.\end{aligned}$$

Clearly $\inf \mathcal{C}'$ is comparable to every element of $\mathcal{C}$, and we showed that it lies in $\mathcal{Y}'$. Thus $\mathcal{C} \cup \{\inf \mathcal{C}'\}$ is a chain in $\mathcal{Y}'$, which by maximality of $\mathcal{C}$ requires that $\inf \mathcal{C}' \in \mathcal{C}$.

Step 3(b): $\inf \mathcal{C}'$ *exists in* $\mathcal{Y}$*.* By essentially the same construction as we used to embed $Y([0,1])$ in $\mathcal{Y}'$ in step 1, $\mathcal{C}'$ may be embedded in a chain $\mathcal{C}'' \subseteq \mathcal{C}$ that is order-dense-in-itself such that for every $c'' \in \mathcal{C}''$, we have $c'_- \lesssim c'' \lesssim c'_+$ for some $c'_-, c'_+ \in \mathcal{C}'$. By order-separability of $\mathcal{C}$, $\mathcal{C}''$ has a countable order-dense subset $\mathcal{C}'''$, necessarily a chain. By countable chain-completeness of $\mathcal{Y}$, $\inf \mathcal{C}'''$ exists in $\mathcal{Y}$. We will show that it is the greatest lower bound of $\mathcal{C}'$.

Observe that $\inf \mathcal{C}'''$ is a lower bound of $\mathcal{C}''$ since $\mathcal{C}'''$ is order-dense in $\mathcal{C}''$. There can be no greater lower bound of $\mathcal{C}''$ since $\mathcal{C}''' \subseteq \mathcal{C}''$. Thus $\inf \mathcal{C}''$ exists in $\mathcal{Y}$ and equals $\inf \mathcal{C}'''$.

Since $\inf \mathcal{C}''$ is a lower bound of $\mathcal{C}'' \supseteq \mathcal{C}'$, it is a lower bound of $\mathcal{C}'$. On the other hand, by construction of $\mathcal{C}''$, we may find for every $c'' \in \mathcal{C}''$ a $c' \in \mathcal{C}'$ such that $c' \lesssim c''$, so there cannot be a greater lower bound of $\mathcal{C}'$. Thus $\inf \mathcal{C}''$ is the greatest lower bound of $\mathcal{C}'$ in $\mathcal{Y}$.

Step 4: identification of $\mathcal{C}$ *with* $[0,1]$. Since $\mathcal{C}$ is an order-separable chain, it is order-isomorphic to a subset $\mathcal{S}$ of $\mathbf{R}$ (see e.g. Theorem 24 in Birkhoff (1967, p. 200)). It follows that $\mathcal{C}$ with the order topology is homeomorphic to $\mathcal{S}$ with its order topology.

The set $\mathcal{S}$ is dense in an interval $\mathcal{S}' \supseteq \mathcal{S}$ since $\mathcal{S}$ is order-dense-in-itself (because $\mathcal{C}$ is). The interval $\mathcal{S}'$ must be closed and bounded since it contains its infimum and supremum (because $\mathcal{C}$ contains $Y(0)$ and $Y(1)$). Since $\mathcal{S}$ is order-complete (because $\mathcal{C}$ is), it must coincide with its closure, so that $\mathcal{S}' = \mathcal{S}$. Finally, $\mathcal{S}$ is a proper interval since $\mathcal{C}$ is neither empty nor a singleton. In sum, we may identify $\mathcal{C}$ with a closed and bounded proper interval of $\mathbf{R}$—without loss of generality, the unit interval $[0,1]$.

We may therefore treat Y as an increasing function $[0,1] \to [0,1]$. With this simplification, it is straightforward to construct a sequence $(Y_n)_{n\in\mathbf{N}}$ with the desired properties; we omit the details. ■

N Preference regularity in selling information (§4.4)

In this appendix, we show that the joint continuity part of preference regularity (p. 11) is satisfied in §4.4. We require two lemmata.

Lemma 9. Let $\mathcal{Y}$ be the set of Borel probability distributions with mean μ_0, equipped with the Blackwell informativeness order (as in §4.4). Give $\mathcal{Y}$ the order topology, and let $\mathcal{C} \subseteq \mathcal{Y}$ be a chain. If a sequence $(y_n)_{n\in\mathbf{N}}$ in $\mathcal{C}$ converges to $y \in \mathcal{C}$ in the relative topology on $\mathcal{C}$, then

$$\sup_{\substack{v^+,v^-:\Delta(\Omega)\to\mathbf{R} \\ \text{continuous convex} \\ \text{s.t. } |v^+-v^-|\leq 1}} \left| \int_{\Delta(\Omega)} \left(v^+ - v^-\right) \mathrm{d}(y_n - y) \right| \to 0 \quad \text{as } n \to \infty.$$

Corollary 1. Under the same hypotheses,

$$\sup_{\substack{v:\Delta(\Omega)\to[-1,1] \\ \text{continuous convex}}} \left| \int_{\Delta(\Omega)} v \mathrm{d}(y_n - y) \right| \to 0 \quad \text{as } n \to \infty.$$

Proof of Lemma 9. Define $d : \mathcal{Y} \times \mathcal{Y} \to \mathbf{R}_+$ by

$$d(y,y') \coloneqq \sup_{\substack{v^+,v^-:\Delta(\Omega)\to\mathbf{R} \\ \text{continuous convex} \\ \text{s.t. } |v^+-v^-|\leq 1}} \left| \int_{\Delta(\Omega)} \left(v^+ - v^-\right) \mathrm{d}(y - y') \right|.$$

(d is in fact a metric on $\mathcal{Y}$.) Let $(y_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{C}$ that converges to some

$y \in \mathcal{C}$ in the relative topology on $\mathcal{C}$ inherited from the order topology on $\mathcal{Y}$; we will show that $d(y_n, y)$ vanishes as $n \to \infty$.

Let $B_\varepsilon := \{y' \in \mathcal{Y} : d(y, y') < \varepsilon\}$ denote the open d-ball of radius $\varepsilon > 0$ around y. Call $I \subseteq \mathcal{Y}$ an *open order interval* iff either (1) $I = \{y' \in \mathcal{Y} : y' < y^+\}$ for some $y^+ \in \mathcal{Y}$, or (2) $I = \{y' \in \mathcal{Y} : y^- < y'\}$ for some $y^- \in \mathcal{Y}$, or (3) $I = \{y' \in \mathcal{Y} : y^- < y' < y^+\}$ for some $y^- < y^+$ in $\mathcal{Y}$. Open order intervals are obviously open in the order topology on $\mathcal{Y}$.

It suffices to show that for every $\varepsilon > 0$, there is an open order interval $I_\varepsilon \subseteq \mathcal{Y}$ such that $y \in I_\varepsilon \subseteq B_\varepsilon$. For then given any $\varepsilon > 0$, we know that y_n lies in $I_\varepsilon \cap \mathcal{C} \subseteq B_\varepsilon$ for all sufficiently large $n \in \mathbf{N}$ because (in the relative topology on $\mathcal{C}$) $I_\varepsilon \cap \mathcal{C}$ is an open set containing y and $y_n \to y$. And this clearly implies that $d(y_n, y)$ vanishes as $n \to \infty$.

So fix an $\varepsilon > 0$; we will construct an open order interval $I \subseteq \mathcal{Y}$ such that $y \in I \subseteq B_\varepsilon$. There are three cases.

Case 1: $y' < y$ for no $y' \in \mathcal{Y}$. Let $y^{++} \in \mathcal{Y}$ be such that $y < y^{++}$. Define

$$y^+ := (1 - \varepsilon/2)y + (\varepsilon/2)y^{++} \in \mathcal{Y} \quad \text{and} \quad I := \left\{y' \in \mathcal{Y} : y' < y^+\right\}.$$

We have $y < y^+$ and thus $y \in I$ since

$$\int_{\Delta(\Omega)} v \mathrm{d}\left(y^+ - y\right) = \frac{\varepsilon}{2} \int_{\Delta(\Omega)} v \mathrm{d}\left(y^{++} - y\right)$$

is weakly (strictly) positive for every (some) continuous and convex $v : \Delta(\Omega) \to \mathbf{R}$ by $y < y^{++}$. To establish that $I \subseteq B_\varepsilon$, it suffices to show that $d\left(y, y^+\right) < \varepsilon$, and this holds because

$$d\left(y, y^+\right) = \frac{\varepsilon}{2} \sup_{\substack{v^+, v^- : \Delta(\Omega) \to \mathbf{R} \\ \text{continuous convex} \\ \text{s.t. } |v^+ - v^-| \leq 1}} \left| \int_{\Delta(\Omega)} \left(v^+ - v^-\right) \mathrm{d}(y - y') \right| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Case 2: $y < y'$ for no $y' \in \mathcal{Y}$. This case is analogous to the first: choose a $y^{--} \in \mathcal{Y}$ such that $y^{--} < y$, and let

$$y^- := (1 - \varepsilon/2)y + (\varepsilon/2)y^{--} \quad \text{and} \quad I := \left\{y' \in \mathcal{Y} : y^- < y'\right\}.$$

The same arguments as in Case 1 yield $y \in I \subseteq B_\varepsilon$.

Case 3: $y' < y < y''$ for some $y', y'' \in \mathcal{Y}$. Define y^+ as in Case 1 and y^- as in Case 2, and let $I := \left\{y' \in \mathcal{Y} : y^- < y' < y^+\right\}$. We have $y \in I \subseteq B_\varepsilon$ by the same arguments as in Cases 1 and 2. ■

Lemma 10. For any continuous function $c : \Delta(\Omega) \to \mathbf{R}$ and any $\varepsilon > 0$, there are continuous convex $w^+, w^- : \Delta(\Omega) \to \mathbf{R}$ such that $w := w^+ - w^-$ satisfies

$\sup_{\mu\in\Delta(\Omega)}|c(\mu)-w(\mu)|<\varepsilon$.

Proof. Write $\mathcal{W}$ for the space of functions $\Delta(\Omega)\to\mathbf{R}$ that can be written as the difference of continuous convex functions. Since the sum of convex functions is convex, $\mathcal{W}$ is a vector space. It is furthermore closed under pointwise multiplication (Hartman, 1959, p. 708), and thus an algebra. Clearly $\mathcal{W}$ contains the constant functions, and it separates points in the sense that for any distinct $\mu,\mu'\in\Delta(\Omega)$ there is a $w\in\mathcal{W}$ with $w(\mu)\neq w(\mu')$. It follows by the Stone–Weierstrass theorem[54] that $\mathcal{W}$ is dense in the space of continuous functions $\Delta(\Omega)\to\mathbf{R}$ when the latter has the sup metric. ∎

With the lemmata in hand, we can verify the continuity hypothesis.

Proposition 5. Consider the setting in §4.4. Let $\mathcal{C}\subseteq\mathcal{Y}$ be a chain, and equip it with the relative topology inherited from the order topology on $\mathcal{Y}$. Then f is (jointly) continuous on $\mathcal{C}\times\mathbf{R}\times[0,1]$.

Proof. Fix a chain $\mathcal{C}\subseteq\mathcal{Y}$, and equip it with the relative topology on $\mathcal{C}$ induced by the order topology on $\mathcal{Y}$. Define $h:\mathcal{C}\times[0,1]\to\mathbf{R}$ by $h(y,t):=\int_{\Delta(\Omega)}V(\mu,t)y(\mathrm{d}\mu)$, so that $f(y,p,t)=g(h(y,t),p)$. Since g is jointly continuous, we need only show that h is jointly continuous.

It suffices to prove that $h(\cdot,0)$ is continuous and that $\{h_2(\cdot,t)\}_{t\in[0,1]}$ is equi-continuous.[55] To see why, take (y,t) and (y',t') in $\mathcal{C}\times[0,1]$ with (wlog) $t\leq t'$, and apply Lebesgue's fundamental theorem of calculus to obtain

$$\begin{aligned}|h(y',t')-h(y,t)|&=\left|h(y',0)+\int_0^{t'}h_2(y',s)\mathrm{d}s-h(y,0)-\int_0^{t}h_2(y,s)\mathrm{d}s\right|\\&\leq|h(y',0)-h(y,0)|+\int_0^{t}|h_2(y',s)-h_2(y,s)|\mathrm{d}s+\int_t^{t'}|h_2(y',s)|\mathrm{d}s.\end{aligned}$$

Given continuity of $h(\cdot,0)$ (equi-continuity of $\{h_2(\cdot,s)\}_{s\in[0,1]}$), the first term (second term) can be made arbitrarily small by taking y and y' sufficiently close (formally, choosing y' in a neighbourhood of y that is small in the sense of set inclusion). By boundedness of h_2, the third term can similarly be made small by choosing t and t' close.

So take a sequence $(y_n)_{n\in\mathbf{N}}$ in $\mathcal{C}$ converging to some $y\in\mathcal{C}$; we must show that

$$|h(y_n,0)-h(y,0)|\quad\text{and}\quad\sup_{t\in[0,1]}|h_2(y_n,t)-h_2(y,t)|$$

[54] See e.g. Folland (1999, Theorem 4.45).

[55] A detail: equi-continuity is a property of functions on a *uniformisable* topological space. To see that $\mathcal{C}$ is uniformisable, we need only convince ourselves that the relative topology on $\mathcal{C}$ inherited from the order topology on $\mathcal{Y}$ is completely regular. This topology is obviously finer than the order topology on $\mathcal{C}$, so it suffices to show that the latter is completely regular. And that is (a consequence of) a standard result; see e.g. Cater (2006).

both vanish as $n \to \infty$. The former is easy: since $V(\cdot, 0)$ is continuous (hence bounded) and convex, we have

$$\begin{aligned}|h(y_n, 0) - h(y, 0)| &= \left| \int_{\Delta(\Omega)} V(\cdot, 0) \mathrm{d}(y_n - y) \right| \\ &\leq \left(\sup_{\mu \in \Delta(\Omega)} |V(\mu, 0)| \right) \times \sup_{\substack{v : \Delta(\Omega) \to [-1,1] \\ \text{continuous convex}}} \left| \int_{\Delta(\Omega)} v \mathrm{d}(y_n - y) \right|\end{aligned}$$

for every $n \in \mathbf{N}$, and the right-hand side vanishes as $n \to \infty$ by Corollary 1.

For the latter, fix an $\varepsilon > 0$; we seek an $N \in \mathbf{N}$ such that

$$|h_2(y_n, t) - h_2(y, t)| < \varepsilon \quad \text{for all } t \in [0, 1] \text{ and } n \geq N.$$

For each $t \in [0, 1]$, since $V_2(\cdot, t)$ is continuous, Lemma 10 permits us to choose continuous and convex functions $w_t^+, w_t^- : \Delta(\Omega) \to \mathbf{R}$ such that $w_t \coloneqq w_t^+ - w_t^-$ is uniformly $\varepsilon/3$-close to $V_2(\cdot, t)$. Write K for the constant bounding V_2, and observe that $\{w_t\}_{t \in [0,1]}$ is uniformly bounded by $K' \coloneqq K + \varepsilon/3$. By Lemma 9, there is an $N \in \mathbf{N}$ such that

$$\sup_{\substack{v^+, v^- : \Delta(\Omega) \to \mathbf{R} \\ \text{continuous convex} \\ \text{s.t. } |v^+ - v^-| \leq 1}} \left| \int_{\Delta(\Omega)} \left(v^+ - v^- \right) \mathrm{d}(y_n - y) \right| < \varepsilon / 3K' \quad \text{for all } n \geq N,$$

and thus

$$\sup_{t \in [0,1]} \left| \int_{\Delta(\Omega)} w_t \mathrm{d}(y_n - y) \right| \leq K' \times \varepsilon / 3K' = \varepsilon / 3 \quad \text{for } n \geq N.$$

Hence for every $t \in [0, 1]$ and $n \geq N$, we have

$$\begin{aligned}|h_2(y_n, t) - h_2(y, t)| &= \left| \int_{\Delta(\Omega)} V_2(\cdot, t) \mathrm{d}(y_n - y) \right| \\ &\leq \left| \int_{\Delta(\Omega)} w_t \mathrm{d}(y_n - y) \right| + \left| \int_{\Delta(\Omega)} [V_2(\cdot, t) - w_t] \mathrm{d}(y_n - y) \right| \\ &\leq \left| \int_{\Delta(\Omega)} w_t \mathrm{d}(y_n - y) \right| + 2 \sup_{\mu \in \Delta(\Omega)} |V_2(\mu, t) - w_t(\mu)| \\ &\leq \varepsilon/3 + 2\varepsilon/3 = \varepsilon,\end{aligned}$$

as desired. ■

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