# A Planner's problem

**Proof of Proposition 1.** The planner's problem is to choose a nonnegative sequence

$$\left\{y_{Ct}, y_{Pt}, (x_{it}, h_{it})_{i\in\{B,C,P\}}\right\}_{t=0}^{\infty}$$

that maximizes

$$\sum_{t=0}^{\infty}\beta^t\left\{u\left(y_{Ct}\right)-\kappa y_{Pt}+\sum_{i\in\{B,C,P\}}\left[v(x_{it})-h_{it}\right]\right\}$$

$$\text{s.t. } y_{Ct}\leq y_{Pt} \text{ and } \sum_{i\in\{B,C,P\}}\left(x_{it}-h_{it}\right)=\underline{\kappa}\left(y_{Pt}-y_{Ct}\right).$$

The first-order necessary and sufficient conditions for optimization are $u'\left(y_{Ct}\right)=\kappa$ and $v'\left(x_{it}\right)=1$, so the planner's solution is $y_{Ct}=y^{\star}$, $y^{\star}_{Pt}=y^{\star}$, and $x_{it}=h_{it}=x^{\star}$ for all $i\in\{B,C,P\}$ and all $t$. ■

# B Nonmonetary economy

The following remark will be useful in the characterization of equilibrium.

**Remark 1** *For $i\in\{B,C,P\}$, the second-subperiod value functions can be written as*

$$W_t^i(a_t^m,a_t^g)=\frac{a_t^m}{p_{2t}}+a_t^g+\bar{W}_t^i, \tag{39}$$

$$\bar{W}_t^i\equiv\frac{T_t^m}{p_{2t}}\mathbb{I}_{\{i=C\}}+v\left(x^{\star}\right)-x^{\star}+\max_{a_{t+1}^m\in\mathbb{R}_+}\left[\beta V_{t+1}^i(a_{t+1}^m)-\frac{a_{t+1}^m}{p_{2t}}\right]. \tag{40}$$

For what follows, it is useful to introduce the following notation. For any $z\in\mathbb{R}$, define the correspondences $\varkappa:\mathbb{R}\rightrightarrows\mathbb{R}$ and $\zeta:\mathbb{R}\rightrightarrows[0,1]$ by[28]

$$\varkappa_{(z)}\begin{cases}=\infty & \text{if } z<0\\ \in[0,\infty] & \text{if } z=0\\ =0 & \text{if } 0<z\end{cases}\quad\text{and}\quad\zeta_{(z)}\begin{cases}=1 & \text{if } 0<z\\ \in[0,1] & \text{if } z=0\\ =0 & \text{if } z<0.\end{cases}$$

Let $\varphi_t^n$ denote the relative price of good 1 in terms of the bond in the first subperiod of period $t$. The following lemma characterizes the first-subperiod outcomes in a nonmonetary economy

[28]Below, we use the variants $\bar{\zeta}_{(z)}$ and $\tilde{\zeta}_{(z)}$ to denote correspondences with $\bar{\zeta}_{(z)}=\tilde{\zeta}_{(z)}=\zeta_{(z)}$ for all $z\neq0$, but possibly $\bar{\zeta}_{(0)}\neq\tilde{\zeta}_{(0)}\neq\zeta_{(0)}$. Similarly, the variants $\{\varkappa_{it(z)}^m\}_{i\in\{B,C,P\}}$ and $\varkappa_{(z)}^p$, denote correspondences with $\varkappa_{it(z)}^m=\varkappa_{(z)}^p=\varkappa_{(z)}$ for all $z\neq0$ and all $i\in\{B,C,P\}$ and $t\in\mathbb{T}$, but possibly $\varkappa_{it(0)}^m\neq\varkappa_{jt(0)}^m\neq\varkappa_{(0)}^p\neq\varkappa_{(0)}$ for some $t\in\mathbb{T}$ and $i,j\in\{B,C,P\}$ with $i\neq j$.

taking the price path $\{\varphi_t^n\}_{t=0}^{\infty}$ as given. The unique price path and consumption/production allocation of good 1 consistent with equilibrium are characterized in Proposition 2. Given this price path and allocation, the rest of the equilibrium is given by Lemma 1.

**Lemma 1** *Consider the first subperiod of period $t$ of an economy with no money. (i) The solution to the banker's portfolio problem (i.e., (5)) is $\bar{a}_{Bt}^{b}=0$. (ii) A consumer's trade (i.e., the solution to (6)) is $\bar{y}_{Ct}=\mathrm{D}\left(\varphi_t^n\right)$ and $\bar{a}_{Ct}^{b}=-\varphi_t^n\mathrm{D}(\varphi_t^n)$. (iii) The post-production trade of a producer who carries inventory $y_t$ and does not contact a banker (i.e., (7)) is $\tilde{y}_{Pt}\left(y_t\right)=0$. The post-production trade of a producer who carries inventory $y_t$ and contacts a banker (i.e., the solution to (8)) is $\bar{y}_{Pt}(y_t)=\zeta_{(\varphi_t^n-\underline{\kappa})}y_t$, $\bar{a}_{Pt}^{b}(y_t)=\varphi_t^n\bar{y}_{Pt}(y_t)$, and $k_{Pt}(y_t)=(1-\theta)(\varphi_t^n-\underline{\kappa})\bar{y}_{Pt}(y_t)$. (iv) A producer's pre-trade production is $y_{Pt}=\varkappa_{(\kappa-R^n(\varphi_t^n))}$, where*

$$R^n\left(\varphi_t^n\right)\equiv\underline{\kappa}+\alpha\theta(\varphi_t^n-\underline{\kappa})\zeta_{(\varphi_t^n-\underline{\kappa})}. \tag{41}$$

**Proof of Lemma 1.** Consider a nonmonetary economy, i.e., $M_t=0$ for all $t$. With a slight abuse, we keep the notation for the value functions of the monetary economy, but simply omit an agent's money holding as an argument in the relevant functions. For example, (39) becomes

$$W_t^i(a_t^g)=a_t^g+\bar{W}_t^i, \tag{42}$$

where $\bar{W}_t^i\equiv v\left(x^{\star}\right)-x^{\star}+\beta V_{t+1}^i$. $(i)$ Problem (5) becomes

$$\hat{W}_t^B(a_t^g)=\max_{\bar{a}_t^b\in\mathbb{R}}W_t^B(a_t^g+\bar{a}_t^b)\text{ s.t. }\bar{a}_t^b\leq 0.$$

With (42), we have $\bar{a}_{Bt}^b=\arg\max_{\bar{a}_t^b\in\mathbb{R}_-}\bar{a}_t^b=0$. $(ii)$ With (42), problem (6) becomes

$$\max_{(\bar{y}_t,\bar{a}_t^b)\in\mathbb{R}_+\times\mathbb{R}}\left[u(\bar{y}_t)+\bar{a}_t^b+\bar{W}_t^i\right]\text{ s.t. }\varphi_t^n\bar{y}_t+\bar{a}_t^b\leq 0,$$

and the solution is $\bar{y}_{Ct}=\mathrm{D}(\varphi_t^n)$ and $\bar{a}_{Ct}^b=-\varphi_t^n\mathrm{D}(\varphi_t^n)$. So the gain from trade to the consumer is

$$\bar{\Gamma}_{Ct}\equiv u\left(\bar{y}_{Ct}\right)+\bar{a}_{Ct}^b=u\left(\mathrm{D}\left(\varphi_t^n\right)\right)-\varphi_t^n\mathrm{D}\left(\varphi_t^n\right).$$

$(iii)$ $(a)$ With (42), condition (7) implies $\tilde{y}_{Pt}\left(y_t\right)=\arg\max_{\tilde{y}_t\in[0,y_t]}W_t^P[(y_t-\tilde{y}_t)\underline{\kappa}]=\arg\max_{\tilde{y}_t\in[0,y_t]}(y_t-\tilde{y}_t)\underline{\kappa}=0$. $(b)$ With (42), problem (6) becomes

$$\max_{(\bar{y}_t,k_t,\bar{a}_t^b)\in\mathbb{R}_+^2\times\mathbb{R}}(\bar{a}_t^b-k_t-\underline{\kappa}\bar{y}_t)^{\theta}k_t^{1-\theta}$$

$$\begin{aligned} \text{s.t. } \bar{a}_t^b &\leq \varphi_t^n \bar{y}_t \\ \bar{y}_t &\leq y_t \\ 0 &\leq \bar{a}_t^b - k_t - \underline{\kappa}\bar{y}_t. \end{aligned}$$

The solution is $\bar{a}_{Pt}^b(y_t) = \varphi_t^n \bar{y}_{Pt}(y_t)$ and $k_{Pt}(y_t) = (1-\theta)(\varphi_t^n - \underline{\kappa})\bar{y}_{Pt}(y_t)$, with

$$\bar{y}_{Pt}(y_t) \begin{cases} 0 & \text{if } \varphi_t^n < \underline{\kappa} \\ \in [0, y_t] & \text{if } \varphi_t^n = \underline{\kappa} \\ y_t & \text{if } \underline{\kappa} < \varphi_t^n. \end{cases}$$

So the gain from trade to the producer is

$$\begin{aligned} \bar{\Gamma}_{Pt} &\equiv \bar{a}_{Pt}^b(y_t) - k_{Pt}(y_t) - \underline{\kappa}\bar{y}_{Pt}(y_t) \\ &= \theta(\varphi_t^n - \underline{\kappa})\bar{y}_{Pt}(y_t). \end{aligned}$$

$(iv)$ After substituting the bargaining outcomes, (11) becomes

$$V_t^P = \max_{y_t \in \mathbb{R}_+} \left[ R^n\left(\varphi_t^n\right) y_t - \kappa y_t + W_t^P(0) \right],$$

where $R^n\left(\varphi_t^n\right)$ as defined in (41). Hence, an individual producer produces

$$y_{Pt} = \arg\max_{y_t \in \mathbb{R}_+} \left[ R^n\left(\varphi_t^n\right) - \kappa \right] y_t$$

units of good 1 at the beginning of the first subperiod. ■

**Proof of Proposition 2.** Part $(iv)$ of Lemma 1 implies

$$y_{Pt} = \arg\max_{y_t \in \mathbb{R}_+} \left[ R^n\left(\varphi_t^n\right) - \kappa \right] y_t \equiv \Upsilon\left(\varphi_t^n\right),$$

so $R^n\left(\varphi_t^n\right) - \kappa \leq 0$, or equivalently,

$$\varphi_t^n \leq \bar{\varphi}^n \equiv \kappa + \frac{1-\alpha\theta}{\alpha\theta}(\kappa - \underline{\kappa}) \tag{43}$$

is a necessary condition for equilibrium. Hence the solution to the producer's beginning-of-period production decision is

$$\Upsilon\left(\varphi_t^n\right) \begin{cases} = 0 & \text{if } \varphi_t^n < \bar{\varphi}^n \\ \in [0, \infty) & \text{if } \varphi_t^n = \bar{\varphi}^n. \end{cases} \tag{44}$$

Lemma 1 also implies $\tilde{Y}_{Pt} = 0$, $\bar{Y}_{Ct} =$D$(\varphi_t^n)$, and $\bar{Y}_{Pt} = \alpha\zeta_{(\varphi_t^n - \underline{\kappa})}$Y$(\varphi_t^n)$. Given (44), and since $\underline{\kappa} < \bar{\varphi}^n$, we can write $\bar{Y}_{Pt} = \alpha$Y$(\varphi_t^n)$. Thus, the market-clearing condition for the goods market can be written as $X_D(\varphi_t^n) = 0$, where

$$X_D(\varphi_t^n) \equiv \text{D}(\varphi_t^n) - \alpha \text{Y}(\varphi_t^n). \tag{45}$$

For all $\varphi_t^n \in [0, \bar{\varphi}^n)$, $0 < X_D(\varphi_t)$, so equilibrium requires $\bar{\varphi}^n \leq \varphi_t^n$, which together with the necessary condition (43), implies $\bar{\varphi}^n = \varphi_t^n \equiv \varphi^n$ must hold in any equilibrium. From part (*ii*) of Lemma 1, $\bar{y}_{Ct}$ satisfies $u'(\bar{y}_{Ct}) = \varphi^n$ (the solution is strictly positive since $\varphi^n < u'(0)$), and from the market-clearing condition for good 1, $y_{Pt} = \bar{y}_{Ct}/\alpha$. ■

# C Monetary economy

The following lemma characterizes the first-subperiod outcomes in a monetary economy.

**Lemma 2** *Let $\varphi_t \equiv (1+\rho_t)\varphi_t^m$. Consider the first subperiod of period $t$ of an economy with money. In each case, focus on an agent who enters the period with money holding $a_t^m$. (i) The solution to the banker's portfolio problem, (i.e., (5)), is $q_t\bar{a}_{Bt}^b(a_t^m) = a_t^m - \bar{a}_{Bt}^m(a_t^m)$ and $\bar{a}_{Bt}^m(a_t^m) = \varkappa_{Bt(\rho_t)}^m$. (ii) The trade of a consumer (i.e., the solution to (6)) is $\bar{y}_{Ct}(a_t^m) =$D$(\varphi_t)$, $\bar{a}_{Ct}^m(a_t^m) = \varkappa_{Ct(\rho_t)}^m$, $q_t\bar{a}_{Ct}^b(a_t^m) = a_t^m - [\bar{a}_{Ct}^m(a_t^m) + p_{1t}\bar{y}_{Ct}(a_t^m)]$. (iii) The post-production trade of a producer who carries inventory $y_t$ and does not contact a banker (i.e., (7)) is $\tilde{y}_{Pt}(y_t, a_t^m) = \tilde{\zeta}_{(\varphi_t^m - \underline{\kappa})}y_t$ with $\tilde{a}_{Pt}^m(y_t, a_t^m) = a_t^m + p_{1t}\tilde{y}_{Pt}(y_t, a_t^m)$. The post-production trade of a producer who carries inventory $y_t$ and contacts a banker (i.e., the solution to (8)) is $\bar{y}_{Pt}(y_t, a_t^m) = \bar{\zeta}_{(\varphi_t - \underline{\kappa})}y_t$, $\bar{a}_{Pt}^m(y_t, a_t^m) = \varkappa_{Pt(\rho_t)}^m$, $q_t\bar{a}_{Pt}^b(y_t, a_t^m) = a_t^m + p_{1t}\bar{y}_{Pt}(y_t, a_t^m) - \bar{a}_{Pt}^m(y_t, a_t^m)$, and*

$$k_{Pt}(y_t, a_t^m) = (1-\theta)\left\{\rho_t \frac{a_t^m}{p_{2t}} + [(\varphi_t - \underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t\}} - (\varphi_t^m - \underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t^m\}}]y_t\right\}.$$

*(iv) A producer's pre-trade production is $y_{Pt}(a_t^m) = \varkappa^p_{(\kappa - R^m(\varphi_t^m, \varphi_t))}$, where*

$$R^m(\varphi_t^m, \varphi_t) \equiv \underline{\kappa} + \alpha\theta(\varphi_t - \underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t\}} + (1-\alpha\theta)(\varphi_t^m - \underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t^m\}}. \tag{46}$$

**Proof of Lemma 2.** (*i*) With (39), (5) can be written as

$$\hat{W}_t^B(a_t^m, a_t^g) = \max_{\bar{\boldsymbol{a}}_t \in \mathbb{R}_+ \times \mathbb{R}} \left(\frac{\bar{a}_t^m}{p_{2t}} + \bar{a}_t^b + a_t^g + \bar{W}_t^B\right) \text{ s.t. } \bar{a}_t^m + q_t\bar{a}_t^b \leq a_t^m,$$

and the solution is $q_t\bar{a}^b_{Bt}(a^m_t) = a^m_t - \bar{a}^m_{Bt}(a^m_t)$, with

$$\bar{a}^m_{Bt}(a^m_t)\begin{cases} = \infty & \text{if } \rho_t < 0 \\ \in [0, \infty] & \text{if } \rho_t = 0 \\ = 0 & \text{if } 0 < \rho_t. \end{cases}$$

$(ii)$ With (39), (6) can be written as

$$\hat{W}^C_t(a^m_t) \equiv \max_{(\bar{y}_t, \bar{\boldsymbol{a}}_t)\in\mathbb{R}^2_+\times\mathbb{R}} \left[ u(\bar{y}_t) + \frac{\bar{a}^m_t}{p_{2t}} + \bar{a}^b_t + \bar{W}^C_t \right] \text{ s.t. } \bar{a}^m_t + p_{1t}\bar{y}_t + q_t\bar{a}^b_t \leq a^m_t.$$

The solution is $\bar{y}_{Ct}(a^m_t) = \mathrm{D}(\varphi_t)$ and $q_t\bar{a}^b_{Ct}(a^m_t) = a^m_t - [\bar{a}^m_{Ct}(a^m_t) + p_{1t}\mathrm{D}(\varphi_t)]$, with

$$\bar{a}^m_{Ct}(a^m_t)\begin{cases} = \infty & \text{if } \rho_t < 0 \\ \in [0, \infty] & \text{if } \rho_t = 0 \\ = 0 & \text{if } 0 < \rho_t. \end{cases}$$

Hereafter specialize the analysis to $\rho_t \geq 0$, since $\rho_t < 0$ entails an arbitrage opportunity inconsistent with equilibrium. The value of the consumer's problem in the first subperiod is

$$\hat{W}^C_t(a^m_t) = u(\mathrm{D}(\varphi_t)) - \varphi_t\mathrm{D}(\varphi_t) + (1+\rho_t)\frac{a^m_t}{p_{2t}} + \bar{W}^C_t.$$

$(iii)$ $(a)$ With (39), (7) can be written as

$$(\tilde{y}_{Pt}(y_t, a^m_t), \tilde{a}^m_{Pt}(y_t, a^m_t)) = \arg \max_{(\tilde{y}_t, \tilde{a}^m_t)\in\mathbb{R}^2_+} \frac{\tilde{a}^m_t}{p_{2t}} + (y_t - \tilde{y}_t)\underline{\kappa}$$

subject to $\frac{1}{p_{1t}}(\tilde{a}^m_t - a^m_t) = \tilde{y}_t \leq y_t$, and therefore $\tilde{a}^m_{Pt}(y_t, a^m_t) = a^m_t + p_{1t}\tilde{y}_{Pt}(y_t, a^m_t)$, with

$$\tilde{y}_{Pt}(y_t, a^m_t)\begin{cases} = y_t & \text{if } \underline{\kappa} < \varphi^m_t \\ \in [0, y_t] & \text{if } \varphi^m_t = \underline{\kappa} \\ = 0 & \text{if } \varphi^m_t < \underline{\kappa}. \end{cases}$$

$(iii)$ $(b)$ With (39), (8) can be written as

$$\max_{(\bar{y}_t, \bar{a}^m_t, \bar{a}^b_t, k_t)\in\mathbb{R}^2_+\times\mathbb{R}\times\mathbb{R}_+} \left[ \frac{\bar{a}^m_t}{p_{2t}} + \bar{a}^b_t - k_t + (y_t - \bar{y}_t)\underline{\kappa} - \frac{\tilde{a}^m_t}{p_{2t}} - [y_t - \tilde{y}_{Pt}(y_t, a^m_t)]\,\underline{\kappa} \right]^\theta k_t^{1-\theta}$$

subject to $\bar{a}^m_t + q_t\bar{a}^b_t \leq a^m_t + p_{1t}\bar{y}_t$ and $\bar{y}_t \leq y_t$. The solution is

$$\bar{a}^b_{Pt}(y_t, a^m_t) = \frac{1}{q_t}\left[a^m_t + p_{1t}\bar{y}_{Pt}(y_t, a^m_t) - \bar{a}^m_{Pt}(y_t, a^m_t)\right],$$

with

$$\bar{y}_{Pt}(y_t,a_t^m)\begin{cases} =y_t & \text{if } \underline{\kappa}<\varphi_t\\ \in[0,y_t] & \text{if } \varphi_t=\underline{\kappa}\\ =0 & \text{if } \varphi_t<\underline{\kappa}\end{cases}$$

$$\bar{a}_{Pt}^m(y_t,a_t^m)\begin{cases} \infty & \text{if } \rho_t<0\\ \in[0,\infty] & \text{if } \rho_t=0\\ =0 & \text{if } 0<\rho_t.\end{cases}$$

Specialize the analysis to $\rho_t \geq 0$, since $\rho_t < 0$ is inconsistent with equilibrium. The intermediation fee is

$$\begin{aligned}
\frac{k_{Pt}(y_t,a_t^m)}{1-\theta} &= \frac{1}{p_{2t}}\bar{a}_{Pt}^m(y_t,a_t^m)+\bar{a}_{Pt}^b(y_t,a_t^m)+\left[y_t-\bar{y}_{Pt}(y_t,a_t^m)\right]\underline{\kappa}\\
&\quad -\left[\frac{1}{p_{2t}}\tilde{a}_{Pt}^m(y_t,a_t^m)+\left[y_t-\tilde{y}_{Pt}(y_t,a_t^m)\right]\underline{\kappa}\right]\\
&= \frac{1}{p_{2t}}\bar{a}_{Pt}^m(y_t,a_t^m)+\bar{a}_{Pt}^b(y_t,a_t^m)-\bar{y}_{Pt}(y_t,a_t^m)\underline{\kappa}+\tilde{y}_{Pt}(y_t,a_t^m)\underline{\kappa}-\frac{1}{p_{2t}}\tilde{a}_{Pt}^m(y_t,a_t^m)\\
&= \frac{1}{q_t}a_t^m+(\varphi_t-\underline{\kappa})\bar{y}_{Pt}(y_t,a_t^m)+\tilde{y}_{Pt}(y_t,a_t^m)\underline{\kappa}-\frac{1}{p_{2t}}\tilde{a}_{Pt}^m(y_t,a_t^m)-\rho_t\frac{1}{p_{2t}}\bar{a}_{Pt}^m(y_t,a_t^m)\\
&= \frac{1}{q_t}a_t^m+(\varphi_t-\underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t\}}y_t+\tilde{y}_{Pt}(y_t,a_t^m)\underline{\kappa}-\frac{1}{p_{2t}}\tilde{a}_{Pt}^m(y_t,a_t^m)\\
&= \rho_t\frac{1}{p_{2t}}a_t^m+\left[(\varphi_t-\underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t\}}-(\varphi_t^m-\underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t^m\}}\right]y_t.
\end{aligned}$$

The gain from trade to the producer in this case is $\bar{\Gamma}_{Pt}(y_t,a_t^m)\equiv\frac{\theta}{1-\theta}k_{Pt}(y_t,a_t^m)$. (*iv*) With (39), and substituting the bargaining outcomes from part (*iii*) above, the value function (11) can be written as

$$\begin{aligned}
V_t^P(a_t^m) &= \max_{y_t\in\mathbb{R}_+}\Big\{-\kappa y_t+\frac{1}{p_{2t}}a_t^m+[\underline{\kappa}+(\varphi_t^m-\underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t^m\}}]y_t+\bar{W}_t^P\\
&\quad +\alpha\theta\{\rho_t\frac{1}{p_{2t}}a_t^m+[(\varphi_t-\underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t\}}-(\varphi_t^m-\underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t^m\}}]y_t\}\Big\},
\end{aligned}$$

or equivalently,

$$V_t^P(a_t^m)=\max_{y_t\in\mathbb{R}_+}\left[R^m(\varphi_t^m,\varphi_t)-\kappa\right]y_t+(1+\alpha\theta\rho_t)\frac{1}{p_{2t}}a_t^m+\bar{W}_t^P, \tag{47}$$

with $R^m(\varphi_t^m,\varphi_t)$ as defined in (46). Hence, an individual producer produces

$$y_{Pt}(a_t^m)=\arg\max_{y_t\in\mathbb{R}_+}\left[R^m(\varphi_t^m,\varphi_t)-\kappa\right]y_t$$

units of good 1 at the beginning of the first subperiod. ■

The following result characterizes the beginning-of-period payoffs.

**Lemma 3** *For an agent of type $i \in \{B, C, P\}$, the beginning-of-period value function, $V_t^i(a_t^m)$, can be written as follows. (i) For a producer,*

$$V_t^P(a_t^m) = \max_{y_t \in \mathbb{R}_+} [R^m(\varphi_t^m, \varphi_t) - \kappa] y_t + (1 + \alpha\theta\rho_t)\frac{1}{p_{2t}}a_t^m + \bar{W}_t^P.$$

*(ii) For a banker,*

$$V_t^B(a_t^m) = (1 + \rho_t)\frac{1}{p_{2t}}a_t^m + \bar{W}_t^B + \alpha \int k_{Pt}(\tilde{a}_t^m) dH_t(\tilde{a}_t^m).$$

*(iii) For a consumer,*

$$V_t^C(a_t^m) = u(\mathrm{D}(\varphi_t)) - \varphi_t \mathrm{D}(\varphi_t) + (1 + \rho_t)\frac{1}{p_{2t}}a_t^m + \bar{W}_t^C.$$

**Proof of Lemma 3.** $(i)$ The value function $V_t^P(a_t^m)$ is given in (47). $(ii)$ With (39), and part $(i)$ of Lemma 2, (9) can be written as

$$\begin{aligned} V_t^B(a_t^m) &= \frac{1}{p_{2t}}\bar{a}_{Bt}^m(a_t^m) + \bar{a}_{Bt}^b(a_t^m) + \bar{W}_t^B + \alpha \int k_{Pt}(\tilde{a}_t^m) dH_t(\tilde{a}_t^m) \\ &= (1 + \rho_t)\frac{1}{p_{2t}}a_t^m + \bar{W}_t^B + \alpha \int k_{Pt}(\tilde{a}_t^m) dH_t(\tilde{a}_t^m). \end{aligned}$$

$(iii)$ The value function (10) can be written as $V_t^C(a_t^m) = \hat{W}_t^C(a_t^m)$, where $\hat{W}_t^C(a_t^m)$ is defined in part $(ii)$ of Lemma 2. ■

The following result characterizes the end-of-period portfolio choice for each type of agent.

**Lemma 4** *Consider the money-demand problem at the end-of-period $t$ (i.e., the maximization on the right side of (40)), and let $a_{it+1}^m$ denote the individual money demand of an agent of type $i \in \{B, C, P\}$. Then $\{a_{it+1}^m\}_{i\in\{B,C,P\}}$ must satisfy the following Euler equations:*

$$-\frac{1}{p_{2t}} + \beta\bar{v}_{t+1}^i \frac{1}{p_{2t+1}} \leq 0, \text{ with "= " if } 0 < a_{it+1}^m \text{ for } i \in \{B, C, P\}, \tag{48}$$

*where $\bar{v}_{t+1}^i \equiv 1 + \rho_{t+1}$ for $i \in \{B, C\}$, and $\bar{v}_{t+1}^P \equiv 1 + \alpha\theta\rho_{t+1}$.*

**Proof of Lemma 4.** Take the first-order conditions for the maximization in (40) using the expressions for the value functions reported in Lemma 3. ■

The following result summarizes the equilibrium conditions that define a monetary equilibrium.

**Lemma 5** *A monetary equilibrium is a sequence*

$$\left\{Z_{1t}, Z_{2t}, \rho_t, Y_{Pt}, \tilde{Y}_{Pt}, \bar{Y}_{Pt}, \bar{Y}_{Ct}, \tilde{\omega}_{Pt}, [\bar{\omega}_{it}, \omega_{it+1}]_{i\in\{B,C,P\}}\right\}_{t=0}^{\infty}$$

*that satisfies the market-clearing conditions*

$$0 = \sum\nolimits_{i\in\{B,C,P\}} \omega_{it+1} - 1 \tag{49}$$

$$0 = \bar{Y}_{Ct} - \left(\bar{Y}_{Pt} + \tilde{Y}_{Pt}\right) \tag{50}$$

$$\begin{aligned} 0 = {} & \left(\omega_{Bt} - \bar{\omega}_{Bt}\right) Z_{1t} \\ & + \left(\omega_{Ct} - \bar{\omega}_{Ct}\right) Z_{1t} - \bar{Y}_{Ct} \\ & + \left(\alpha\omega_{Pt} - \bar{\omega}_{Pt}\right) Z_{1t} + \bar{Y}_{Pt} \end{aligned} \tag{51}$$

*and the optimality conditions*

$$0 = \left(-\mu Z_{2t} + \beta \bar{v}_{t+1}^{i} Z_{2t+1}\right) \omega_{it+1} \geq -\mu Z_{2t} + \beta \bar{v}_{t+1}^{i} Z_{2t+1} \text{ for } i \in \{B, C, P\} \tag{52}$$

$$Y_{Pt} = \begin{cases} \infty & \text{if } \kappa - R^m\left(\varphi_t^m, \varphi_t\right) < 0 \\ [0, \infty] & \text{if } \kappa - R^m\left(\varphi_t^m, \varphi_t\right) = 0 \\ 0 & \text{if } 0 < \kappa - R^m\left(\varphi_t^m, \varphi_t\right) \end{cases} \tag{53}$$

$$\tilde{Y}_{Pt} = \begin{cases} (1-\alpha)Y_{Pt} & \text{if } 0 < \varphi_t^m - \underline{\kappa} \\ [0, (1-\alpha)Y_{Pt}] & \text{if } \varphi_t^m - \underline{\kappa} = 0 \\ 0 & \text{if } \varphi_t^m - \underline{\kappa} < 0 \end{cases} \tag{54}$$

$$\bar{Y}_{Pt} = \begin{cases} \alpha Y_{Pt} & \text{if } 0 < \varphi_t - \underline{\kappa} \\ [0, \alpha Y_{Pt}] & \text{if } \varphi_t - \underline{\kappa} = 0 \\ 0 & \text{if } \varphi_t - \underline{\kappa} < 0 \end{cases} \tag{55}$$

$$\bar{Y}_{Ct} = \mathrm{D}\left(\varphi_t\right) \tag{56}$$

$$\tilde{\omega}_{Pt} = (1-\alpha)\,\omega_{Pt} + \frac{\tilde{Y}_{Pt}}{Z_{1t}} \tag{57}$$

$$\bar{\omega}_{it} = \begin{cases} \infty & \text{if } \rho_t < 0 \\ [0, \infty] & \text{if } \rho_t = 0 \\ 0 & \text{if } 0 < \rho_t \end{cases} \quad \text{for } i \in \{B, C, P\} \tag{58}$$

*where*

$$\varphi_t^m \equiv \frac{Z_{2t}}{Z_{1t}} \tag{59}$$

$$\varphi_t \equiv \left(1 + \rho_t\right) \varphi_t^m \tag{60}$$

$$\bar{v}_{t+1}^{P} \equiv 1 + \alpha\theta\rho_{t+1}$$

$$\bar{v}_{t+1}^{i} \equiv 1 + \rho_{t+1} \text{ for } i \in \{B, C\}$$

$$R^m\left(\varphi_t^m, \varphi_t\right) \equiv \underline{\kappa} + \alpha\theta(\varphi_t - \underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t\}} + (1-\alpha\theta)\left(\varphi_t^m - \underline{\kappa}\right)\mathbb{I}_{\{\underline{\kappa}<\varphi_t^m\}}.$$

**Proof of Lemma 5.** By using Definition 1, Lemma 2, and Lemma 4, we know a monetary equilibrium is a sequence

$$\left\{p_{1t}, p_{2t}, q_t, Y_{Pt}, \tilde{Y}_{Pt}, \bar{Y}_{Pt}, \bar{Y}_{Ct}, \tilde{A}^m_{Pt}, \left[\bar{A}^m_{it}, \bar{A}^b_{it}, A^m_{it+1}\right]_{i\in\{B,C,P\}}\right\}_{t=0}^{\infty}$$

that satisfies the market-clearing conditions

$$0 = \sum\nolimits_{i\in\{B,C,P\}} A^m_{it+1} - M_{t+1} \tag{61}$$

$$0 = \bar{Y}_{Ct} - \left(\bar{Y}_{Pt} + \tilde{Y}_{Pt}\right) \tag{62}$$

$$0 = \sum\nolimits_{i\in\{B,C,P\}} \bar{A}^b_{it} \tag{63}$$

and the optimality conditions

$$0 = \left(-\frac{1}{p_{2t}} + \beta\bar{v}^i_{t+1}\frac{1}{p_{2t+1}}\right) A^m_{it+1} \geq -\frac{1}{p_{2t}} + \beta\bar{v}^i_{t+1}\frac{1}{p_{2t+1}} for i \in \{B,C,P\} \tag{64}$$

$$Y_{Pt} = \begin{cases} \infty & \text{if } \kappa - R^m\left(\varphi^m_t, \varphi_t\right) < 0 \\ [0,\infty] & \text{if } \kappa - R^m\left(\varphi^m_t, \varphi_t\right) = 0 \\ 0 & \text{if } 0 < \kappa - R^m\left(\varphi^m_t, \varphi_t\right) \end{cases} \tag{65}$$

$$\tilde{Y}_{Pt} = \begin{cases} (1-\alpha)Y_{Pt} & \text{if } 0 < \varphi^m_t - \underline{\kappa} \\ [0,(1-\alpha)Y_{Pt}] & \text{if } \varphi^m_t - \underline{\kappa} = 0 \\ 0 & \text{if } \varphi^m_t - \underline{\kappa} < 0 \end{cases} \tag{66}$$

$$\bar{Y}_{Pt} = \begin{cases} \alpha Y_{Pt} & \text{if } 0 < \varphi_t - \underline{\kappa} \\ [0,\alpha Y_{Pt}] & \text{if } \varphi_t - \underline{\kappa} = 0 \\ 0 & \text{if } \varphi_t - \underline{\kappa} < 0 \end{cases} \tag{67}$$

$$\bar{Y}_{Ct} = \mathrm{D}\left(\varphi_t\right) \tag{68}$$

$$\tilde{A}^m_{Pt} = (1-\alpha)\, A^m_{Pt} + p_{1t}\tilde{Y}_{Pt} \tag{69}$$

$$\bar{A}^m_{it} = \begin{cases} \infty & \text{if } \rho_t < 0 \\ [0,\infty] & \text{if } \rho_t = 0 \quad \text{for } i \in \{B,C,P\} \\ 0 & \text{if } 0 < \rho_t \end{cases} \tag{70}$$

$$\bar{A}^b_{Pt} = \frac{1}{q_t}\left(\alpha A^m_{Pt} + p_{1t}\bar{Y}_{Pt} - \bar{A}^m_{Pt}\right) \tag{71}$$

$$\bar{A}^b_{Bt} = \frac{1}{q_t}\left(A^m_{Bt} - \bar{A}^m_{Bt}\right) \tag{72}$$

$$\bar{A}^b_{Ct} = \frac{1}{q_t}\left[A^m_{Ct} - \left(\bar{A}^m_{Ct} + p_{1t}\bar{Y}_{Ct}\right)\right], \tag{73}$$

with

$$
\begin{aligned}
\bar{v}_{t+1}^P &\equiv 1+\alpha\theta\rho_{t+1} \\
\bar{v}_{t+1}^i &\equiv 1+\rho_{t+1} \text{ for } i \in \{B,C\} \\
R^m\left(\varphi_t^m,\varphi_t\right) &\equiv \underline{\kappa}+\alpha\theta(\varphi_t-\underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi_t\}}+(1-\alpha\theta)\left(\varphi_t^m-\underline{\kappa}\right)\mathbb{I}_{\{\underline{\kappa}<\varphi_t^m\}} \\
\rho_t &\equiv \frac{p_{2t}}{q_t}-1 \\
\varphi_t^m &\equiv \frac{p_{1t}}{p_{2t}} \\
\varphi_t &\equiv (1+\rho_t)\,\varphi_t^m.
\end{aligned}
$$

With $\omega_{it} \equiv A_{it}^m/M_t$, (61) can be written as (49). By using (70)-(73), $\omega_{it} \equiv A_{it}^m/M_t$, $\bar{\omega}_{it} \equiv \bar{A}_{it}^m/M_t$, and $Z_{1t} \equiv M_t/p_{1t}$, (63) can be written as (51). With $Z_{2t} \equiv M_t/p_{2t}$ and $M_{t+1}/M_t = \mu$, (64) can be written as (52). Condition (50) is the same as (62), and conditions (53)-(56) are the same as (65)-(68). With $\tilde{\omega}_{Pt} \equiv \tilde{A}_{Pt}^m/M_t$, $\omega_{Pt} \equiv A_{Pt}^m/M_t$, and $Z_{1t} \equiv M_t/p_{1t}$, (69) can be written as (57). With $\bar{\omega}_{it} \equiv \bar{A}_{it}^m/M_t$, (70) can be written as (58). ■

**Corollary 3** *Given the real equilibrium variables described in Lemma 5, the nominal equilibrium variables are obtained as follows:*

$$
\begin{aligned}
p_{jt} &= \frac{M_t}{Z_{jt}} \textit{ for } j \in \{1,2\} \\
q_t &= \frac{p_{2t}}{1+\rho_t} \\
\tilde{A}_{Pt}^m &= \tilde{\omega}_{Pt}M_t \\
\bar{A}_{it}^m &= \bar{\omega}_{it}M_t \textit{ for } i \in \{B,C,P\} \\
A_{it+1}^m &= \omega_{it+1}M_{t+1} \textit{ for } i \in \{B,C,P\} \\
\bar{A}_{Pt}^b &= \frac{1}{q_t}\left(\alpha A_{Pt}^m + p_{1t}\bar{Y}_{Pt} - \bar{A}_{Pt}^m\right) \\
\bar{A}_{Bt}^b &= \frac{1}{q_t}\left(A_{Bt}^m - \bar{A}_{Bt}^m\right) \\
\bar{A}_{Ct}^b &= \frac{1}{q_t}\left[A_{Ct}^m - \left(\bar{A}_{Ct}^m + p_{1t}\bar{Y}_{Ct}\right)\right].
\end{aligned}
$$

## C.1 Stationary monetary equilibrium

**Proof of Proposition 3.** From Lemma 5, a stationary monetary equilibrium is a vector

$$
\left(Z_1, Z_2, \rho, Y_P, \tilde{Y}_P, \bar{Y}_P, \bar{Y}_C, \tilde{\omega}_P, [\omega_i,\bar{\omega}_i]_{i\in\{B,C,P\}}\right)
$$

with $Z_j > 0$ for $j \in \{1, 2\}$ that satisfies the market-clearing conditions

$$\begin{aligned} 0 &= \sum\nolimits_{i \in \{B,C,P\}} \omega_i - 1 && (74) \\ 0 &= \bar{Y}_C - \left(\bar{Y}_P + \tilde{Y}_P\right) && (75) \\ 0 &= (\omega_B - \bar{\omega}_B) Z_1 \\ &\quad + (\omega_C - \bar{\omega}_C) Z_1 - \bar{Y}_C \\ &\quad + (\alpha\omega_P - \bar{\omega}_P) Z_1 + \bar{Y}_P && (76) \end{aligned}$$

and the optimality conditions

$$\begin{aligned} 0 &= \left(-\mu + \beta \bar{v}^i\right) \omega_i \text{ for } i \in \{B, C, P\}, \text{ with } 0 \leq fi\omega_i && (77) \\ Y_P &= \begin{cases} \infty & \text{if } \kappa - R^m(\varphi^m, \varphi) < 0 \\ [0, \infty] & \text{if } \kappa - R^m(\varphi^m, \varphi) = 0 \\ 0 & \text{if } 0 < \kappa - R^m(\varphi^m, \varphi) \end{cases} && (78) \\ \tilde{Y}_P &= \begin{cases} (1-\alpha)Y_P & \text{if } 0 < \varphi^m - \underline{\kappa} \\ [0, (1-\alpha)Y_P] & \text{if } \varphi^m - \underline{\kappa} = 0 \\ 0 & \text{if } \varphi^m - \underline{\kappa} < 0 \end{cases} && (79) \\ \bar{Y}_P &= \begin{cases} \alpha Y_P & \text{if } 0 < \varphi - \underline{\kappa} \\ [0, \alpha Y_P] & \text{if } \varphi - \underline{\kappa} = 0 \\ 0 & \text{if } \varphi - \underline{\kappa} < 0 \end{cases} && (80) \\ \bar{Y}_C &= \text{D}(\varphi) && (81) \\ \tilde{\omega}_P &= (1-\alpha)\omega_P + \frac{\tilde{Y}_P}{Z_1} && (82) \\ \bar{\omega}_i &= \begin{cases} \infty & \text{if } \rho < 0 \\ [0, \infty] & \text{if } \rho = 0 \\ 0 & \text{if } 0 < \rho \end{cases} \quad \text{for } i \in \{B, C, P\} && (83) \end{aligned}$$

where

$$\begin{aligned} \varphi^m &\equiv \frac{Z_2}{Z_1} && (84) \\ \varphi &\equiv (1+\rho)\varphi^m && (85) \\ \bar{v}^P &\equiv 1 + \alpha\theta\rho && (86) \\ \bar{v}^i &\equiv 1 + \rho \text{ for } i \in \{B, C\} && (87) \\ R^m(\varphi^m, \varphi) &\equiv \underline{\kappa} + \alpha\theta(\varphi - \underline{\kappa})\mathbb{I}_{\{\underline{\kappa} < \varphi\}} + (1-\alpha\theta)(\varphi^m - \underline{\kappa})\mathbb{I}_{\{\underline{\kappa} < \varphi^m\}}. && (88) \end{aligned}$$

First, we know that $\varphi^m \leq \varphi$, since $0 \leq \rho$ must hold in any equilibrium. Second, in any equilibrium in which good 1 is produced, we must have: $(a)$ $\kappa = R^m(\varphi^m, \varphi)$ (this follows from

(78)), or equivalently,

$$\kappa = \underline{\kappa} + \alpha\theta(\varphi - \underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi\}} + (1-\alpha\theta)(\varphi^m - \underline{\kappa})\mathbb{I}_{\{\underline{\kappa}<\varphi^m\}}. \tag{89}$$

$(b)$ $\underline{\kappa} < \varphi$, i.e., banked producers never store output. To see why, notice that if $\varphi \leq \underline{\kappa}$, then we know that $\varphi^m \leq \varphi \leq \underline{\kappa}$, and therefore $R^m(\varphi^m, \varphi) = \underline{\kappa} < \kappa$ which implies good 1 is never produced. $(c)$ If $\varphi^m = \varphi$, then (89) implies $\varphi^m = \varphi = \kappa > \underline{\kappa}$. Third, $\bar{v}^P \leq \bar{v}^B = \bar{v}^C$ (with "$<$" unless $\alpha\theta = 1$ or $\rho = 0$), so the Euler equations (77) imply that if either $\alpha\theta = 1$ or $\rho = 0$, then any triple $\omega_B, \omega_C, \omega_P \in [0,1]$ with $\omega_B + \omega_C + \omega_P = 1$ is consistent with equilibrium; otherwise, $\omega_P = 0$ and any pair $\omega_B, \omega_C \in [0,1]$ with $\omega_B + \omega_C = 1$ is consistent with a monetary equilibrium. In the remainder of the proof we assume $\alpha\theta < 1$, but will consider the limiting case $\alpha\theta \to 1$ in Corollary 2. From the previous observations we know $\underline{\kappa} < \varphi$, $0 \leq \rho$, and $\varphi - \varphi^m$ has the same sign as $\rho$. Hence, there are only three possible equilibrium configurations in which good 1 is produced: (1) $0 < \rho$ and $\varphi^m < \underline{\kappa}$, (2) $0 < \rho$ and $\underline{\kappa} \leq \varphi^m$, and (3) $\rho = 0$ and $\underline{\kappa} < \varphi^m = \varphi = \kappa$. Next, we consider each configuration in turn.

**Configuration 1.** $0 < \rho$ and $\varphi^m < \underline{\kappa}$. Under this conjecture, the equilibrium conditions (74)-(83) imply $Z_1 = 0$, so this configuration is inconsistent with monetary equilibrium.

**Configuration 2.** $0 < \rho$ and $\underline{\kappa} \leq \varphi^m$. Under this conjecture, the equilibrium conditions (74)-(83) together with the definitions (84)-(88) imply the equilibrium is:

$$\begin{aligned}
\rho &= \iota \\
\varphi^m &= \frac{\kappa}{1+\alpha\theta\iota} \\
\varphi &\equiv \frac{1+\iota}{1+\alpha\theta\iota}\kappa \\
Z_1 &= (1-\alpha)\,\mathrm{D}(\varphi) \\
Z_2 &= \varphi^m Z_1 \\
Y_P &= \bar{Y}_C = \mathrm{D}(\varphi) \\
\tilde{Y}_P &= (1-\alpha)\mathrm{D}(\varphi) \\
\bar{Y}_P &= \alpha\mathrm{D}(\varphi) \\
\tilde{\omega}_P &= 1 \\
\bar{\omega}_i &= 0 \text{ for } i \in \{B, C, P\} \\
\omega_P &= 0 \\
\omega_B, \omega_C &\in [0,1] \text{ with } \omega_B + \omega_C = 1.
\end{aligned}$$

For this to be an equilibrium, it only remains to check that $\underline{\kappa} \leq \varphi^m$ and that $\mathrm{D}(\varphi) \geq 0$. The former is equivalent to $\iota \leq \bar{\iota}$, with $\bar{\iota}$ as defined in (16). The fact that the latter holds for all $\iota \in [0, \bar{\iota}]$ is implied by the assumption $\varphi^n < u'(0)$.

**Configuration 3.** $\rho = 0$ and $\underline{\kappa} < \varphi^m = \varphi = \kappa$. Under this conjecture, the equilibrium conditions (74)-(83) together with the definitions (84)-(88) imply the equilibrium is:

$$\begin{aligned}
\rho &= \iota = 0 \\
\varphi^m &= \varphi = \kappa \\
Y_P &= \bar{Y}_C = \frac{\tilde{Y}_P}{1-\alpha} = \frac{\bar{Y}_P}{\alpha} = \mathrm{D}(\kappa) \\
Z_1 &= \frac{1}{\tilde{\omega}_P - (1-\alpha)\,\omega_P}(1-\alpha)\,\mathrm{D}(\kappa) \\
Z_2 &= \kappa Z_1 \\
\omega_i &\in [0,\infty] \text{ for } i \in \{B,C,P\}\,, \text{ with } \omega_B + \omega_C + \omega_P = 1 \\
\tilde{\omega}_P, \bar{\omega}_i &\in [0,\infty] \text{ for } i \in \{B,C,P\}\,, \text{ with } (1-\alpha)\,\omega_P < \tilde{\omega}_P = 1 - \sum_{i\in\{B,C,P\}} \bar{\omega}_i.
\end{aligned}$$

Since $\kappa < \varphi^n$, $\mathrm{D}(\kappa) \geq 0$ is implied by the assumption $\varphi^n < u'(0)$. This concludes the proof. ■

## C.2 Dynamic monetary equilibrium

In this section we characterize deterministic dynamic monetary equilibria for an economy with production of good 1. The following result offers a characterization of the set of deterministic dynamic monetary equilibria with production of good 1.

**Proposition 6** *Assume $\varphi^n < u'(0)$. Define $z_{it} \equiv \frac{1}{1-\alpha} Z_{it}$ for $i \in \{1,2\}$, and for any $z \in [\underline{\kappa}\mathrm{D}(\varphi^n), \kappa\mathrm{D}(\kappa)]$, let $f(z)$ denote the unique value $\varphi \in [\kappa, \varphi^n]$ that satisfies*

$$z = \frac{\kappa - \alpha\theta\varphi}{1-\alpha\theta}\mathrm{D}(\varphi)\,.$$

*A dynamic monetary equilibrium is a bounded sequence $\{z_{1t}, z_{2t}, \varphi_t^m, \rho_t, y_t^m\}_{t=0}^{\infty}$, where $\{z_{2t}\}_{t=0}^{\infty}$ satisfies*

$$z_{2t} = \begin{cases} \frac{1}{1+\iota} z_{2t+1} & \textit{if } \kappa\mathrm{D}(\kappa) \leq z_{2t+1} \\ \frac{1}{1+\iota}\frac{(1-\alpha\theta)f(z_{2t+1})}{\kappa-\alpha\theta f(z_{2t+1})} z_{2t+1} & \textit{if } \underline{\kappa}\mathrm{D}(\varphi^n) < z_{2t+1} < \kappa\mathrm{D}(\kappa) \\ \frac{1+\bar{\iota}}{1+\iota} z_{2t+1} & \textit{if } z_{2t+1} \leq \underline{\kappa}\mathrm{D}(\varphi^n)\,. \end{cases} \tag{90}$$

*Given the equilibrium path* $\{z_{2t}\}_{t=0}^{\infty}$*,*

$$\varphi_t^m = \begin{cases} \kappa & \text{if } \kappa \mathrm{D}(\kappa) \le z_{2t} \\ \frac{\kappa - \alpha\theta f(z_{2t})}{1-\alpha\theta} & \text{if } \underline{\kappa}\mathrm{D}(\varphi^n) < z_{2t} < \kappa\mathrm{D}(\kappa) \\ \underline{\kappa} & \text{if } z_{2t} \le \underline{\kappa}\mathrm{D}(\varphi^n) \end{cases}$$

$$\rho_t = \begin{cases} 0 & \text{if } \kappa \mathrm{D}(\kappa) \le z_{2t} \\ \frac{f(z_{2t}) - \kappa}{\kappa - \alpha\theta f(z_{2t})} & \text{if } \underline{\kappa}\mathrm{D}(\varphi^n) < z_{2t} < \kappa\mathrm{D}(\kappa) \\ \bar{\iota} & \text{if } z_{2t} \le \underline{\kappa}\mathrm{D}(\varphi^n) \end{cases}$$

$$z_{1t} = \begin{cases} \frac{1}{\kappa} z_{2t} & \text{if } \kappa \mathrm{D}(\kappa) \le z_{2t} \\ y_t^m & \text{if } \underline{\kappa}\mathrm{D}(\varphi^n) < z_{2t} < \kappa\mathrm{D}(\kappa) \\ \frac{1}{\underline{\kappa}} z_{2t} & \text{if } z_{2t} \le \underline{\kappa}\mathrm{D}(\varphi^n) \end{cases}$$

*and* $y_t^m = \mathrm{D}[(1+\rho_t)\varphi_t^m]$ *is the consumption of good 1. Nominal prices are* $p_{1t} = \varphi_t^m p_{2t} = \frac{M_t}{(1-\alpha) z_{1t}}$ *and* $q_t = \frac{p_{2t}}{1+\rho_t}$*, and velocity is* $\mathcal{V}_t = \frac{y_t^m}{(1-\alpha) z_{1t}}$*.*

Proposition 6 reduces the task of finding dynamic monetary equilibria to finding a bounded solution $\{z_{2t}\}_{t=0}^{\infty}$ to the difference equation (90).

**Corollary 4** *In any dynamic monetary equilibrium,* $\mathrm{D}(\varphi^n) \le \mathrm{D}[(1+\rho_t)\varphi_t^m]$ *for all* $t$*, with "="* *only if* $z_{2t} \le \underline{\kappa}\mathrm{D}(\varphi^n)$ *or* $\alpha\theta = 1$*.*

Corollary 4 of Proposition 6 establishes that in any dynamic monetary equilibrium, consumers face an effective relative price of good 1 (in terms of good 2), i.e., $(1+\rho_t)\varphi_t^m$, that is lower than the relative price they would face in the equilibrium of the same economy without money, i.e., $\varphi^n$. Thus, consumption of good 1 (and therefore welfare) is higher in the economy with money than in the nonmonetary economy—*strictly higher* if the equilibrium path has $z_{2t} > \underline{\kappa}\mathrm{D}(\varphi^n)$ for at least one $t$.

### C.2.1 Cashless limit

The following corollary of Proposition 6 describes the cashless limit (as $\alpha \to 1$) of the dynamical system that characterizes any dynamic monetary equilibrium path.

**Corollary 5** *Assume* $\varphi^{n*} < u'(0)$*. For any* $z \in [\underline{\kappa}\mathrm{D}(\varphi^{n*}), \kappa\mathrm{D}(\kappa)]$*, let* $g(z)$ *denote the unique value* $\varphi \in [\kappa, \varphi^{n*}]$ *that satisfies*

$$z = \frac{\kappa - \theta\varphi}{1-\theta}\mathrm{D}(\varphi). \tag{91}$$

*Let* $\{z_{1t}, z_{2t}, \varphi_t^m, \rho_t, y_t^m\}_{t=0}^{\infty}$ *be a dynamic monetary equilibrium. Then:*

*(i) As* $\alpha \to 1$, $\{z_{1t}, z_{2t}, \varphi_t^m, \rho_t, y_t^m\}_{t=0}^{\infty} \to \{z_{1t}^*, z_{2t}^*, \varphi_t^{m*}, \rho_t^*, y_t^{m*}\}_{t=0}^{\infty}$*, where* $\{z_{2t}^*\}_{t=0}^{\infty}$ *satisfies*

$$z_{2t}^* = \begin{cases} \frac{1}{1+\iota} z_{2t+1}^* & \text{if } \kappa \mathrm{D}(\kappa) \leq z_{2t+1}^* \\ \frac{1}{1+\iota} \frac{(1-\theta) g\left(z_{2t+1}^*\right)}{\kappa - \theta g\left(z_{2t+1}^*\right)} z_{2t+1}^* & \text{if } \underline{\kappa} \mathrm{D}(\varphi^{n*}) < z_{2t+1}^* < \kappa \mathrm{D}(\kappa) \\ \frac{1+\bar{\iota}^*}{1+\iota} z_{2t+1}^* & \text{if } z_{2t+1}^* \leq \underline{\kappa} \mathrm{D}(\varphi^{n*}). \end{cases}$$

*Given the equilibrium path* $\{z_{2t}^*\}_{t=0}^{\infty}$*,*

$$\varphi_t^{m*} = \begin{cases} \kappa & \text{if } \kappa \mathrm{D}(\kappa) \leq z_{2t}^* \\ \frac{\kappa - \theta g\left(z_{2t}^*\right)}{1-\theta} & \text{if } \underline{\kappa} \mathrm{D}(\varphi^{n*}) < z_{2t}^* < \kappa \mathrm{D}(\kappa) \\ \underline{\kappa} & \text{if } z_{2t}^* \leq \underline{\kappa} \mathrm{D}(\varphi^{n*}) \end{cases}$$

$$\rho_t^* = \begin{cases} 0 & \text{if } \kappa \mathrm{D}(\kappa) \leq z_{2t}^* \\ \frac{g\left(z_{2t}^*\right) - \kappa}{\kappa - \theta g\left(z_{2t}^*\right)} & \text{if } \underline{\kappa} \mathrm{D}(\varphi^{n*}) < z_{2t}^* < \kappa \mathrm{D}(\kappa) \\ \bar{\iota}^* & \text{if } z_{2t}^* \leq \underline{\kappa} \mathrm{D}(\varphi^{n*}) \end{cases}$$

$$z_{1t}^* = \begin{cases} \frac{1}{\kappa} z_{2t}^* & \text{if } \kappa \mathrm{D}(\kappa) \leq z_{2t}^* \\ y_t^{m*} & \text{if } \underline{\kappa} \mathrm{D}(\varphi^{n*}) < z_{2t}^* < \kappa \mathrm{D}(\kappa) \\ \frac{1}{\underline{\kappa}} z_{2t}^* & \text{if } z_{2t}^* \leq \underline{\kappa} \mathrm{D}(\varphi^{n*}) \end{cases}$$

*and* $y_t^{m*} = \mathrm{D}[(1+\rho_t^*)\varphi_t^{m*}]$ *is the consumption of good 1.*

*(ii)* $\mathrm{D}(\varphi^{n*}) \leq \mathrm{D}[(1+\rho_t^*)\varphi_t^{m*}]$ *for all* $t$*, with "=" only if* $z_{2t}^* \leq \underline{\kappa}\mathrm{D}(\varphi^{n*})$ *or* $\theta = 1$.

Part $(i)$ of Corollary 5 describes the set of conditions that characterize the "cashless limiting path" to which the path corresponding to any given dynamic monetary equilibrium converges (pointwise, for each $t$) as $\alpha \to 1$. Part $(ii)$ establishes a key result that generalizes the main result in Corollary 2: As long as bankers have market power against producers, i.e., $\theta < 1$, in the cashless limit of any dynamic monetary equilibrium, consumers face an effective relative price of good 1 (in terms of good 2) that is lower than the relative price they would face in the equilibrium of the same economy without money. Thus, consumption of good 1, and therefore welfare, is higher in the pure-credit cashless limit of a dynamic monetary equilibrium of the economy with money than in the pure-credit limit of the economy without money. Welfare is *strictly* higher in the former than the latter if $\theta < 1$ and the equilibrium path has $z_{2t}^* > \underline{\kappa}\mathrm{D}(\varphi^{n*})$ for some $t$. The equilibrium conditions in Corollary 5 are stated in terms of real balances normalized by the number of producers who have no access to bankers, i.e., $z_{it}^* \equiv \lim_{\alpha \to 1} z_{it}$, where $z_{it} \equiv \frac{Z_{it}}{1-\alpha}$ for $i \in \{1, 2\}$. Hence, in the cashless limit of a dynamic monetary equilibrium characterized in the corollary, we have

$$\lim_{\alpha \to 1} \frac{1}{p_{it}} = \lim_{\alpha \to 1} \frac{1}{\mathcal{V}_t} = \lim_{\alpha \to 1} Z_{it} = \lim_{\alpha \to 1} (1-\alpha) z_{it}^* = 0 \text{ for } i \in \{1, 2\}.$$

### C.2.2 Proofs

**Proof of Proposition 6.** The proof builds on Lemma 5. We seek to characterize deterministic monetary equilibria in which good 1 is produced in every period. An equilibrium is *monetary* if $Z_{it} > 0$ for $i \in \{1, 2\}$ and all $t$.

We first establish that a monetary equilibrium has production of good 1 in every period only if $\underline{\kappa} \leq \varphi_t^m$ for all $t$. To this end, suppose there is a monetary equilibrium (i.e., $Z_{it} > 0$ for $i \in \{1, 2\}$ and all $t$) with $\varphi_t^m < \underline{\kappa}$ for some $t$. There are two possibilities: either $\rho_t = 0$, or $0 < \rho_t$. If $\rho_t = 0$, (60) implies $\varphi_t = \varphi_t^m < \underline{\kappa}$, but then (53) implies $Y_{Pt} = 0$ (good 1 is not produced). If $0 < \rho_t$, (49) and (52) imply $\omega_{Pt} = 0$ and $\omega_{Bt} + \omega_{Ct} = 1$, and (58) implies $\bar{\omega}_{it} = 0$ for $i \in \{B, C, P\}$. Hence, the bond-market clearing condition (51) becomes $\bar{Y}_{Ct} - Z_{1t} = \bar{Y}_{Pt}$, which together with (50) (the market-clearing condition for good 1), implies $Z_{1t} = \tilde{Y}_{Pt}$. But since this conjectured monetary equilibrium has $\varphi_t^m < \underline{\kappa}$, (54) implies $\tilde{Y}_{Pt} = 0$, and therefore $Z_{1t} = 0$, a contradiction. Next, we characterize the set of deterministic monetary equilibria in which good 1 is produced in every period by considering three possible equilibrium configurations from some arbitrary period $t$ onwards: $(i)$ $\rho_{t+1} = 0$; $(ii)$ $0 < \rho_{t+1}$ and $\underline{\kappa} < \varphi_{t+1}^m$; $(iii)$ $0 < \rho_{t+1}$ and $\varphi_{t+1}^m = \underline{\kappa}$.

$(i)$ Suppose $\rho_{t+1} = 0$. Then, (60) implies $\varphi_{t+1} = \varphi_{t+1}^m$, and (53) implies that in an equilibrium with production of good 1,

$$\varphi_{t+1} = \varphi_{t+1}^m = \kappa. \tag{92}$$

Then, (53), (54), and (55) imply $Y_{Pt+1} \in [0, \infty]$, $\tilde{Y}_{Pt+1} = (1 - \alpha) Y_{Pt+1}$, and $\bar{Y}_{Pt+1} = \alpha Y_{Pt+1}$. Since $\tilde{Y}_{Pt+1} + \bar{Y}_{Pt+1} = Y_{Pt+1}$, (50), (56), and (92) imply

$$Y_{Pt+1} = \bar{Y}_{Ct+1} = \mathrm{D}(\kappa), \tag{93}$$

and therefore

$$\tilde{Y}_{Pt+1} = (1 - \alpha)\mathrm{D}(\kappa) \tag{94}$$

$$\bar{Y}_{Pt+1} = \alpha \mathrm{D}(\kappa). \tag{95}$$

Together with (59), the fact that $\varphi_{t+1}^m = \kappa$ implies

$$Z_{1t+1} = \frac{Z_{2t+1}}{\kappa}. \tag{96}$$

Condition (52) implies

$$Z_{2t} = \frac{1}{1 + \iota} Z_{2t+1} \tag{97}$$

and $\omega_{it+1} \in [0, \infty]$ for $i \in \{B, C, P\}$, which together with (49) implies $(\omega_{it+1})_{i\in\{B,C,P\}}$ is only restricted to satisfy

$$\omega_{it+1} \in [0, 1], \text{ with } \sum\nolimits_{i\in\{B,C,P\}} \omega_{it+1} = 1. \tag{98}$$

Condition (58) implies

$$\bar{\omega}_{it+1} \in [0, 1] \text{ for } i \in \{B, C, P\}. \tag{99}$$

Together with $\tilde{Y}_{Pt+1} = (1-\alpha)\text{D}(\kappa)$, (57) implies

$$\tilde{\omega}_{Pt+1} = (1-\alpha)\,\omega_{Pt+1} + \frac{(1-\alpha)\text{D}\,(\kappa)}{Z_{1t+1}}. \tag{100}$$

Together with $\bar{Y}_{Ct+1} = \frac{\bar{Y}_{Pt+1}}{\alpha} = \text{D}(\kappa)$, (51) yields

$$Z_{1t+1} = \frac{(1-\alpha)\text{D}\,(\kappa)}{\omega_{Bt+1} + \omega_{Ct+1} + \alpha\omega_{Pt+1} - \bar{\omega}_{Bt+1} - \bar{\omega}_{Ct+1} - \bar{\omega}_{Pt+1}}.$$

The only restriction that this condition implies on $Z_{1t+1}$ for it to be part of a monetary equilibrium is that $(1-\alpha)\text{D}(\kappa) \leq Z_{1t+1}$, or equivalently, since $\kappa Z_{1t+1} = Z_{2t+1}$, this inequality is equivalent to

$$\kappa\text{D}\,(\kappa) \leq z_{2t+1}, \tag{101}$$

where

$$z_{jt+1} \equiv \frac{Z_{jt+1}}{1-\alpha} \text{ for } j \in \{1, 2\}.$$

To summarize, given a value $z_{2t} \in \mathbb{R}_{++}$, under the conjecture that $\rho_{t+1} = 0$, and provided condition (101) holds, the rest of equilibrium allocation at $t+1$ is obtained as follows: $Y_{Pt+1}$ and $\bar{Y}_{Ct+1}$ are given by (93), $\tilde{Y}_{Pt+1}$ is given by (94), $\bar{Y}_{Pt+1}$ by (95), $z_{1t+1}$ by (96), $z_{2t+1}$ by (97) with (101), and $\left([\omega_{it+1}, \bar{\omega}_{it+1},]_{i\in\{B,C,P\}}, \tilde{\omega}_{Pt+1}\right)$ by (98)-(100).

(*ii*) Suppose $0 < \rho_{t+1}$ and $\underline{\kappa} < \varphi^m_{t+1}$. Then, (49) and (52) imply

$$Z_{2t} = \frac{1+\rho_{t+1}}{1+\iota} Z_{2t+1} \tag{102}$$

and

$$\omega_{Pt+1} = 0 \tag{103}$$

$$\omega_{Bt+1}, \omega_{Ct+1} \in \mathbb{R}_+, \text{ with } \omega_{Bt+1} + \omega_{Ct+1} = 1. \tag{104}$$

Since $0 < \rho_{t+1}$, conditions (58) imply

$$\bar{\omega}_{it+1} = 0 \text{ for } i \in \{B, C, P\}. \tag{105}$$

Given (60), the assumptions $0 < \rho_{t+1}$ and $\underline{\kappa} < \varphi^m_{t+1}$ imply $\underline{\kappa} < \varphi^m_{t+1} < \varphi_{t+1}$, so (54) and (55) imply $\tilde{Y}_{Pt+1} = (1-\alpha)Y_{Pt+1}$ and $\bar{Y}_{Pt+1} = \alpha Y_{Pt+1}$, and (53) implies $Y_{Pt} \in [0, \infty]$ and

$$\alpha\theta\varphi_{t+1} + (1-\alpha\theta)\,\varphi^m_{t+1} = \kappa.$$

This last condition is equivalent to

$$\varphi^m_{t+1} = \frac{\kappa - \alpha\theta\varphi_{t+1}}{1-\alpha\theta}, \tag{106}$$

and together with (60), it implies

$$\rho_{t+1} = \frac{\varphi_{t+1} - \kappa}{\kappa - \alpha\theta\varphi_{t+1}}. \tag{107}$$

Condition (107) is equivalent to

$$\varphi_{t+1} = \frac{1+\rho_{t+1}}{1+\alpha\theta\rho_{t+1}}\kappa,$$

which together with (106) yields

$$\varphi^m_{t+1} = \frac{\kappa}{1+\alpha\theta\rho_{t+1}}.$$

From this last condition it is easy to see that

$$\underline{\kappa} < \varphi^m_{t+1} \Leftrightarrow \rho_{t+1} < \bar{\iota}. \tag{108}$$

Together with (50) and (56), $\tilde{Y}_{Pt+1} = (1-\alpha)Y_{Pt+1}$ and $\bar{Y}_{Pt+1} = \alpha Y_{Pt+1}$ imply

$$\bar{Y}_{Ct+1} = Y_{Pt+1} = \frac{\tilde{Y}_{Pt+1}}{1-\alpha} = \frac{\bar{Y}_{Pt+1}}{\alpha} = \mathrm{D}\left(\varphi_{t+1}\right). \tag{109}$$

Conditions (103)-(105) together with (51) imply

$$Z_{1t+1} = (1-\alpha)\,\mathrm{D}\left(\varphi_{t+1}\right), \tag{110}$$

which together with (59) can be written as

$$Z_{2t+1} = (1-\alpha)\,\varphi^m_{t+1}\mathrm{D}\left(\varphi_{t+1}\right). \tag{111}$$

Conditions (106) and (111) imply $z_{2t+1} = h\left(\varphi_{t+1}\right)$, where

$$h\left(\varphi_{t+1}\right) \equiv \frac{\kappa - \alpha\theta\varphi_{t+1}}{1-\alpha\theta}\mathrm{D}\left(\varphi_{t+1}\right).$$

Notice that $h' < 0$, and

$$h\left(\varphi^n\right) = \underline{\kappa}\mathrm{D}\left(\varphi^n\right) < h\left(\kappa\right) = \kappa\mathrm{D}\left(\kappa\right),$$

so for every $z_{2t+1} \in [\underline{\kappa}\mathrm{D}(\varphi^n), \kappa\mathrm{D}(\kappa)]$, there exists a unique $\varphi_{t+1} \in [\kappa, \varphi^n]$ given by $\varphi_{t+1} = f(z_{2t+1})$, where $f(z_{2t+1}) \equiv h^{-1}(z_{2t+1})$. By substituting (107) and $\varphi_{t+1} = f(z_{2t+1})$ into (102), we obtain

$$z_{2t} = \frac{1}{1+\iota} \frac{(1-\alpha\theta) f(z_{2t+1})}{\kappa - \alpha\theta f(z_{2t+1})} z_{2t+1}. \tag{112}$$

Conditions (57), (103), (110), and (109) imply

$$\tilde{\omega}_{Pt+1} = 1. \tag{113}$$

The two conditions for case $(ii)$ are $0 < \rho_{t+1}$ and $\underline{\kappa} < \varphi^m_{t+1}$, which with (107), (108), and $\varphi_{t+1} = f(z_{2t+1})$, can be written as

$$0 < \frac{f(z_{2t+1}) - \kappa}{\kappa - \alpha\theta f(z_{2t+1})} < \bar{\iota}. \tag{114}$$

Since $f$ is a strictly decreasing function, with $f(\kappa\mathrm{D}(\kappa)) = \kappa$ and $f(\underline{\kappa}\mathrm{D}(\varphi^n)) = \varphi^n$, (114) is equivalent to

$$\underline{\kappa}\mathrm{D}(\varphi^n) < z_{2t+1} < \kappa\mathrm{D}(\kappa). \tag{115}$$

To summarize, given a value $z_{2t} \in \mathbb{R}_{++}$, under the conjecture that $0 < \rho_{t+1}$ and $\underline{\kappa} < \varphi^m_{t+1}$, and provided conditions (115) hold, the rest of equilibrium allocation at $t+1$ is obtained as follows: $(Y_{Pt}, \tilde{Y}_{Pt}, \bar{Y}_{Pt}, \bar{Y}_{Ct})$ is given by (109), $\left([\omega_{it+1}, \bar{\omega}_{it+1},]_{i\in\{B,C,P\}}, \tilde{\omega}_{Pt+1}\right)$ by (103), (104), (105) and (113), $z_{2t+1}$ by (112), $\varphi_{t+1}$ by $\varphi_{t+1} = f(z_{2t+1})$, $z_{1t+1}$ by (110), $\rho_{t+1} = \frac{f(z_{2t+1})-\kappa}{\kappa-\alpha\theta f(z_{2t+1})}$ by (107), and $\varphi^m_{t+1} = \frac{\kappa-\alpha\theta f(z_{2t+1})}{1-\alpha\theta}$ from (106).

$(iii)$ Suppose $0 < \rho_{t+1}$ and

$$\varphi^m_{t+1} = \underline{\kappa}. \tag{116}$$

Then, $Z_{2t+1}$ satisfies (102), $\{\omega_{it+1}\}_{i\in\{B,C,P\}}$ satisfies (103) and (104), and $\{\bar{\omega}_{it+1}\}_{i\in\{B,C,P\}}$ satisfies (105). The assumptions $0 < \rho_{t+1}$ and $\varphi^m_{t+1} = \underline{\kappa}$ imply $\underline{\kappa} = \varphi^m_{t+1} < \varphi_{t+1}$, so (54) and (55) imply

$$\tilde{Y}_{Pt+1} \in [0, (1-\alpha)Y_{Pt+1}] \tag{117}$$

and

$$\bar{Y}_{Pt+1} = \alpha Y_{Pt+1}, \tag{118}$$

and (53) implies $Y_{Pt+1} \in [0, \infty]$ and

$$\varphi_{t+1} = \varphi^n. \tag{119}$$

Hence, (60) implies

$$\rho_{t+1} = \bar{\iota}, \tag{120}$$

and condition (56) implies

$$\bar{Y}_{Ct+1} = \mathrm{D}\left(\varphi^n\right). \tag{121}$$

Thus, (102) becomes

$$z_{2t} = \frac{1+\bar{\iota}}{1+\iota} z_{2t+1}. \tag{122}$$

Given $z_{2t+1}$, (59) and (116) can be used to obtain

$$Z_{1t+1} = \frac{(1-\alpha)\, z_{2t+1}}{\underline{\kappa}}. \tag{123}$$

Condition (51), together with (56), (103)-(105), (119), and $\bar{Y}_{Pt+1} = \alpha Y_{Pt+1}$, implies

$$Y_{Pt+1} = \frac{\mathrm{D}\left(\varphi^n\right) - Z_{1t+1}}{\alpha}. \tag{124}$$

Then (50) implies

$$\begin{aligned} \tilde{Y}_{Pt+1} &= \mathrm{D}\left(\varphi^n\right) - \bar{Y}_{Pt+1} \\ &= Z_{1t+1}. \end{aligned} \tag{125}$$

Thus, the optimality condition (117) requires

$$0 \leq Z_{1t+1} \leq (1-\alpha) Y_{Pt+1},$$

which using (124) is equivalent to

$$0 \leq Z_{1t+1} \leq (1-\alpha)\frac{\mathrm{D}\left(\varphi^n\right) - Z_{1t+1}}{\alpha}.$$

With (123), these inequalities become

$$0 \leq z_{2t+1} \leq \underline{\kappa}\mathrm{D}\left(\varphi^n\right). \tag{126}$$

To summarize, given a value $z_{2t} \in \mathbb{R}_{++}$, under the conjecture that $0 < \rho_{t+1}$ and $\varphi^m_{t+1} = \underline{\kappa}$, and provided conditions (126) hold, the rest of equilibrium allocation at $t+1$ is obtained as follows: $z_{2t+1}$ is given by (122), $z_{1t+1}$ by (123), $\rho_{t+1}$ by (120), $\varphi_{t+1}$ by (119), $Y_{Pt+1}$ by (124), $\tilde{Y}_{Pt+1}$ by (125), $\bar{Y}_{Pt+1}$ by (118), $\bar{Y}_{Ct+1}$ by (121), $\{\omega_{it+1}\}_{i\in\{B,C,P\}}$ by (103) and (104), $\{\bar{\omega}_{it+1}\}_{i\in\{B,C,P\}}$ by (105), and $\tilde{\omega}_{Pt+1} = 1$ (by (103) and (125)).

From the previous analysis of cases $(i)$-$(iii)$, it follows that a dynamic deterministic monetary equilibrium with production of good 1 consists of a sequence of real balances, interest rates, relative prices, and consumption, production, and sales of good 1,

$$\left\{z_{1t}, z_{2t}, \rho_t, \varphi_t, \varphi_t^m, Y_{Pt}, \tilde{Y}_{Pt}, \bar{Y}_{Pt}, \bar{Y}_{Ct}\right\}_{t=0}^{\infty},$$

with $z_{it} > 0$ for $i \in \{1, 2\}$ and all $t$, that satisfies the following conditions:

$$\begin{aligned}
z_{2t} &= \begin{cases} \frac{1}{1+\iota} z_{2t+1} & \text{if } \kappa \mathrm{D}(\kappa) \leq z_{2t+1} \\ \frac{1}{1+\iota} \frac{(1-\alpha\theta) f(z_{2t+1})}{\kappa - \alpha\theta f(z_{2t+1})} z_{2t+1} & \text{if } \underline{\kappa} \mathrm{D}(\varphi^n) < z_{2t+1} < \kappa \mathrm{D}(\kappa) \\ \frac{1+\bar{\iota}}{1+\iota} z_{2t+1} & \text{if } z_{2t+1} \leq \underline{\kappa} \mathrm{D}(\varphi^n) \end{cases} \\
\varphi_t &= \begin{cases} \kappa & \text{if } \kappa \mathrm{D}(\kappa) \leq z_{2t} \\ f(z_{2t}) & \text{if } \underline{\kappa} \mathrm{D}(\varphi^n) < z_{2t} < \kappa \mathrm{D}(\kappa) \\ \varphi^n & \text{if } z_{2t} \leq \underline{\kappa} \mathrm{D}(\varphi^n) \end{cases} \\
z_{1t} &= \begin{cases} \frac{1}{\kappa} z_{2t} & \text{if } \kappa \mathrm{D}(\kappa) \leq z_{2t} \\ \mathrm{D}(\varphi_t) & \text{if } \underline{\kappa} \mathrm{D}(\varphi^n) < z_{2t} < \kappa \mathrm{D}(\kappa) \\ \frac{1}{\underline{\kappa}} z_{2t} & \text{if } z_{2t} \leq \underline{\kappa} \mathrm{D}(\varphi^n) \end{cases} \\
\varphi_t^m &= \frac{\kappa - \alpha\theta\varphi_t}{1 - \alpha\theta} \\
\rho_t &= \frac{\varphi_t - \kappa}{\kappa - \alpha\theta\varphi_t} \\
\bar{Y}_{Ct} &= \mathrm{D}(\varphi_t) \\
\tilde{Y}_{Pt} &= \begin{cases} (1-\alpha) \mathrm{D}(\varphi_t) & \text{if } \underline{\kappa} \mathrm{D}(\varphi^n) < z_{2t} \\ (1-\alpha) z_{1t} & \text{if } z_{2t} \leq \underline{\kappa} \mathrm{D}(\varphi^n) \end{cases} \\
\bar{Y}_{Pt} &= \begin{cases} \alpha \mathrm{D}(\varphi_t) & \text{if } \underline{\kappa} \mathrm{D}(\varphi^n) < z_{2t} \\ \mathrm{D}(\varphi^n) - (1-\alpha) z_{1t} & \text{if } z_{2t} \leq \underline{\kappa} \mathrm{D}(\varphi^n) \end{cases} \\
Y_{Pt} &= \begin{cases} \mathrm{D}(\varphi_t) & \text{if } \underline{\kappa} \mathrm{D}(\varphi^n) < z_{2t} \\ \frac{\mathrm{D}(\varphi^n) - (1-\alpha) z_{1t}}{\alpha} & \text{if } z_{2t} \leq \underline{\kappa} \mathrm{D}(\varphi^n) \end{cases}
\end{aligned} \tag{127}$$

where for any $z \in [\underline{\kappa} \mathrm{D}(\varphi^n), \kappa \mathrm{D}(\kappa)]$, $f(z)$ denotes the unique value $\varphi \in [\kappa, \varphi^n]$ that satisfies

$$z = \frac{\kappa - \alpha\theta\varphi}{1 - \alpha\theta} \mathrm{D}(\varphi).$$

The equilibrium nominal prices are

$$\begin{aligned}
p_{1t} &= \frac{M_t}{(1-\alpha) z_{1t}} \\
p_{2t} &= \frac{p_{1t}}{\varphi_t^m} \\
q_t &= \frac{p_{2t}}{1 + \rho_t}.
\end{aligned}$$

This concludes the proof. ■

**Proof of Corollary 4.** From the definition of $f$ in the statement of Proposition 6 it follows that $f(z_{2t}) \leq \varphi^n$ for all $z_{2t} \geq \underline{\kappa}\text{D}(\varphi^n)$, with "=" only if $z_{2t} = \underline{\kappa}\text{D}(\varphi^n)$. Then (127) implies $\varphi_t \equiv (1+\rho_t)\varphi_t^m \leq \varphi^n$, with "=" only if $z_{2t} \leq \underline{\kappa}\text{D}(\varphi^n)$. Since $\text{D}'(\cdot) < 0$, it follows that $\text{D}(\varphi^n) \leq \text{D}[(1+\rho_t)\varphi_t^m]$ for all $t$, with "=" only for $t \in \mathbb{T}$ such that $z_{2t} \leq \underline{\kappa}\text{D}(\varphi^n)$. ■

## C.3 Sunspot equilibria

In this section we construct equilibria where prices and allocations are time-invariant functions of a *sunspot*, i.e., a random variable on which agents may coordinate actions but that does not directly affect any primitives, including endowments, preferences, and production or trading possibilities. We focus on equilibria where only consumers hold money between periods, which is without loss for our purposes. In Section C.3.2 (Corollary 7), we provide the equilibrium conditions for a set of sunspot states $\mathbb{S} = \{s_1, ..., s_N\}$, where the time path of the sunspot state, $s_t \in \mathbb{S}$, follows a Markov chain with $\eta_{ij} = \Pr(s_{t+1} = s_j | s_t = s_i)$. In this context we describe equilibrium with time-invariant functions of the sunspot state, i.e., for any $s_t \in \mathbb{S}$ we use $\varphi^m(s_t)$, $\rho(s_t)$, $\{Z_i(s_t), p_i(s_t, M_t)\}_{i\in\{1,2\}}$, $\mathcal{V}(s_t)$, and $y^m(s_t)$, to denote the prices $\varphi_t^m$, $\rho_t$, and $\{Z_{it}, p_{it}\}_{i\in\{1,2\}}$, velocity, $\mathcal{V}_t$, and consumption of good 1, $y_t^m \equiv \text{D}[(1+\rho_t)\varphi_t^m]$, respectively.

The following result characterizes a family of sunspot equilibria that contains the nonmonetary equilibrium of Proposition 2 and the monetary equilibrium of Proposition 3.

**Proposition 7** *Assume $\varphi^n < u'(0)$, and $\mathbb{S} = \{s_1, s_2\}$, with $\eta_{11} \equiv \eta \in [0,1]$ and $\eta_{22} = 1$. For any arbitrary $\eta \in (0,1]$, provided $1 \leq 1+\iota < \eta(1+\bar{\iota})$, there exists a sunspot equilibrium given by $\varphi^m(s_2) = Z_1(s_2) = Z_2(s_2) = 0$, $y^m(s_2) = \text{D}(\varphi^n)$,*

$$\begin{aligned}
\rho(s_1) &= \frac{\iota + 1 - \eta}{\eta} \\
\varphi^m(s_1) &= \frac{\eta}{1 + \alpha\theta\iota - (1-\eta)(1-\alpha\theta)}\kappa \\
\frac{Z_1(s_1)}{1-\alpha} &= \frac{Z_2(s_1)}{(1-\alpha)\varphi^m(s_1)} = y^m(s_1) = \text{D}\left[(1+\rho(s_1))\varphi^m(s_1)\right] \\
\mathcal{V}(s_1) &= \frac{1}{1-\alpha},
\end{aligned}$$

*and $p_i(s, M_t) = \frac{M_t}{Z_i(s)}$ for $i \in \{1,2\}$ and $s \in \mathbb{S}$.*

For $\eta = 0$, the equilibrium described in Proposition 7 reduces to the nonmonetary equilibrium of Proposition 2. Conversely, for $\eta = 1$, it reduces to the monetary equilibrium of Proposition 3. By varying $\eta$ from 0 to 1, we can generate a continuum of proper sunspot equilibria that "convexify" the equilibrium set spanned by the monetary and the nonmonetary equilibrium.

### C.3.1 Cashless limit

For every $\alpha \in [0,1]$, the set of equilibria indexed by the sunspot probability $\eta$ described in Proposition 7, defines an equilibrium correspondence that is continuous. The following corollary of Proposition 7 characterizes the limit of this equilibrium correspondence as $\alpha \to 1$.

**Corollary 6** *Consider the set of monetary equilibria indexed by $\eta \in (0,1]$ characterized in Proposition 7. Assume $\varphi^{n*} < u'(0)$ and $1 \leq 1+\iota < \eta(1+\bar{\iota}^*)$. For any arbitrary $\eta \in (0,1]$,*

$$\begin{aligned}
\lim_{\alpha\to 1} y^m(s_2) &= \mathrm{D}(\varphi^{n*}) \\
\lim_{\alpha\to 1} \rho(s_1) &= \frac{\iota+1-\eta}{\eta} \\
\lim_{\alpha\to 1} \varphi^m(s_1) &= \frac{\eta}{1+\theta\iota-(1-\eta)(1-\theta)}\kappa \\
\lim_{\alpha\to 1} \frac{Z_1(s_1)}{1-\alpha} &= \lim_{\alpha\to 1} \frac{Z_2(s_1)}{(1-\alpha)\varphi^m(s_1)} = \lim_{\alpha\to 1} y^m(s_1) = \mathrm{D}\left(\frac{1+\iota}{1+\theta\iota-(1-\eta)(1-\theta)}\kappa\right) \\
\lim_{\alpha\to 1} \frac{1}{p_i(s,M_t)} &= \lim_{\alpha\to 1} \frac{1}{\mathcal{V}(s_1)} = 0 \textit{ for } i \in \{1,2\}.
\end{aligned}$$

Corollary 6 contains two insights. First, it generalizes the result (e.g., in Corollary 2) that the allocation implemented by the pure-credit cashless limit of a monetary equilibrium is generically different from the allocation implemented by the pure-credit limit of a nonmonetary economy. This is clear from the fact that, provided $1 \leq 1+\iota < \eta(1+\bar{\iota}^*)$, $\lim_{\alpha\to 1} y^m(s_1) > \mathrm{D}(\varphi^{n*})$ for all $\eta \in (0,1]$ and all $\theta \in [0,1)$. Second, Corollary 6 formalizes the idea that since the equilibrium correspondence for the set of sunspot equilibria is continuous, by adopting a particular equilibrium selection scheme, it is possible to construct a sunspot monetary equilibrium whose pure-credit cashless limit converges to the pure-credit limit of the nonmonetary economy. The selection involves decreasing the probability $\eta$ toward zero as $\alpha$ approaches 1, i.e., intuitively, agents' expectations that money will lose its value forever (purely due to self-fulfilling beliefs) must converge to 1 along with $\alpha$. More formally, the equilibrium selection scheme is to focus on the particular joint limit on credit *and beliefs*, $\alpha(1-\eta) \to 1$, and in this case, even if $\theta < 1$, one would indeed find $\lim_{\alpha(1-\eta)\to 1} \varphi(s_1) = \varphi^{n\star}$, and therefore $\lim_{\alpha \to 1} y^m(s_1) = \text{D}(\varphi^{n*})$. It is our view that this kind of approximation result based on an arbitrary equilibrium selection from a large set of equilibria is too frail to offer a compelling basis for a moneyless approach to monetary economics.

### C.3.2 Proofs

The following corollary of Lemma 5 summarizes the conditions that characterize a recursive monetary sunspot equilibrium. Without relevant loss of generality, we focus on equilibria where only consumers hold money between periods, and only unbanked producers hold money between the first and second subperiod of a given period.

**Corollary 7** *A recursive monetary sunspot equilibrium is a collection of functions of $s \in \mathbb{S}$,*

$$\left\langle Z_1(s), Z_2(s), \rho(s), Y_P(s), \tilde{Y}_P(s), \bar{Y}_P(s), \bar{Y}_C(s) \right\rangle,$$

*that, for all $s \in \mathbb{S}$, satisfies the market-clearing conditions*

$$\begin{aligned} 0 &= \bar{Y}_C(s) - \bar{Y}_P(s) - \tilde{Y}_P(s) \\ 0 &= Z_1(s) - \bar{Y}_C(s) + \bar{Y}_P(s) \end{aligned}$$

*and the optimality conditions*

$$\begin{aligned} Z_2(s_i) &= \frac{1}{1+\iota} \sum_{j=1}^{N} \eta_{ij} \left[1 + \rho(s_j)\right] Z_2(s_j) \;\; \textit{for all } s_i \in \mathbb{S} \\ Y_P(s) &= \begin{cases} \infty & \textit{if } \kappa - R^m(s) < 0 \\ [0, \infty] & \textit{if } \kappa - R^m(s) = 0 \\ 0 & \textit{if } 0 < \kappa - R^m(s) \end{cases} \\ \tilde{Y}_P(s) &= \begin{cases} (1-\alpha) Y_P(s) & \textit{if } 0 < \varphi^m(s) - \underline{\kappa} \\ [0, (1-\alpha) Y_P(s)] & \textit{if } \varphi^m(s) - \underline{\kappa} = 0 \\ 0 & \textit{if } \varphi^m(s) - \underline{\kappa} < 0 \end{cases} \\ \bar{Y}_P(s) &= \begin{cases} \alpha Y_P(s) & \textit{if } 0 < \varphi(s) - \underline{\kappa} \\ [0, \alpha Y_P(s)] & \textit{if } \varphi(s) - \underline{\kappa} = 0 \\ 0 & \textit{if } \varphi(s) - \underline{\kappa} < 0 \end{cases} \\ \bar{Y}_C(s) &= \text{D}(\varphi(s)) \end{aligned}$$

*where*

$$\varphi^m(s) \equiv \frac{Z_2(s)}{Z_1(s)}$$
$$\varphi(s) \equiv [1+\rho(s)]\,\varphi^m(s)$$
$$R^m(s) \equiv \underline{\kappa} + \alpha\theta\,[\varphi(s) - \underline{\kappa}]\,\mathbb{I}_{\{\underline{\kappa}<\varphi(s)\}} + (1-\alpha\theta)\,[\varphi^m(s) - \underline{\kappa}]\,\mathbb{I}_{\{\underline{\kappa}<\varphi^m(s)\}}.$$

**Proof of Proposition 7.** Conjecture the following sunspot equilibrium:

$$\begin{aligned}
\rho(s_1) &= \frac{\iota+1-\eta}{\eta}\\
\varphi^m(s_1) &= \frac{\eta}{1+\alpha\theta\iota-(1-\eta)(1-\alpha\theta)}\kappa\\
Z_1(s_1) &= \frac{Z_2(s_1)}{\varphi^m(s_1)} = (1-\alpha)\mathrm{D}(\varphi(s_1)),\ \text{with}\ \varphi(s_1)\equiv[1+\rho(s_1)]\,\varphi^m(s_1)\\
\mathcal{V}(s_1) &= \frac{1}{1-\alpha}\\
Y_P(s_1) &= \bar{Y}_C(s_1) = \frac{\bar{Y}_P(s_1)}{\alpha} = \frac{\tilde{Y}_P(s_1)}{1-\alpha} = \mathrm{D}(\varphi(s_1))\\
\varphi^m(s_2) &= Z_1(s_2) = Z_2(s_2) = \tilde{Y}_P(s_2) = 0\\
\bar{Y}_C(s_2) &= \bar{Y}_P(s_2) = \alpha Y_P(s_2) = \mathrm{D}(\varphi^n)\\
p_i(s,M_t) &= \frac{M_t}{Z_i(s)}\ \text{for}\ i\in\{1,2\}\ \text{and}\ s\in\mathbb{S}.
\end{aligned}$$

It is easy to verify that the conjectured allocations and prices satisfy the equilibrium conditions in Corollary 7. ■

# D Welfare

**Lemma 6** *Consider an economy with $v(x)=x$.*

*(i) Along the stationary monetary equilibrium, welfare is*

$$(1-\beta)\,\mathcal{W}^m = u(\mathrm{D}(\varphi)) - \kappa\mathrm{D}(\varphi),$$

*with $\varphi \equiv (1+\iota)\,\varphi^m$, and $\varphi^m$ as given in part (i) of Proposition 3.*

*(ii) Along the nonmonetary equilibrium, welfare is*

$$(1-\beta)\,\mathcal{W}^n = u(\mathrm{D}(\varphi^n)) - \left[\kappa + \frac{1-\alpha}{\alpha\theta}(\kappa-\underline{\kappa})\right]\mathrm{D}(\varphi^n),$$

*with $\varphi^n$ as given in Proposition 2.*

**Proof of Lemma 6.** $(i)$ From Lemma 3,

$$\begin{aligned}
V_t^B(a_t^m) &= (1+\rho_t)\frac{1}{p_{2t}}a_t^m + \alpha(1-\theta)\rho_t\left[\int \frac{1}{p_{2t}}\tilde{a}_t^m dH_t(\tilde{a}_t^m) + \varphi_t^m \mathrm{D}(\varphi_t)\right] + \bar{W}_t^B \\
V_t^C(a_t^m) &= (1+\rho_t)\frac{1}{p_{2t}}a_t^m + u(\mathrm{D}(\varphi_t)) - \varphi_t \mathrm{D}(\varphi_t) + \bar{W}_t^C \\
V_t^P(a_t^m) &= (1+\alpha\theta\rho_t)\frac{1}{p_{2t}}a_t^m + \bar{W}_t^P,
\end{aligned}$$

where $\varphi_t \equiv (1+\rho_t)\varphi_t^m$, and

$$\bar{W}_t^i = v(x^\star) - x^\star + \mathbb{I}_{\{i=C\}}\frac{1}{p_{2t}}(T_t^m - M_{t+1}) + \beta V_{t+1}^i\left(\mathbb{I}_{\{i=C\}}M_{t+1}\right)$$

for $i \in \{B, C, P\}$. (The expression for $\bar{W}_t^i$ follows from (40) and the fact that only consumers carry cash across periods; the expression for $V_t^B(a_t^m)$ uses the fee that a banker charges a producer reported in part $(iii)$ of Lemma 2; and the expression for $V_t^P(a_t^m)$ uses part $(iv)$ of Lemma 2.)

Along the equilibrium path only consumers hold money at the beginning of the period, so the relevant beginning-of-period payoffs are:

$$\begin{aligned}
V_t^B(0) &= \alpha(1-\theta)\rho_t\varphi_t^m \mathrm{D}(\varphi_t) + \bar{W}_t^B \\
V_t^C(M_t) &= (1+\rho_t)\frac{1}{p_{2t}}M_t + u(\mathrm{D}(\varphi_t)) - \varphi_t \mathrm{D}(\varphi_t) + \bar{W}_t^C \\
V_t^P(0) &= \bar{W}_t^P.
\end{aligned}$$

Also, along a stationary monetary equilibrium, we have $\frac{1}{p_{2t}}M_t = Z_2$, $\varphi_t^m = \varphi^m$, $\rho_t = \rho$, $\varphi_t = \varphi \equiv (1+\rho)\varphi^m$, and $\frac{1}{p_{2t}}T_t^m = \frac{1}{p_{2t}}(M_{t+1} - M_t) = (\mu - 1)Z_2$, so

$$\bar{W}_t^B = v(x^\star) - x^\star + \beta V^B \equiv \bar{W}^B \tag{128}$$

$$\bar{W}_t^C = v(x^\star) - x^\star - Z_2 + \beta V^C(Z_2) \equiv \bar{W}^C \tag{129}$$

$$\bar{W}_t^P = v(x^\star) - x^\star + \beta V^P \equiv \bar{W}^P \tag{130}$$

and the beginning-of-period payoffs are

$$V_t^B(0) = \alpha(1-\theta)\rho\varphi^m \mathrm{D}(\varphi) + \bar{W}^B \equiv V^B \tag{131}$$

$$V_t^C(M_t) = (1+\rho)Z_2 + u(\mathrm{D}(\varphi)) - \varphi \mathrm{D}(\varphi) + \bar{W}^C \equiv V^C(Z_2) \tag{132}$$

$$V_t^P(0) = \bar{W}^P \equiv V^P. \tag{133}$$

By substituting (128)-(130) into (131)-(133), the beginning-of-period values become

$$V^B = \alpha(1-\theta)\rho\varphi^m \mathrm{D}(\varphi) + v(x^\star) - x^\star + \beta V^B \tag{134}$$

$$V^C(Z_2) = \rho Z_2 + u(\mathrm{D}(\varphi)) - \varphi \mathrm{D}(\varphi) + v(x^\star) - x^\star + \beta V^C(Z_2) \tag{135}$$

$$V^P = [R^m(\varphi^m, \varphi) - \kappa]\, \mathrm{D}(\varphi) + v(x^\star) - x^\star + \beta V^P, \tag{136}$$

where $R^m(\varphi^m, \varphi) - \kappa = \varphi^m + \alpha\theta(\varphi - \varphi^m) - \kappa = 0$. Consider the (equally weighted) welfare function, $\mathcal{W}^m \equiv V^B + V^C(Z_2) + V^P$. With (134)-(136), we have

$$\mathcal{W}^m = \rho Z_2 + u(\mathrm{D}(\varphi)) - [\kappa + (1-\alpha)\rho\varphi^m]\, \mathrm{D}(\varphi) + 3[v(x^\star) - x^\star] + \beta\mathcal{W}^m.$$

After substituting the equilibrium condition $Z_2 = (1-\alpha)\varphi^m \mathrm{D}(\varphi)$ ((19) in Proposition 3), we get

$$(1-\beta)\mathcal{W}^m = u(\mathrm{D}(\varphi)) - \kappa \mathrm{D}(\varphi) + 3[v(x^\star) - x^\star], \tag{137}$$

where $\varphi = (1+\iota)\varphi^m = \frac{1+\iota}{1+\alpha\theta\iota}\kappa$ (from (17) and (18) in Proposition 3). To conclude, set $v(x) = x$ in (137) to obtain (29).

($ii$) In the nonmonetary equilibrium, from Lemma 3 and Lemma 1, the value functions are

$$V^B = \alpha(1-\theta)(\varphi^n - \underline{\kappa})\mathrm{D}(\varphi^n) + v(x^\star) - x^\star + \beta V^B$$

$$V^C = u(\mathrm{D}(\varphi^n)) - \varphi^n \mathrm{D}(\varphi^n) + v(x^\star) - x^\star + \beta V^C$$

$$V^P = [R^n(\varphi^n) - \kappa]\, \mathrm{D}(\varphi^n) + v(x^\star) - x^\star + \beta V^P,$$

where

$$R^n(\varphi^n) - \kappa = \underline{\kappa} + \alpha\theta(\varphi^n - \underline{\kappa}) - \kappa = 0.$$

for $i \in \{B, C, P\}$. The (equally weighted) welfare function, $\mathcal{W}^n \equiv V^B + V^C + V^P$, is

$$(1-\beta)\mathcal{W}^n = u(\mathrm{D}(\varphi^n)) - \left[\kappa + \frac{1-\alpha}{\alpha\theta}(\kappa - \underline{\kappa})\right] \mathrm{D}(\varphi^n) + 3[v(x^\star) - x^\star], \tag{138}$$

with $\varphi^n = \kappa + \frac{1-\alpha\theta}{\alpha\theta}(\kappa - \underline{\kappa})$. To conclude, set $v(x) = x$ in (138) to obtain (28). ■

**Proof of Proposition 4.** ($i$) From Proposition 3 we know that $(1+\iota)\varphi^m = \kappa$ if $\iota = 0$, so (27) and (29) imply $\mathcal{W}^m = \mathcal{W}^\star$ if $\iota = 0$. Also, given $\alpha < 1$, $\partial[(1+\iota)\varphi^m]/\partial\iota > 0$ (which implies $\partial \mathrm{D}((1+\iota)\varphi^m)/\partial\iota < 0$), and $\kappa < u'(\mathrm{D}((1+\iota)\varphi^m))$ for $\iota > 0$, so it follows from (29) that $\partial\mathcal{W}^m/\partial\iota < 0$ and therefore $\mathcal{W}^m < \mathcal{W}^\star$ for all $\iota \in (0, \bar{\iota}]$.

Notice that $\mathcal{W}^n$ and $\mathcal{W}^m$ can be written as

$$\begin{aligned}(1-\beta)\,\mathcal{W}^n &= u(\mathrm{D}\,(\varphi^n)) - \varphi^n \mathrm{D}\,(\varphi^n) + \frac{1-\theta}{\theta}(\kappa - \underline{\kappa})\mathrm{D}\,(\varphi^n) \\ (1-\beta)\,\mathcal{W}^m &= u(\mathrm{D}\,(\varphi)) - \varphi \mathrm{D}\,(\varphi) + \frac{(1-\alpha\theta)\,\iota}{1+\alpha\theta\iota}\kappa \mathrm{D}\,(\varphi)\,,\end{aligned}$$

so

$$\begin{aligned}(1-\beta)\,(\mathcal{W}^m - \mathcal{W}^n) &= u(\mathrm{D}\,(\varphi)) - \varphi \mathrm{D}\,(\varphi) - [u(\mathrm{D}\,(\varphi^n)) - \varphi^n \mathrm{D}\,(\varphi^n)] \\ &\quad + \frac{(1-\alpha\theta)\,\iota}{1+\alpha\theta\iota}\kappa \mathrm{D}\,(\varphi) - \frac{1-\theta}{\theta}(\kappa - \underline{\kappa})\mathrm{D}\,(\varphi^n) \\ &= u(\mathrm{D}\,(\varphi)) - \varphi \mathrm{D}\,(\varphi) - [u(\mathrm{D}\,(\varphi^n)) - \varphi \mathrm{D}\,(\varphi^n)] + (\varphi^n - \varphi)\,\mathrm{D}\,(\varphi^n) \\ &\quad + \frac{1-\alpha\theta}{\alpha\theta}\frac{\kappa - \underline{\kappa} - \alpha\theta\,(\bar{\iota}-\iota)\,\underline{\kappa}}{\kappa - \alpha\theta\,(\bar{\iota}-\iota)\,\underline{\kappa}}\kappa \mathrm{D}\,(\varphi) - \frac{1-\theta}{\theta}(\kappa - \underline{\kappa})\mathrm{D}\,(\varphi^n)\,,\end{aligned}$$

where $\varphi = (1+\iota)\,\varphi^m$. From Proposition 3 we know that $\varphi = \varphi^n$ if $\iota = \bar{\iota}$. Hence,

$$(1-\beta)\,(\mathcal{W}^m - \mathcal{W}^n) = \frac{1-\alpha}{\alpha\theta}(\kappa - \underline{\kappa})\mathrm{D}\,(\varphi^n) \text{ if } \iota = \bar{\iota}.$$

From this we learn that $\mathcal{W}^n < \mathcal{W}^m$ if $\iota = \bar{\iota}$ (provided $\alpha < 1$). Then $\partial \mathcal{W}^m/\partial\iota < 0$ implies $\mathcal{W}^m > \mathcal{W}^n$ for all $\iota \in [0, \bar{\iota})$.

($ii$) Notice that $\mathcal{W}^\star$ is independent of $\alpha$, while (28) and (29) imply

$$\begin{aligned}(1-\beta)\lim_{\alpha\to1}\mathcal{W}^n &= u(\mathrm{D}\,(\varphi^{n*})) - \kappa \mathrm{D}\,(\varphi^{n*}) \\ (1-\beta)\lim_{\alpha\to1}\mathcal{W}^m &= u(\mathrm{D}\,((1+\iota)\,\varphi^{m*})) - \kappa \mathrm{D}\,((1+\iota)\,\varphi^{m*})\,,\end{aligned}$$

with $\varphi^{n*}$ and $\varphi^{m*}$ as defined in Corollary 1 and Corollary 2, respectively. From (26), it is clear that $\lim_{\alpha\to1}(\mathcal{W}^m - \mathcal{W}^n) \geq 0$, with "=" only if either $\iota = \lim_{\alpha\to1}\bar{\iota}$ or $\theta = 1$. Finally, from (24), it is clear that $\kappa \leq (1+\iota)\,\varphi^{m*}$ (and therefore $\lim_{\alpha\to1}\mathcal{W}^m \leq \mathcal{W}^\star$), with "=" only if $\iota = 0$ or $\theta = 1$. ■

**Proof of Proposition 5.** With a slight abuse of notation, let $\varphi\,(\iota) \equiv (1+\iota)\,\varphi^m$, with $\varphi^m$ as defined in part ($i$) of Proposition 3, i.e., $\varphi\,(\iota) = \frac{1+\iota}{1+\alpha\theta\iota}\kappa$, so

$$\ln\varphi\,(\iota) = \ln\frac{1+\iota}{1+\alpha\theta\iota} + \ln\kappa. \tag{139}$$

From (29), $\tau\,(\iota)$ is defined by

$$u(\mathrm{D}\,(\varphi\,(0))) - \kappa \mathrm{D}\,(\varphi\,(0)) = u(\mathrm{D}\,(\varphi\,(\iota))\,(1+\tau\,(\iota))) - \kappa \mathrm{D}\,(\varphi\,(\iota)\,(1+\tau\,(\iota)))\,,$$

so

$$1+\tau(\iota)=\frac{\mathrm{D}(\varphi(0))}{\mathrm{D}(\varphi(\iota))},$$

and for $\iota \approx 0$,

$$\tau(\iota) \approx \ln \mathrm{D}(\varphi(0)) - \ln \mathrm{D}(\varphi(\iota)). \tag{140}$$

Also, for $\iota \approx 0$, $\ln \frac{1+\iota}{1+\alpha\theta\iota} \approx (1-\alpha\theta)\,\iota$, so (139) implies $\ln \varphi(\iota) \approx (1-\alpha\theta)\,\iota + \ln \kappa$, and therefore

$$\frac{d \ln \varphi(\iota)}{d\iota} = 1-\alpha\theta. \tag{141}$$

Hence, (140) and (141) imply

$$\frac{d\tau(\iota)}{d\iota} \approx -\frac{d \ln \mathrm{D}(\varphi(\iota))}{d\iota} = -\frac{d \ln \mathrm{D}(\varphi(\iota))}{d \ln \varphi(\iota)} \frac{d \ln \varphi(\iota)}{d\iota} = -\epsilon(1-\alpha\theta).$$

In the cashless limit, $\alpha \to 1$, and we obtain the expression in the statement. ■

# E Money-in-the-utility formulation

**Lemma 7** *A stationary monetary equilibrium of the reduced-form model with money in the utility function (described by (30)-(34)) is a vector* $\left((c_j, h_j, y_j, \mathcal{Z}_j)_{j\in\{1,2\}}, \phi, \pi\right)$ *that satisfies*

$$\begin{aligned}
\phi &= \frac{\varepsilon}{\varepsilon-1}B && (142)\\
c_1 &= h_1 = y_1 = \mathrm{D}(\phi) && (143)\\
c_2 &= h_2 = y_2 = x^{\star} && (144)\\
\pi &= \frac{1}{\varepsilon}\phi \mathrm{D}(\phi) && (145)\\
\iota &= \frac{A}{\phi}\ell'(\mathcal{Z}_1) && (146)\\
\mathcal{Z}_2 &= \phi \mathcal{Z}_1. && (147)
\end{aligned}$$

**Proof of Lemma 7.** The Lagrangian for (31) with the preference specification (34) is

$$\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty}\beta^t \Bigg\{ & u(c_{1t}) + v(c_{2t}) + A\ell\left(\frac{m_t}{P_{1t}}\right) - Bh_{1t} - h_{2t} \\
& + \varsigma_t m_{t+1} + \lambda_t \left[w_{1t}h_{1t} + P_{2t}h_{2t} + m_t + \Pi_{1t} + T_t - (P_{1t}c_{1t} + P_{2t}c_{2t} + m_{t+1})\right] \Bigg\},
\end{aligned}$$

where $\varsigma_t$ is the multiplier on the constraint $0 \leq m_{t+1}$, and $\lambda_t$ is the multiplier on the budget constraint. The first-order conditions for this problem are:

$$u'(c_{1t}) = \lambda_t P_{1t} \tag{148}$$

$$v'(c_{2t}) = \lambda_t P_{2t} \tag{149}$$

$$B = \lambda_t w_{1t} \tag{150}$$

$$1 = \lambda_t P_{2t} \tag{151}$$

$$\lambda_t \geq \beta \left[ A \frac{1}{P_{1t+1}} \ell' \left( \frac{m_{t+1}}{P_{1t+1}} \right) + \lambda_{t+1} \right], \text{ with `` = '' if } 0 < m_{t+1}. \tag{152}$$

Conditions (148)-(151) imply

$$v'(c_{2t}) = u'(c_{1t}) \frac{P_{2t}}{P_{1t}} = B \frac{P_{2t}}{w_{1t}} = 1. \tag{153}$$

From (153) it is immediate that $c_{2t} = x^{\star}$, which together with the market-clearing condition for good 2, i.e., $c_{2t} = y_{2t}$, and the production technology for good 2, i.e., $y_{2t} = h_{2t}$, gives (144). In an equilibrium where money is held (i.e., $m_{t+1} = M_{t+1} > 0$) we can use (151) to write the Euler equation (152) as

$$\frac{\frac{\mathcal{Z}_{2t}}{\mathcal{Z}_{2t+1}} \mu - \beta}{\beta} = \frac{A}{\phi_{t+1}} \ell' (\mathcal{Z}_{1t+1}). \tag{154}$$

In a stationary monetary equilibrium, (154) reduces to (146). Condition (147) is immediate from the definitions $\mathcal{Z}_{jt} \equiv \frac{M_t}{P_{jt}}$ and $\phi_t \equiv \frac{P_{1t}}{P_{2t}}$.

The first-order condition for the problem of the firm that produces the final good 1 (i.e., problem (32)) implies that the firm's demand for the intermediate good of type $i \in [0,1]$ is

$$y_t(i) = \left( \frac{P_{1t}}{p_t(i)} \right)^{\varepsilon} y_{1t}, \tag{155}$$

where $y_{1t}$ is the total output of good 1 given by (30). This condition in turn implies that the nominal price of the final good 1 satisfies

$$P_{1t} = \left( \int_0^1 p_t(i)^{1-\varepsilon} \, di \right)^{\frac{1}{1-\varepsilon}}. \tag{156}$$

The problem of the firm that produces intermediate good $i \in [0,1]$ (i.e., problem (32)) is equivalent to

$$\Pi_t(i) = \max_{p_t(i)} \left[ p_t(i) - w_{1t} \right] \Upsilon_t(p_t(i)) \tag{157}$$

$$\text{with } h_t(i) = \Upsilon_t(p_t(i)). \tag{158}$$

The first-order condition for this problem is

$$\mathrm{Y}_t\left(p_t\left(i\right)\right)+\left[p_t\left(i\right)-w_{1t}\right]\frac{\partial \mathrm{Y}_t\left(p_t\left(i\right)\right)}{\partial p_t\left(i\right)}=0. \tag{159}$$

From (155) we know that

$$\mathrm{Y}_t\left(p_t\left(i\right)\right)=\left(\frac{P_{1t}}{p_t\left(i\right)}\right)^{\varepsilon}y_{1t}, \tag{160}$$

so

$$\frac{\partial \mathrm{Y}_t\left(p_t\left(i\right)\right)}{\partial p_t\left(i\right)}=-\varepsilon\left(p_t\left(i\right)\right)^{-\varepsilon-1}\left(P_{1t}\right)^{\varepsilon}y_{1t}. \tag{161}$$

Substitute (160) and (161) into (159) to get

$$p_t\left(i\right)=\frac{\varepsilon}{\varepsilon-1}w_{1t}\text{ for all }i\in\left[0,1\right]. \tag{162}$$

Together, conditions (156) and (162) imply

$$P_{1t}=p_t\left(i\right)=\frac{\varepsilon}{\varepsilon-1}w_{1t}\text{ for all }i\in\left[0,1\right]. \tag{163}$$

Then (160) and (163) imply

$$\mathrm{Y}_t\left(p_t\left(i\right)\right)=y_{1t}\text{ for all }i\in\left[0,1\right]. \tag{164}$$

With (164), (158) implies

$$h_t\left(i\right)=y_{1t}=h_{1t}\text{ for all }i\in\left[0,1\right]. \tag{165}$$

To obtain the profit of the firm that produces intermediate good $i\in[0,1]$, substitute (162) and (164) into the intermediate producer firm's objective function (157) to get

$$\Pi_t(i)=\frac{1}{\varepsilon-1}w_{1t}y_{1t}=\frac{1}{\varepsilon}P_{1t}y_{1t}=\Pi_{1t}\text{ for all }i\in\left[0,1\right]. \tag{166}$$

The last equality in (166) implies (145).

Condition (163) together with the last two equalities in (153) imply

$$u'\left(c_{1t}\right)=\frac{P_{1t}}{P_{2t}}=\frac{\varepsilon}{\varepsilon-1}B. \tag{167}$$

Conditions (142) and (143) in the statement of the lemma follow from (167) (the fact that $c_{1t}=h_{1t}$ follows from the last equality in (165) and the market-clearing condition for good 1, $c_{1t}=y_{1t}$). ■