Online Appendix: Reputation Effects under Interdependent Values

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A Generalization & Proof of Theorem 2

I state a generalized version of Theorem 2 by allowing for arbitrary correlations between the two dimensions of player 1’s private information in $\mu$. For every $\theta \in \Theta$, let $\mu(\theta)$ be the probability of commitment type $\theta$. For every $a^*_1 \in A^*_1$, let $\mu(a^*_1)$ be the probability of commitment type $a^*_1$. I say that $\mu$ is optimistic if

$$
\mu(\overline{a}_1)D(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta_\gamma \cup \Theta_p} \mu(\theta)D(\theta, \overline{a}_1) > 0,
$$

(A.1)

and is pessimistic otherwise, which generalize the optimistic and pessimistic belief conditions in the main text.

Theorem 2’. If the game satisfies Assumptions 1, 2 and 3, $\mu$ has full support and satisfies Assumption 4 and (A.1), then

$$
\liminf_{\delta \to 1} v_\delta(\delta, \mu, u_1, u_2) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) \quad \text{for every } \theta \in \Theta^*.
$$

A.1 Defining Useful Constants

Recall the definitions of $\Theta_\gamma$, $\Theta_p$ and $\Theta_n$ in Appendix D of the main text. Let $\theta_\gamma$, $\theta_p$, and $\theta_n$ be typical elements of these sets and recall that Lemma D.1 in the main text shows that $\theta_\gamma > \theta_p > \theta_n$.

My proof starts from defining several useful constants which only depend on $\mu$, $u_1$ and $u_2$, but are independent of $\sigma$ and $\delta$. Let $M \equiv \max_{\theta, a_1, a_2} |u_1(\theta, a_1, a_2)|$ and

$$
K \equiv \max_{\theta \in \Theta} \left\{ u_1(\theta, \overline{a}_1, \overline{a}_2) - u_1(\theta, a_1, a_2) \right\} = \min_{\theta \in \Theta} \left\{ u_1(\theta, \overline{a}_1, \overline{a}_2) - u_1(\theta, a_1, a_2) \right\}.
$$

Since $D(\phi_{\overline{a}_1}, \overline{a}_1) > 0$, the optimistic belief condition (A.1) implies the existence of $\kappa \in (0, 1)$ such that:

$$
\kappa \mu(\overline{a}_1)D(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta} \mu(\theta)D(\theta, \overline{a}_1) > 0.
$$
For any $\kappa \in (0, 1)$, let
\[
\rho_0(\kappa) \equiv \frac{(1 - \kappa)\mu(\pi_1)D(\phi_{\pi_1}, \pi_1)}{2\max_{(\theta, a_1) \in \Theta \times A_1} D(\theta, a_1)} > 0
\] (A.2)
and
\[
T_0(\kappa) \equiv \lceil 1/\rho_0(\kappa) \rceil.
\] (A.3)

Let
\[
\rho_1(\kappa) \equiv \frac{\kappa\mu(\pi_1)D(\phi_{\pi_1}, \pi_1)}{\max_{(\theta, a_1) \in \Theta \times A_1} D(\theta, a_1)}
\] (A.4)
and
\[
T_1(\kappa) \equiv \lceil 1/\rho_1(\kappa) \rceil.
\] (A.5)

Let $\delta \in (0, 1)$ be close enough to 1 such that for every $\delta \in [\delta, 1)$ and $\theta_p \in \Theta_p$,
\[
(1 - \delta T_0(\delta))u_1(\theta_p, a_1, a_2) + \delta T_0(\delta)u_1(\theta_p, a_1, a_2) > \frac{1}{2}\left(u_1(\theta_p, a_1, a_2) + u_1(\theta_p, a_1, a_2)\right).
\] (A.6)

### A.2 Random History & Random Path

Let $\Omega \equiv A_1^t \cup \Theta$ be the set of types with $\omega$ a typical element of $\Omega$. Abusing notation, I write $\mu$ as a full support distribution on $\Omega$. Let $h^t \equiv (a^t, r^t)$, with $a^t \equiv (a_{1,s})_{s \leq t-1}$ and $r^t \equiv (a_{2,s}, \xi_s)_{s \leq t-1}$. Let $a_s^t \equiv (\overline{a}_1, ..., \overline{a}_1)$. I call $h^t$ a public history, $r^t$ a random history and $r^\infty$ a random path. Let $\mathcal{H}$ and $\mathcal{R}$ be the set of public histories and random histories, respectively, with $\succ, \preceq, \prec$ and $\succeq$ naturally defined. Recall that a strategy profile $\sigma$ consists of $(\sigma_\theta)_{\theta \in \Theta}$ with $\sigma_\theta : \mathcal{H} \to \Delta(A_1)$ and $\sigma_\omega : \mathcal{H} \to \Delta(A_2)$. Let $\mathcal{P}^\sigma(\theta)$ be the probability measure over public histories induced by $(\sigma_\theta, \sigma_\omega)$. Let $\mathcal{P}^\sigma \equiv \sum_{\omega \in \Omega} \mu(\omega)\mathcal{P}^\sigma(\omega)$ be the probability measure induced by $\sigma$, taken into account the possibilities of commitment types. Let $v^\sigma(h^t) \equiv \left\{v^\sigma_\theta(h^t)\right\}_{\theta \in \Theta} \in \mathbb{R}^{(\Theta)}$ be the continuation payoff vector for strategic types at $h^t$ under strategy profile $\sigma$.

Let $\mathcal{H}_\sigma \subset \mathcal{H}$ be the set of histories $h^t$ such that $\mathcal{P}^\sigma(h^t) > 0$, and let $\mathcal{H}_\omega(\omega) \subset \mathcal{H}$ be the set of histories $h^t$ such that $\mathcal{P}^\sigma(\omega)(h^t) > 0$. Let
\[
\mathcal{R}^\sigma \equiv \left\{r^\infty \middle| (a^t, r^t) \in \mathcal{H}_\sigma \text{ for all } t \text{ and } r^t \prec r^\infty\right\}
\]
be the set of random paths consistent with player 1 playing $\pi_1$ in every period. For every $h^t = (a^t, r^t)$, let $\overline{a}_1[h^t] : \mathcal{H} \to A_1$ be a strategy in the continuation game at $h^t$ that satisfies $\overline{a}_1[h^t](h^s) = \overline{a}_1$ for all $h^s \succeq h^t$ with $h^s = (a^t, \overline{a}_1, ..., \overline{a}_1, r^s) \in \mathcal{H}_\sigma$. Let $\overline{a}_1[h^t] : \mathcal{H} \to A_1$ be a strategy in the continuation game at $h^t$ that satisfies $\overline{a}_1[h^t](h^s) = \overline{a}_1$ for all $h^s \succeq h^t$ with $h^s = (a^t, \overline{a}_1, ..., \overline{a}_1, r^s) \in \mathcal{H}_\sigma$. For every $\theta \in \Theta$, let
\[
\overline{\mathcal{R}}^\sigma(\theta) \equiv \left\{r^t \middle| \overline{a}_1[a^t, r^t] \text{ is type } \theta \text{'s best reply to } \sigma_2 \right\} \text{ and } \overline{\mathcal{R}}^\sigma(\theta) \equiv \left\{r^t \middle| \overline{a}_1[a^t, r^t] \text{ is type } \theta \text{'s best reply to } \sigma_2 \right\}.
\]
A.3 Beliefs & Best Response Sets

Let $\mu(a^i, r^i) \in \Delta(\Omega)$ be player 2’s posterior belief at $(a^i, r^i)$ and specifically, let $\mu^*(r^i) \equiv \mu(a^i, r^i)$. Let

$$B_\kappa \equiv \left\{ \tilde{\mu} \in \Delta(\Omega) \middle| \kappa \tilde{\mu}(\bar{\pi}_1) D(\phi_{\pi_1}, \bar{\pi}_1) + \sum_{\theta \in \Theta} \tilde{\mu}(\theta) D(\theta, \bar{\pi}_1) \geq 0 \right\}. \quad (A.7)$$

By definition, $B_{\kappa'} \subseteq B_\kappa$ for every $\kappa, \kappa' \in [0, 1]$ with $\kappa' < \kappa$.

For every $r^t \in \mathcal{R}^t$ and $\omega \in \Omega$, let $q^*(r^t)(\omega)$ be the ex ante probability that (1) player 1 is type $\omega$; (2) player 1 has played $\pi_1$ from period 0 to $t - 1$, conditional on the realization of random history being $r^t$. Let $q^*(r^t) \equiv \left\{ q^*(r^t)(\omega) \right\}_{\omega \in \Omega}$. For every $\delta \in (0, 1)$ and strategy profile $\sigma \in \text{NE}(\delta, \mu)$,

1. For every $a^i$ and $r^t, \hat{r}^t > r^{t-1}$ satisfying $(a^i, r^t), (a^i, \hat{r}^t) \in \mathcal{H}^\sigma$, we have $\mu(a^i, r^t) = \mu(a^i, \hat{r}^t)$.

2. For every $r^t, \hat{r}^t > r^{t-1}$ with $(a^i, r^t), (a^i, \hat{r}^t) \in \mathcal{H}^\sigma$, we have $q^*(r^t) = q^*(\hat{r}^t)$.

This is because player 1’s action in period $t - 1$ depends on $r^t$ only through $r^{t-1}$, so is player 2’s belief at every on-path history. Since the commitment type plays $\bar{\pi}_1$ in every period, we have $q^*(r^t)(\bar{\pi}_1) = \mu_0(\bar{\pi}_1)$.

For future reference, I introduce two sets of random histories based on player 2’s posterior beliefs. Let

$$\mathcal{R}^\sigma_g \equiv \left\{ r^t \middle| (a^i, r^t) \in \mathcal{H}^\sigma \text{ and } \mu^*(r^t)(\Theta_\rho \cup \Theta_n) = 0 \right\}. \quad (A.8)$$

and let

$$\hat{\mathcal{R}}^\sigma_g \equiv \left\{ r^t \middle| \text{there exists } r^T \succ r^t \text{ such that } r^T \in \mathcal{R}^\sigma_g \right\}. \quad (A.9)$$

Intuitively, $\hat{\mathcal{R}}^\sigma_g$ is the set of on-path random histories under which all the strategic types in $\Theta_\rho \cup \Theta_n$ will be separated from commitment type $\bar{\pi}_1$ at some random histories in the future.

A.4 Four Useful Lemmas

Recall that $\sigma_\theta : \mathcal{H} \rightarrow \Delta(A_1)$ is type $\theta$’s strategy. The first lemma outlines the implications of monotone-supermodularity on different types of player 1’s equilibrium strategies:

**Lemma A.1.** Suppose $\sigma$ is an equilibrium under $(\delta, \mu)$, $\theta \succ \tilde{\theta}$ and $h^i_\theta = (a^i, r^i) \in \mathcal{H}^\sigma(\theta) \cap \mathcal{H}^\sigma(\tilde{\theta})$.

1. If $r^t \in \hat{\mathcal{R}}^\sigma(\tilde{\theta})$, then $\sigma_\theta(a^i, r^i)(\bar{\pi}_1) = 1$ for every $(a^i, r^i) \in \mathcal{H}^{(\sigma_1(h^i_\theta), \sigma_2)}(\theta)$ with $r^s \succ r^t$.

2. If $r^t \in \hat{\mathcal{R}}^\sigma(\theta)$, then $\sigma_\tilde{\theta}(a^i, r^i)(\bar{\pi}_1) = 1$ for every $(a^i, r^i) \in \mathcal{H}^{(\sigma_1(h^i_\theta), \sigma_2)}(\tilde{\theta})$ with $(a^i, r^i) \succ (a^i, r^i)$. 

3
PROOF OF LEMMA A.1: I only need to show the first part, as the second part is symmetric after switching signs. Without loss of generality, I focus on history \( h^0 \). For notation simplicity, let \( \sigma_1[h^0] = \sigma_1 \). For every \( \sigma_\omega \) and \( \sigma_2 \), let \( P^{(\sigma_\omega, \sigma_2)} : A_1 \times A_2 \to [0, 1] \) be defined as:

\[
P^{(\sigma_\omega, \sigma_2)}(a_1, a_2) \equiv \sum_{t=0}^{+\infty} (1 - \delta)^t p_t^{(\sigma_\omega, \sigma_2)}(a_1, a_2)
\]

where \( p_t^{(\sigma_\omega, \sigma_2)}(a_1, a_2) \) is the probability of \( (a_1, a_2) \) occurring in period \( t \) under \( (\sigma_\omega, \sigma_2) \). Let \( P_i^{(\sigma_1, \sigma_2)} \in \Delta(A_2) \) be \( P^{(\sigma_1, \sigma_2)} \)'s marginal distribution on \( A_i \), for \( i \in \{1, 2\} \).

Suppose towards a contradiction that \( \sigma_1 \) is type \( \tilde{\theta} \)'s best reply and there exists \( \sigma_\theta \) with \( P_1^{(\sigma_\theta, \sigma_2)}(\sigma_1) < 1 \) such that \( \sigma_\theta \) is type \( \theta \)'s best reply, then type \( \tilde{\theta} \) and \( \theta \)'s incentive constraints require that:

\[
\sum_{a_2 \in A_2} \left( P_2^{(\sigma_1, \sigma_2)}(a_2) - P_2^{(\sigma_\theta, \sigma_2)}(a_2) \right) u_1(\tilde{\theta}, a_1, a_2) \geq \sum_{a_2 \in A_2, a_1 \neq \sigma_1} P^{(\sigma_\theta, \sigma_2)}(a_1, a_2) \left( u_1(\tilde{\theta}, a_1, a_2) - u_1(\tilde{\theta}, \sigma_1, a_2) \right),
\]

and

\[
\sum_{a_2 \in A_2} \left( P_2^{(\sigma_1, \sigma_2)}(a_2) - P_2^{(\sigma_\theta, \sigma_2)}(a_2) \right) u_1(\theta, a_1, a_2) \leq \sum_{a_2 \in A_2, a_1 \neq \sigma_1} P^{(\sigma_\theta, \sigma_2)}(a_1, a_2) \left( u_1(\theta, a_1, a_2) - u_1(\theta, \sigma_1, a_2) \right).
\]

Since \( P_1^{(\sigma_\theta, \sigma_2)}(\sigma_1) < 1 \) and \( u_1 \) has strictly increasing differences in \( \theta \) and \( a_1 \), we have:

\[
\sum_{a_2 \in A_2, a_1 \neq \sigma_1} P^{(\sigma_\theta, \sigma_2)}(a_1, a_2) \left( u_1(\tilde{\theta}, a_1, a_2) - u_1(\tilde{\theta}, \sigma_1, a_2) \right) > \sum_{a_2 \in A_2, a_1 \neq \sigma_1} P^{(\sigma_\theta, \sigma_2)}(a_1, a_2) \left( u_1(\theta, a_1, a_2) - u_1(\theta, \sigma_1, a_2) \right)
\]

which implies that:

\[
\sum_{a_2 \in A_2} \left( P_2^{(\sigma_\theta, \sigma_2)}(a_2) - P_2^{(\sigma_1, \sigma_2)}(a_2) \right) \left( u_1(\tilde{\theta}, a_1, a_2) - u_1(\tilde{\theta}, \sigma_1, a_2) \right) > 0. \tag{A.10}
\]

On the other hand, since \( u_1 \) is strictly decreasing in \( a_1 \), we have:

\[
\sum_{a_2 \in A_2, a_1 \neq \sigma_1} P^{(\sigma_\theta, \sigma_2)}(a_1, a_2) \left( u_1(\tilde{\theta}, a_1, a_2) - u_1(\tilde{\theta}, \sigma_1, a_2) \right) > 0
\]

Strategic type \( \tilde{\theta} \)'s incentive constraint implies that:

\[
\sum_{a_2 \in A_2} \left( P_2^{(\sigma_1, \sigma_2)}(a_2) - P_2^{(\sigma_\theta, \sigma_2)}(a_2) \right) u_1(\tilde{\theta}, \sigma_1, a_2) > 0. \tag{A.11}
\]
Since both $P_2^{(σ_0, σ_2)}$ and $P_2^{(σ_1, σ_2)}$ are probability distributions, we have

$$\sum_{a_2 ∈ A_2} \left( P_2^{(σ_0, σ_2)}(a_2) - P_2^{(σ_1, σ_2)}(a_2) \right) = 0.$$ 

Since $u_1(θ, a_1, a_2) - u_1(θ, a_1, a_2)$ is weakly increasing in $a_2$, inequality (A.10) implies that $P_2^{(σ_0, σ_2)}(a_2) - P_2^{(σ_1, σ_2)}(a_2) > 0$. Since $u_1(θ, a_1, a_2)$ is strictly increasing in $a_2$, (A.11) implies that $P_2^{(σ_0, σ_2)}(a_2) - P_2^{(σ_1, σ_2)}(a_2) < 0$, leading to a contradiction.

The next Lemma establishes a uniform upper bound on the number of periods in which $a_2$ is not a strict best reply although $a_1$ has been played in all previous periods and $μ^*(r^t) ∈ B_κ$.

**Lemma A.2.** If $μ^*(r^t) ∈ B_κ$ and $a_2$ is not a strict best reply at $(a_1^t, r^t)$, then for every $r^{t+1} > r^t$ with $(a_1^{t+1}, a_2^{t+1}) ∈ H^σ$, we have:

$$\sum_{θ ∈ Θ} \left( q^*(r^t)(θ) - q^*(r^{t+1})(θ) \right) ≥ ρ_0(κ). \quad (A.12)$$

**Proof of Lemma A.2:** If $μ^*(r^t) ∈ B_κ$, then:

$$κμ(a_1^t)D(φ_{a_1}, a_1) + \sum_{θ ∈ Θ} q^*(r^t)(θ)D(θ, a_1) ≥ 0.$$

Suppose $a_2$ is not a strict best reply at $(a_1^t, r^t)$, then,

$$μ(a_1^t)D(φ_{a_1}, a_1) + \sum_{θ ∈ Θ} q^*(r^{t+1})(θ)D(θ, a_1) + \sum_{θ ∈ Θ} \left( q^*(r^t)(θ) - q^*(r^{t+1})(θ) \right)D(θ, a_1) ≤ 0,$$

for every $r^{t+1} > r^t$ with $(a_1^{t+1}, r^{t+1}) ∈ H^σ$, or equivalently

$$κμ(a_1^t)D(φ_{a_1}, a_1) + \sum_{θ ∈ Θ} q^*(r^t)(θ)D(θ, a_1) + \frac{(1 - κ)μ(a_1^t)D(φ_{a_1}, a_1)}{>0}$$

$$+ \sum_{θ ∈ Θ} \left( q^*(r^{t+1})(θ) - q^*(r^t)(θ) \right)D(θ, a_1) + \sum_{θ ∈ Θ} \left( q^*(r^t)(θ) - q^*(r^{t+1})(θ) \right)D(θ, a_1) ≤ 0,$$

According to (A.2), we have:

$$\sum_{θ ∈ Θ} \left( q^*(r^t)(θ) - q^*(r^{t+1})(θ) \right) ≥ \frac{(1 - κ)μ(a_1^t)D(φ_{a_1}, a_1)}{2 max(θ, a_1) ∈ Θ × A_1 |D(θ, a_1)|} = ρ_0(κ).$$
Lemma A.2 implies that for every \( \sigma \in \text{NE}(\delta, \mu) \) and along every \( r^\infty \in \mathbb{R}_+^\infty \), the number of \( r^t \) such that \( \mu^t(r^t) \in B_k \) but \( \bar{s}_2 \) is not a strict best reply is at most \( T_0(\kappa) \). The next lemma obtains an upper bound for player 1’s continuation payoff after separating from commitment type \( \bar{s}_1 \) at a history with a pessimistic posterior belief.

**Lemma A.3.** For every \( \sigma \in \text{NE}(\delta, \mu) \) and \( h^t \in \mathcal{H}^\sigma \) with

\[
\mu(h^t)(\bar{s}_1)\mathcal{D}(\phi_{\bar{s}_1}, \bar{s}_1) + \sum_{\theta \in \Theta} \mu(h^t)(\theta)\mathcal{D}(\theta, \bar{s}_1) < 0,
\]

(A.13)

we have \( v_\bar{\theta}(h^t) = u_1(\theta, a_1, a_2) \) with \( \bar{\theta} \equiv \min \left\{ \text{supp}(\mu(h^t)) \right\} \).

**Proof of Lemma A.3:** Let

\[
\Theta' \equiv \left\{ \bar{\theta} \in \Theta_\rho \cup \Theta_\sigma \mid \mu(h^t)(\bar{\theta}) > 0 \right\}.
\]

Since \( \mathcal{D}(\phi_{\bar{s}_1}, \bar{s}_1) > 0 \), A.13 implies that \( \Theta' \neq \{ \emptyset \} \). The rest of the proof is done via induction on \( |\Theta'| \). When \( |\Theta'| = 1 \), there exists a pure strategy \( \sigma^*_\bar{\theta} : \mathcal{H} \rightarrow A_1 \) in the support of \( \sigma^*_\bar{\theta} \) such that (A.13) holds for all \( h^s \) satisfying \( h^s \in \mathcal{H}^{(\sigma^*_\bar{\theta}, \sigma^*_2)} \) and \( h^s \succeq h^t \). At every such \( h^s, a_2 \) is player 2’s strict best reply. When playing \( \sigma^*_\bar{\theta} \), type \( \bar{\theta}' \)’s stage game payoff is no more than \( u_1(\bar{\theta}, a_1, a_2) \) in every period.

Suppose towards a contradiction that the conclusion holds when \( |\Theta'| \leq k - 1 \) but fails when \( |\Theta'| = k \), then there exists \( h^s \in \mathcal{H}^\sigma(\bar{\theta}) \) with \( h^s \succeq h^t \) such that

1. \( \mu(h^s) \notin B_k \) for all \( h^s \succeq h^t \).
2. \( v_{\bar{\theta}}(h^s) > u_1(\bar{\theta}, a_1, a_2) \).
3. For all \( a_1 \) such that \( \mu(h^s, a_1) \notin B_k, \sigma^s_{\bar{\theta}}(h^s)(a_1) = 0 \).

Since belief is a martingale, there exists \( a_1 \) such that \( (h^s, a_1) \in \mathcal{H}^\sigma \) and \( \mu(h^s, a_1) \) satisfies A.13. Since \( \mu(h^s, a_1)(\bar{\theta}) = 0 \), there exists \( \bar{\theta} \in \Theta_\rho \setminus \{ \emptyset \} \) such that \( (h^s, a_1) \in \mathcal{H}^\sigma(\bar{\theta}) \). My induction hypothesis suggests that \( v_{\bar{\theta}}(h^s) = u_1(\bar{\theta}, a_1, a_2) \). The incentive constraints of type \( \bar{\theta} \) and type \( \bar{\theta} \) at \( h^s \) require the existence of \( (\alpha_{1, \tau}, \alpha_{2, \tau})_{\tau=0}^\infty \) with \( \alpha_{i, \tau} \in \Delta(A_i) \) such that:

\[
\mathbb{E}\left[ \sum_{\tau=0}^\infty (1-\delta)^\tau \left( u_1(\bar{\theta}, \alpha_{1, \tau}, \alpha_{2, \tau}) - u_1(\bar{\theta}, a_1, a_2) \right) \right] > 0 \geq \mathbb{E}\left[ \sum_{\tau=0}^\infty (1-\delta)^\tau \left( u_1(\bar{\theta}, \alpha_{1, \tau}, \alpha_{2, \tau}) - u_1(\bar{\theta}, a_1, a_2) \right) \right],
\]

where \( \mathbb{E}[\cdot] \) is taken over probability measure \( \mathcal{P}^\sigma \). However, the supermodularity condition implies that,

\[
u_1(\bar{\theta}, \alpha_{1, \tau}, \alpha_{2, \tau}) - u_1(\bar{\theta}, a_1, a_2) \leq u_1(\bar{\theta}, \alpha_{1, \tau}, \alpha_{2, \tau}) - u_1(\bar{\theta}, a_1, a_2). \]
This leads to a contradiction. □

The next lemma outlines an important implication of $r^t \notin \hat{R}_g^\sigma$.

**Lemma A.4.** If $r^t \notin \hat{R}_g^\sigma$ and $(a^t_*, r^t) \in \mathcal{H}^\sigma$, then there exists $\theta \in (\Theta_p \cup \Theta_n) \cap \text{supp}(\mu^*(r^t))$ such that $r^t \in \mathcal{R}^\sigma(\theta)$.

**Proof of Lemma A.4:** Suppose towards a contradiction that $r^t \notin \hat{R}_g^\sigma$ but no such $\theta$ exists. Let

$$
\theta_1 \equiv \max \left\{ (\Theta_p \cup \Theta_n) \cap \text{supp}(\mu^*(r^t)) \right\}.
$$

The set on the RHS is non-empty according to the definition of $\hat{R}_g^\sigma$ and $\mathcal{R}_g^\sigma$.

Let $(a^t_1, r^{t_1}) \geq (a^t_*, r^t)$ be the history at which type $\theta_1$ has a strict incentive not to play $\pi_1$ with $(a^t_1, r^{t_1}) \in \mathcal{H}^\sigma$. For any $(a^t_{k+1}, r^{t_{k+1}}) \succ (a^t_* , r^t)$ with $(a^t_{k+1}, r^{t_{k+1}}) \in \mathcal{H}^\sigma$, on one hand, we have $\mu^*(r^{t_{k+1}})(\theta_1) = 0$. On the other hand, the fact that $r^t \notin \hat{R}_g^\sigma$ implies that $\mu^*(r^{t_{k+1}})(\Theta_n \cup \Theta_p) > 0$. Let

$$
\theta_2 \equiv \max \left\{ (\Theta_p \cup \Theta_n) \cap \text{supp}(\mu^*(r^{t_{k+1}})) \right\},
$$

and let us examine type $\theta_1$ and $\theta_2$’s incentive constraints at $(a^t_*, r^t)$. According to Lemma [A.1], there exists $r^{t_2} \succ r^{t1}$ such that type $\theta_2$ has a strict incentive not to play $\overline{a}_1$ at $(a^t_2, r^{t_2}) \in \mathcal{H}^\sigma$. One can iterate the above process and construct $r^{t_3} \succ r^{t_4} \ldots$ Since

$$
\left| \text{supp}(\mu^*(r^{t_{k+1}})) \right| \leq \left| \text{supp}(\mu^*(r^{t_{k}})) \right| - 1,
$$

for any $k \in \mathbb{N}$, there exists $m \leq |\Theta_p \cup \Theta_n|$ such that $(a^t_{m}, r^{t_m}) \in \mathcal{H}^\sigma$, $r^{t_m} \succ r^t$ and $\mu^*(r^{t_m})(\Theta_n \cup \Theta_p) = 0$, which contradicts $r^t \notin \hat{R}_g^\sigma$. □

**A.5  Proof of Theorem 2:** $\Theta_n = \{ \emptyset \}$

**Proposition A.1.** If $\Theta_n = \{ \emptyset \}$ and $\mu \in \mathcal{B}_\kappa$, then for every $\theta \in \Theta$, we have:

$$
v_\theta(a^0_*, r^0) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) - 2M(1 - \delta T_0(\kappa)).
$$

Despite Proposition [A.1] is stated in terms of player 1’s guaranteed payoff at $h^0$, the conclusion applies to all $r^t$ and $\theta \in \Theta_g \cup \Theta_p$ as long as $\mu^*(r^t) \in \mathcal{B}_\kappa$ and $(a^t_*, r^t) \in \mathcal{H}^\sigma(\theta)$ but $(a^t_*, r^t) \notin \bigcup_{\theta_n \in \Theta_n} \mathcal{H}^\sigma(\theta_n)$. I show Lemma [A.5] and Lemma [A.6] which together imply Proposition [A.1].

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Lemma A.5. For every $\sigma \in \text{NE}(\delta, \mu)$, if $\mu^*(r^t) \in B_\kappa$ for all $r^t \in \widehat{R}_g^\sigma$, then for every $r^\infty \in R_\sigma$,

$$\left| \left\{ t \in \mathbb{N} \left| r^\infty \succ r^t \text{ and } \overline{a}_2 \text{ is not a strict best reply at } (a^*_s, r^t) \right. \right\} \right| \leq T_0(\kappa).$$

(A.14)

Proof of Lemma A.5: Pick any $r^\infty \in R_\sigma$, if $r^0 \notin \widehat{R}_g^\sigma$, then let $t^* = -1$. Otherwise, let

$$t^* \equiv \max \left\{ t \in \mathbb{N} \cup \{+\infty\} \left| r^t \in \widehat{R}_g^\sigma \text{ and } r^\infty \succ r^t \right. \right\}.$$

According to Lemma A.2, for every $t \leq t^*$, if $\overline{a}_2$ is not a strict best reply at $(a^*_s, r^t)$, then we have inequality (A.12).

Next, I show that $\mu^*(r^{t^*+1}) \in B_\kappa$. If $t^* = -1$, this is a direct implication of (A.1). If $t^* \geq 0$, then there exists $r^{t^*+1} \succ r^{t^*}$ such that $r^{t^*+1} \in \widehat{R}_g^\sigma$. Let $r^{t^*+1} \prec r^\infty$, we have $q^*(r^{t^*+1}) = q^*(r^{t^*+1})$. Moreover, since $\mu^*(r^t) \in B_\kappa$ for every $r^t \in \widehat{R}_g^\sigma$, we have $\mu^*(r^{t^*+1}) = \mu^*(r^{t^*+1}) \in B_\kappa$.

Since $r^{t^*+1} \notin \widehat{R}_g^\sigma$, Lemma A.4 implies the existence of

$$\theta \in (\Theta_p \cup \Theta_n) \bigcap \text{supp}(\mu^*(r^{t^*+1}))$$

such that $r^{t^*+1} \in \mathcal{P}^\sigma(\theta)$. Since $\theta_g \succ \theta$ for all $\theta_g \in \Theta_g$, Lemma A.1 implies that for every $\theta_g$ and $r^\infty \succ r^t \succ r^{t^*+1}$, we have $\sigma_{\theta_g}(a^*_s, r^t) = 1$, and therefore, $q^*(r^t)(\theta_g) = q^*(r^{t^*+1})(\theta_g)$. This implies that $\mu^*(r^t) \in B_\kappa$ for every $r^\infty \succ r^t \succ r^{t^*+1}$. If $\overline{a}_2$ is not a strict best reply at $(a^*_s, r^t)$ for any $t > t^*$, inequality (A.12) again applies.

To sum up, for every $t \in \mathbb{N}$, if $\overline{a}_2$ is not a strict best reply at $(a^*_s, r^t)$, then:

$$\sum_{\theta \in \Theta} \left( q^*(r^t)(\theta) - q^*(r^{t^*+1})(\theta) \right) \geq \rho_0(\kappa),$$

from which we obtain (A.14).

Next, I show that the condition required in Lemma A.5 holds in every equilibrium when $\delta$ is large enough.

Lemma A.6. For every $\sigma \in \text{NE}(\delta, \mu)$ with $\delta > \overline{\delta}$, $\mu^*(r^t) \in B_0$ for every $r^t \in \widehat{R}_g^\sigma$ with $\mu^*(r^t)(\Theta_n) = 0$.

Proof of Lemma A.6: For any given $\delta > \overline{\delta}$, according to (A.6), there exists $\kappa^* \in (0, 1)$ such that:

$$\left(1 - \delta^{T_0(\kappa^*)}\right) u_1(\theta_p, a_1, a_2) + \delta^{T_0(\kappa^*)} u_1(\theta_p, a_1, a_2) > \frac{1}{2} \left( u_1(\theta_p, a_1, a_2) + u_1(\theta_p, a_1, a_2) \right).$$

(A.15)

Suppose toward a contradiction that there exist $r^{t^1}$ and $r^{T_1}$ such that: $r^{T_1} \succ r^{t^1}$, $r^{T_1} \in R_\sigma^\delta$ and $\mu^*(r^{T_1}) \notin B_0$. Since $\mu^*(r^{T_1}) \in B_0$, let $t^{t^1}_1$ be the largest $t \in \mathbb{N}$ such that $\mu^*(r^t) \notin B_0$ for $r^{T_1} \succ r^{t^1} \succ r^{t_1}$. Then there exists $a^* = a_1$ and $r^{t^{t_1}+1} \succ r^{t^1}$ such that $\mu((a^*_{t^1}, a^*_{t^{t_1}+1}) \notin B_0$ and $((a^*_{t^1}, a^*_{t^{t_1}+1}) \in \mathcal{H}_\sigma$. This also implies the
existence of $\theta_p \in \Theta_p \cap \text{supp}(\mu((a^t_1, a_1), r^{t+1})).$

According to Lemma A.3, type $\theta_p$’s continuation payoff at $(a^t_1, r^t)$ by playing $a_1$ is at most

$$(1 - \delta)u_1(\theta_p, a_1, \bar{a}_2) + \delta u_1(\theta_p, a_1, \bar{a}_2). \quad \text{(A.16)}$$

His incentive constraint at history $(a^t_1, r^t)$ requires that his expected payoff from $\bar{a}_1$ is weakly lower than (A.16), i.e. there exists $r^{t+1} > r^t$ satisfying $(a^t_1, r^{t+1}) \in \mathcal{H}^\sigma$ and type $\theta_p$’s continuation payoff at $(a^t_1, r^{t+1})$ is no more than:

$$\frac{1}{2} \left( u_1(\theta_p, \bar{a}_1, \bar{a}_2) + u_1(\theta_p, a_1, \bar{a}_2) \right). \quad \text{(A.17)}$$

If $\mu^*(r^t) \in \mathcal{B}_{\kappa^*}$ for every $r^t \in \hat{\mathcal{R}}_g^G \cap \{r^t \succeq r^{t^1}\}$, then according to Lemma A.5, his continuation payoff at $(a^t_1, r^t)$ by playing $\bar{a}_1$ is at least:

$$(1 - \delta T_0(\kappa^*))u_1(\theta_p, \bar{a}_1, \bar{a}_2) + \delta T_0(\kappa^*)u_1(\theta_p, a_1, \bar{a}_2),$$

which is strictly larger than (A.17) by the definition of $\kappa^*$ in (A.15), leading to a contradiction.

Suppose on the other hand, there exists $r^{t_2} > r^{t_1}$ such that $r^{T_2} \in \hat{\mathcal{R}}_g^G$ while $\mu^*(r^{t_2}) \notin \mathcal{B}_{\kappa^*}$. There exists $r^{T_2} > r^{t_2}$ such that $r^{T_2} \in \mathcal{R}_g^G$ and $r^{T_2} > r^{t_2}$. Again, we can find $r^{t_2}$ such that $t_2$ be the largest $t \in \{t_2, t_2+1, ..., T_2\}$ such that $\mu^*(r^t) \notin \mathcal{B}_0$ for $r^{t_2} > r^t \succeq r^{t_2}$. Then there exists $a_1 \neq \bar{a}_1$ and $r^{t_2+1} > r^{t_2}$ such that $\mu((a^{t_2}_1, a_1), r^{t_2+1}) \notin \mathcal{B}_0$ and $(\{a^{t_2}_1, a_1\}, r^{t_2+1}) \in \mathcal{H}^\sigma$.

Iterate the above process and repeatedly apply the aforementioned argument, we know that for every $k \geq 1$, in order to satisfy player 1’s incentive constraint to play $a_1 \neq \bar{a}_1$ at $(a^t_k, r^t_k)$, we can find a triple $(a^t_{k+1}, r^t_{k+1}, r^{T_{k+1}})$. It implies that this process cannot stop after a finite number of rounds. Since $\mu^*(r^{t_k}) \notin \mathcal{B}_{\kappa^*}$ but $\mu^*(r^{t_{k+1}}) \in \mathcal{B}_0$ as well as $r^{t_{k+1}} \succeq r^{t^{k+1}}$, we have:

$$\sum_{\theta \in \Theta} q^* (r^{t_k}) (\theta) - q^* (r^{t_{k+1}}) (\theta) \geq \sum_{\theta \in \Theta} q^* (r^{t_k}) (\theta) - q^* (r^{t_{k+1}}) (\theta) \geq \rho_1 (\kappa^*) \quad \text{(A.18)}$$

for every $k \geq 2$. (A.18) and (A.5) together suggest that this iteration process cannot last for more than $T_1(\kappa^*)$ rounds, which is an integer independent of $\delta$, leading to a contradiction. \boxend

**Lemma A.7.** For every $\delta \geq \delta$ and $\sigma \in \text{NE}(\delta, \mu)$. If $r^t$ satisfies $(a^t_1, r^t) \in \mathcal{H}^\sigma$, $\mu^*(r^t)(\Theta_0) = 0$, $r^t \notin \hat{\mathcal{R}}_g^G$ and

$$\mu(\bar{a}_1) \mathcal{D}(\phi_{\bar{a}_1}, \bar{a}_1) + \sum_{\theta \in \Theta} q^* (r^t) (\theta) \mathcal{D}(\theta, \bar{a}_1) > 0, \quad \text{(A.19)}$$

then $\bar{a}_2$ is player 2’s strict best reply at every $(a^t_1, r^t) \succeq (a^t_1, r^t)$ with $(a^t_1, r^t) \in \mathcal{H}^\sigma$.  

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Proof of Lemma A.7: Since $\mu^*(r^t)(\Theta_n) = 0$ and $r^t \notin \overline{R}_g$, Lemma A.4 implies the existence of $\theta_p \in \Theta_p \cap \text{supp}(\mu^*(r^t))$ such that $r^t \in \overline{R}_g(\theta_p)$. According to Lemma A.1, $\sigma_p(a^*_s, r^s)(a_1) = 1$ for every $(a^*_s, r^s) \in \mathcal{H}^\sigma(\theta)$ with $r^s \succeq r^t$. From (A.19), we know that $\overline{a}_2$ is not a strict best reply only if there exists type $\theta_p \in \Theta_p$ who plays $a_1 \neq \overline{a}_1$ with positive probability. In particular, (A.19) implies the existence of $\kappa \in (0, 1)$ such that

$$
\kappa \mu(\overline{a}_1)D(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta} q^*(r^t)(\theta)D(\theta, \overline{a}_1) > 0.
$$

According to (A.12), we have:

$$
\sum_{\theta \in \Theta} \left( q^*(r^s)(\theta) - q^*(r^{s+1})(\theta) \right) \geq \rho_0(\kappa)
$$

whenever $\overline{a}_2$ is not a strict best reply at $(a^*_s, r^s) \succ (a^*_s, r^t)$. Therefore, there can be at most $\overline{T}_0(\pi)$ such periods. Hence, there exists $r^N$ with $(a^*_s, r^N) \in \mathcal{H}^\sigma$ such that:

1. $\overline{a}_2$ is not a strict best reply at $(a^*_s, r^N)$.

2. $\overline{a}_2$ is a strict best reply for all $(a^*_s, r^s) \succ (a^*_s, r^N)$ with $(a^*_s, r^s) \in \mathcal{H}^\sigma$.

Then there exists $\theta_p \in \Theta_p$ that plays $a_1 \neq \overline{a}_1$ in equilibrium at $(a^*_s, r^N)$, his continuation payoff by playing $\overline{a}_1$ in every subsequent period is at least $(1 - \delta)u_1(\theta_p, \overline{a}_1, \overline{a}_2) + \delta u_1(\theta_p, \overline{a}_1, \overline{a}_2)$ while his equilibrium continuation payoff from playing $a_1$ is at most $(1 - \delta)u_1(\theta_p, \overline{a}_1, \overline{a}_2) + \delta u_1(\theta_p, \overline{a}_1, \overline{a}_2)$ according to Lemma A.3. The latter is strictly less than the former when $\delta > \delta$. This leads to a contradiction. \(\square\)

### A.6 Proof of Theorem 2: Incorporating Types in $\Theta_n$

Next, we extend the proof in section A.5 by allowing for types in $\Theta_n$. Lemmas A.5 and A.6 imply the following result in this general environment:

**Proposition A.2.** For every $\delta > \delta$ and $\sigma \in \text{NE}(\delta, \mu)$, there exists no $\theta_p \in \Theta_p$, random histories $r^{t+1}$ and $r^t$ with $r^{t+1} \succ r^t$ and $a_1 \neq \overline{a}_1$ that simultaneously satisfy the following three requirements:

1. $r^{t+1} \in \overline{R}_g^\sigma$.

2. $((a^*_s, a_1), r^{t+1}) \in \mathcal{H}^\sigma(\theta_p)$.

3. $v_{\theta_p}((a^*_s, a_1), r^{t+1}) = u_1(\theta_p, a_1, a_2)$ for all $r^{t+1} \succ r^t$.

---

1There are two reasons for why one cannot directly apply the conclusion in Lemma A.2. First, a stronger conclusion is required for Lemma A.7. Second, $\pi$ can be arbitrarily close to 1, while $\kappa$ is uniformly bounded below 1 for any given $\mu$. 

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PROOF OF PROPOSITION A.2: Suppose towards a contradiction that such $\theta_p \in \Theta_p$, $r^t$, $r^{t+1}$ and $a_1$ exist. From requirement 3, we know that $r^t \in \mathcal{R}^\sigma(\theta_p)$. According to Lemma D.1 in the main text, $\theta_n < \theta_p$ for all $\theta_n \in \Theta_n$. The second part of Lemma A.1 then implies that $\mu^*(r^{t+1})(\Theta_n) = 0$ for all $r^{t+1} > r^t$ with $(a^t_{\ast}, r^t) \in \mathcal{H}^\sigma$.

If $\mu^*(r^{t+1}) \in \mathcal{B}_\kappa$, then requirement 2 and Proposition A.1 result in a contradiction when examining type $\theta_p$’s incentive at $(a^t_{\ast}, r^t)$ to play $a_1$ as opposed to $\pi_1$. If $\mu^*(r^{t+1}) \notin \mathcal{B}_\kappa$, since $\delta > \bar{\delta}$ and $r^{t+1} \in \mathcal{R}^\sigma_g$, we obtain a contradiction from Lemma A.6.

The rest of the proof considers a given $\sigma \in \text{NE}(\delta, \mu)$ when $\delta$ is large enough. First,

$$\mu(\pi_1)D(\phi_{\pi_1}, \pi_1) + \sum_{\theta \in \Theta} q^*(r^t)(\theta)D(\theta, \pi_1) \geq 0 \quad (A.20)$$

for all $t \geq 1$ and $r^t$ satisfying $(a^t_{\ast}, r^t) \in \mathcal{H}^\sigma$. This is because otherwise, according to Lemma A.3, there exists $\theta \in \text{supp}(\mu^*(r^t))$ such that $v_\theta(a^t_{\ast}, r^t) = u_1(\theta, \pi_1, \pi_2)$. But then, at $(a^{t-1}_{\ast}, r^{t-1})$ with $r^{t-1} < r^t$, he could obtain strictly higher payoff by playing $a_1$ instead of $\pi_1$, leading to a contradiction.

**Lemma A.8.** If $\mu$ is optimistic, then $v_\theta(a^t_{\ast}, r^t) \geq u_1(\theta, \pi_1, \pi_2) - 2M(K + 1)(1 - \delta)$ for every $\theta$ and $r^t \notin \mathcal{R}^\sigma_g$ satisfying the following two requirements:

1. $(a^t_{\ast}, r^t) \in \mathcal{H}^\sigma$.

2. Either $t = 0$ or $t \geq 1$ but there exists $\tilde{r}^t$ such that $r^t, \tilde{r}^t > r^{t-1}$, $(a^t_{\ast}, \tilde{r}^t) \in \mathcal{H}^\sigma$ and $\tilde{r}^t \in \mathcal{R}^\sigma_g$.

PROOF OF LEMMA A.8: If $\mu^*(r^t) \in \mathcal{B}_\kappa$ and $r^t \notin \mathcal{R}^\sigma_g$, then Lemmas A.1 and A.4 suggest that $\mu^*(r^s) \in \mathcal{B}_\kappa$ for all $r^s \succeq r^t$ and the conclusion is straightforward from Lemma A.2.

Therefore, for the rest of the proof, I consider the adverse circumstance in which $\mu^*(r^t) \notin \mathcal{B}_\kappa$. I consider two cases. First, when $\mu^*(r^t)(\Theta_n) > 0$, then according to (A.20),

$$\mu(\pi_1)D(\phi_{\pi_1}, \pi_1) + \sum_{\theta \in \Theta_p \cup \Theta_n} q^*(r^t)(\theta)D(\theta, \pi_1) > 0.$$ 

Since $r^t \notin \mathcal{R}^\sigma_g$, according to Lemma A.4, there exists $\theta \in \Theta_p \cup \Theta_n$ with $(a^t_{\ast}, r^t) \in \mathcal{H}^\sigma(\theta)$ such that $r^t \in \mathcal{R}^\sigma(\theta)$. According to Lemma A.1, for all $\theta_g \in \Theta_g$ with $(a^t_{\ast}, r^t) \in \mathcal{H}^\sigma(\theta_g)$ and every $(a^s_{\ast}, r^s) \in \mathcal{H}^\sigma(\theta)$ with $r^s \succeq r^t$, we have $\sigma_{\theta_g}(a^s_{\ast}, r^s)(\pi_1) = 1$. This implies that for every $h^s = (a^s, r^s) \succ (a^t_{\ast}, r^t)$ with $a^s \neq a^t_{\ast}$ and $h^s \in \mathcal{H}^\sigma$, we have $\mu(h^s)(\Theta_g) = 0$. Therefore,

$$v_\theta(h^s) = u_1(\theta, \pi_1, \pi_2) \text{ for every } \theta \in \Theta. \quad (A.21)$$


Let \( \tau : R^\sigma_* \to N \cup \{+\infty\} \) be such that for \( r^\tau < r^{\tau+1} < r^\infty \), we have: \( \mu^*(r^\tau)(\Theta_n) > 0 \) while \( \mu^*(r^{\tau+1})(\Theta_n) = 0 \).

Let

\[
\bar{\theta}_n \equiv \max \left\{ \text{supp}(\mu^*(r^t)) \cap \Theta_n \right\}.
\]

The second part of Lemma \[A.1\] and \[A.21\] together imply that \( \mu^*(r^\tau)(\bar{\theta}_n) > 0 \). Let us examine type \( \bar{\theta}_n \)'s incentive at \((a^t_1, r^t)\) to play his equilibrium strategy as opposed to play \( a_1 \) in every period. This requires that:

\[
\mathbb{E} \left[ \sum_{s=t}^{\tau-1} (1-\delta)\delta^{s-t}\sum_{s=t}^t u_1(\bar{\theta}_n, \bar{a}_1, \alpha_2, s) + (\delta^{s-t} - \delta^{s+1-t})u_1(\bar{\theta}_n, a_1, \alpha_2, s) + \delta^{s+1-t}u_1(\bar{\theta}_n, a_1, a_2) \right] \geq u_1(\bar{\theta}_n, a_1, a_2).
\]

where \( \mathbb{E}[\cdot] \) is taken over \( \mathcal{P}^\sigma \) and \( \alpha_2, s \in \Delta(A_2) \) is player 2's action in period \( s \).

Using the fact that \( u_1(\bar{\theta}_n, a_1, a_2) \geq u_1(\bar{\theta}_n, \bar{a}_1, \bar{a}_2) \), the above inequality implies that:

\[
\mathbb{E} \left[ \sum_{s=t}^{\tau} (1-\delta)\delta^{s-t}\left( u_1(\theta_n, \bar{a}_1, \alpha_2, s) - u_1(\theta_n, \alpha_1, \alpha_2, s) \right) \right] \leq 0.
\]

According to the definitions of \( K \) and \( M \), we know that for all \( \theta \),

\[
\mathbb{E} \left[ \sum_{s=t}^{\tau} (1-\delta)\delta^{s-t}\left( u_1(\theta_n, \bar{a}_1, \alpha_2, s) - u_1(\theta_n, \alpha_1, \alpha_2, s) \right) \right] \leq 2M(K+1)(1-\delta). \tag{A.22}
\]

This bounds the loss (relative to the payoff from the highest action profile) from above in periods before all types in \( \Theta_n \) separate from the commitment type. For every \( r^\infty \in R^\sigma_* \), since \( r^t \notin \hat{R}^\sigma_\theta \), we have:

\[
\mu(\bar{a}_1)D(\phi_1, \bar{a}_1) + \sum_{\theta \in \Theta} \pi^*(r^{\tau(r^\infty)+1})(\theta)D(\theta, \bar{a}_1) \geq \mu(\bar{a}_1)D(\phi_1, \bar{a}_1) + \sum_{\theta \in \Theta} \pi^*(r^t)(\theta)D(\theta, \bar{a}_1) > \mu(\bar{a}_1)D(\phi_1, \bar{a}_1) + \sum_{\theta \in \Theta} \pi^*(r^t)(\theta)D(\theta, \bar{a}_1) \geq 0.
\]

According to Lemma \[A.7\], we know that \( \nu_\theta(\pi^*(r^{\tau(r^\infty)+1}), r^{\tau(r^\infty)+1}) = u_1(\theta, \bar{a}_1, \bar{a}_2) \) for all \( \theta \in \Theta_G \cup \Theta_p \) and \( r^{\tau(r^\infty)+1} \in R^\sigma_* \).

This together with \[A.22\] gives the conclusion.

Second, when \( \mu^*(r^t)(\Theta_n) = 0 \). If \( t = 0 \), the conclusion directly follows from Proposition \[A.1\]. If \( t \geq 1 \) and there exists \( r^t \) such that \( r^t, r^t > r^{t-1}, (a^t_1, \hat{r}^t) \in \mathcal{H}^\sigma \) and \( \hat{r}^t \in \hat{R}^\sigma_\theta \). Then, since

\[
\mu^*(r^t) = \mu^*(\hat{r}^t),
\]

we have \( \mu^*(\hat{r}^t)(\Theta_n) = 0 \). Since \( \hat{r}^t \in \hat{R}^\sigma_\theta \), according to Lemma \[A.6\], \( \mu^*(\hat{r}^t) = \mu^*(r^t) \in \mathcal{B}_\mu \). The conclusion then follows from Lemma \[A.7\]. 

\[ \square \]
The next Lemma puts an upper bound on type \( \theta_n \in \Theta_n \)’s continuation payoff at \((a^*_s, r^t)\) with \( r^t \notin \hat{R}^\sigma_g \).

**Lemma A.9.** For every \( \theta_n \in \Theta_n \) such that \( \bar{a}_2 \notin BR_2(\theta_n, a_1|u_2) \) and \( r^t \notin \hat{R}^\sigma_g \) with \((a^*_s, r^t) \in H^\sigma_{\theta_n} \) and \( \mu^*(r^t) \notin B_\kappa \), we have:

\[
v_{\theta_n}(a^*_s, r^t) \leq u_1(\theta_n, a_1, a_2) + 2(1 - \delta)M. \quad (A.23)
\]

This is implied by Lemma [A.8](Part I). Let

\[
A(\delta) \equiv 2M(K + 1)(1 - \delta), \quad B(\delta) \equiv 2M(1 - \delta \bar{T}_n(\kappa))
\]

and

\[
C(\delta) \equiv 2MK|\Theta_n|(1 - \delta).
\]

Notice that when \( \delta \to 1 \), all three functions converge to 0. The next lemma establishes a uniform upper bound on player 1’s payoff when \( r^t \in \hat{R}^\sigma_g \).

**Lemma A.10.** When \( \delta > \delta \) and \( \sigma \in NE(\delta, \mu) \), for every \( r^t \in \hat{R}^\sigma_g \),

\[
v_{\theta}(a^*_s, r^t) \geq u_1(\theta, \bar{a}_1, \bar{a}_2) - \left( A(\delta) + B(\delta) \right) - 2\bar{T}_1(\kappa)\left( A(\delta) + B(\delta) + C(\delta) \right). \quad (A.24)
\]

for all \( \theta \) such that \((a^*_s, r^t) \in H^\sigma(\theta)\).

**Proof of Lemma A.10:** The non-trivial part of the proof deals with situations where \( \mu^*(r^t) \notin B_\kappa \). Since \( r^t \in \hat{R}^\sigma_g \), Lemma A.6 implies that \( \mu^*(r^t)(\Theta_n) \neq 0 \). Without loss of generality, assume \( \Theta_n \subset \text{supp} \left( \mu^*(r^t) \right) \). Let me introduce \(|\Theta_n| + 1 \) integer valued random variables on the space \( \mathcal{R}^\sigma_g \).

- Let \( \tau : \mathcal{R}^\sigma_g \to \mathbb{N} \cup \{ +\infty \} \) be the first period \( s \in \mathbb{N} \) along random path \( r^\infty \) such that either one of the following two conditions are met.
  1. \( \mu^*(r^{s+1}) \in B_{\kappa/2} \) for \( r^{s+1} \succ r^s \) with \((a^*_s, r^{s+1}) \in H^\sigma \).
  2. \( r^s \notin \hat{R}^\sigma_g \).

In the first case, there exists \( a_1 \neq \bar{a}_1 \) and \( r^{\tau+1} \succ r^{\tau} \) such that

\[
-(a^*_s, a_1, r^{\tau+1}) \in H^\sigma(\tilde{\theta}) \text{ for some } \tilde{\theta} \in \Theta_p \cup \Theta_n, \text{ and moreover, } \mu((a^*_s, a_1, r^{\tau+1}) \notin B_0.
\]

Lemma A.3 implies the existence of \( \theta \in \Theta_p \cup \Theta_n \) with \((a^*_s, a_1, r^{\tau+1}) \in H^\sigma(\theta) \) such that \( v_{\theta}(a^*_s, a_1, r^{\tau+1}) = u_1(\theta, a_1, a_2) \).
Suppose toward a contradiction that $\theta \in \Theta_\mu$, then Lemma A.1 implies that $\mu^*(r^{\tau+1})(\Theta_n) = 0$. Since $\mu^*(r^{\tau+1}) \in B_{\kappa/2}$, Proposition A.1 implies that type $\theta$’s continuation payoff by playing $\bar{a}_1$ in all subsequent periods is at least:

$$(1 - \delta^{T_0(\kappa/2)})u_1(\theta, \bar{a}_1) + \delta^{T_0(\kappa/2)}u_1(\theta, \bar{a}_2),$$

which is strictly larger than his payoff from playing $a_1$, which is at most $2M(1 - \delta) + u_1(\theta, a_1, a_2)$. This leads to a contradiction. Hence, there exists $\theta_n \in \Theta_n$ such that $v_{\theta_n}(\bar{a}_1, a_1, a_2) = u_1(\theta_n, a_1, a_2)$, which implies that $v_{\theta_n}(\bar{a}_1, r^\tau) \leq u_1(\theta_n, a_1, a_2) + 2(1 - \delta)M$. In the second case, Lemma A.9 implies that $v_{\theta_n}(\bar{a}_1, r^\tau) \leq u_1(\theta_n, a_1, a_2) + 2(1 - \delta)M$ for all $\theta_n \in \Theta_n$ with $r^\tau \in H^\sigma(\theta_n)$.

- For every $\theta_n \in \Theta_n$, let $\tau_{\theta_n}: \mathcal{R}_s \rightarrow \mathbb{N} \cup \{+\infty\}$ be the first period $s$ along random path $r^\infty$ such that either one of the following three conditions is met.

1. $\mu^*(r^{s+1}) \in B_{\kappa/2}$ for $r^{s+1} \succ r^s$ with $(a^{s+1}, r^{s+1}) \in \mathcal{H}^\sigma$.
2. $r^s \not\in \mathcal{R}_s$.
3. $\mu^*(r^{s+1})(\theta_n) = 0$ for $r^{s+1} \succ r^s$ with $(a^{s+1}, r^{s+1}) \in \mathcal{H}^\sigma$.

By definition, $\tau \geq \tau_{\theta_n}$, so $\tau \geq \max_{\theta_n \in \Theta_n}\{\tau_{\theta_n}\}$. Next, I show that

$$\tau = \max_{\theta_n \in \Theta_n}\{\tau_{\theta_n}\}.$$  \hfill (A.25)

Suppose toward a contradiction that $\tau > \max_{\theta_n \in \Theta_n}\{\tau_{\theta_n}\}$ for some $r^\infty \in \mathcal{R}_s^\sigma$. Then there exists $(a^s, r^s) \succeq (a^t, r^t)$ such that $r^s \in \mathcal{R}_s^\sigma$, $\mu^*(r^s) \not\in B_{\kappa}$ and $\mu^*(r^s)(\theta_n) = 0$. This contradicts Lemma A.6 when $\delta > \delta_0$.

Next, I show by induction on the number of states in $\Theta_n$ that:

$$\mathbb{E}\left[\sum_{s=t}^{\tau} (1 - \delta)\delta^{s-t}\left(u_1(\theta, \bar{a}_1, \bar{a}_2) - u_1(\theta, \bar{a}_1, \hat{a}_{2,s})\right)\right] \leq 2MK|\Theta_n|(1 - \delta),$$  \hfill (A.26)

for all $\theta \in \Theta$ and

$$v_{\theta_n}(a^\tau_{\theta_n}, r^\tau_{\theta_n}) \leq u_1(\theta_n, a_1, a_2) + 2(1 - \delta)M,$$  \hfill (A.27)

for

$$\hat{\theta} \equiv \min\left\{\Theta_n \bigcap \text{supp}\left(\mu^*(r^\tau_{\theta_n} + 1)\right)\right\}$$

with $\theta_n, \hat{\theta}_n \in \Theta_n$, where $\mathbb{E}[\cdot]$ is taken over $\mathcal{P}^\sigma$ and $\hat{a}_{2,s} \in \Delta(A_2)$ is player 2’s (mixed) action at $(a^s, r^s)$.

When $|\Theta_n| = 1$, let $\theta_n$ be its unique element. Consider player 1’s pure strategy of playing $\bar{a}_1$ until $r^\tau$ and then
play \( a_1 \) forever. This is one of type \( \theta_n \)'s best responses according to (A.25), which results in payoff at most:

\[
E \left[ \sum_{s=t}^{\tau-1} (1 - \delta)\delta^{s-t}u_1(\theta_n, a_1, a_2) + \delta^{\tau-t}(u_1(\theta_n, a_1, a_2) + 2(1 - \delta)M) \right].
\]

The above expression cannot be smaller than \( u_1(\theta_n, a_1, a_2) \), which is the payoff he can guarantee by playing \( a_2 \) in every period. Since \( u_1(\theta_n, a_1, a_2) \geq u_1(\theta_n, \bar{a}_1, \bar{a}_2) \), and from the definition of \( K \), we get for all \( \theta \),

\[
E \left[ \sum_{s=t}^{\tau-1} (1 - \delta)\delta^{s-t}\left(u_1(\theta, a_1, a_2) - u_1(\theta, \bar{a}_1, \bar{a}_2)\right) \right] \leq 2MK(1 - \delta).
\]

We can then obtain (A.27) for free since \( \tau = \tau_{\theta_n} \) and type \( \theta_n \)'s continuation value at \( (a^*_s, r^\tau) \) is at most \( u_1(\theta_n, a_1, a_2) + 2(1 - \delta)M \) by Lemma A.3.

Suppose the conclusion holds for all \( |\Theta_n| \leq k - 1 \), consider when \( |\Theta_n| = k \) and let \( \theta_n \equiv \min \Theta_n \). If \( (a^*_s, r^\tau) \notin \mathcal{H}^\sigma(\theta_n) \), then there exists \( (a^*_{s_n}, r^\tau_{s_n}) < (a^*_s, r^\tau) \) with \( (a^*_{s_n}, r^\tau_{s_n}) \in \mathcal{H}^\sigma(\theta_n) \) at which type \( \theta_n \) plays \( \bar{a}_1 \) with probability 0. I put an upper bound on type \( \theta_n \)'s continuation payoff at \( (a^*_{s_n}, r^\tau_{s_n}) \) by examining type \( \tilde{\theta}_n \in \Theta_n \setminus \{\theta_n\} \)'s incentive to play \( \bar{a}_1 \) at \( (a^*_{s_n}, r^\tau_{s_n}) \), where

\[
\tilde{\theta} \equiv \min \left\{ \Theta_n \cap \text{supp}(\mu^*(r^\tau_{s_n} + 1)) \right\}
\]

This requires that:

\[
E \left[ \sum_{s=0}^{\infty} (1 - \delta)\delta^{s}u_1(\tilde{\theta}_n, a_1, a_2) \right] \leq u_1(\tilde{\theta}_n, a_1, a_2) + 2(1 - \delta)M,
\]

by induction hypothesis

where \( \{(\alpha_{1,s}, \alpha_{2,s})\}_{s \in \mathbb{N}} \) is the equilibrium continuation play following \( (a^*_{s_n}, r^\tau_{s_n}) \). By definition, \( \tilde{\theta}_n > \theta_n \), the supermodularity condition implies that:

\[
u_{\tilde{\theta}_n}(a^*_{s_n}, r^\tau_{s_n}) = E \left[ \sum_{s=0}^{\infty} (1 - \delta)\delta^{s}u_1(\tilde{\theta}_n, \alpha_{1,s}, \alpha_{2,s}) \right] \leq E \left[ \sum_{s=0}^{\infty} (1 - \delta)\delta^{s}(u_1(\tilde{\theta}_n, \alpha_{1,s}, \alpha_{2,s}) + u_1(\theta_n, a_1, a_2) - u_1(\tilde{\theta}_n, a_1, a_2)) \right] \leq u_1(\theta_n, a_1, a_2) + 2(1 - \delta)M.
\]
Since it is optimal for type $\theta_n$ to play $a_1$ until $r_\tau\theta_n$ and then play $a_1$ forever, doing so must give him a higher payoff than playing $a_1$ forever starting from $r^t$, which gives:

$$
\mathbb{E} \left[ \sum_{s=t}^{\tau \theta_n - 1} (1 - \delta) \delta^{s-t} u_1(\theta_n, a_1, \alpha_2, s) + \delta^{\tau \theta_n} \left( u_1(\theta_n, a_1, a_2) + 2(1 - \delta)M \right) \right] \geq u_1(\theta_n, a_1, a_2).
$$

This implies that:

$$
\mathbb{E} \left[ \sum_{s=t}^{\tau \theta_n - 1} (1 - \delta) \delta^{s-t} \left( u_1(\theta_n, a_1, a_2) - u_1(\theta_n, a_1, \alpha_2, s) \right) \right] \leq 2M(1 - \delta),
$$

which also implies that for every $\theta \in \Theta$,

$$
\mathbb{E} \left[ \sum_{s=t}^{\tau \theta_n - 1} (1 - \delta) \delta^{s-t} \left( u_1(\theta, a_1, a_2) - u_1(\theta, a_1, \alpha_2, s) \right) \right] \leq 2MK(1 - \delta). \tag{A.28}
$$

When $\tau > \tau \theta_n$, the induction hypothesis implies that:

$$
\mathbb{E} \left[ \sum_{s=t}^{\tau - \tau \theta_n - 1} (1 - \delta) \delta^{s-t} \left( u_1(\theta, a_1, a_2) - u_1(\theta, a_1, \alpha_2, s) \right) \right] \leq 2MK(k - 1)(1 - \delta). \tag{A.29}
$$

According to (A.28) and (A.29),

$$
\mathbb{E} \left[ \sum_{s=t}^{\tau} (1 - \delta) \delta^{s-t} \left( u_1(\theta, a_1, a_2) - u_1(\theta, a_1, \alpha_2, s) \right) \right] \leq 2MKk(1 - \delta),
$$

which establishes (A.26) when $|\Theta_n| = k$. (A.27) can be obtained directly from the induction hypothesis.

I examine player 1’s continuation payoff at on-path histories after $(a_{\tau+1}^\tau, r_{\tau+1}^\tau) \in \mathcal{H}^\sigma$ in three cases.

1. If $r_{\tau+1}^\tau \notin \hat{\mathcal{R}}_g^\sigma$, by Lemma A.8, then for every $\theta$,

$$
v_\theta(a_{\tau+1}^\tau, r_{\tau+1}^\tau) \geq u_1(\theta, \alpha_1, a_2) - A(\delta).
$$

2. If $r_{\tau+1}^\tau \in \hat{\mathcal{R}}_g^\sigma$ and $\mu^*(r^s) \in \mathcal{B}_n$ for all $r^s$ satisfying $r^s \succ r_{\tau+1}^\tau$ and $r^s \in \hat{\mathcal{R}}_g^\sigma$, then for every $\theta$,

$$
v_\theta(a_{\tau+1}^\tau, r_{\tau+1}^\tau) \geq u_1(\theta, \alpha_1, a_2) - B(\delta).
$$

3. If there exists $r^s$ such that $\mu^*(r^s) \notin \mathcal{B}_n$ with $r^s \succ r_{\tau+1}^\tau$ and $r^s \in \hat{\mathcal{R}}_g^\sigma$, then repeat the procedure in the
beginning of this proof by defining random variables

- \( \tau' : \mathcal{R}_\sigma^* \to \{ n \in \mathbb{N} \cup \{+\infty\} | n \geq s \} \)

- \( \tau'_{\theta_n} : \mathcal{R}_\sigma^* \to \{ n \in \mathbb{N} \cup \{+\infty\} | n \geq s \} \)

similarly as we have defined \( \tau \) and \( \tau_{\theta_n} \), and then examine continuation payoffs at \( r^{\tau'+1} \).

Since \( \mu^*(r^{\tau'+1}) \in \mathcal{B}_{\kappa/2} \) but \( \mu^*(r^s) \notin \mathcal{B}_{\kappa} \), then

\[
\sum_{\theta \in \Theta} \left( q^*(r^{\tau'+1})(\theta) - q^*(r^s)(\theta) \right) \geq \frac{\rho_1(\kappa)}{2}.
\]

(A.30)

Therefore, such iterations can last for at most \( 2T_1(\kappa) \) rounds.

Next, I establish the payoff lower bound in case 3. For future reference, I introduce the notion of trees. Let

\[
\mathcal{R}_g^\sigma = \left\{ r^t \mid \mu^*(r^t) \notin \mathcal{B}_\kappa \text{ and } r^t \in \mathcal{R}_g^\sigma \right\}
\]

For \( k \in \mathbb{N} \), I define set \( \mathcal{R}(k) \subset \mathcal{R} \) recursively as follows. Let

\[
\mathcal{R}(1) = \left\{ r^t \mid r^t \in \mathcal{R}_g^\sigma \text{ and there exists no } r^s \prec r^t \text{ such that } r^s \in \mathcal{R}_g^\sigma \right\}.
\]

For every \( r^t \in \mathcal{R}(1) \), let \( \tau[r^t] : \mathcal{R}_\sigma^* \to \mathbb{N} \cup \{+\infty\} \) be the first period \( s > t \) (starting from \( r^t \)) such that either one of the following two conditions is met:

1. \( \mu^*(r^{s+1}) \in \mathcal{B}_{\kappa/2} \) for \( r^{s+1} \succ r^s \) with \( (a^{s+1}_t, r^{s+1}) \in \mathcal{H}^\sigma \),

2. \( r^s \notin \mathcal{R}_g^\sigma \),

then

\[
\mathcal{T}(r^t) = \left\{ r^s \mid r^s \succ r^{[r^t]} \succ r^t \right\}
\]

is called a tree with root \( r^t \). For any \( k \geq 2 \), let

\[
\mathcal{R}(k) = \left\{ r^t \mid r^t \in \mathcal{R}_g^\sigma, \ r^t \succ r^{[r^t]} \text{ for some } r^s \in \mathcal{R}(k-1) \text{ and there exists no } r^s \prec r^t \text{ that satisfy these two conditions} \right\}.
\]

Let \( T \) be the largest integer such that \( \mathcal{R}(T) \neq \emptyset \). According to (A.30), we know that \( T \leq 2T_1(\kappa) \). Similarly, one can define trees with roots in \( \mathcal{R}(k) \) for every \( k \leq T \).
In what follows, I show that for every \( \theta \) and every \( r^t \in \mathcal{R}^\theta(k) \),

\[
v_\theta(a^*_s, r^t) \geq u_1(\theta, \alpha_1, \alpha_2) - (T + 1 - k)(A(\delta) + B(\delta) + C(\delta)). \tag{A.31}
\]

The proof is done by inducting on \( k \) from \( T \) to 0. When \( k = T \), player 1’s continuation value at \((a^*_s, r^T+1, r^T+1)\) is at least \( u_1(\theta, \alpha_1, \alpha_2) - A(\delta) - B(\delta) \) according to Lemma [A.2] and Lemma [A.8]. His continuation value at \( r^t \) is at least:

\[
u_1(\theta, \alpha_1, \alpha_2) - A(\delta) - B(\delta) - C(\delta).
\]

Suppose the conclusion holds for all \( k \geq n + 1 \), then when \( k = n \), type \( \theta \)'s continuation payoff at \((a^*_s, r^t)\) is at least:

\[
\mathbb{E}\left[ (1 - \delta^{r^t})^{-1} u_1(\theta, \alpha_1, \alpha_2) + \delta^{r^t} V_\theta(a^*_s, r^t+1, r^t+1) \right] - C(\delta)
\]

Pick any \((a^*_s, r^t+1, r^t+1)\), consider the set of random paths \( r^\infty \) that it is consistent with. Denote this set by \( \mathcal{R}^\infty(a^*_s, r^t+1, r^t+1) \).

Partition it into the following two subsets:

1. \( \mathcal{R}^\infty(a^*_s, r^t+1, r^t+1) \) consists of \( r^\infty \) such that for all \( s \geq r^t \) and \( r^s \prec r^\infty \), we have \( r^s \notin \mathcal{R}^\theta_0 \).

2. \( \mathcal{R}^\infty(a^*_s, r^t+1, r^t+1) \) consists of \( r^\infty \) such that there exists \( s \geq r^t \) and \( r^s \prec r^\infty \) at which \( r^s \in \mathcal{R}^\theta(n+1) \).

Conditional on \( r^\infty \in \mathcal{R}^\infty(a^*_s, r^t+1, r^t+1) \), we have:

\[
v_\theta(a^*_s, r^t+1, r^t+1) \geq u_1(\theta, \alpha_1, \alpha_2) - A(\delta) - B(\delta).
\]

Conditional on \( r^\infty \in \mathcal{R}^\infty(a^*_s, r^t+1, r^t+1) \), type \( \theta \)'s continuation payoff is no less than

\[
v_\theta(a^*_s, r^s) \geq u_1(\theta, \alpha_1, \alpha_2) - (T - n)(A(\delta) + B(\delta) + C(\delta))
\]

after reaching \( r^s \in \mathcal{R}^\theta(n) \) according to the induction hypothesis. Moreover, since his payoff lost is at most \( A(\delta) + B(\delta) \) before reaching \( r^s \) (according to Lemmas [A.2] and [A.8]), we have:

\[
v_\theta(a^*_s, r^t+1, r^t+1) \geq u_1(\theta, \alpha_1, \alpha_2) - (T + 1 - n)(A(\delta) + B(\delta) + C(\delta)).
\]
which obtains (A.31). (A.24) is implied by (A.31) since player 1’s loss is bounded above by $A(\delta) + B(\delta)$ from $r^0$ to every $r^t \in R^\sigma(0)$.

Theorem 2’ is implied by Lemmas A.8, A.9 and A.10.

B Proof of Theorem 3

B.1 Proof of Theorem 3: Equilibrium Payoff

First, I show that for every $\theta \in \Theta$, strategic type $\theta$ secures payoff $w_\theta(\phi)$ in all equilibria. Let $\kappa \in (0, 1)$. Given $\delta > \delta$ and $\sigma \in \text{NE}(\delta, \mu)$, let us examine $r^1$ such that $(a^1_x, r^1) \in H^\sigma$. If $\mu^*(r^1) \in B_\kappa$, then for every $r^1$ with $(a^1_x, r^1) \in H^\sigma$, we have $\mu^*(r^1) \in B_\kappa$. The conclusion is then implied by Theorem 2. If $\mu^*(r^1) \notin B_\kappa$, then we still have:

$$\mu(\bar{a}_1)D(\phi_{\bar{a}_1}, \bar{a}_1) + \sum_{\theta \in \Theta} q^*(r^1)(\theta)D(\theta, \bar{a}_1) \geq 0.$$  

This is because otherwise, there exists $\theta \in \text{supp}\mu^*(r^1)$ such that $v_\theta(a^1_x, r^1) = u_1(\theta, a^1_x, a^1_2)$ according to Lemma A.3, contradicting type $\theta$’s incentive to play $a^1_x$ in period 0. I consider two cases separately.

1. If $\Theta_n \cap \text{supp}\mu^*(r^1) = \{\emptyset\}$, then Lemma A.6 implies that $r^1 \notin \hat{R}^g_\theta$. According to Lemma A.4, there exists $\theta \in (\Theta_p \cup \Theta_n) \cap \text{supp}\mu^*(r^1)$ such that $r^1 \in \hat{R}^g$. According to Lemma A.1, for every $\theta_g \in \Theta_g$, type $\theta_g$ will play $\bar{a}_1$ at every $(a^t_x, r^t) \succ (a^1_x, r^1)$ with $(a^t_x, r^t) \in H^\sigma(\theta_g)$.

According to the definition of $w_\theta(\phi)$, and given that the two dimensions of player 1’s private information are independently distributed, we know that type $\theta$ can secure payoff $w_\theta(\phi)$ at $r^1$ for every $\theta \in \Theta$. Since $\mu^*(r^1) \notin B_\kappa, \mu^*(r^1) \notin B_\kappa$ for every $r^1$ with $(a^1_x, r^1) \in H^\sigma$. The argument in the previous paragraph applies symmetrically, which implies that type $\theta$’s discounted average payoff at $h^0$ is at least

$$(1 - \delta)u_1(\theta, \bar{a}_1, a^1_2) + \delta w_\theta(\phi).$$

2. If $\Theta_n \cap \text{supp}\mu^*(r^1) \neq \{\emptyset\}$, then according to Lemma A.10, type $\theta$ can guarantee payoff at least the RHS of (A.24), which leads to the same conclusion.

Next, I uniquely pin down every type’s equilibrium payoff when the total probability of commitment types is arbitrarily small. The key is to show that for every Nash equilibrium $\sigma$, we have:

$$\mu(\bar{a}_1)D(\phi_{\bar{a}_1}, \bar{a}_1) + \sum_{\theta \in \Theta} q^*(r^1)(\theta)D(\theta, \bar{a}_1) = 0,$$
for every $r^1$ such that $(a^1_s, r^1) \in \mathcal{H}^\sigma$. This is because when the total probability of commitment types is small enough and $\phi$ is pessimistic,

$$\mu(\overline{a}_1) D(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta^*} q_0(\theta) D(\theta, \overline{a}_1) < 0.$$  

Suppose towards a contradiction that

$$\mu(\overline{a}_1) D(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta} q^*(r^1)(\theta) D(\theta, \overline{a}_1) > 0.$$ 

On one hand, Theorem 2 suggests that every type $\theta \in \Theta^*$ receives continuation payoff at least $u_1(\theta, \overline{a}_1, a_2)$ after playing $\overline{a}_1$ in period 0. On the other hand, it also implies that there exists type $\theta \in \Theta^*$ that plays actions other than $\overline{a}_1$ with positive probability, and according to Lemma C.3, this type's continuation payoff in period 1 is $u_1(\theta, a_1, a_2)$. As a result, this type has a strict incentive to deviate by playing $\overline{a}_1$ in period 1, which leads to a contradiction. Similarly, one can show by induction that for every $t \geq 1$ and $(a^t_s, r^t) \in \mathcal{H}^\sigma$,

$$\mu(\overline{a}_1) D(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta^*} q^*(r^t)(\theta) D(\theta, \overline{a}_1) = 0.$$ 

The rest of proof follows the same steps as Appendix D in the main text.

### B.2 Proof of Theorem 3: On-Path Behavior

#### Step 1: 

Let

$$X(h^t) \equiv \mu(\overline{a}_1) D(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta_p \cup \Theta^p} q(h^t)(\theta) D(\theta, \overline{a}_1).$$  \hspace{1cm} \text{(B.2)}$$

and

$$Y(h^t) \equiv \mu(\overline{A}^*_1) D(\overline{a}, \overline{a}_1) + \sum_{\theta \in \Theta_p \cup \Theta^p} q(h^t)(\theta) D(\theta, \overline{a}_1).$$  \hspace{1cm} \text{(B.3)}$$

When belief is pessimistic, $X(h^0) < 0$ and $Y(h^0) < 0$. Moreover, at every $h^t \in \mathcal{H}^\sigma$ with $Y(h^t) < 0$, player 2 has a strict incentive to play $a_2$. According to Lemma A.3, there exists $\theta_p \in \Theta_p$ with $h^t \in \mathcal{H}(\theta_p)$ such that type $\theta_p$’s continuation value at $h^t$ is $u_1(\theta_p, a_1, a_2)$, which further implies that playing $a_1$ in every period is one of his best replies. According to Lemma A.1 and using the implication that $Y(h^0) < 0$, every $\theta_n \in \Theta_n$ plays $a_1$ with probability 1 at every $h^t \in \mathcal{H}(\theta_n)$. 

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Step 2: Let us examine the equilibrium behaviors of the types in $\Theta_p \cup \Theta_g$. I claim that for every $h^1 = (\overline{a}_1, r^1) \in \mathcal{H}^{\sigma}$, we have:

$$\sum_{\theta \in \Theta_g \cup \Theta_p} q(h^1)(\theta)D(\theta, \overline{a}_1) < 0.$$ (B.4)

Suppose towards a contradiction that $\sum_{\theta \in \Theta_g \cup \Theta_p} q(h^1)(\theta)D(\theta, \overline{a}_1) \geq 0$, then $X(h^1) \geq \mu(\overline{a}_1)D(\phi_{\overline{a}_1}, \overline{a}_1)$. According to Proposition [A.1], there exists $K \in \mathbb{R}_+$ independent of $\delta$ such that type $\theta$'s continuation payoff is at least $u_1(\theta, \overline{a}_1, \overline{a}_2) - (1 - \delta)K$ at every $h^1 \in \mathcal{H}^{\sigma}$. When $\delta$ is large enough, this contradicts the conclusion in the previous step that there exists $\theta_g \in \Theta_p$ such that type $\theta_g$’s continuation value at $h^0$ is $u_1(\theta_g, a_1, a_2)$, as he can profitably deviate by playing $\overline{a}_1$ in period 0.

Step 3: According to [B.4], we have $\mu^*(r^1) \notin B_0$. Step 1 also implies that $\mu^*(r^1)(\Theta_n) = 0$. According to Lemma [A.6], we have $r^1 \notin \hat{R}(a_g)$. According to Lemma [A.1], type $\theta_g$ plays $\overline{a}_1$ at every $h^1 \in \mathcal{H}(\theta_g)$ with $t \geq 1$ for every $\theta_g \in \Theta_g$. Next, I show that $r^0 \notin \hat{R}(a_g)$. Suppose towards a contradiction that $r^0 \in \hat{R}(a_g)$, then there exists $h^T = (a^T, r^T) \in \mathcal{H}^{\sigma}$ such that $\mu(h^T)(\Theta_p \cup \Theta_n) = 0$. If $T \geq 2$, it contradicts our previous conclusion that $r^1 \notin \hat{R}(a_g)$. If $T = 1$, then it contradicts (B.4). Therefore, we have $r^0 \notin \hat{R}(a_g)$. This implies that type $\theta_g$ plays $\overline{a}_1$ at every $h^1 \in \mathcal{H}(\theta_g)$ with $t \geq 0$ for every $\theta_g \in \Theta_g$.

Step 4: In the last step, I pin down the strategies of type $\theta_p$ by showing that $X(h^t) = 0$ for every $h^t = (a^t, r^t) \in \mathcal{H}^{\sigma}$ with $t \geq 1$. First, I show that $X(h^1) = 0$. The argument at other histories follows similarly. Suppose first that $X(h^1) > 0$, then according to Lemma [A.7], type $\theta_p$’s continuation payoff at $(a^1, r^t+1)$ is $u_1(\theta_p, \overline{a}_1, \overline{a}_2)$ by playing $\overline{a}_1$ in every period, while his continuation payoff at $(a^t, a_1, r^t+1)$ is $u_1(\theta_p, a_1, a_2)$, leading to a contradiction. Suppose next that $X(h^1) < 0$, similar to the previous argument, there exists type $\theta_p \in \Theta_p$ with $h^1 \in \mathcal{H}(\theta_p)$ such that his incentive constraint is violated. Similarly, one can show that $X(h^t) = 0$ for every $t \geq 1$, $h^t = (a^t, r^t) \in \mathcal{H}^{\sigma}$. This establishes the uniqueness of player 1’s equilibrium behavior.

C Highest Guaranteed Payoff in Binary-Action MS Games

I show that the payoff lower bound in Theorem 2 is tight in the sense that when the total probability of commitment types is sufficiently small, and the set $\Theta_p$ is nonempty, no type of strategic player 1 can guarantee payoff strictly higher than $\max\{u_1(\theta, \overline{a}_1, \overline{a}_2), u_1(\theta, a_1, a_2)\}$.

Assumption C.1. There exists $\theta \in \Theta^*$ such that $BR_2(\theta, a_2) = \{a_2\}$.

Intuitively, Assumption C.1 implies that there exists a state $\theta$ under which (1) playing $\overline{a}_1$ is individually rational,
and (2) player 2 does not have an incentive to play the desirable action when she knew that player 1 is strategic type $\theta$. The result is stated as Proposition C.1

**Proposition C.1.** Suppose the game satisfies Assumptions 2 and C.1. For every $\phi \in \Delta(\Theta)$, there exist $\bar{\varepsilon} \in (0, 1)$ and $\bar{\delta} \in (0, 1)$, such that for every $\delta > \bar{\delta}$, and every $\mu$ with state distribution $\phi$ and attaches probability less than $\bar{\varepsilon}$ to all commitment types, there exists an equilibrium such that for all $\theta \in \Theta$, strategic type $\theta$’s payoff is no more than $\max\{u_1(\theta, \bar{a}_1, \bar{a}_2), u_1(\theta, \bar{a}_1, \bar{a}_2)\}$.

This proposition applies regardless of the set of commitment actions $A_1^*$, as well as the distributions of the states conditional on each commitment type $\{\phi_{a_1^*}\}_{a_1^* \in A_1^*}$. This contrasts to the private value benchmark, in which a patient player can guarantee his commitment payoff from $a_1 \in A_1$ when $a_1$ is one of the commitment actions.

**Proof of Proposition C.1.** Since $w_\theta(\phi) \leq \max\{u_1(\theta, \bar{a}_1, \bar{a}_2), u_1(\theta, \bar{a}_1, \bar{a}_2)\}$ for every $\theta \in \Theta$, the case in which $\phi$ is pessimistic is implied by the payoff uniqueness result of Theorem 3. When $\phi$ is optimistic, let

$$\bar{\theta} \equiv \min \Theta^* \text{ and } \bar{\theta} \equiv \max \Theta^*.$$  

Assumption C.1 and Assumption 2 in the main text together imply that $\text{BR}_2(\bar{\theta}, \bar{a}_1) = \{a_2\}$. The assumption that $\phi$ is optimistic implies that $\text{BR}_2(\bar{\theta}, \bar{a}_1) = \{\bar{a}_2\}$. For every full support $\phi \in \Delta(\Theta)$, let $\bar{\varepsilon}$ be bounded from above by:

$$\bar{\varepsilon} < \min \left\{ \frac{\phi(\bar{\theta}) D(\bar{\theta}, \bar{a}_1)}{D(\bar{\theta}, \bar{a}_1)}, \frac{\phi(\bar{\theta}) D(\bar{\theta}, \bar{a}_1)}{D(\min \Theta, \bar{a}_1)} \right\}. \quad (C.1)$$

Recall that $A_1^*$ is the set of commitment actions. For every $a_1^* \in A_1^*$, let $\phi_{a_1^*} \in \Delta(\Theta)$ be the distribution of $\theta$ conditional on player 1 being commitment type $a_1^*$. Let $A_1^0$ is the subset of $A_1^*$ such that

$$\text{BR}_2(\phi_{a_1^*}, a_1^*) = \{\bar{a}_2\}.$$  

When $A_1^0$ is nonempty, consider the following equilibrium:

- **Strategic types outside $\Theta^*$** plays $\bar{a}_1$ in every period on the equilibrium path.
- **Strategic types in $\Theta^* \setminus \{\theta\}$** plays $\bar{a}_1$ in every period on the equilibrium path.
- **Strategic type $\theta$** mixes between actions in $\{\bar{a}_1\} \cup A_1^0$ in period 0, and on the equilibrium path, repeats the same action that he has played in period 0 in all subsequent periods. The probability with which he plays $a_1^*$
in period 0 is denoted by \( p(a^*_1) \), given by:

\[
p(a^*_1) = \begin{cases} 
\mu(a^*_1) \frac{D(\phi_{a^*_1,a^*_1})}{(1-\varepsilon)^k D(\bar{a}_1)} & \text{if } a^*_1 \in A^g_1 \setminus \{a_1, \bar{a}_1\} \\
1 - \sum_{\hat{a}_1 \in A^g_1 \setminus \{a_1, \bar{a}_1\}} p(\hat{a}_1) & \text{if } a^*_1 = \bar{a}_1.
\end{cases}
\]

(C.2)

where \( \mu(a^*_1) \) denotes the probability that player 2’s prior belief attaches to commitment type \( a^*_1 \), and \( \varepsilon \) denotes the probability it attaches to all the commitment types. Intuitively, after player 2 observes \( a^*_1 \in A^g_1 \setminus \{a_1, \bar{a}_1\} \) in period 0, her posterior belief makes her indifferent between \( a_2 \) and \( a_2 \) against \( a^*_1 \).

- Starting from period 1, player 2 plays \( \bar{a}_2 \) in every period if player 1 has played \( \bar{a}_1 \) in all previous period; she mixes between \( \bar{a}_2 \) and \( a_2 \) if player 1 has played \( a^*_1 \in A^g_1 \setminus \{a_1, \bar{a}_1\} \) in all previous period, with the probability of playing \( \bar{a}_2 \) is such that type \( \theta \) is indifferent between playing \( \bar{a}_1 \) in every period and playing \( a^*_1 \) in every period at period 0. At all other histories, she plays \( a_2 \) with probability 1.

In the above equilibrium, all types in \( \Theta^* \) receives payoff approximately \( u_1(\theta, a_1, a_2) \), and all types outside \( \Theta^* \) receives payoff approximately \( u_1(\theta, a_1, a_2) \). This establishes Proposition C.1.

D Counterexamples

D.1 Conflicts Between Reputation Building and Signaling under MS Stage-Game Payoff

I show that when Assumptions 1-4 are satisfied, and the prior belief about \( \theta \) is optimistic, there exist equilibria such that player 1’s highest action signals the low state at some on-path history. Players’ stage-game payoffs are:

<table>
<thead>
<tr>
<th>( \theta = \theta_h )</th>
<th>( T )</th>
<th>( N )</th>
<th>( \theta = \theta_l )</th>
<th>( T )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G )</td>
<td>1, 1</td>
<td>-1, 0</td>
<td>( G )</td>
<td>( 1 - \eta ), -1</td>
<td>-1 - \eta, 0</td>
</tr>
<tr>
<td>( B )</td>
<td>2, -1</td>
<td>0, 0</td>
<td>( B )</td>
<td>2, -2</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

There is only one commitment plan, given by:

\[
\gamma(\theta) = \begin{cases} 
G & \text{if } \theta = \theta_h \\
B & \text{if } \theta = \theta_l.
\end{cases}
\]

The equilibrium construction focus on settings in which \( \eta \in (0, 1) \) and the prior probability of state \( \theta_h \), denoted by \( \phi_h \), is strictly between \( 10/11 \) and 1.

Consider the following strategy profile. In period 0, player 2 plays \( N \), strategic type \( \theta_h \) plays \( G \) with probability

\[
\frac{2\phi_h \varepsilon}{3\phi_h (1 - \varepsilon)},
\]

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and strategic type \( \theta_l \) plays \( G \) with probability

\[
\frac{\phi_h \varepsilon}{6(1 - \phi_h)(1 - \varepsilon)}.
\]

According to Bayes Rule, the probability of state \( \theta \) after observing \( G \) in period 0 is \( 10/11 \), which is strictly less than \( \phi_h \). Namely, observing player 1 playing his highest action \( G \) leads to negative inferences about \( \theta \). In period 1, 

- If the history is \( (B, N) \), then future play is dictated by the realization of the public randomization device. With probability \( (1 - \delta)/\delta \), players play \( (B, N) \) in every subsequent period on the equilibrium path; with complementary probability, players play \( (G, T) \) in every subsequent period on the equilibrium path. Off-path deviations are deterred by grim-trigger strategies, namely, whenever player 2 observes player 1 playing \( B \) in periods in which he is supposed to play \( G \), player 2 plays \( N \) in all subsequent periods.

- If the history is \( (G, N) \), then both strategic types play \( B \) with probability 1 and player 2 plays \( T \). This is incentive compatible for player 2 since at history \( (G, N) \), the probability of commitment type \( G \) is \( 6/11 \), the probability of strategic type \( \theta_h \) is \( 4/11 \), and the probability of strategic type \( \theta_l \) is \( 1/11 \).

In period 2, players’ actions at history \( (B, N, B, N), (B, N, G, T), (B, N, B, T) \) have been specified. At history \( (G, N, G, T) \), players play \( (G, T) \) in every subsequent period on the equilibrium path, with off-path deviations deterred via grim trigger strategies. At history \( (G, N, B, T) \), then

- With probability \( (1 - \delta)/\delta \), players play \( (B, N) \) in every subsequent period on the equilibrium path.

- With probability \( 1 - \frac{1 - \delta}{\delta^2} \), play \( (G, T) \) in every subsequent period on the equilibrium path, with off-path deviations deterred via grim trigger strategies.

- With probability \( (1 - \delta)/\delta^2 \), type \( \theta_l \) plays \( B \) for sure and type \( \theta_h \) plays \( B \) with probability \( 1/4 \) and \( G \) with probability \( 3/4 \). Player 2 plays \( T \).

In period 3, 

- At history \( (G, N, B, T, G, T) \), play \( (G, T) \) in every subsequent period on the equilibrium path, with off-path deviations deterred via grim trigger strategies.

- At history \( (G, N, B, T, B, T) \), future play is determined by the realization of public randomization. With probability \( (1 - \delta)/\delta \), play \( (B, N) \) in every subsequent period on the equilibrium path. With complementary probability, play \( (G, T) \) in every subsequent period on the equilibrium path, with off-path deviations deterred via grim trigger strategies.
The above strategy profile an equilibrium when $\delta$ is large enough. Despite the game satisfies Assumptions 1-4 and the prior belief about state is optimistic, playing $G$ in period 0 signals state $\theta_1$.

**D.2 Reputation Failure in Common Interest Games**

I present an example of a *common interest game* with nontrivial interdependent values, under which there exists equilibrium such that all strategic types’ equilibrium payoffs are *arbitrarily low* compared to their commitment payoffs. Consider the following game:

<table>
<thead>
<tr>
<th>$\theta = \theta_1$</th>
<th>$h$</th>
<th>$l$</th>
<th>$\theta = \theta_2$</th>
<th>$h$</th>
<th>$l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>1, 1</td>
<td>0, 0</td>
<td>$H$</td>
<td>0, 0</td>
<td>$\epsilon, \epsilon$</td>
</tr>
<tr>
<td>$L$</td>
<td>0, 0</td>
<td>$\epsilon, \epsilon$</td>
<td>$L$</td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

with $\epsilon \in (0, 1)$ being a parameter. Suppose $\Gamma \equiv \{\gamma\}$ in which the committed player 1 plays his Stackelberg action in every state:

$$\gamma(\theta) \equiv \begin{cases} 
H & \text{if } \theta = \theta_1 \\
L & \text{if } \theta = \theta_2.
\end{cases} \quad (D.1)$$

**Proposition D.1.** *For every full support $\phi \in \Delta\{\theta_1, \theta_2\}$ and $\epsilon \in (0, 1)$, there exists $\bar{\epsilon} > 0$, such that when player 1 is committed with probability less than $\bar{\epsilon}$, there exists an equilibrium in which strategic player 1’s payoff is $\epsilon$ in every state.*

*Proof of Proposition D.1:* Let

$$\bar{\epsilon} \equiv \min\{\phi(\theta_1), \phi(\theta_2)\} \frac{\epsilon}{1 + \epsilon}. \quad (D.2)$$

I verify that the following strategy profile is an equilibrium for every $\delta \in (0, 1)$:

- **Player 2** plays $l$ at every history.
- **Strategic type** $\theta_1$ plays $L$ at every history. **Strategic type** $\theta_2$ plays $H$ at every history.

First, given player 2’s strategy, player 1’s strategy maximizes his payoff at each state and at each history. Second, given player 1’s strategy, I show that player 2 has a strict incentive to play $l$ for all histories.

This is because if player 1 plays $L$, then he is either strategic type $\theta_1$ or commitment type $L$. The likelihood ratio between these two types is strictly greater than $\frac{\phi(\theta_1)-\bar{\epsilon}}{\bar{\epsilon}}$, which according to (D.2) is at least $1/\epsilon$. This implies that player 2 strictly prefers $l$ to $h$ in the event that player 1 plays $L$. Similarly, in the event that player 1 plays $H$, player 2 strictly prefers $l$ to $h$. \hfill $\Box$