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SUPPLEMENT TO "ON THE EFFICIENCY OF SOCIAL LEARNING"

Dinah Rosenberg and Nicolas Vieille[1]

This file contains the proofs of the statements of "on the efficiency of online learning", with the exception of Theorem 2, which is proven in the main paper.

1. THE BENCHMARK: PROOF OF THEOREM 1

We here provide the missing arguments in the analysis of the benchmark case.

Lemma 1 *One has $\psi_*(0) > 0$.*

Proof: Recall that for $\lambda \geq 0$, $\psi(\lambda) = \ln \mathbf{E}_L \left[\exp\left(\lambda \ln \frac{\tilde{q}}{1-\tilde{q}}\right)\right]$ which by virtue of Claim 1 in the paper, is equal to

$$\psi(\lambda) = \ln \mathbf{E}_L \left[\left(\frac{f_H}{f_L}(\tilde{q})\right)^\lambda\right] = \ln \int_0^1 f_H^\lambda(q) f_L^{1-\lambda}(q) dq.$$

This readily yields $\psi(0) = \psi(1) = 0$.

Since $t \mapsto e^{tx}$ is convex, the set $\Lambda := \{\lambda \geq 0, \psi(\lambda) < +\infty\}$ is an interval. Since the private belief $\tilde{q}$ is not a.s. constant, and since $t \mapsto e^{tx}$ is strictly convex whenever $x \neq 0$, ψ is strictly convex on Λ. Since $\psi(0) = \psi(1) = 0$, this implies that $\psi(\lambda) < 0$ for each $\lambda \in (0,1)$ and $\psi(\lambda) > 0$ for each $\lambda > 1$. Thus,

$$\psi_*(0) = \sup_{[0,1]} (-\psi) > 0.$$

Q.E.D.

Lemma 2 *If $F(q) = q$ for each q, one has $\psi_*(0) = -\ln \frac{\pi}{4}$.*

Proof: Recall from the proof of Lemma 1 that $\psi_*(0) = -\min_{[0,1]} \psi$. Since $F(q) = q$, one has $f_H(q) = f_L(1-q) = 2q$ for each $q \in [0,1]$, hence $\psi(\lambda) = \psi(1-\lambda)$ for each $\lambda \in [0,1]$. Since ψ is convex on $[0,1]$, this implies that

$$\min_{[0,1]} \psi = \psi\left(\frac{1}{2}\right)$$

Economics and Decision Sciences, HEC Paris, 1, rue de la Libération, 78351 Jouy en Josas, France. rosenberg@hec.fr; vieille@hec.fr

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and therefore,

$$\psi_*(0) = -\psi\left(\frac{1}{2}\right) = -\ln \int_0^1 2\sqrt{q(1-q)}dq = -\ln\frac{\pi}{4},$$

where the last equality follows using routine computations. *Q.E.D.*

2. THE LAST-OBSERVED SETUP: PROOF OF THEOREM 3

In this section, we prove Theorem 3, following closely the outline in Section 3.1. We thus assume that $\phi_n(a_1,\ldots,a_{n-1}) = a_n$. The social belief at time n is given by $\pi_n = \mathbf{P}(\tilde\theta = H \mid a_n)$.

For $n \geq 1$, we denote by $x_n := \mathbf{P}_L(a_n = h)$ the probability that agent n makes the wrong choice (given L). Under assumptions **A1** and **A2**, we show that

$$\sum_{n\geq 1} x_n < +\infty \Leftrightarrow \int_0^1 \frac{q}{\int_0^q F(x)dx}dx < \infty. \tag{2.1}$$

Since $\mathbf{E}_L[W] = \sum_{n=1}^{+\infty} x_n$, Theorem 3 will follow.

We start with some simple properties of the sequence (x_n). The core of the argument is in Section 2.2.

2.1. *The sequence* (x_n)

Lemma 3 *For all* $n \geq 1$, *one has* $x_{n+1} - x_n = -2\int_0^{x_n} F(q)dq$.

Proof: Fix $n \geq 1$. By Bayes' rule and for each $a \in \{l, h\}$, one has

$$\frac{\mathbf{P}(\tilde\theta = H \mid a_n = a)}{\mathbf{P}(\tilde\theta = L \mid a_n = a)} = \frac{\mathbf{P}_H(a_n = a)}{\mathbf{P}_L(a_n = a)}. \tag{2.2}$$

Since $a_{n+1} = h$ if and only if $q_{n+1} + \pi_{n+1} \geq 1$, it follows from (2.2) that

$$\begin{aligned}\mathbf{P}_\theta(a_{n+1} = h) &= \sum_{a\in\{l,h\}} \mathbf{P}_\theta(a_n = h)\mathbf{P}_\theta\left(q_{n+1} + \mathbf{P}_\theta(H \mid a_n = a) \geq 1\right) \\ &= \sum_{a\in\{l,h\}} \mathbf{P}_\theta(a_n = a)\mathbf{P}_\theta\left(q_{n+1} \geq 1 - \frac{\mathbf{P}_H(a_n = a)}{\mathbf{P}_L(a_n = a) + \mathbf{P}_H(a_n = a)}\right). \qquad (2.3)\end{aligned}$$

The symmetry assumption **A1** implies inductively that $\mathbf{P}_H(a_n = l) = \mathbf{P}_L(a_n = h)$ for each n or, equivalently, $\mathbf{P}_L(a_n = a) + \mathbf{P}_H(a_n = a) = 1$ for each a and

$n \in \mathbf{N}$. Equation (2.3) thus yields

$$\begin{aligned} x_{n+1} &= \sum_{a\in\{l,h\}} \mathbf{P}_L(a_n = a) \times \mathbf{P}_L\left(q_{n+1} \geq \mathbf{P}_L(a_n = a)\right) \\ &= x_n\left(1 - F_L(x_n)\right) + (1 - x_n)\left(1 - F_L(1 - x_n)\right) \end{aligned}$$

Substituting $F_L(q) = 2(1-q)F(q) + 2\int_0^q F(x)dx$ (see Section A), elementary manipulations lead to the desired result. *Q.E.D.*

Since F_L and F_H have the same support, the set of guesses that agent n makes with positive probability is independent of θ. By symmetry, both guesses are made with positive probability: $\mathbf{P}_\theta(a_n = a) > 0$, for each a, θ and n. Thus, $x_n > 0$.

LEMMA 4 *The sequence (x_n) is nonincreasing, with $\lim x_n = 0$ if and only if $q_{min} = 0$.*

PROOF: Denote by $l \geq 0$ the limit of the nonnegative, nonincreasing sequence (x_n), and observe that l solves $\int_0^l F(t)dt = 0$. If $q_{min} = 0$, one has $F(q) > 0$ for all $q > 0$; therefore, $l = 0$. If $q_{min} > 0$, either $x_n > q_{min}$ for all n and $l \geq q_{min}$ or $x_{\bar{n}} \leq q_{min}$ for some $\bar{n}$, in which case $x_n = x_{\bar{n}}$ for all $n \geq \bar{n}$ and, thus, $l = x_{\bar{n}} > 0$. *Q.E.D.*

Lemma 4 allows us to dispose of the case where $q_{min} > 0$. In that case, the sequence (x_n) is bounded away from zero; therefore, $\sum_n x_n = +\infty$: learning is inefficient and the equivalence (2.1) holds.

In the rest of the proof, we assume that $F(q) > 0$ for all $q > 0$, and we set $G(x) := 2\int_0^x F(t)dt$.

2.2. *The continuous-time approximation*

We use a time-change technique to assess the convergence of $\sum x_n$. Fix $a > 0$ such that $a\alpha > 1$, where $\alpha > 0$ is given by **A2**, and for $k \geq 1$, set

$$\omega_k := \inf\left\{n : x_n < \frac{1}{k^a}\right\} \ (\text{ with } \inf\emptyset = +\infty).$$

Note that $\omega_{k+1} \geq \omega_k$ and that $\omega_k < +\infty$ for each k since $(x_n) \to 0$.

Heuristically, the derivation of the continuous-time approximation is sufficiently simple. For $\omega_k \leq n < \omega_{k+1}$, x_n is of the order of $1/k^a$ and $x_n - x_{n+1}$ is of the order of $G(1/k^a)$. Therefore, $\omega_{k+1} - \omega_k$ is approximately equal to $\frac{\frac{1}{k^a} - \frac{1}{(k+1)^a}}{G(1/k^a)}$, which is of the order of $\dfrac{1}{k^{a+1}G(1/k^a)}$. Thus, $\sum_{n=1}^{+\infty} x_n = \sum_{k=1}^{+\infty} \sum_{\omega_k}^{\omega_{k+1}-1} x_n$ is of the order of $\sum_{k=1}^{+\infty} \dfrac{1}{k^{2a+1}} \dfrac{1}{G(\frac{1}{k^a})}$ (Lemmas 6, 7 and 8). We then conclude with a simple series-integral comparison argument. The details are, however, somewhat cumbersome.

Lemma 5 is the only place in the proof where Assumption **A2** is used.

LEMMA 5 *The sequence* (ω_k) *is eventually strictly increasing.*

PROOF: When integrating **A2**, one obtains $G(x) \leq \dfrac{2C}{\alpha+1} x^{\alpha+1}$ for x sufficiently close to 0. In particular, $G\left(\dfrac{1}{k^a}\right) \leq \dfrac{2C}{\alpha+1} \left(\dfrac{1}{k}\right)^{a+a\alpha}$ for k large; thus,

$$G\left(\frac{1}{k^a}\right) = o\left(\frac{1}{k^{a+1}}\right) \text{ as } k \to +\infty,$$

since $a\alpha > 1$.

Since $\dfrac{1}{k^a} - \dfrac{1}{(k+1)^a} \sim \dfrac{a}{k^{a+1}}$ as $k \to +\infty$, this implies the existence of $K_0 \in \mathbf{N}$ such that

$$G\left(\frac{1}{k^a}\right) < \frac{1}{k^a} - \frac{1}{(k+1)^a} \text{ for each } k \geq K_0.$$

On the other hand, let $\bar{q}$ be the median of F: $F(\bar{q}) = \frac{1}{2}$. Since $1 - G'(x) = 1 - 2F(x)$, the map $x \mapsto x - G(x)$ is nondecreasing on $[0, \bar{q}]$. Let N be such that $x_n < \bar{q}$ for each $n \geq N$ and K_1 be s.t. $\omega_{K_1} > N + 1$. Finally, set $K_* := \max(K_0, K_1)$. We will prove that $\omega_{k+1} > \omega_k$ for each $k \geq K_*$.

Let $k \geq K_*$ be arbitrary and set $n := \omega_k - 1$. Since $k \geq K_1$, one has $n > N$, so

$$\frac{1}{k^a} \leq x_{\omega_k - 1} = x_n < \bar{q}.$$

Thus,

$$x_{n+1} = x_n - G(x_n) \geq \frac{1}{k^a} - G\left(\frac{1}{k^a}\right) > \frac{1}{(k+1)^a},$$

where the first inequality holds since G is nondecreasing on $[0, \bar{q}]$ and the second holds since $k \geq K_0$. Since $n+1 = \omega_k$, we have thus proven that $x_{\omega_k} > \dfrac{1}{(k+1)^a}$, which implies $\omega_{k+1} > \omega_k$. *Q.E.D.*

LEMMA 6 *One has*

$$x_{\omega_k} - x_{\omega_{k+1}} \sim \frac{a}{k^{a+1}}, \text{ as } k \to +\infty.$$

PROOF: We let K_* be defined as in the proof of Lemma 5. For $k \geq K_*$, one has

$$\frac{1}{k^a} \geq x_{\omega_k} = x_{\omega_k - 1} - G(x_{\omega_k - 1}) \geq \frac{1}{k^a} - G\left(\frac{1}{k^a}\right), \tag{2.4}$$

where the first inequality holds by definition of ω_k and the second holds since $x_{\omega_k - 1} \in \left[\frac{1}{k^a}, \bar{q}\right]$ and since $x \mapsto x - G(x)$ is nonincreasing on $[0, \bar{q}]$.

For the same reason,

$$\frac{1}{(k+1)^a} \geq x_{\omega_{k+1}} \geq \frac{1}{(k+1)^a} - G\left(\frac{1}{(k+1)^a}\right). \tag{2.5}$$

By combining (2.4) and (2.5), one obtains

$$\frac{1}{k^a} - \frac{1}{(k+1)^a} - G\left(\frac{1}{k^a}\right) \leq x_{\omega_k} - x_{\omega_{k+1}} \leq \frac{1}{k^a} - \frac{1}{(k+1)^a} + G\left(\frac{1}{(k+1)^a}\right).$$

Since $\frac{1}{k^a} - \frac{1}{(k+1)^a} \sim \frac{a}{k^{a+1}}$ and $G\left(\frac{1}{k^a}\right) = o\left(\frac{1}{k^{a+1}}\right)$ as $k \to +\infty$ (see the proof of Lemma 5), the result follows. *Q.E.D.*

LEMMA 7 *One has* $\sum_{n=1}^{+\infty} x_n < +\infty \Leftrightarrow \sum_{k=1}^{+\infty} \frac{\omega_{k+1} - \omega_k}{k^a} < +\infty.$

PROOF: Since $\frac{1}{(k+1)^a} < x_n \leq \frac{1}{k^a}$ when $\omega_k \leq n < \omega_{k+1}$, one has

$$\sum_{k=1}^{+\infty} \frac{\omega_{k+1} - \omega_k}{(k+1)^a} < \sum_{n=\omega_1}^{+\infty} x_n \leq \sum_{k=K_*}^{+\infty} \frac{\omega_{k+1} - \omega_k}{k^a}.$$

Since $\frac{1}{k^a} \sim \frac{1}{(k+1)^a}$ as $k \to +\infty$, the result follows. *Q.E.D.*

LEMMA 8 *One has* $\sum_{n=1}^{+\infty} x_n < +\infty \Leftrightarrow \sum_{k=1}^{+\infty} \frac{1}{k^{2a+1}} \frac{1}{G\left(\frac{1}{k^a}\right)} < +\infty.$

PROOF: For each k and n such that $\omega_k \leq n < \omega_{k+1}$,

$$G\left(\frac{1}{(k+1)^a}\right) \leq x_n - x_{n+1} \leq G\left(\frac{1}{k^a}\right);$$

hence, by summing over n,

$$(\omega_{k+1}-\omega_k)G\left(\frac{1}{(k+1)^a}\right) \leq x_{\omega_k} - x_{\omega_{k+1}} \leq (\omega_{k+1}-\omega_k)G\left(\frac{1}{k^a}\right). \tag{2.6}$$

We note that without further information about F, it is unclear whether $G\left(\frac{1}{(k+1)^a}\right) \sim G\left(\frac{1}{k^a}\right)$ as $k \to +\infty$. Hence, it is not possible to derive from (2.6) an asymptotic equivalent for $\omega_{k+1}-\omega_k$; more work is needed.

If $\sum_n x_n < +\infty$, then $\sum_k \frac{\omega_{k+1}-\omega_k}{k^a} < +\infty$ by Lemma 7; hence, $\sum_k \frac{x_{\omega_k}-x_{\omega_{k+1}}}{k^a G(\frac{1}{k^a})} < +\infty$ by (2.6), which by Lemma 6 implies $\sum_k \frac{1}{k^{2a+1}G(1/k^a)} < +\infty$.

Conversely, if $\sum_k \frac{1}{k^{2a+1}G(1/k^a)} < +\infty$, then $\sum_k \frac{1}{(k-1)^{2a+1}G(1/k^a)} < +\infty$ since $\frac{1}{(k-1)^{2a+1}} \sim \frac{1}{k^a}$ as $k \to +\infty$, hence $\sum_k \frac{1}{k^{2a+1}G(1/(k+1)^a)} < +\infty$, which by Lemma 6 implies $\sum_k \frac{x_{\omega_k}-x_{\omega_{k+1}}}{k^a G(1/(k+1)^a)} < +\infty$ and, therefore, $\sum_k \frac{\omega_{k+1}-\omega_k}{k^a} < +\infty$ by (2.6), which yields $\sum_n x_n < +\infty$ by Lemma 7. *Q.E.D.*

To simplify the following statement, we introduce

$$a(t) := \frac{1}{t^{2a+1}} \text{ and } b(t) := G\left(\frac{1}{k^a}\right) \ (t>0).$$

Lemma 9 *One has* $\sum_{k=1}^{+\infty} \frac{a(k)}{b(k)} < +\infty \Leftrightarrow \int_1^{+\infty} \frac{a(t)}{b(t)} < +\infty.$

Proof: Since $a(\cdot)$ and $b(\cdot)$ are decreasing on $[1,+\infty)$,

$$\frac{a(k+1)}{b(k)} \leq \int_k^{k+1} \frac{a(t)}{b(t)}dt \leq \frac{a(k)}{b(k+1)}$$

for each k and, therefore,

$$\sum_{k=1}^{+\infty} \frac{a(k+1)}{b(k)} \leq \int_1^{+\infty} \frac{a(t)}{b(t)}dt \leq \sum_{k=1}^{+\infty} \frac{a(k)}{b(k+1)}. \tag{2.7}$$

Since $a(k) \sim a(k+1)$ as $k \to +\infty$, the three series $\sum \frac{a(k+1)}{b(k)}$, $\sum \frac{a(k)}{b(k+1)}$ and $\sum \frac{a(k)}{b(k)}$ are simultaneously convergent or divergent; hence, the result follows from (2.7). *Q.E.D.*

Observe now that $\int_1^{+\infty} \frac{a(t)}{b(t)}dt = \int_0^1 \frac{q}{G(q)}dq$, using the change of variables $q = 1/t^a$. We have thus proven that $\sum x_n < +\infty$ if and only if $\int_0^1 \frac{q}{G(q)} < +\infty$. This concludes the proof of Theorem 3.

3. ILLUSTRATIONS: PROOFS OF PROPOSITIONS 1 AND 2

Denote by Φ the c.d.f. of the standard normal distribution. We start with the proof of Proposition 1. Assume w.l.o.g. that $\Delta\mu := \mu_H - \mu_L > 0$, and denote by g_θ the conditional density of s_n given $\tilde{\theta} = \theta$. Following a signal realization $\tilde{s}$ and by Bayes rule, the private belief $\tilde{q}$ is given by

$$\ln \frac{\tilde{q}}{1-\tilde{q}} = \ln \frac{g_H(\tilde{s})}{g_L(\tilde{s})} = \frac{\Delta\mu}{\sigma^2}\left(\tilde{s} - \frac{\mu_H + \mu_L}{2}\right)$$

and is therefore increasing in $\tilde{s}$. Thus, for $q \in (0,1)$, one has

$$F_L(q) = \mathbf{P}_L(\tilde{q} \le q) = \mathbf{P}_L\left(\ln \frac{\tilde{q}}{1-\tilde{q}} \le \ln \frac{q}{1-q}\right) = \mathbf{P}_L\left(\tilde{s} \le \frac{\mu_H + \mu_L}{2} + \frac{\sigma^2}{\Delta\mu} \ln \frac{q}{1-q}\right).$$

Since the r.v. $\dfrac{\tilde{s} - \mu_L}{\sigma}$ follows a standard normal distribution conditional on $\tilde{\theta} = L$, this yields $F_L(q) = \Phi(x(q))$ for each q, where

$$x(q) := \frac{\Delta\mu}{2\sigma} + \frac{\sigma}{\Delta\mu} \ln \frac{q}{1-q}.$$

We will use the inequality $\Phi(x) \le e^{-x^2/2}$, which holds for all $x < 0$ such that $|x|$ is large enough. Since

$$x(q)^2 = \left(\frac{\Delta\mu}{2\sigma}\right)^2 + \ln \frac{q}{1-q} + \left(\frac{\sigma}{\Delta\mu}\right)^2 \left(\ln \frac{q}{1-q}\right)^2 \ge \ln q + \frac{\sigma^2}{(\Delta\mu)^2}\left(\ln \frac{q}{1-q}\right)^2,$$

one obtains

$$F_L(q) \le e^{-(x(q))^2/2} \le \frac{1}{\sqrt{q}} \exp\left\{-\frac{\sigma^2}{2(\Delta\mu)^2}\left(\ln \frac{q}{1-q}\right)^2\right\} \tag{3.1}$$

for all q close enough to zero. The right-hand side of (3.1) is equivalent to $\dfrac{1}{\sqrt{q}} \exp\left(-\dfrac{\sigma^2}{2(\Delta\mu)^2}(\ln q)^2\right)$ in the neighborhood of zero,[1] which around zero is negligible relative to any polynomial function of q. Proposition 1 follows.

We turn to the proof of Proposition 2, which is similar. We assume w.l.o.g. that $\mu_H = \mu_L = 0$. Following a signal realization $\tilde{s}$ and by Bayes rule, the private belief $\tilde{q}$ is given by

$$\frac{\tilde{q}}{1-\tilde{q}} = \frac{\sigma_L}{\sigma_H} e^{-\frac{\tilde{s}^2}{2\delta}},$$

[1] Indeed, the ratio of these two quantities is given by $\exp\left(\frac{1}{2a^2}\ln(1-q)\ln\frac{2q}{1-q}\right)$. Around zero, the expression within the exponential is equivalent to $-\frac{1}{2a^2} \times q \ln q$, which converges to zero as $q \to 0$.

with $\dfrac{1}{\delta} = \dfrac{1}{\sigma_H^2} - \dfrac{1}{\sigma_L^2} > 0$. Since the likelihood ratio $\frac{\tilde{q}}{1-\tilde{q}}$ does not exceed $\frac{\sigma_L}{\sigma_H}$, the private belief $\tilde{q}$ cannot possibly exceed $q_{max} := \frac{\sigma_L}{\sigma_L+\sigma_H} < 1$. For $q \in (0, q_{\max}]$, one has

$$\begin{aligned} F_L(q) &= \mathbf{P}_L\left(\frac{\sigma_L}{\sigma_H} e^{-\frac{\tilde{s}^2}{2\delta}} \le \frac{q}{1-q}\right) \\ &= \mathbf{P}_L\left(\tilde{s}^2 \ge 2\delta \ln \frac{1-q}{q} + 2\delta \ln \frac{\sigma_L}{\sigma_H}\right) \\ &= 2\mathbf{P}_L\left(\frac{\tilde{s}}{\sigma_L} \ge \frac{1}{\sigma_L}\sqrt{2\delta \ln\left(\frac{1-q}{q} \times \frac{\sigma_L}{\sigma_H}\right)}\right). \end{aligned}$$

Since the r.v. $\dfrac{\tilde{s}}{\sigma_L}$ follows a standard normal distribution, one has

$$F_L(q) = 2\left(1 - \Phi(z(q))\right),$$

with $z(q) := \dfrac{1}{\sigma_L}\sqrt{2\delta \ln\left(\dfrac{1-q}{q} \times \dfrac{\sigma_L}{\sigma_H}\right)}$. Recall from Section A that $F_L(q) \sim_0 2F(q)$ as $q \to 0$. Using the inequalities

$$\frac{z}{z^2+1} e^{-z^2/2} \le 1 - \Phi(z) \le \frac{1}{z} e^{-z^2/2} \text{ for } z > 0,$$

see *e.g.* Revuz and Yor (1999) p. 30, it follows that $F(q) \sim \dfrac{1}{z(q)} e^{-z(q)^2/2}$ as $q \to 0$, and thus, that $\displaystyle\int_0^1 \frac{1}{F(q)} dq < +\infty$ is equivalent to $\displaystyle\int_0^1 z(q) e^{z(q)^2/2} dq < +\infty$.

Next, observe that $z(q) \sim \dfrac{\sqrt{2\delta}}{\sigma_L}\sqrt{|\ln q|}$ as $q \to 0$, and that

$$e^{z(q)^2/2} = \exp\left(\frac{\delta}{\sigma_L^2} \ln\left(\frac{1-q}{q} \times \frac{\sigma_L}{\sigma_H}\right)\right) = \left(\frac{1-q}{q}\right)^{\delta/\sigma_L^2} \left(\frac{\sigma_L}{\sigma_H}\right)^{\delta/\sigma_L^2},$$

hence

$$z(q) e^{z(q)^2/2} \sim C_2 \frac{\sqrt{|\ln q|}}{q^{\delta/\sigma_L^2}}$$

as $q \to 0$, for some constant $C_2 > 0$. It follows that the integral $\displaystyle\int_0^1 z(q) e^{z(q)^2/2} dp$ is finite if and only if $\delta/\sigma_L^2 < 1$ or equivalently, $\sigma_L^2 > 2\sigma_H^2$, as desired.

4. RATES OF CONVERGENCE: PROOF OF THEOREM 4

The proof of Lemma 1 relies on Lemma 10 below, which is a classical result on asymptotic expansions of sequences. An equivalent statement appears in Francinou, Gianella and Nicolas (2013, in French). Related analysis may be found in de Bruijn (1961).

LEMMA 10 *Let $g : \mathbf{R}_+ \to \mathbf{R}_+$, and (u_n) a sequence be given, such that $u_{n+1} = g(u_n)$ for each n. Assume that $\lim u_n = 0$ and that $g(x) = x - ax^\beta + o(x^\beta)$ in the neighborhood of zero, with $a > 0$ and $\beta > 1$. Then*

$$u_n \sim \left(\frac{1}{a(\beta-1)}\frac{1}{n}\right)^{1/(\beta-1)} \quad \text{as } n \to +\infty.$$

PROOF: We follow the proof in Francinou, Gianella and Nicolas (2013). For $x > 0$,

$$\begin{aligned} g(x)^{1-\beta} - x^{1-\beta} &= \left(x - ax^\beta + o(x^\beta)\right)^{1-\beta} - x^{1-\beta} \\ &= x^{1-\beta}\left(\left(1 - ax^{\beta-1} + o(x^{\beta-1})\right)^{1-\beta} - 1\right) \\ &= x^{1-\beta}\left(-a(1-\beta)x^{\beta-1} + o(x^{\beta-1})\right) = a(\beta-1) + o(1), \end{aligned}$$

hence $\lim_{x\to 0}\left(g(x)^{1-\beta} - x^{1-\beta}\right) = a(\beta-1)$. Since $\lim u_n = 0$ and $u_{n+1} = g(u_n)$, this implies $\lim\left(u_{n+1}^{1-\beta} - u_n^{1-\beta}\right) = a(\beta-1)$. By Cesaro Theorem, one has therefore $\lim \dfrac{u_n^{1-\beta}}{n} = a(\beta-1)$ as well, hence $u_n \sim (a(\beta-1)n)^{1/(1-\beta)}$ as $n \to +\infty$, as desired.

Q.E.D.

PROOF OF LEMMA 1: We assume first that all choices are public, and recall that $\mathbf{P}_L(\tau > n) \sim (1 - \pi_n^*)\mathbf{P}_H(a_m = h \text{ for all } m)$ as $n \to +\infty$, using the notations of Section B.2. Set $u_n := 1 - \pi_n^*$. From (B.6), one has

$$\frac{u_{n+1}}{1-u_{n+1}} = \frac{u_n}{1-u_n} \times \frac{1-F_L(u_n)}{1-F_H(u_n)},$$

or equivalently,

$$u_{n+1} = g(u_n) := \frac{u_n(1-F_L(u_n))}{u_n(1-F_L(u_n)) + (1-u_n)(1-F_H(u_n))}. \tag{4.1}$$

Under the assumption that $F(q) = aq^\alpha + o(q^\alpha)$ as $q \to 0$, Section A yields $F_L(q) = 2aq^\alpha + o(q^\alpha)$ and $F_H(q) = o(q^\alpha)$ as $q \to 0$. Plugging into (4.1), we obtain

$$g(x) = x - 2ax^{\alpha+1} + o(x^{\alpha+1}) \text{ as } x \to 0.$$

The result then follows from Lemma 10.

Assume now that only the previous choice is observed. From Lemma 14, and the assumption on F, one has

$$x_{n+1} = x_n - \int_0^{x_n} F(q)dq = x_n - \frac{2a}{\alpha+1}x_n^{\alpha+1} + o(x_n^{\alpha+1}) \text{ as } n \to +\infty.$$

The result again follows from Lemma 10. *Q.E.D.*

Proof of Theorem 4: We rely on the following elementary observation on divergent series. Let (x_n) and (u_n) be two bounded sequences such that $x_n \sim u_n$ as $n \to +\infty$. Assume that $u_n > 0$ for each n and that the series $\sum u_n$ is divergent. Then $\displaystyle\sum_{k=1}^{n} x_k \sim \sum_{k=1}^{n} u_k$ as $n \to +\infty$, and $\displaystyle\sum_{k=1}^{+\infty} \delta^{k-1} x_k \sim \sum_{k=1}^{+\infty} \delta^{k-1} u_k$ as $\delta \to 1$.

Assume as stated that $F(q) \sim a q^\alpha$ as $q \to 0$, with $\alpha \geq 1$. In the all-observed setup, let $x_n := \mathbf{P}_L(\tau > n)$ and $u_n := c_1 1/n^{1/\alpha}$. Since $\sum u_n$ is divergent, and since $\mathbf{E}_L\left[\min(\tau, n)\right] = 1 + \displaystyle\sum_{k=1}^{n-1} x_k$, one has

$$\mathbf{E}_L\left[\min(\tau, n)\right] \sim c_1 \sum_{k=1}^{n} \frac{1}{k^{1/\alpha}}, \text{ as } n \to +\infty.$$

Since $\alpha \geq 1$ and since $\sum 1/k^{1/\alpha}$ is divergent, a usual series-integral comparison argument yields $\displaystyle\sum_{k=1}^{n} \frac{1}{k^{1/\alpha}} \sim \int_1^n \frac{1}{x^{1/\alpha}} dx$ as $\to +\infty$, and the first claim follows.

In the last-observed setup, we let $x_n := \mathbf{P}_L(a_n = h)$ and $u_n := c_2 1/n^{1/\alpha}$. Since $\mathbf{E}_L[W_n] = \sum_{k=1}^n x_k$ it follows as in the previous paragraph that

$$\mathbf{E}_L[W_n] \sim c_2 \int_1^n \frac{1}{x^{1/\alpha}} dx \text{ as } n \to +\infty.$$

We are left with the estimate of $\mathbf{E}_L[W_\delta]$. Using the notations of the previous paragraph, one has

$$\mathbf{E}_L[W_\delta] = \sum_{k=1}^{+\infty} \delta^{k-1} x_k \sim c_2 \sum_{k=1}^{+\infty} \frac{\delta^{k-1}}{k^{1/\alpha}},$$

which in turn is equivalent to $\displaystyle c_2 \sum_{k=1}^{+\infty} \frac{\delta^k}{k^{1/\alpha}}$ as $\delta \to 1$.

Since

$$\frac{\delta^{k+1}}{(k+1)^{1/\alpha}} \leq \int_k^{k+1} \frac{\delta^x}{x^{1/\alpha}} dx \leq \frac{\delta^k}{k^{1/\alpha}} \text{ for each } k \geq 1,$$

one gets by summation over k

$$\sum_{k=2}^{+\infty} \frac{\delta^k}{k^{1/\alpha}} \leq \int_1^{+\infty} \frac{\delta^x}{x^{1/\alpha}} dx \leq \sum_{k=1}^{+\infty} \frac{\delta^k}{k^{1/\alpha}},$$

and therefore, $\sum_{k=1}^{+\infty} \frac{\delta^k}{k^{1/\alpha}} \sim \int_1^{+\infty} \frac{\delta^x}{x^{1/\alpha}} dx$ as $\delta \to 1$, since $\lim_{\delta \to 1} \int_1^{+\infty} \frac{\delta^x}{x^{1/\alpha}} dx = +\infty$.

Using the change of variable $y = -x \ln \delta$, the latter integral is equal to

$$\int_1^{+\infty} \frac{\delta^x}{x^{1/\alpha}} dx = (-\ln \delta)^{1/\alpha - 1} \times \int_{-\ln \delta}^{+\infty} e^{-y} y^{-1/\alpha} dy. \tag{4.2}$$

If $\alpha > 1$, the desired estimate follows from equation (4.2) since $-\ln(\delta) \sim (1-\delta)$ and since $\int_{-\ln \delta}^{+\infty} e^{-y} y^{-1/\alpha} dy$ converges to $\int_0^{+\infty} e^{-y} y^{-1/\alpha} dy = \Gamma(1 - 1/\alpha)$ as $\delta \to 1$.

If $\alpha = 1$, the integral $\int_0^{+\infty} e^{-y} y^{-1/\alpha} dy$ is infinite. Since $e^{-y}/y \sim 1/y$ as $y \to 0$, routine arguments show that

$$\int_{-\ln \delta}^{+\infty} \frac{e^{-y}}{y} dy \sim \int_{-\ln \delta}^{1} \frac{1}{y} dy = -\ln \ln \frac{1}{\delta},$$

and the result also follows from equation (4.2). *Q.E.D.*

For completeness, we give a quick proof that the constants c_1 and c_2 in Lemma 1 are equal to $\frac{1}{\pi}$ and to 1, when private beliefs are uniformly distributed.

When all guesses are public, one has $u_n := \mathbf{P}_L(\tau > n) = \prod_{k=1}^{n} (1 - F_L(1 - \pi_k))$.

With $F(p) = p$, one has $F_L(p) = p(2-p)$ and $F_H(p) = p^2$, hence $u_n = \prod_{k=1}^{n} \pi_k^2$ and the belief updating equation (3.6) reduces to $\frac{\pi_{n+1}}{1 - \pi_{n+1}} = \frac{2 - \pi_n}{1 - \pi_n}$, from which it follows that $\left(\frac{1}{1 - \pi_n}\right)_n$ is an arithmetic sequence, and $\pi_n = 1 - \frac{1}{2n}$ for each $n \geq 1$.

Consequently,

$$u_n = \left(\prod_{k=1}^{n} \left(1 - \frac{1}{2k}\right) \right)^2 = \left(\frac{(2n)!}{2^{2n} (n!)^2} \right)^2.$$

Using Stirling formula, it follows that $u_n \sim \frac{1}{\pi n}$ as $n \to +\infty$.

When only the previous guess is observed, the probability $x_n := \mathbf{P}_L(a_n = h)$ of a wrong guess is given by $x_{n+1} - x_n = -2 \int_0^{x_n} F(p) dp$, which reduces to a discrete time logistic equation

$$x_{n+1} = x_n (1 - x_n). \tag{4.3}$$

Since $x_1 \in (0,1)$, it is obvious from (4.3) that (x_n) is decreasing and must converge to zero. An easy induction shows that $x_n < \frac{1}{n+1}$ for all $n \geq 2$. Set now $y_n := nx_n$, and observe that

$$y_{n+1} - y_n = x_n\left(1 - (n+1)x_n\right) \geq 0. \tag{4.4}$$

The sequence (y_n) being non-decreasing with $y_n \leq 1$, it has a positive limit, which we denote by $l > 0$.

Equation (4.4) also yields

$$y_{n+1} - y_n = \frac{y_n(1-y_n)}{n} - \frac{y_n^2}{n^2}.$$

Since the sequence (y_n) converges, the series $\sum(y_{n+1} - y_n)$ converges as well, hence $l = 1$.[2] We have thus shown that $x_n \sim \frac{1}{n}$ as $n \to +\infty$.

The latter estimate implies that $\mathbf{E}_L[\tau] < +\infty$, and therefore, that the two efficiency criteria $\mathbf{E}_L[W] < +\infty$ and $\mathbf{E}_L[\tau] < +\infty$ are not equivalent when only the previous action is observed. One indeed has for each n, $\mathbf{P}(\tilde{\theta} = H \mid a_n = h) = \mathbf{P}_H(a_n = h) = 1 - x_n$ which implies,

$$\begin{aligned} \mathbf{P}_L(\tau > n+1 \mid \tau > n) &= \mathbf{P}_L(a_{n+1} = h \mid a_n = h) \\ &= 1 - F_L(x_n) = (1-x_n)^2 \end{aligned}$$

The sequence $(\mathbf{P}_L(\tau > n))_n$ satisfies

$$\frac{\mathbf{P}_L(\tau > n+1)}{\mathbf{P}_L(\tau > n)} = (1-x_n)^2 = 1 - \frac{2}{n} + o\left(\frac{1}{n}\right).$$

This implies that the series $\sum \mathbf{P}_L(\tau > n)$ is convergent, using Raabe-Duhamel rule, and therefore, $\mathbf{E}_L[\tau] < +\infty$.

5. INEFFICIENCY OF RANDOM SAMPLING: PROOF OF THEOREM 5

The proof of Theorem 5 follows closely the proof of Theorem 3 and we refer to Section C for notations. In addition, we will denote by $\bar{x}_n := \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_n$ the expected proportion of wrong choices among the first n agents, and by α_n the random action observed by agent $n+1$. Thus, the social belief is here equal to $\pi_n = \mathbf{P}\left(\tilde{\theta} = H \mid \alpha_{n-1}\right)$.

Lemma 11 *For each $n \geq 1$, one has* $\bar{x}_{n+1} - \bar{x}_n = -\dfrac{2}{n+1}\displaystyle\int_0^{\bar{x}_n} F(q)dq$.

[2] Otherwise, $y_{n+1} - y_n$ would be equivalent to $l(1-l)/n$.

PROOF: Since agent $n+1$ samples uniformly among all previous agents, one has

$$\mathbf{P}_\theta(\alpha_n = a) = \frac{1}{n}\sum_{k=1}^{n}\mathbf{P}_\theta(a_k = a) \text{ for each } \theta \text{ and } a.$$

On the event $\alpha_n = a$, Bayes rule leads to $\dfrac{\pi_{n+1}}{1-\pi_{n+1}} = \dfrac{\pi_n}{1-\pi_n} \times \dfrac{\mathbf{P}_H(\alpha_n = a)}{\mathbf{P}_L(\alpha_n = a)}$. Using $\mathbf{P}_\theta(a_{n+1} = h) = \mathbf{P}_\theta(q_{n+1} \geq 1-\pi_{n+1})$ and the symmetry assumption **A1**, elementary manipulations similar to those in the proof of Lemma 14 lead to

$$\begin{aligned} x_{n+1} &= \bar{x}_n\left(1 - F_L(\bar{x}_n)\right) + (1-\bar{x}_n)\left(1 - F_L(1-\bar{x}_n)\right) \\ &= \bar{x}_n - 2\int_0^{\bar{x}_n} F(q)dq. \end{aligned}$$

Since $\bar{x}_{n+1} = \dfrac{n}{n+1}\bar{x}_n + \dfrac{1}{n+1}x_{n+1}$, the result follows. *Q.E.D.*

LEMMA 12 *One has* $\sum_{n=1}^{+\infty} x_n < +\infty \Leftrightarrow \sum_{n=1}^{+\infty} \bar{x}_n < +\infty$.

PROOF: The argument that $x_n > 0$ applies without change, and yields $\bar{x}_n > 0$ for each n. The proof of Lemma 15 requires minor changes. Set $l := \lim \bar{x}_n$. Since (x_n) is non-increasing, one has $\lim x_n = l$ as well. As in the proof of Lemma 15, and if $q_{min} > 0$, either $\bar{x}_n > q_{min}$ for all n, and then $l \geq q_{min}$, or $\bar{x}_{n_1} \leq q_{min}$ for some n_1, in which case $\bar{x}_n = \bar{x}_{n_1}$ for all $n \geq n_1$, and thus $l = \bar{x}_{n_1} > 0$. In that case, both $\sum x_n$ and $\sum \bar{x}_n$ are divergent.

In the rest of the proof, we may thus assume that $F(q) > 0$ for each $q > 0$. We claim that $l = 0$. Otherwise indeed, one would have $\bar{x}_{n+1} - \bar{x}_n \sim -\dfrac{1}{n} \times \int_0^l F(q)dq$ as $n \to +\infty$. Since the series $\sum \frac{1}{n}$ is divergent, this would imply $\lim \bar{x}_n = -\infty$, a contradiction. Hence $l = 0$, as claimed.

Using again Lemma 11, $|x_{n+1} - \bar{x}_n| \leq 2\bar{x}_n F(\bar{x}_n)$, hence $x_{n+1} \sim \bar{x}_n$ as $n \to +\infty$ since $\lim F(\bar{x}_n) = 0$. Hence, the convergence of the series $\sum x_n$ is equivalent to that of $\sum \bar{x}_n$. *Q.E.D.*

By **A2** (and when possibly lowering α), one has $F(q) \leq \frac{1}{2}(\alpha+1)q^\alpha$ in a neighborhood of zero. Using Lemma 11, there is $N_0 \in \mathbf{N}$ s.t.

$$\bar{x}_{n+1} \geq \bar{x}_n - \frac{1}{n+1}\bar{x}_n^{1+\alpha}, \text{ for all } n \geq N_0. \tag{5.1}$$

On the other hand, the map $y \mapsto y - y^{1+\alpha}$ is increasing over the interval $\left[0, \dfrac{1}{(\alpha+1)^{1/\alpha}}\right]$. Choose N_1 s.t. $\bar{x}_n \in \left[0, \frac{1}{(\alpha+1)^{1/\alpha}}\right]$ for each $n \geq N_1$, and set $N := \max(N_0, N_1)$.

Introduce now a sequence (y_n) s.t. $y_N = x_N$ and $y_{n+1} - y_n = -\frac{1}{n+1} y_n^{1+\alpha}$ for each $n \geq N$. From the choice of N, it follows by induction that $\bar{x}_n \geq y_n$ for all $n \geq N$. It is thus sufficient to prove that the series $\sum y_n$ is divergent.

Obviously, the sequence (y_n) is positive, decreasing, with $\lim y_n = 0$.[3] Hence

$$\frac{y_{n+1}}{y_n} = 1 - \frac{1}{n} y_n^\alpha = 1 + o\left(\frac{1}{n}\right).$$

It follows from the Raabe-Duhamel criterion that $\sum y_n$ is divergent.

6. ALTERNATIVE SETUP: PROOF OF THEOREM 5

Since $F(q) = q$ satisfies **A1**, $\mathbf{E}_\theta[\tau]$ is independent of θ. We choose $\theta = L$ for concreteness.

Let C_2 be an upper bound for the sequence (d_{k+1}/d_k). For $k \geq 1$, denote by $\Delta_k := d_1 + \cdots + d_k$ the cumulative size of the first k generations, with $\Delta_0 = 1$. We will prove that $\sum_{k=1}^{+\infty} d_k \mathbf{P}_L(\tau > \Delta_k) = +\infty$. Since

$$\mathbf{E}_L[\tau] = \sum_{k=1}^{+\infty} \sum_{n=\Delta_{k-1}+1}^{\Delta_k} \mathbf{P}_L(\tau \geq n) \geq \sum_{k=1}^{+\infty} d_k \mathbf{P}_L(\tau > \Delta_k),$$

the result will follow.

Since $F(q) = q$, one has $F_H(q) = q^2$ and $F_L(q) = q(2-q)$ for each q (see Section A), and thus $1 - F_L(1-\rho) = \rho^2$ for each ρ. For $k \geq 1$, we denote by

$$\rho_k := \mathbf{P}_L(\tilde{\theta} = H \mid a_1 = \cdots = a_{\Delta_{k-1}} = h)$$

the (common) social belief of agents from the k-th generation, in the event $\tau > \Delta_{k-1}$ where all agents from all previous generations have chosen h.

Conditional on $\tau > \Delta_{k-1}$, agent n from the k-th generation chooses $a_n = h$ if and only if $q_n \geq 1 - \rho_k$, which occurs with probability $1 - F_L(1-\rho_k) = \rho_k^2$ in state L. Since there are d_k such agents, $\mathbf{P}_L(\tau > \Delta_k \mid \tau > \Delta_{k-1}) = \rho_k^{2d_k}$ and thus,

$$\mathbf{P}_L(\tau > \Delta_k) = \prod_{i=1}^{k} \rho_i^{2d_i}. \tag{6.1}$$

[3] If (y_n) instead had a positive limit l, we would have $y_{n+1} - y_n \leq -\frac{l^\alpha}{n}$ for each n, which by summation would imply $\lim y_n = -\infty$.

On the other hand, Bayes rule leads to the belief updating formula

$$\frac{\rho_{k+1}}{1-\rho_{k+1}} = \frac{\rho_k}{1-\rho_k} \times \left(\frac{1-F_H(1-\rho_k)}{1-F_L(1-\rho_k)}\right)^{d_k} = \frac{\rho_k}{1-\rho_k} \times \left(\frac{2-\rho_k}{\rho_k}\right)^{d_k}. \tag{6.2}$$

Setting $u_k := \frac{1}{2}\frac{\rho_k}{1-\rho_k}$, we have $\rho_k = 1 - \frac{1}{1+2u_k}$, and (6.2) rewrites

$$u_{k+1} = u_k\left(1+\frac{1}{u_k}\right)^{d_k}. \tag{6.3}$$

We proceed with a series of claims.

CLAIM 1 *One has $u_{k+1} \geq \Delta_k + 1$ for all k.*

PROOF: The inequality $(1+x)^\alpha \geq 1 + \alpha x$ (valid for $\alpha > 1$, $x > 0$) yields $u_{n+1} \geq u_n + d_n$. The result then follows by induction. *Q.E.D.*

CLAIM 2 *The series $\sum \frac{d_k}{(u_k)^2}$ is convergent.*

PROOF: Thanks to Claim 1, since $u_1 = \frac{1}{2}$ and since $\Delta_k = \Delta_{k-1} + d_k \leq \Delta_{k-1}(1+C_2)$, one has

$$\sum_{k=1}^{\infty}\frac{d_k}{(u_k)^2} \leq 4d_1 + \sum_{k=2}^{\infty}\frac{d_k}{(\Delta_{k-1})^2} \leq 4d_1 + (1+C_2)^2\sum_{k=1}^{+\infty}\frac{d_k}{(\Delta_k)^2}.$$

Observe finally that the series $\sum \frac{d_k}{(\Delta_k)^2}$ is convergent, since

$$\sum_{k=2}^{+\infty}\frac{d_k}{(\Delta_k)^2} = \sum_{k=2}^{+\infty}\frac{\Delta_k - \Delta_{k-1}}{(\Delta_k)^2} \leq \sum_{k=2}^{+\infty}\int_{\Delta_{k-1}}^{\Delta_k}\frac{1}{x^2}dx = \int_{d_1}^{+\infty}\frac{1}{x^2}dx.$$

Q.E.D.

CLAIM 3 *The series $\sum \frac{d_k}{u_k}$ is divergent.*

PROOF: Observe that $\frac{u_{k+1}}{u_k} = \left(1+\frac{1}{u_k}\right)^{d_k} \leq e^{d_k/u_k}$ (since $\ln(1+x) \leq x$ for $x > 0$). Taking products over k, this implies

$$\frac{1}{2}u_{n+1} \leq \exp\left(\sum_{k=1}^{n}\frac{d_k}{u_k}\right).$$

The result follows, since $\lim u_n = \infty$ by Claim 1. *Q.E.D.*

Claim 4 *The series* $\sum d_n e^{-\sum_{k=1}^n d_k/u_k}$ *is divergent.*

Proof: Since $\ln(1+x) \geq x - x^2$ for $x \geq 0$, one has $\frac{u_{k+1}}{u_k} = \left(1 + \frac{1}{u_k}\right)^{d_k} \geq \exp\left(\frac{d_k}{u_k} - \frac{d_k}{u_k^2}\right)$ or equivalently,

$$\exp\left(-\frac{d_k}{u_k}\right) \geq \frac{u_k}{u_{k+1}} \times \exp\left(-\frac{d_k}{u_k^2}\right).$$

Taking products over k, and multiplying by d_n, one obtains

$$d_n \exp\left\{-\sum_{k=1}^{n} \frac{d_k}{u_k}\right\} \geq \frac{d_n}{2u_n} \exp\left\{-\sum_{k=1}^{+\infty} \frac{d_k}{(u_k)^2}\right\}. \tag{6.4}$$

The result follows from Claims 2 and 3. *Q.E.D.*

We now conclude. Since $\lim \rho_k = 1$ and $\ln(1+x) \geq x - x^2$ for $x > -\frac{1}{2}$, one has

$$\ln \rho_k \geq \rho_k - 1 - (\rho_k - 1)^2 = -\frac{1}{1+2u_k} - \left(\frac{1}{1+2u_k}\right)^2 \geq -\frac{1}{2u_k} - \frac{1}{(2u_k)^2} \tag{6.5}$$

for all i large enough. Plugging into (6.1), one gets

$$\mathbf{P}_L(\tau > \Delta_k) = \prod_{i=1}^{k} \rho^i 2d_i = \exp\left(\sum_{i=1}^{k} 2d_i \ln \rho_i\right) \geq \exp\left\{-\sum_{i=1}^{k-1} \frac{d_i}{u_i}\right\} \times \exp\left\{-\frac{1}{2}\sum_{i=1}^{k-1} \frac{d_i}{(u_i)^2}\right\}$$

for some $C_3 > 0$ and all $k \geq 1$.[4] From Claims 2 and 4, it follows that the series $\sum d_k \mathbf{P}_L(\tau > \Delta_k)$ is divergent, as desired.

[4]The additional C_3 accounts for the first values of i where (6.5) need not hold.