Supplement to

When Moving-Average Models Meet High-Frequency Data: Uniform Inference on Volatility

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Abstract

This supplement contains lemmas supporting Appendix A and proofs of Corollary 1 and Proposition 1.

Appendix B Auxiliary Lemmas

Lemma B1. *Suppose Assumptions 1 - 5 hold. Then we have that for $\widehat{q}_n$ described in the statement of Theorem 1,*

(i) Under $n^{1/2}\iota^{(n)} \to \infty$, it holds that $\widehat{\mathcal{R}}_n(\widehat{q}_n, b) = o_{\mathrm{P}}(1)$ and $\mathcal{R}^{(n)}(\widehat{q}_n, b) = o_{\mathrm{P}}(1)$.

(ii) Under $n^{1/2}\iota^{(n)} \leq K$, it holds that $\widehat{\mathcal{R}}_n(\widehat{q}_n, s) = o_{\mathrm{P}}(1)$ and $\mathcal{R}^{(n)}(\widehat{q}_n, s) = o_{\mathrm{P}}(1)$.

Proof. Step 1. (Technical preparation) We establish in this step some technical results. We start by introducing some notation:

$$\begin{aligned}
\mathcal{R}_a &:= 2n^{-1}\big(L_n(\widehat{\sigma}_n^2(\widehat{q}_n), \widehat{\gamma}_n(\widehat{q}_n)) - \bar{L}_n^\star(\widehat{\sigma}_n^2(\widehat{q}_n), \widehat{\gamma}_n(\widehat{q}_n))\big),\\
\mathcal{R}_b &:= -2n^{-1}\big(L_n(\widehat{\sigma}_n^2(q_n^\star), \widehat{\gamma}_n(q_n^\star)) - \bar{L}_n^\star(\sigma^{(n)}(q_n^\star)^2, \gamma^{(n)}(q_n^\star))\big),\\
\mathcal{R}_c &:= -2n^{-1}\big(\bar{L}_n^\star(\sigma^{(n)}(q_n^\star)^2, \gamma^{(n)}(q_n^\star)) - \bar{L}_n^\star(C_T, \gamma^{(n)})\big).
\end{aligned}$$

Note that $q_n^\star$ is defined in Assumption 5 and here we suppress the dependence of $q_n^\star$ on k. We recall the relevant definitions

$$\widehat{q}_{n,\mathrm{AIC}} = \arg\min_q \big(2q - 2L_n(\widehat{\sigma}_n^2(q), \widehat{\gamma}_n(q))\big) \quad \text{and} \quad L_n(\widehat{\sigma}_n^2(q), \widehat{\gamma}_n(q)) = \max_{(\sigma^2,\gamma)\in\Pi_n^{(\sigma^2,\gamma)}(q)} L_n(\sigma^2, \gamma),$$

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and notice that the requirement $\widehat{q}_n \geq \widehat{q}_{n,\mathrm{AIC}}$ and the definition of $\Pi_n^{(\sigma^2,\gamma)}$ indicate $L_n(\widehat{\sigma}_n^2(\widehat{q}_n), \widehat{\gamma}_n(\widehat{q}_n)) \geq L_n(\widehat{\sigma}_n^2(\widehat{q}_{n,\mathrm{AIC}}), \widehat{\gamma}_n(\widehat{q}_{n,\mathrm{AIC}}))$. We then obtain

$$-2n^{-1}(\bar{L}_n^\star(\widehat{\sigma}_n^2(\widehat{q}_n), \widehat{\gamma}_n(\widehat{q}_n)) - \bar{L}_n^\star(C_T, \gamma^{(n)})) \leq \mathcal{R}_a + \mathcal{R}_b + \mathcal{R}_c + 2n^{-1}(q_n^\star - \widehat{q}_n). \tag{B.1}$$

We now study the properties of $\mathcal{R}_a$, $\mathcal{R}_b$, and $\mathcal{R}_c$. First, we define Ω_n' as the set of all ω such that $K^{-1} \leq n\Delta_n \leq K$ (it shall not be confused with the matrix Ω_n) and observe that

$$n^{-1}n_t = \frac{1}{T}\int_0^t \xi_s^{-1} ds + o_{\mathrm{P}}(1) \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}(\Omega_n') = 1, \tag{B.2}$$

which are direct results of Lemma 14.1.5 of Jacod and Protter (2011) and Assumption 2. Moreover, step 2 of the proof of Lemma A2 of Da and Xiu (2021) shows that, uniformly over $-\pi \leq \lambda \leq \pi$ and $(\sigma^2, \gamma) \in \Pi_n^{(\sigma^2,\gamma)}$,

$$\frac{1}{K} \leq \sigma^2 \leq K, \quad K^{-1}\chi^2 \leq \sigma^2\Delta_n + f(\lambda;\gamma) \leq K\chi^2, \quad \text{and} \quad \sum_{j=1}^{\infty} j^2|\gamma_j| \leq K\chi^2, \tag{B.3}$$

where $\chi^2 = \chi^2(\sigma^2, \gamma, \Delta_n)$. Straightforwardly, Lemma A9 of Da and Xiu (2021) indicates that for some $\alpha_n \to 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\big|\mathcal{R}_a\big| \leq \alpha_n\Big(n^{-1}\bar{L}_n^\star(C_T, \gamma^{(n)}) - n^{-1}\bar{L}_n^\star(\widehat{\sigma}_n^2(\widehat{q}_n), \widehat{\gamma}_n(\widehat{q}_n)) + 1\Big)\Big) = 1 \quad \text{and} \quad \mathcal{R}_b = o_{\mathrm{P}}(1). \tag{B.4}$$

Note that $\bar{L}_n^\star(C_T, \gamma^{(n)}) - \bar{L}_n^\star(\sigma^2, \gamma)$ is always positive over $\Pi_n^{(\sigma^2,\gamma)}$. Here for the second result we additionally use that $|\mathcal{R}_c| \leq K$ in probability, because of (B.2) and that (B.3) indicates $\frac{1}{n_T}\bar{L}_n^\star(C_T, \gamma^{(n)}) - \frac{1}{n_T}\bar{L}_n^\star(\sigma^{(n)}(q_n)^2, \gamma^{(n)}(q_n)) \leq K$ for all $\{q_n\}$. Further, according to Lemma A10 of Da and Xiu (2021), it holds that for any two sequences $\{q_n\}$ and $\{q_n'\}$ and with probability approaching one,

$$\frac{1}{n_T}\bar{L}_n^\star(\sigma^{(n)}(q_n)^2, \gamma^{(n)}(q_n)) - \frac{1}{n_T}\bar{L}_n^\star(\sigma^{(n)}(q_n')^2, \gamma^{(n)}(q_n')) \sim \psi_n^4\big(\|\widetilde{\kappa}^{(n)}\|_{(q_n')}^2 - \|\widetilde{\kappa}^{(n)}\|_{(q_n)}^2\big). \tag{B.5}$$

On the other hand, Assumption 5 indicates $\psi_n^4\|\widetilde{\kappa}^{(n)}\|_{(q_n)}^2 \to 0$, which, combined with (B.5) and (B.2), shows $\mathcal{R}_c = o_{\mathrm{P}}(1)$. Therefore, in view of (B.1), (B.4), and that $q_n^\star - \widehat{q}_n \leq q_n^\star = o(n)$, we can write for some $\alpha_n \to 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\Big|\bar{L}_n^\star(C_T, \gamma^{(n)}) - \bar{L}_n^\star(\widehat{\sigma}_n^2(\widehat{q}_n), \widehat{\gamma}_n(\widehat{q}_n)\Big| \leq \alpha_n\Big|\bar{L}_n^\star(C_T, \gamma^{(n)}) - \bar{L}_n^\star(\widehat{\sigma}_n^2(\widehat{q}_n), \widehat{\gamma}_n(\widehat{q}_n)) + n\Big|\Big) = 1,$$

which immediately indicates

$$\bar{L}_n^\star(C_T, \gamma^{(n)}) - \bar{L}_n^\star(\widehat{\sigma}_n^2(\widehat{q}_n), \widehat{\gamma}_n(\widehat{q}_n)) = o_{\mathrm{P}}(n). \tag{B.6}$$

Step 2. (Main proof) We start by proving the convergence of $\widehat{\mathcal{R}}_n(\widehat{q}_n, b)$ under $n^{1/2}\iota^{(n)} \to \infty$

and $\widehat{\mathcal{R}}_n(\widehat{q}_n, s)$ under $n^{1/2}\iota^{(n)} \le K$. Since both (σ_n^2, γ_n) and $(C_T, \gamma^{(n)})$ belong to $\Pi_n^{(\sigma^2,\gamma)}$, according to Theorem 4.1.1, Proposition 4.5.3, Proposition 3.2.1, and Theorem 3.1.2 in Brockwell and Davis (1991), there exist unique (χ_n^2, ϕ_n) and $((\chi^{(n)})^2, \phi^{(n)})$ such that for all $-\pi \le \lambda \le \pi$,

$$f(\lambda; \widehat{\sigma}_n^2(\widehat{q}_n), \widehat{\gamma}_n(\widehat{q}_n), \Delta_n) = \chi_n^2 g(\lambda; \phi_n) \quad \text{and} \quad f(\lambda; C_T, \gamma^{(n)}, \Delta_n) = (\chi^{(n)})^2 g(\lambda; \phi^{(n)}), \tag{B.7}$$

where we recall that $f(\lambda; \sigma^2, \gamma, \Delta_n)$ is defined in Section 3.1, and

$$1 + \inf_{z\in\mathbb{C},|z|\le 1} \sum_{j=1}^{\infty} \phi_{n,j} z^j > 0 \quad \text{and} \quad 1 + \inf_{z\in\mathbb{C},|z|\le 1} \sum_{j=1}^{\infty} \phi_j^{(n)} z^j > 0. \tag{B.8}$$

In view of (B.7) and the definition of $\bar{L}_n^\star$, the bound (B.6) can be rewritten in terms of (χ_n^2, ϕ_n) and $((\chi^{(n)})^2, \phi^{(n)})$, which leads to

$$\log \frac{\chi_n^2}{(\chi^{(n)})^2} = o_{\mathrm{P}}(1) \quad \text{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\lambda; C_T, \gamma^{(n)}, \Delta_n)}{f(\lambda; \sigma_n^2, \gamma_n, \Delta_n)} d\lambda - 1 = o_{\mathrm{P}}(1). \tag{B.9}$$

Here we use (B.2) and the fact that $(2\pi)^{-1}\int_{-\pi}^{\pi} g(\lambda; \phi^{(n)})/g(\lambda; \phi_n) d\lambda \ge 1$, indicated by (B.8). With $\chi^{(n)}$ calculated using Assumption 4, the first part of (B.9) indicates that $\log \chi_n^2 = \log(\iota^{(n)})^2 + o_{\mathrm{P}}(1)$ under $n^{1/2}\iota^{(n)} \to \infty$ and that $\mathbb{P}(\chi_n^2 \sim n^{-1}) \to 1$ under $n^{1/2}\iota^{(n)} \le K$. Substituting the estimate of χ_n^2 back into (B.3), plus using the second part of (B.9), plus (B.2), immediately allows us to prove the convergence of $\widehat{\mathcal{R}}_n(\widehat{q}_n, b)$ and $\widehat{\mathcal{R}}_n(\widehat{q}_n, s)$. Now we prove the convergence of $\mathcal{R}^{(n)}(\widehat{q}_n, b)$ and $\mathcal{R}^{(n)}(\widehat{q}_n, s)$. We let

$$\mathcal{R}_d(q) := \bar{L}_n^\star(C_T, \gamma^{(n)})) - \bar{L}_n^\star(\sigma^{(n)}(q)^2, \gamma^{(n)}(q)). \tag{B.10}$$

If we compare (B.10) with (B.6) and compare $\widehat{\mathcal{R}}_n(\widehat{q}_n, b)$ and $\widehat{\mathcal{R}}_n(\widehat{q}_n, s)$ with $\mathcal{R}^{(n)}(\widehat{q}_n, b)$ and $\mathcal{R}^{(n)}(\widehat{q}_n, s)$, a scrutiny of the reasoning above reveals that it is sufficient to prove that $\mathcal{R}_d(\widehat{q}_n) = o_{\mathrm{P}}(1)$ holds under either $n^{1/2}\iota^{(n)} \to \infty$ or $n^{1/2}\iota^{(n)} \le K$. Since according to (B.2) and (B.5) the violation of $\mathcal{R}_d(\widehat{q}_n) = o_{\mathrm{P}}(n)$ indicates the violation of $\bar{\psi}_n^4 \|\bar{\kappa}^{(n)}\|_{(\widehat{q}_n)}^2 = o_{\mathrm{P}}(1)$, which, in view of Assumption 4, contradicts the established fact that $\widehat{\mathcal{R}}_n(\widehat{q}_n, b) = o_{\mathrm{P}}(1)$ under $n^{1/2}\iota^{(n)} \to \infty$ and $\widehat{\mathcal{R}}_n(\widehat{q}_n, s) = o_{\mathrm{P}}(1)$ under $n^{1/2}\iota^{(n)} \le K$. We then indeed have that $\mathcal{R}_d(\widehat{q}_n) = o_{\mathrm{P}}(1)$ holds under either $n^{1/2}\iota^{(n)} \to \infty$ or $n^{1/2}\iota^{(n)} \le K$ and conclude the proof. ∎

Lemma B2. *Suppose Assumptions 1 - 4 hold. Let $\mathcal{U}_n(j)$, $\overline{\mathcal{U}}_n(j)$, $\mathcal{V}_n(j)$, and $\overline{\mathcal{V}}_n(j)$ be defined by (A.4) and (A.6), where q_n is deterministic and we set $\beta_n(\sigma^2, \gamma) = (\sigma^2, \gamma)$. Then*

$$\sum_{j=1}^{J_d} (\mathcal{U}_n(j) - \overline{\mathcal{U}}_n(j) - \mathcal{V}_n(j) + \overline{\mathcal{V}}_n(j)) = o_{\mathrm{P}}(n^{1/2}(q_n+1)^{1/2} + n^{3/4}(\iota^{(n)})^{1/2}).$$

holds if either of the two following conditions is true:

(i) We have $n^{1/2}\iota^{(n)} \to \infty$, and $q_n \le K n^{1/3}$.

(ii) We have $n^{1/2}\iota^{(n)} \le K$, $q_n \le K n^{1/3}$, and $q_n \to \infty$.

Proof. Step 1. (Characterization of $\mathcal{U}_n(j) - \overline{\mathcal{U}}_n(j) - \mathcal{V}_n(j) + \overline{\mathcal{V}}_n(j)$) We start with some notation:

$$\begin{aligned}
\mathcal{R}_{a1}(j) &= \sum_{i=1}^{n_d}\sum_{k=1}^{n_d} \Theta_{i,k}(\Delta_i^n X^{B,r}(j)\Delta_k^n X^{B,r}(j) - \Delta_i^n X^C(j)\Delta_k^n X^C(j) - \Omega_n^B(j)_{ik} + \Omega_n^C(j)_{ik}),\\
\mathcal{R}_{a2}(j) &= \sum_{i=1}^{n_d}\sum_{k=1}^{n_d} \Theta_{i,k}(\Delta_i^n X(j)\Delta_k^n U(j) - \Delta_i^n X^C(j)\Delta_k^n U^C(j)),\\
\mathcal{R}_{a3}(j) &= \sum_{i=1}^{n_d}\sum_{k=1}^{n_d} \Theta_{i,k}(\Delta_i^n U(j)\Delta_k^n U(j) - \Delta_i^n U^C(j)\Delta_k^n U^C(j) - \Omega_n^U(j)_{ik} + \Omega_n^{U,C}(j)_{ik}).
\end{aligned}$$

Here Θ is defined in (A.29). By definition we have for $1 \le j \le J_d$,

$$\mathcal{U}_n(j) - \overline{\mathcal{U}}_n(j) - \mathcal{V}_n(j) + \overline{\mathcal{V}}_n(j) = \mathcal{R}_{a1}(j) + 2\mathcal{R}_{a2}(j) + \mathcal{R}_{a3}(j). \tag{B.11}$$

The lemma then follows if it holds for all $s \in \{1,2,3\}$ that

$$\sum_{j=1}^{J_d} \mathcal{R}_{as}(j) = o_{\mathrm{P}}(n^{1/2}(q_n+1)^{1/2} + n^{3/4}(\iota^{(n)})^{1/2}). \tag{B.12}$$

Step 2. (Decompositions of $\mathcal{R}_{as}(j)$) This step is devoted to decompositions of $\mathcal{R}_{as}(j)$. Let

$$\begin{aligned}
\mathcal{R}_{b1}(j)_{k,l} &= \Delta_k^n X^{B,r}(j)\Delta_l^n X^{B,r}(j) - \Delta_k^n X^C(j)\Delta_l^n X^C(j) - \Omega_n^B(j)_{kl} + \Omega_n^C(j)_{kl},\\
\overline{\mathcal{R}}_{b1}(j,m,p) &= \sum_{k=1}^{n_d}\sum_{l=1}^{n_d} \widetilde{O}(m,p)_{k,l}\mathcal{R}_{b1}(j)_{k,l},
\end{aligned}$$

where $\widetilde{O}(m,p)$ introduced in (A.7). We can then write that for $1 \le j \le J_d$,

$$\mathcal{R}_{a1}(j) = \sum_{m=0}^{\widetilde{J}_d - 1} \sum_{p=1-\bar{n}_d(m)}^{\bar{n}_d(m)} \widetilde{\Theta}(m,p)\overline{\mathcal{R}}_{b1}(j,m,p). \tag{B.13}$$

Here $\widetilde{J}_d$ and $\bar{n}_d(m)$ are defined above (A.7), and $\widetilde{\Theta}(m,p)$ is defined in (A.30). Now we further decompose $\overline{\mathcal{R}}_{b1}(j,m,p)$. To do so, we define

$$\begin{aligned}
\mathcal{R}_{b2}(j)_{k,l} &= \int_{t(j)_{k-1}}^{t(j)_k} \mu_s^r ds \int_{t(j)_{l-1}}^{t(j)_l} \mu_s^r ds, & \overline{\mathcal{R}}_{b2}(j,m,p) &= \sum_{k=1}^{n_d}\sum_{l=1}^{n_d} \widetilde{O}(m,p)_{k,l}\mathcal{R}_{b2}(j)_{k,l},\\
\mathcal{R}_{b3}(j)_{k,l} &= 2\int_{t(j)_{k-1}}^{t(j)_k} \mu_s^r ds \Delta_l^n \bar{X}^B(j), & \overline{\mathcal{R}}_{b3}(j,m,p) &= \sum_{k=1}^{n_d}\sum_{l=1}^{n_d} \widetilde{O}(m,p)_{k,l}\mathcal{R}_{b3}(j)_{k,l},\\
\mathcal{R}_{b4}(j)_k &= \Delta_k^n \bar{X}^B(j)^2 - \int_{t(j)_{k-1}}^{t(j)_k} \sigma_s^2 ds, & \overline{\mathcal{R}}_{b4}(j,m,p) &= \sum_{k=1}^{n_d} \widetilde{O}(m,p)_{k,k}\mathcal{R}_{b4}(j)_k,
\end{aligned}$$

$$\mathcal{R}_{b5}(j)_k = -\sigma_C^2(j)\xi_C(j)\left(\frac{(\Delta_k^n W(j))^2}{\xi(j)_{k-1}} - \frac{T}{n}\right), \qquad \overline{\mathcal{R}}_{b5}(j,m,p) = \sum_{k=1}^{n_d} \widetilde{O}(m,p)_{k,k}\mathcal{R}_{b5}(j)_k,$$

$$\mathcal{R}_{b6}(j)_{k,l} = 2\mathcal{R}_{b8}(j)_k\Delta_l^n \bar{X}^B(j), \qquad \overline{\mathcal{R}}_{b6}(j,m,p) = \sum_{k=1}^{n_d}\sum_{l=k+1}^{n_d} \widetilde{O}(m,p)_{k,l}\mathcal{R}_{b6}(j)_{k,l},$$

$$\mathcal{R}_{b7}(j)_{k,l} = 2\Delta_k^n X^C(j)\mathcal{R}_{b8}(j)_l, \qquad \overline{\mathcal{R}}_{b7}(j,m,p) = \sum_{k=1}^{n_d}\sum_{l=k+1}^{n_d} \widetilde{O}(m,p)_{k,l}\mathcal{R}_{b7}(j)_{k,l},$$

where we use the notation $\Delta_i^n \bar{X}^B(j) = \int_{t(j)_{i-1}}^{t(j)_i} \sigma_s dW_s$ and $\mathcal{R}_{b8}(j)_i = \Delta_i^n \bar{X}^B(j) - \Delta_i^n X^C(j)$. Using the definitions of $\Omega_n^B(j)$ and $\Omega_n^C(j)$, we obtain that for $1 \le j \le J_d$,

$$\overline{\mathcal{R}}_{b1}(j,m,p) = \sum_{s=2}^{7} \overline{\mathcal{R}}_{bs}(j,m,p). \tag{B.14}$$

Next, we decompose $\mathcal{R}_{a2}(j)$. Moreover, we set $\bar{\eta}(j)_k = \eta(j)_k - \eta_C(j)$ and define

$$\bar{U}(j)_k = \iota^{(n)} \sum_{m=-\infty}^{k} \bar{\eta}(j)_k \theta_{k-m}^{(n)} \varepsilon_C(j)_m, \quad \widetilde{U}(j)_k = \iota^{(n)}\eta(j)_k \sum_{m=-\infty}^{0} \theta_{k-m}^{(n)}(\widetilde{\varepsilon}(j)_m - \varepsilon(j)_m). \tag{B.15}$$

We further define

$$\begin{aligned}
\overline{\mathcal{R}}_{c1}(j,m,p) &= \sum_{i=1}^{n_d}\sum_{k=1}^{n_d} \widetilde{O}(m,p)_{i,k} \int_{t(j)_{i-1}}^{t(j)_i} \mu_s^r ds \Delta_k^n U(j), \\
\overline{\mathcal{R}}_{c2}(j,m,p) &= \sum_{i=1}^{n_d}\sum_{k=1}^{n_d} \widetilde{O}(m,p)_{i,k} \Delta_i^n \bar{X}^B(j)\Delta_k^n \bar{U}(j), \\
\overline{\mathcal{R}}_{c3}(j,m,p) &= \sum_{i=1}^{n_d}\sum_{k=1}^{n_d} \widetilde{O}(m,p)_{i,k} (\Delta_i^n \bar{X}^B(j) - \Delta_i^n X^C(j))\Delta_k^n U^C(j), \\
\mathcal{R}_{c4}(j) &= \sum_{i=1}^{n_d} \Delta_i^n \bar{X}^B(j)\Big(-\sum_{k=1}^{n_d} \Delta\Theta_{i,k}\widetilde{U}(j)_k + \Theta_{i,n_d}\widetilde{U}(j)_{n_d} - \Theta_{i,1}\widetilde{U}(j)_0\Big).
\end{aligned}$$

This leads to, by observing the relation $U(j)_k - U^C(j)_k = \bar{U}(j)_k + \widetilde{U}(j)_k$, which in turn is a direct result of Assumption 3 and the definition of $U^C(j)_k$, that for $1 \le j \le J_d$,

$$\mathcal{R}_{a2}(j) = \sum_{s=1}^{4} \mathcal{R}_{cs}(j), \tag{B.16}$$

where $\mathcal{R}_{cs}(j) = \sum_{m=0}^{\widetilde{J}_d-1}\sum_{p=1-\bar{n}_d(m)}^{\bar{n}_d(m)} \widetilde{\Theta}(m,p)\overline{\mathcal{R}}_{cs}(j,m,p)$ for $s \in \{1,2,3\}$. We now decompose $\mathcal{R}_{a3}(j)$. For any double-indexed variable $A_{i,k}$, we set $\Delta A_{i,k} = A_{i,k+1} - A_{i,k}$ and $\widetilde{\Delta} A_{i,k} = \Delta A_{i+1,k} - \Delta A_{i,k}$.

Next we introduce shorthand notation $\bar{\kappa}_j^{(n)} = (\iota^{(n)})^2\kappa_j$ and define

$$
\begin{aligned}
\mathcal{R}_{d1}(j) &= \sum_{i=1}^{n_d-1}\sum_{k=1}^{n_d-1}\widetilde{\Delta}\Theta_{i,k}\eta(j)_i\eta(j)_k\bigg(\sum_{l=-\infty}^{i\wedge k}\theta_{i-l}^{(n)}\theta_{k-l}^{(n)}\varepsilon(j)_l\varepsilon(j)_l - \kappa_{|i-k|}^{(n)}\bigg),\\
\mathcal{R}_{d2}(j) &= -\sum_{i=1}^{n_d-1}\sum_{k=1}^{n_d-1}\widetilde{\Delta}\Theta_{i,k}\eta_C^2(j)\bigg(\sum_{l=-\infty}^{i\wedge k}\theta_{i-l}^{(n)}\theta_{k-l}^{(n)}\varepsilon_C(j)_l\varepsilon_C(j)_l - \kappa_{|i-k|}^{(n)}\bigg),\\
\mathcal{R}_{d3}(j) &= 2\sum_{i=1}^{n_d-1}\sum_{k=1}^{n_d-1}\sum_{l=1}^{i\wedge k}\sum_{m=l+1}^{k}\widetilde{\Delta}\Theta_{i,k}(\eta(j)_i\eta(j)_k - \eta_C^2(j))\theta_{i-l}^{(n)}\theta_{k-m}^{(n)}\varepsilon(j)_l\varepsilon(j)_m,\\
\mathcal{R}_{d4}(j) &= 2\sum_{i=1}^{n_d-1}\sum_{k=1}^{n_d-1}\sum_{l=-\infty}^{0}\sum_{m=l+1}^{k}\eta(j)_i\eta(j)_k\widetilde{\Delta}\Theta_{i,k}\theta_{i-l}^{(n)}\theta_{k-m}^{(n)}\varepsilon(j)_l\varepsilon(j)_m,\\
\mathcal{R}_{d5}(j) &= -2\sum_{i=1}^{n_d-1}\sum_{k=1}^{n_d-1}\sum_{l=-\infty}^{0}\sum_{m=l+1}^{k}\eta_C(j)\eta_C(j)\widetilde{\Delta}\Theta_{i,k}\theta_{i-l}^{(n)}\theta_{k-m}^{(n)}\varepsilon_C(j)_l\varepsilon_C(j)_m,\\
\mathcal{R}_{d6}(j) &= 2\sum_{k=1}^{n_d-1}(\Delta\Theta_{1,k}U(j)_0 - \Delta\Theta_{n_d,k}U(j)_{n_d})U(j)_k,\\
\mathcal{R}_{d7}(j) &= -2\sum_{k=1}^{n_d-1}(\Delta\Theta_{1,k}\eta(j)_0\eta(j)_k\bar{\kappa}_k^{(n)} - \Delta\Theta_{n_d,k}\eta(j)_{n_d}\eta(j)_k\bar{\kappa}_{n_d-k}^{(n)}),\\
\mathcal{R}_{d8}(j) &= -2\sum_{k=1}^{n_d-1}(\Delta\Theta_{1,k}U^C(j)_0 - \Delta\Theta_{n_d,k}U^C(j)_{n_d})U^C(j)_k,\\
\mathcal{R}_{d9}(j) &= 2\sum_{k=1}^{n_d-1}(\Delta\Theta_{1,k}\eta_C^2(j)\bar{\kappa}_k^{(n)} - \Delta\Theta_{n_d,k}\eta_C^2(j)\bar{\kappa}_{n_d-k}^{(n)}),\\
\mathcal{R}_{d10}(j) &= \Theta_{1,1}(U_0U_0 + U_{n_d}U_{n_d} - U_0^CU_0^C - U_{n_d}^CU_{n_d}^C) - 2\Theta_{n_d,1}(U_0U_{n_d} - U_0^CU_{n_d}^C)\\
&\quad -\Theta_{1,1}(\eta(j)_0\eta(j)_0 + \eta(j)_{n_d}\eta(j)_{n_d} - 2\eta_C^2(j))\bar{\kappa}_0^{(n)} + 2\Theta_{n_d,1}(\eta(j)_{n_d}\eta(j)_0 - \eta_C^2(j))\bar{\kappa}_{n_d}^{(n)},\\
\mathcal{R}_{d11}(j) &= -2\sum_{i=1}^{n_d}\sum_{k=1}^{n_d-1}\Delta\Theta_{i,k+1}(\bar{\kappa}_{i+k+1}^{(n)} - \bar{\kappa}_{i+k}^{(n)}) + 2\sum_{k=1}^{n_d-1}\Delta\Theta_{1,k+1}(\bar{\kappa}_{k+1}^{(n)} + \bar{\kappa}_{k+n_d+1}^{(n)})\\
&\quad -(4\Theta_{n_d,1}\bar{\kappa}_{n_d+1}^{(n)} - 2\Theta_{1,1}\bar{\kappa}_1^{(n)} - 2\Theta_{n_d,n_d}\bar{\kappa}_{2n_d+1}^{(n)}).
\end{aligned}
$$

Using the definitions of $\Omega_n^U(j)$ and $\Omega_n^{U,C}(j)$, one can verify that for $1 \le j \le J_d$,

$$\mathcal{R}_{a3}(j) = (\iota^{(n)})^2\sum_{l=1}^{5}\mathcal{R}_{dl}(j) + \sum_{l=6}^{11}\mathcal{R}_{dl}(j). \tag{B.17}$$

Step 3. (Bounds of $\widetilde{O}(m,p)$ and $\widetilde{\Theta}(m,p)$) We start with $\widetilde{O}(m,p)$. In the rest of the proof, we omit mentioning the argument m of $\widetilde{n}_d$ (defined above (A.7)) and $\bar{n}_d$ unless necessary. It holds by definition that for all $1 \le k, l \le n_d$, all $0 \le m \le \widetilde{J}_d - 1$, and all $1 - \bar{n}_d \le p \le \bar{n}_d$,

$$|\widetilde{O}(m,p)_{k,l}| \le K n_d^{-1}\bar{n}_d\Big(1 \wedge \big(\big|n_d^{-1}\bar{n}_d|k-l| - |p|\big|^{-1} + \big|n_d^{-1}\bar{n}_d(k+l) - |p|\big|^{-1}\big)\Big). \tag{B.18}$$

Now we provide the bound of $\widetilde{\Theta}(m,p)$. From the definition of $\Theta_{i,k}$, we can write

$$\Theta_{i,k} = -\frac{\partial \Omega_{n_d}(\beta^{(n)})^{-1}_{ik}}{\partial \beta}\partial\bar{\Xi}^{\star}_n(\bar{\beta}^{(n)})^{-1}(\partial\sigma^2_n)^{\intercal}.$$

We further notice $\partial\bar{\Xi}^{\star}_n(\bar{\beta}^{(n)}) = \frac{n_T}{4\pi n}\int_{-\pi}^{\pi}\big(\partial \log f(\lambda;\bar{\beta}^{(n)},\Delta_n)/\partial\beta\big)^{\intercal}\big(\partial \log f(\lambda;\bar{\beta}^{(n)},\Delta_n)/\partial\beta\big)d\lambda$. We set the bijection β_n to be identity. Following the rule of matrix differentiation, and using the definition of Ω, we can further write

$$\Theta = 2\Omega_{n_d}((\sigma^{(n)})^2,\gamma^{(n)},\Delta_n)^{-2}\Big(\Delta_n\partial\bar{\Xi}^{\star}_n(\bar{\beta}^{(n)})^{-1}_{1,1}\mathbb{I}_{n_d} + \sum_{j=0}^{q_n}\partial\bar{\Xi}^{\star}_n(\bar{\beta}^{(n)})^{-1}_{1,j+2}(2\mathbb{I}_{n_d} - \mathbb{F}^1_{n_d})\mathbb{F}^j_{n_d}\Big). \tag{B.19}$$

Then the definition of $\widetilde{\Theta}$ given by (A.29) indicates

$$\widetilde{\Theta} = 2V_{n_d}((\sigma^{(n)})^2,\gamma^{(n)},\Delta_n)^{-2}\Big(\Delta_n\partial\bar{\Xi}^{\star}_n(\bar{\beta}^{(n)})^{-1}_{1,1}\mathbb{I}_{n_d} + \sum_{j=0}^{q_n}\partial\bar{\Xi}^{\star}_n(\bar{\beta}^{(n)})^{-1}_{1,j+2}(2\mathbb{I}_{n_d} - \mathbb{D}^1_{n_d})\mathbb{D}^j_{n_d}\Big). \tag{B.20}$$

This is the direct result of $\mathbb{D}^j_m = O_m\mathbb{F}^j_m O_m$ from Lemma A1 of Da and Xiu (2021). Now we define a function $\check{\Theta}(\lambda)$ as

$$\check{\Theta}(\lambda) = 2f(\lambda;(\sigma^{(n)})^2,\gamma^{(n)},\Delta_n)^{-2}\Psi(\lambda;(\sigma^{(n)})^2,\gamma^{(n)},\Delta_n),$$

where $\Psi(\lambda;\sigma^2,\gamma,\Delta_n) := \big(\partial f(\lambda;\sigma^2,\gamma,\Delta_n)/\partial(\sigma^2,\gamma)\big)\partial\bar{\Xi}^{\star}_n(\bar{\beta}^{(n)})^{-1}(1,0_{q_n+1})$. We note $\widetilde{\Theta}_{i,j} = \delta_{i,j}\check{\Theta}\big(\frac{j\pi}{n_d+1}\big)$. Now we further define for $-\pi \le \lambda \le \pi$,

$$\bar{\Theta}(\lambda;m) = \check{\Theta}\Big(\frac{\widetilde{n}_d\pi + \bar{n}_d|\lambda| + 1/2}{n_d+1}\Big) \quad \text{and} \quad \rho_{\Theta}(m)_h = \frac{1}{2\pi}\int_{-\pi}^{\pi}\bar{\Theta}(\lambda;m)e^{\mathbb{i}h\lambda}d\lambda.$$

Then we can write, in view of the definition of $\widetilde{\Theta}(m,p)$, that for all $1 - \bar{n}_d \le p \le \bar{n}_d$,

$$\widetilde{\Theta}(m,p) = (4\bar{n}_d)^{-1}\sum_{i=1-\bar{n}_d}^{\bar{n}_d}\bar{\Theta}\Big(\frac{\pi(i-1/2)}{\bar{n}_d};m\Big)e^{\mathbb{i}\pi\frac{(i-1/2)p}{\bar{n}_d}} = \frac{1}{2}\sum_{h=-\infty}^{\infty}\rho_{\Theta}(m)_{2h\bar{n}_d+p}. \tag{B.21}$$

Here we use Theorem II.8.1 of Zygmund (2002) and the fact that $\frac{1}{2\bar{n}_d}\sum_{i=1-\bar{n}_d}^{\bar{n}_d}\exp\Big(\pi\mathbb{i}\frac{ik}{\bar{n}_d}\Big) = \delta_{k,0}$ for $-\bar{n}_d \le k \le \bar{n}_d$. Now we provide bounds on $\rho_{\Theta}(m)_h$. We first consider the case $n^{1/2}\iota^{(n)} \to \infty$ and define a $(q_n+2)\times(q_n+2)$ matrix

$$C_\Psi^{-1}(\lambda;\sigma^2,\gamma)=\frac{\partial(\sigma^2,f(\lambda;\sigma^2,\gamma,\Delta_n),\bar\gamma)}{\partial(z,\phi)}C^{-1}(z,\phi)\Big(\frac{\partial(\sigma^2,f(\lambda;\sigma^2,\gamma,\Delta_n),\bar\gamma)}{\partial(z,\phi)}\Big)^\intercal.$$

Here $\bar\gamma\coloneqq(\gamma_1,\dots,\gamma_{q_n})^\intercal$, (z,ϕ) and $C(z,\phi)$ are both introduced in the proof of Lemma A5 of Da and Xiu (2021). And it apparently holds that $\Psi(\lambda;\sigma^2,\gamma,\Delta_n)=C_\Psi^{-1}(\lambda;\sigma^2,\gamma)_{1,2}$. Hence, following the same reasoning as in that proof and using the definition of $\bar\Theta(\lambda;m)$, plus noting the relation that $\sum_{i=0}^{q_n}(2-\delta_{i,0})\cos i\lambda=\sin\big((q_n+1/2)\lambda\big)/\sin(\lambda/2)$, we obtain that in restriction to Ω_n' (introduced before (B.2)) and for all $0\le m\le\widetilde J_d-1$,

$$(n\bar J_d)^{-1}\sum_{h=0}^{\infty}\big|\rho_\Theta(m)_h\big|\le\frac{K(m+1)}{2^m}\wedge\frac{K(m+1)\bar J_d^2/(n^{1/2}\iota^{(n)})^2}{2^{3m}},\tag{B.22}$$

and that in restriction to Ω_n' and for all h and all fixed m,

$$(n\bar J_d)^{-1}\big|\rho_\Theta(m)_h\big|\le K(h^{-2}\vee1).\tag{B.23}$$

Here we also use the proof of Theorem II.4.7 of Zygmund (2002) and exploit properties of $\sigma^{(n)}(q_n)$ and $\gamma^{(n)}(q_n)$ provided by (B.3). For the case $n^{1/2}\iota^{(n)}\le K$, in view of the proof of Lemma A6 of Da and Xiu (2021), we obtain that (B.22) and (B.23) still hold. Combining (B.22) and (B.23) with (B.21), we conclude that, again in restriction to Ω_n' and for all $0\le m\le\widetilde J_d-1$,

$$(n\bar J_d)^{-1}\sum_{p=1-\bar n_d}^{\bar n_d}\big|\widetilde\Theta(m,p)\big|\le\frac{K(m+1)}{2^m}\wedge\frac{K(m+1)\bar J_d^2/(n^{1/2}\iota^{(n)})^2}{2^{3m}},\tag{B.24}$$

and that in restriction to Ω_n', for all $1-\bar n_d\le p\le\bar n_d$, and for all fixed m,

$$(n\bar J_d)^{-1}\big|\widetilde\Theta(m,p)\big|\le K(p^{-2}\wedge1).\tag{B.25}$$

Step 4. (Bounds of $\Theta_{i,k}$, $\Delta\Theta_{i,k}$, and $\widetilde\Delta\Theta_{i,k}$) Now we provide bounds on $\Theta_{i,k}$, $\Delta\Theta_{i,k}$, and $\widetilde\Delta\Theta_{i,k}$. We start by noting the expression of Θ has been given by (B.19). According to Lemma A2 of Da and Xiu (2021), we can write the expression of $\Omega_{n_d}^{-1}$ as

$$(\Omega_{n_d}^{-1})_{i,k}=\rho_{|i-k|}-\rho_{i+k}-\rho_{2n_d+2-i-k},$$

Both ρ and z_n^* appearing below are functions of $(\sigma_n^2,\gamma_n,\Delta_n)$ and are introduced in the statement of that lemma. Because the lemma has provided precise characterization of ρ_h, $\rho_h-\rho_{h+1}$, and $2\rho_{h+1}-\rho_h-\rho_{h+2}$, plus the observation that $i+k\ge|i-k|$ and $2n_d+2-i-k\ge|i-k|$ for all $1\le i,k\le n_d$, tedious algebra leads to that uniformly over all sequences $\{(\sigma_n^2,\gamma_n)\in\Pi_n^{(\sigma^2,\gamma)}(q_n):n\ge1\}$ which satisfy either $\Delta_n^{-1}\chi^2(\sigma_n^2,\gamma_n,\Delta_n)\to\infty$ or $\Delta_n^{-1}\chi^2(\sigma_n^2,\gamma_n,\Delta_n)\le K$, and for all $1\le i,k\le n_d$,

$$\begin{aligned}
|\Omega_{n_d}(\sigma_n^2,\gamma_n,\Delta_n)^{-1}_{i,k}| &\lesssim \Delta_n^{-1/2}\chi_n^{-1}(1-(z_n^+)^{2k}-(z_n^+)^{2n_d+2-2k})+\frac{1}{\chi_n^2}\wedge\frac{1}{(i-k)^2\Delta_n},\\
|\Delta\Omega_{n_d}(\sigma_n^2,\gamma_n,\Delta_n)^{-1}_{i,k}| &\lesssim \chi_n^{-2}(z_n^+)^{|i-k|}+\Delta_n^{1/2}\chi_n^{-1}\Big(\frac{1}{\chi_n^2}\wedge\frac{1}{(i-k)^2\Delta_n}\Big),\\
|\Delta\Omega_{n_d}(\sigma_n^2,\gamma_n,\Delta_n)^{-2}_{i,k}| &\lesssim \Delta_n^{-1}\chi_n^{-2}(z_n^+)^{|i-k|}+\Delta_n^{-1/2}\chi_n^{-1}\Big(\frac{1}{\chi_n^2}\wedge\frac{1}{(i-k)^2\Delta_n}\Big),\\
|\widetilde{\Delta}\Omega_{n_d}(\sigma_n^2,\gamma_n,\Delta_n)^{-1}_{i,k}| &\lesssim \Delta_n^{1/2}\chi_n^{-3}(z_n^+)^{|i-k|}+\frac{1}{\chi_n^2(i-k)^2},\\
|\widetilde{\Delta}\Omega_{n_d}(\sigma_n^2,\gamma_n,\Delta_n)^{-2}_{i,l}| &\lesssim \Delta_n^{-1/2}\chi_n^{-3}(z_n^+)^{|i-k|}+\Delta_n^{1/2}\chi_n^{-3}\Big(\frac{1}{\chi_n^2}\wedge\frac{1}{(i-k)^2\Delta_n}\Big).
\end{aligned}$$

Here $z_n^+:=\max\{z_n^*,1/2\}$ and $\chi_n^2=\chi^2(\sigma_n^2,\gamma_n,\Delta_n)$. We additionally observe that we can write, for all (σ^2,γ) and m,

$$2\mathbb{I}_m-\mathbb{F}_m^1=(\Omega_m(\sigma^2,\gamma)-\sigma^2\Delta_n\mathbb{I}_m)O_mD_m(\gamma)^{-1}O_m. \tag{B.26}$$

We also notice that $\mathbb{F}_{n_d}^j$ has a very simple structure. Therefore, we can calculate that, in restriction to Ω_n', under either $n^{1/2}\iota^{(n)}\to\infty$ or $n^{1/2}\iota^{(n)}\le K$, and for all $1\le i,k\le n_d$,

$$|\Theta_{i,k}| \lesssim n(z_n^+)^{(|i-k|-q)_+}(1-(z_n^+)^{2k}-(z_n^+)^{2n_d+2-2k})+\frac{n}{(|i-k|-q)_++1}-\frac{n}{|i-k|+2}, \tag{B.27}$$

$$|\Delta\Theta_{i,k}| \lesssim n\Delta_n^{1/2}\chi_n^{-1}(z_n^+)^{(|i-k|-q)_+}+n\Delta_n^{1/2}\chi_n^{-1}\Big(\frac{1}{(|i-k|-q)_++1}-\frac{1}{|i-k|+2}\Big), \tag{B.28}$$

$$|\widetilde{\Delta}\Theta_{i,k}| \lesssim n\Delta_n\chi_n^{-2}(z_n^+)^{(|i-k|-q)_+}+n\Delta_n\chi_n^{-2}\Big(\frac{1}{(|i-k|-q)_++1}-\frac{1}{|i-k|+2}\Big). \tag{B.29}$$

Here z_n^+ and χ_n are evaluated at $((\sigma^{(n)})^2,\gamma^{(n)},\Delta_n)$ and we clearly have $\chi_n\sim\iota^{(n)}+n^{-1/2}$. We also use the properties of $\partial\bar{\Xi}_n^\star(\bar{\beta}^{(n)})^{-1}$ indicated by the proof of Lemmas A5 and A6 of Da and Xiu (2021).

Step 5. (Bound of $\mathcal{R}_{a1}(j)$) According to (B.13), we can write

$$\begin{aligned}
\mathbb{E}\Big|\mathbb{1}_{\Omega_n'}\frac{1}{n}\sum_{j=1}^{J_d}\mathcal{R}_{a1}(j)\Big| &\le \frac{1}{n}\sum_{m=0}^{\widetilde{J}_d-1}\sum_{p=1-\bar{n}_d}^{\bar{n}_d}\mathbb{E}\Big|\mathbb{1}_{\Omega_n'}\widetilde{\Theta}(m,p)\sum_{j=1}^{J_d}\overline{\mathcal{R}}_{b1}(j,m,p)\Big|\\
&\le K\bar{J}_d\sum_{m=0}^{\widetilde{J}_d-1}(m+1)\Big(\frac{1}{2^m}\wedge\frac{\bar{J}_d^2/(n^{1/2}\iota^{(n)})^2}{2^{3m}}\Big)\sup_p\mathbb{E}\Big|\mathbb{1}_{\Omega_n'}\sum_{j=1}^{J_d}\overline{\mathcal{R}}_{b1}(j,m,p)\Big|.
\end{aligned}\tag{B.30}$$

Here the range of p over which the supremum in the last line is taken is clear from the context and is omitted, and the second inequality uses the bound on $\widetilde{\Theta}(m,p)$ provided by (B.24). Now we bound $\sup_{m,p}\mathbb{E}\left|\sum_{j=1}^{J_d}\overline{\mathcal{R}}_{b1}(j,m,p)\right|$. Guided by the relation (B.14), we aim to prove that for all

$s \in \{2, 3, 4, 5, 6, 7\}$,

$$\sup_{m,p} \left(2^{-m/2} \bar{J}_d \mathbb{E} \left| \mathbb{1}_{\Omega'_n} \sum_{j=1}^{J_d} \overline{\mathcal{R}}_{bs}(j, m, p) \right| \right) = o(n^{-1/2}(q_n + 1)^{1/2} + n^{-1/4}(\iota^{(n)})^{1/2}), \tag{B.31}$$

where the supremum is taken over $0 \leq m \leq \widetilde{J}_d - 1$ and $1 - \bar{n}_d \leq p \leq \bar{n}_d$. We start with $\overline{\mathcal{R}}_{b2}(j, m, p)$. We have that (B.31) holds for $s = 2$ because

$$\sup_j \mathbb{E}|\overline{\mathcal{R}}_{b2}(j, m, p)| \leq Kn^{-2} \sup_j \sum_{k=1}^{n_d} \sum_{l=1}^{n_d} |\widetilde{O}(m, p)_{k,l}| \leq Kn^{-2} n_d \log n.$$

The first inequality comes from bounds on μ_s^r and $\mathbb{E}|t(j)_k - t(j)_{k-1}|$ as direct results of Assumption A1. The second inequality comes from the bound on $\widetilde{O}(m, p)_{k,l}$ provided by (B.18). Now we consider $\overline{\mathcal{R}}_{b3}(j, m, p)$. In view of its definition, we write

$$\begin{aligned} \sup_j \mathbb{E}|\overline{\mathcal{R}}_{b3}(j, m, p)|^2 &\leq K \sup_j n^{-2} \sum_{k=1}^{n_d} \sum_{k'=1}^{n_d} \mathbb{E} \left| \sum_{l=1}^{n_d} \sum_{l'=1}^{n_d} \widetilde{O}(m, p)_{k,l} \widetilde{O}(m, p)_{k',l'} \Delta_l^n \bar{X}^B(j) \Delta_{l'}^n \bar{X}^B(j) \right| \\ &\leq Kn^{-2} n_d^2 \sup_{j,k} \mathbb{E}\left[\Big(\sum_{l=1}^{n_d} \widetilde{O}(m, p)_{k,l} \Delta_l^n \bar{X}^B(j) \Big)^2 \right] \leq Kn^{-3} n_d \bar{n}_d. \end{aligned} \tag{B.32}$$

We omit the range of j, which is $1 \leq j \leq J_d$. Using Assumption A1, the first inequality comes from Hölder's inequality, the second inequality from Cauchy-Schwarz, and the last inequality from (B.18) and Burkholder-Davis-Gundy inequality. On the other hand, we have the well-known result (see Section 2.1.5 of Jacod and Protter (2011)) that under Assumption A1 and for two finite stopping times $S \leq S'$ and some $p \geq 0$, and for a process A which is one of μ, σ, ξ, ξ^{-1} and η,

$$\mathbb{E}\Big(\sup_{S \leq s \leq S'} (\|A_s - A_S\|^p) | \mathcal{F}_S \Big) \leq \mathbb{E}((S' - S)^{1 \wedge (p/2)} | \mathcal{F}_S). \tag{B.33}$$

Applying (B.33) for the process μ to the equation

$$\mathbb{E}(\overline{\mathcal{R}}_{b3}(j, m, p)|\mathcal{F}_{t(j)_0}) = 2 \sum_{k=1}^{n_d} \mathbb{E}\Big(\int_{t(j)_{k-1}}^{t(j)_k} (\mu_s^r - \mu_{t(j)_0}) ds \sum_{k=l}^{n_d} \widetilde{O}(m, p)_{k,l} \Delta_l^n \bar{X}^B(j) | \mathcal{F}_{t(j)_0} \Big),$$

we obtain,

$$\sup_j |\mathbb{E}(\overline{\mathcal{R}}_{b3}(j, m, p)|\mathcal{F}_{t(j)_0})| = o(n^{-3/2} n_d^{1/2} \bar{n}_d^{1/2}). \tag{B.34}$$

Using (B.34) and (B.32) and applying Cauchy-Schwarz inequality, we obtain

$$\sup_{j<j'} |\mathbb{E}(\overline{\mathcal{R}}_{b3}(j, m, p) \overline{\mathcal{R}}_{b3}(j', m, p))| = o(n^{-3} n_d \bar{n}_d). \tag{B.35}$$

Combination of (B.32) and (B.35) immediately proves (B.31) for $s = 3$, given the definition of $\bar{n}_d$. Next, we study $\overline{\mathcal{R}}_{b4}(j, m, p)$ and $\overline{\mathcal{R}}_{b5}(j, m, p)$. We notice that for $s \in \{4, 5\}$, $|\mathbb{E}(\mathcal{R}_{bs}(j)_k, \mathcal{R}_{bs}(j')_{k'})| \leq K\delta_{j,j'}\delta_{k,k'}n^{-2}$. Therefore we are able to write for $s \in \{4, 5\}$,

$$\mathbb{E}\Big|\overline{\mathcal{R}}_{bs}(j, m, p)\overline{\mathcal{R}}_{bs}(j', m, p)\Big|^2 \leq Kn^{-2}\sum_{k=1}^{n_d}\sum_{l=1}^{n_d}|\widetilde{O}(m, p)_{k,k}\widetilde{O}(m, p)_{l,l}|\delta_{j,j'}\delta_{k,l} \leq Kn^{-3}\bar{n}_d^2.$$

Here the last step comes from the bound on $\widetilde{O}(m, p)_{k,l}$ provided by (B.18). Combined with Cauchy-Schwarz inequality, this result immediately leads to that (B.31) holds for $s \in \{4, 5\}$. We move to $\overline{\mathcal{R}}_{b6}(j, m, p)$. We have

$$\begin{aligned}
\sup_j \mathbb{E}(\overline{\mathcal{R}}_{b6}(j, m, p)^2) &= 4\sup_j \mathbb{E}\Big[\sum_{l=1}^{n_d}\Big(\sum_{k=1}^{l-1}\widetilde{O}(m, p)_{k,l}\mathcal{R}_{b8}(j)_k\Big)^2\int_{t(j)_{l-1}}^{t(j)_l}\sigma_s^2 ds\Big] \\
&\leq Kn^{-1}n_d\sup_j\sup_{1\leq l\leq n_d}\mathbb{E}\Big[\Big(\sum_{k=1}^{l-1}\widetilde{O}(m, p)_{k,l}\mathcal{R}_{b8}(j)_k\Big)^2\Big] \\
&\leq Kn^{-2}\bar{n}_d\sup_{1\leq j\leq J_d}\sup_{1\leq k\leq n_d}\mathbb{E}\Big[\sup_{t(j)_{k-1}\leq s\leq t(j)_k}|(\sigma_s - \sigma_{t(j)_0})^4 + (\xi_s - \xi_{t(j)_0})^4|\Big]^{1/2} \\
&\leq Kn_d^{1/2}\bar{n}_d n^{-5/2}. \qquad\qquad (B.36)
\end{aligned}$$

Here we follow the same reasoning of (B.32) and the last step utilizes (B.33) for the processes σ and ξ^{-1}. We hence obtain $\sup_j \mathbb{E}(\overline{\mathcal{R}}_{b6}(j, m, p)^2) = o(n^{-2}\bar{n}_d)$. Combined with the observation that $\mathbb{E}\big(\overline{\mathcal{R}}_{b6}(j, m, p)\overline{\mathcal{R}}_{b6}(j', m, p)\big) = 0$ for $j \neq j'$, we obtain, using Cauchy-Schwartz inequality, that (B.31) holds for $s = 6$. A symmetric argument applied to $\overline{\mathcal{R}}_{b7}(j, m, p)$ proves (B.31) holds for $s = 7$ and we have hence proved (B.31) for $s \in \{2, 3, 4, 5, 6, 7\}$. At this stage, combining (B.30), (B.14), and (B.31), plus using (B.2), we are able to claim that (B.12) holds for $s = 1$.

Step 6. (Bound of $\mathcal{R}_{a2}(j)$) Our target is to show that for $s \in \{1, 2, 3, 4\}$

$$\frac{1}{n}\sum_{j=1}^{J_d}\mathcal{R}_{cs}(j) = o_{\mathrm{P}}(n^{-1/2}(q_n + 1)^{1/2} + n^{-1/4}(\iota^{(n)})^{1/2}). \qquad (B.37)$$

In view of (A.30) and (A.7) and following the same reasoning of (B.30), plus using (B.2), we conclude that (B.37) holds for $s \in \{1, 2, 3\}$ as long as we can show that for $s \in \{1, 2, 3\}$,

$$\sup_{m,p}\mathbb{E}\Big|2^{-3m/2}\bar{J}_d^2/(n^{1/2}\iota^{(n)})\mathbb{1}_{\Omega_n'}\sum_{j=1}^{J_d}\overline{\mathcal{R}}_{cs}(j, m, p)\Big| = o(n^{-1/2}(q_n + 1)^{1/2} + n^{-1/4}(\iota^{(n)})^{1/2}). \qquad (B.38)$$

We denote by $\sigma(\chi_i : i \leq j)$ the σ-field generated by the sequence of all χ_i with $i \leq j$ and write

$\widetilde{\mathcal{F}}_\infty = \mathcal{F}_\infty \otimes \bigvee_{j\geq 0}\sigma(\chi_i : i \leq j)$. According to Assumption 3, we can write

$$\mathbb{E}(\Delta_k^n U(j)\Delta_{k'}^n U(j)|\widetilde{\mathcal{F}}_\infty) = \frac{(\iota^{(n)})^2}{2\pi}\int_{-\pi}^{\pi} g(\lambda;\theta^{(n)})(\eta(j)_k e^{\mathrm{i}k\lambda} - \eta(j)_{k-1}e^{\mathrm{i}(k-1)\lambda}) \times(\eta(j)_{k'}e^{-\mathrm{i}k'\lambda} - \eta(j)_{k'-1}e^{-\mathrm{i}(k'-1)\lambda})d\lambda.$$

Using the fact that $K^{-1} \leq g(\lambda;\theta^{(n)}) \leq K$ uniformly over λ as required by Assumption 4, we obtain for all $\{x_k\}_{k=1}^{n_d}$, all $1 \leq j \leq J_d$, and all $i \geq 2$,

$$\sum_{k=1}^{n_d}\sum_{k'=1}^{n_d}\mathbb{E}(\Delta_k^n U(j)\Delta_{k'}^n U(j)|\widetilde{\mathcal{F}}_\infty)x_k x_{k'} \leq K(\iota^{(n)})^2\sum_{k=0}^{n_d}\eta(j)_k^2(x_k - x_{k+1})^2 \leq K(\iota^{(n)})^2\sup_{1\leq l\leq n_d}\eta(j)_l^2\sum_{k=1}^{n_d}\sum_{k'=1}^{n_d}x_k(2\mathbb{I}_{n_d} - \mathbb{F}_{n_d}^1)x_{k'} \quad (B.39)$$

$$\sum_{k=1}^{n_d}\sum_{k'=1}^{n_d}\mathbb{E}(\Delta_k^n U(j)\Delta_{k'}^n U(j+i)|\widetilde{\mathcal{F}}_\infty)x_k x_{k'} \leq K(\iota^{(n)})^2 i^{-2}\sum_{k=1}^{n_d}\sum_{k'=1}^{n_d}x_k(2\mathbb{I}_{n_d} - \mathbb{F}_{n_d}^1)x_{k'} \quad (B.40)$$

Here we set $x_0 = x_{n_d+1} = 0$ by convention. For (B.40) we additionally use the observation that $n_d\left|\int_{-\pi}^{\pi} g(\lambda;\theta^{(n)})e^{\mathrm{i}in_d\lambda}d\lambda\right| \leq i^{-2}$ and Assumption A1. The definition of $\bar{U}(j)_k$ provided by (B.15) and the definition of $\Delta_k^n U^C(j)$ indicate that a completely symmetric argument would yield

$$\sum_{k=1}^{n_d}\sum_{k'=1}^{n_d}\mathbb{E}(\Delta_k^n \bar{U}(j)\Delta_{k'}^n \bar{U}(j)|\widetilde{\mathcal{F}}_\infty)x_k x_{k'} \leq K(\iota^{(n)})^2\sup_{1\leq l\leq n_d}\bar{\eta}(j)_l^2\sum_{k=1}^{n_d}\sum_{k'=1}^{n_d}x_k(2\mathbb{I}_{n_d} - \mathbb{F}_{n_d}^1)x_{k'}, \quad (B.41)$$

$$\sum_{k=1}^{n_d}\sum_{k'=1}^{n_d}\mathbb{E}(\Delta_k^n U^C(j)\Delta_{k'}^n U^C(j)|\widetilde{\mathcal{F}}_\infty)x_k x_{k'} \leq K(\iota^{(n)})^2\eta_C(j)^2\sum_{k=1}^{n_d}\sum_{k'=1}^{n_d}x_k(2\mathbb{I}_{n_d} - \mathbb{F}_{n_d}^1)x_{k'}. \quad (B.42)$$

In view of the definitions of $\overline{\mathcal{R}}_{cs}(j,m,p)$ with $s \in \{1,2,3\}$, plus using (A.7) and $\mathbb{D}_m^j = O_m\mathbb{F}_m^j O_m$, the combination of (B.39), (B.41), and (B.42) directly leads to that for all $1 \leq j \leq J_d$, all $1 \leq m \leq \widetilde{J}_d - 1$ and all $1 - \bar{n}_d \leq p \leq \bar{n}_d$,

$$\mathbb{E}(\overline{\mathcal{R}}_{c1}(j,m,p)^2|\widetilde{\mathcal{F}}_\infty) \leq K(\iota^{(n)})^2\sum_{l=1}^{\bar{n}_d}(2\mathbb{I}_{n_d} - \mathbb{D}_{n_d}^1)_{\widetilde{n}_d+l,\widetilde{n}_d+l}\Big(\sum_{i=1}^{n_d}(O_{n_d})_{\widetilde{n}_d+l,i}\int_{t(j)_{i-1}}^{t(j)_i}\mu_s^r ds\Big)^2,$$

$$\mathbb{E}(\overline{\mathcal{R}}_{c2}(j,m,p)^2|\widetilde{\mathcal{F}}_\infty) \leq K(\iota^{(n)})^2\sup_{1\leq k\leq n_d}\bar{\eta}(j)_k^2\sum_{l=1}^{\bar{n}_d}(2\mathbb{I}_{n_d} - \mathbb{D}_{n_d}^1)_{\widetilde{n}_d+l,\widetilde{n}_d+l}\Big(\sum_{i=1}^{n_d}(O_{n_d})_{\widetilde{n}_d+l,i}\Delta_i^n\bar{X}^B(j)\Big)^2,$$

$$\mathbb{E}(\overline{\mathcal{R}}_{c3}(j,m,p)^2|\widetilde{\mathcal{F}}_\infty) \leq K(\iota^{(n)})^2\sum_{l=1}^{\bar{n}_d}(2\mathbb{I}_{n_d} - \mathbb{D}_{n_d}^1)_{\widetilde{n}_d+l,\widetilde{n}_d+l}\Big(\sum_{i=1}^{n_d}(O_{n_d})_{\widetilde{n}_d+l,i}(\Delta_i^n\bar{X}^B(j) - \Delta_i^n X^C(j))\Big)^2.$$

Here for the first and last lines we additionally use the boundedness of η_s. Further utilizing that $(2\mathbb{I}_{n_d} - \mathbb{D}_{n_d}^1)_{\widetilde{n}_d+l,\widetilde{n}_d+l} \leq K2^{2m}\bar{J}_d^{-2}$, we obtain that for all $1 \leq j \lesssim n^{1/8}$, all $1 \leq m \leq \widetilde{J}_d - 1$ and all

$1-\bar{n}_d \le p \le \bar{n}_d$,

$$\mathbb{E}(\overline{\mathcal{R}}_{c1}(j,m,p)^2) \le K(\iota^{(n)})^2 \bar{n}_d 2^{2m} \bar{J}_d^{-2} n_d n^{-2}, \quad \mathbb{E}(\overline{\mathcal{R}}_{c2}(j,m,p)^2) \le K(\iota^{(n)})^2 \bar{n}_d 2^{2m} \bar{J}_d^{-2} n^{-1} \sup_{1\le k\le n_d} \bar{\eta}(j)_k^2,$$

$$\mathbb{E}(\overline{\mathcal{R}}_{c3}(j,m,p)^2) \le K(\iota^{(n)})^2 \bar{n}_d 2^{2m} \bar{J}_d^{-2} \sup_{1\le i\le n_d} \mathbb{E}\big((\Delta_i^n \bar{X}^B(j) - \Delta_i^n X^C(j))^2\big).$$

In addition, using (B.40) instead of (B.39), we can prove $\sup_{j,p}\left|\mathbb{E}(\overline{\mathcal{R}}_{c1}(j,m,p)\overline{\mathcal{R}}_{c1}(j+i,m,p))\right| \le Ki^{-2}(\iota^{(n)})^2\bar{n}_d 2^{2m}\bar{J}_d^{-2} n_d n^{-2}$ for $i \ge 2$. Applying Cauchy-Schwarz inequality immediately proves (B.38) for $s=1$. On the other hand, we observe that $\mathbb{E}(\overline{\mathcal{R}}_{cs}(j,m,p)\overline{\mathcal{R}}_{cs}(j',m,p)|\widetilde{\mathcal{F}}_\infty) = 0$ for $j \ne j'$ and $s \in \{2,3\}$ because of the definition of $\varepsilon_C(j)_m$. Since (B.33) indicates $\mathbb{E}|\sup_{1\le k\le n_d}\bar{\eta}(j)_k^2| = o(1)$ and $\sup_{1\le i\le n_d}\mathbb{E}\big((\Delta_i^n\bar{X}^B(j) - \Delta_i^n X^C(j))^2\big) = o(n^{-1})$, we obtain (B.38) for $s\in\{2,3\}$. We have proved (B.37) for $s\in\{1,2,3\}$. Now we consider $\mathcal{R}_{c4}(j)$. Firstly one can verify using Assumption 4 that

$$\|\theta^{(n)}\|_{(i)} \le Ki^{-2}. \tag{B.43}$$

Using this result, we can write $\mathbb{E}\big(|\widetilde{U}(j)_k|\big|\widetilde{\mathcal{F}}_\infty\big) \le K\iota^{(n)}\Big(\sum_{m=-\infty}^{0}|\theta_{k-m}^{(n)}|^2\Big)^{1/2} \le \iota^{(n)}(k+1)^{-2}$, where the first inequality comes from the definition of $\widetilde{U}(j)_k$ provided by (B.15), that $\eta_C(j)^2$ is bounded because of Assumption A1, and Cauchy-Schwarz. Therefore, using Hölder's inequality and the fact that $\Delta_i^n\bar{X}^B(j)\Theta_{k,l}$ is $\widetilde{\mathcal{F}}_\infty$-measurable, plus (B.2), we can prove (B.37) for $s=4$ as long as we show that for all $1\le k\le n_d$,

$$\sup_{1\le k\le n_d}\frac{1}{n}\mathbb{1}_{\Omega_n'}\sum_{j=1}^{J_d}\sum_{i=1}^{n_d}\big(\mathbb{E}|\Delta_i^n\bar{X}^B(j)\Delta\Theta_{i,k}\mathbb{1}_{\Omega_n'}| + \mathbb{E}|\Delta_i^n\bar{X}^B(j)\Theta_{i,n_d}\mathbb{1}_{\Omega_n'}| + \mathbb{E}|\Delta_i^n\bar{X}^B(j)\Theta_{i,1}\mathbb{1}_{\Omega_n'}|\big)$$
$$= o(n^{-1/2}(q_n+1)^{1/2}(\iota^{(n)})^{-1} + n^{-1/4}(\iota^{(n)})^{-1/2}).$$

This is indeed true since we have

$$\begin{aligned}
&\sup_{1\le k\le n_d}\sup_j\sum_{i=1}^{n_d}\mathbb{E}\Big(\big|\Delta_i^n\bar{X}^B(j)\big|\big(|\Delta\Theta_{i,k}| + |\Theta_{i,n_d}| + |\Theta_{i,1}|\big)\mathbb{1}_{\Omega_n'}\Big)\\
\le\ & K\sup_{1\le k\le n_d}\sup_j\sum_{i=1}^{n_d}n^{-1/2}\Big(\mathbb{E}\big(|\Delta\Theta_{i,k}\mathbb{1}_{\Omega_n'}|^2 + |\Theta_{i,n_d}\mathbb{1}_{\Omega_n'}|^2 + |\Theta_{i,1}\mathbb{1}_{\Omega_n'}|^2\big)\Big)^{1/2} \le K\bar{J}_d^{1/2}(\iota^{(n)} + n^{-1/2})^{-1}.
\end{aligned}$$

The second inequality comes from (B.28) and (B.27). Having proved (B.37) for all $s\in\{1,2,3,4\}$, using the relation (B.16) we immediately obtain that (B.12) holds for $s=2$.

Step 7. (Bound of $\mathcal{R}_{a3}(j)$ and conclusion) We start with proving that, for $s\in\{1,2,\ldots,5\}$,

$$\frac{1}{n}\sum_{j=1}^{J_d}\mathcal{R}_{ds}(j) = o_{\mathrm{P}}(n^{-1/2}(q_n+1)^{1/2}(\iota^{(n)})^{-2} + n^{-1/4}(\iota^{(n)})^{-3/2}). \tag{B.44}$$

We define $\mathcal{R}_{e1}(j)_{i,k} = \sum_{l=-\infty}^{i\wedge k} \theta^{(n)}_{i-l}\theta^{(n)}_{k-l}\varepsilon(j)_l\varepsilon(j)_l - \kappa^{(n)}_{|i-k|}$, $\mathcal{R}_{e2}(j)_{i,k} = \sum_{l=-\infty}^{i\wedge k} \theta^{(n)}_{i-l}\theta^{(n)}_{k-l}\varepsilon_C(j)_l\varepsilon_C(j)_l - \kappa^{(n)}_{|i-k|}$, and obtain that for $s \in \{1,2\}$ and all $1 \leq j \leq J_d$,

$$\mathbb{E}(\mathcal{R}_{es}(j)_{i,k}\mathcal{R}_{es}(j)_{i',k'}|\widetilde{\mathcal{F}}_\infty) = \mathrm{Cum}_4(\varepsilon)\sum_{l=-\infty}^{i\wedge k\wedge i'\wedge k'} \theta^{(n)}_{i-l}\theta^{(n)}_{k-l}\theta^{(n)}_{i'-l}\theta^{(n)}_{k'-l}.$$

On the other hand, we can write

$$\sum_{l=-\infty}^{i\wedge i'} |\theta^{(n)}_{i-l}\theta^{(n)}_{i'-l}| \leq K\|\theta^{(n)}\|_{(|i-i'|-1)} \leq \frac{K}{|i-i'|^2+1}. \tag{B.45}$$

The first inequality comes from Cauchy-Schwarz and the bound on $\|\theta^{(n)}\|$ required by Assumption 4. The second inequality comes from (B.43). This immediately leads to that, for $s \in \{1,2\}$,

$$\begin{aligned}
&\sup_j |\mathbb{E}(\mathcal{R}_{es}(j)_{i,k}\mathcal{R}_{es}(j)_{i',k'}|\widetilde{\mathcal{F}}_\infty)| \\
\leq\ & K\Big(\sum_{l=-\infty}^{i\wedge i'} |\theta^{(n)}_{i-l}\theta^{(n)}_{i'-l}| \sum_{l=-\infty}^{k\wedge k'} |\theta^{(n)}_{k-l}\theta^{(n)}_{k'-l}|\Big) \wedge \Big(\sum_{l=-\infty}^{i\wedge k} |\theta^{(n)}_{i-l}\theta^{(n)}_{k-l}| \sum_{l=-\infty}^{i'\wedge k'} |\theta^{(n)}_{i'-l}\theta^{(n)}_{k'-l}|\Big) \\
\leq\ & \frac{K}{(|i-i'|^2+1)(|k-k'|^2+1)} \wedge \frac{K}{(|i-k|^2+1)(|i'-k'|^2+1)}.
\end{aligned}\tag{B.46}$$

A symmetric argument leads to that, for $l \geq 2$ and $s \in \{1,2\}$,

$$\sup_j |\mathbb{E}(\mathcal{R}_{es}(j)_{i,k}\mathcal{R}_{es}(j+l)_{i',k'}|\widetilde{\mathcal{F}}_\infty)| \leq \frac{K}{l^4 n_d^4} \wedge \frac{K}{(|i-k|^2+1)(|i'-k'|^2+1)}. \tag{B.47}$$

From the definition of $\mathcal{R}_{d1}(j)$ and $\mathcal{R}_{d2}(j)$, we have that for $s \in \{1,2\}$,

$$\begin{aligned}
\sup_j \mathbb{E}(\mathcal{R}_{ds}(j)^2|\widetilde{\mathcal{F}}_\infty) &\leq K\sup_j \sum_{i=1}^{n_d}\sum_{k=1}^{n_d}\sum_{i'=1}^{n_d}\sum_{k'=1}^{n_d} |\widetilde{\Delta}\Theta_{i,k}||\widetilde{\Delta}\Theta_{i',k'}||\mathbb{E}(\mathcal{R}_{es}(j)_{i,k}\mathcal{R}_{es}(j)_{i',k'}|\widetilde{\mathcal{F}}_\infty)| \\
&\leq K n_d n^2 \Delta_n^2(\iota^{(n)} + \Delta_n^{1/2})^{-4}\log(\Delta_n^{-1/2}\iota^{(n)} + q_n).
\end{aligned}\tag{B.48}$$

The first inequality uses that η_s is bounded from Assumption A1. The second inequality uses the bound on $\mathbb{E}(\mathcal{R}_{es}(j)_{i,k}\mathcal{R}_{es}(j)_{i',k'}|\mathcal{F}_\infty)$ for $s \in \{1,2\}$ provided by (B.46) and the bound on $|\widetilde{\Delta}\Theta_{i,k}|$ provided by (B.29). Following the same reasoning and using (B.47) instead of (B.46), we obtain for $s \in \{1,2\}$ and for $l \geq 2$ and all j,

$$\begin{aligned}
&\sup_j |\mathbb{E}(\mathcal{R}_{ds}(j)\mathcal{R}_{ds}(j+l)|\widetilde{\mathcal{F}}_\infty)| \\
\leq\ & K\sum_{i=1}^{n_d}\sum_{k=1}^{n_d}\sum_{i'=1}^{n_d}\sum_{k'=1}^{n_d} |\widetilde{\Delta}\Theta_{i,k}||\widetilde{\Delta}\Theta_{i',k'}|| \sup_j \mathbb{E}(\mathcal{R}_{es}(j)_{i,k}\mathcal{R}_{es}(j+l)_{i',k'}|\widetilde{\mathcal{F}}_\infty)|
\end{aligned}$$

$$\leq \frac{Kn^2\Delta_n^2}{(\iota^{(n)}+\Delta_n^{1/2})^4}\sum_{i=1}^{n_d}\sum_{k=1}^{n_d}\sum_{i'=1}^{n_d}\sum_{k'=1}^{n_d}\frac{1}{l^4n_d^4} \leq \frac{Kn^2\Delta_n^2}{l^4(\iota^{(n)}+\Delta_n^{1/2})^4}. \tag{B.49}$$

Combining (B.48) and (B.49) and applying Cauchy-Schwarz, plus using (B.2), we prove (B.44) for $s \in \{1,2\}$. Now we move to $\mathcal{R}_{d3}(j)$. We define $\mathcal{R}_{e3}(j)_{i,k} := \sum_{l=1}^{i\wedge k}\sum_{m=l+1}^{k}\theta_{i-l}^{(n)}\theta_{k-m}^{(n)}\varepsilon(j)_l\varepsilon(j)_m$, and obtain that for all $1 \leq j \leq J_d$,

$$|\mathbb{E}(\mathcal{R}_{e3}(j)_{i,k}\mathcal{R}_{e3}(j)_{i',k'}|\widetilde{\mathcal{F}}_\infty)| = \left|\sum_{l=1}^{i\wedge k\wedge i'\wedge k'}\theta_{i-l}^{(n)}\theta_{i'-l}^{(n)}\sum_{m=l+1}^{k\wedge k'}\theta_{k-m}^{(n)}\theta_{k'-m}^{(n)}\right| \leq \frac{K}{(|i-i'|^2+1)(|k-k'|^2+1)}. \tag{B.50}$$

The last inequality comes from (B.45). The definition of $\mathcal{R}_{d3}(j)$ then leads to

$$\begin{aligned}
\sup_j \mathbb{E}(\mathcal{R}_{d3}(j)^2|\widetilde{\mathcal{F}}_\infty) &\leq K\sup_{0\leq l\leq n_d}(\eta(j)_l-\eta_C(j))^2 \\
&\quad \times\sum_{i=1}^{n_d-1}\sum_{k=1}^{n_d-1}\sum_{i'=1}^{n_d-1}\sum_{k'=1}^{n_d-1}|\widetilde{\Delta}\Theta_{i,k}||\widetilde{\Delta}\Theta_{i',k'}|\sup_j|\mathbb{E}(\mathcal{R}_{e3}(j)_{i,k}\mathcal{R}_{e3}(j)_{i',k'}|\widetilde{\mathcal{F}}_\infty)| \\
&\leq Kn^2\Delta_n^2(\iota^{(n)}+\Delta_n^{1/2})^{-4}(\Delta_n^{-1/2}\iota^{(n)}+q_n)n_d\sup_{0\leq l\leq n_d}(\eta(j)_l-\eta_C(j))^2.
\end{aligned} \tag{B.51}$$

The second inequality uses (B.50) and (B.29). Because we have $\mathbb{E}(\mathcal{R}_{d3}(j)\mathcal{R}_{d3}(j')) = 0$ for $j \neq j'$, we immediately conclude, using (B.51) and Cauchy-Schwarz inequality, applying (B.33) to the process η, plus (B.2), that (B.44) holds for $s = 3$. We consider $\mathcal{R}_{ds}(j)$ for $s \in \{4,5\}$ now. We define $\mathcal{R}_{e4}(j)_{i,k} = \sum_{l=-\infty}^{0}\sum_{m=l+1}^{k}\theta_{i-l}^{(n)}\theta_{k-m}^{(n)}\varepsilon(j)_l\varepsilon(j)_m$, $\mathcal{R}_{e5}(j)_{i,k} = \sum_{l=-\infty}^{0}\sum_{m=l+1}^{k}\theta_{i-l}^{(n)}\theta_{k-m}^{(n)}\varepsilon_C(j)_l\varepsilon_C(j)_m$, and calculate that for $s \in \{4,5\}$

$$|\mathbb{E}(\mathcal{R}_{es}(j)_{i,k}\mathcal{R}_{es}(j)_{i',k'}|\widetilde{\mathcal{F}}_\infty)| = \left|\sum_{l=-\infty}^{0}\theta_{i-l}^{(n)}\theta_{i'-l}^{(n)}\sum_{m=l+1}^{k\wedge k'}\theta_{k-m}^{(n)}\theta_{k'-m}^{(n)}\right| \leq \frac{K}{(i^2+1)((i')^2+1)(|k-k'|^2+1)}. \tag{B.52}$$

The last inequality comes from (B.43) and Cauchy-Schwarz inequality. Following the same reasoning, we have for all $l \geq 2$ and all $1 \leq j \leq J_d$,

$$|\mathbb{E}(\mathcal{R}_{e4}(j)_{i,k}\mathcal{R}_{e4}(j+l)_{i',k'}|\widetilde{\mathcal{F}}_\infty)| \leq \frac{K}{l^4n_d^4} \quad \text{and} \quad \mathbb{E}(\mathcal{R}_{e5}(j)_{i,k}\mathcal{R}_{e5}(j+l)_{i',k'}|\widetilde{\mathcal{F}}_\infty) = 0. \tag{B.53}$$

In view of (B.52) and the bound on $|\widetilde{\Delta}\Theta_{i',k'}|$ provided by (B.29), the definitions of $\mathcal{R}_{d4}(j)$ and $\mathcal{R}_{d5}(j)$ then lead to that for $s \in \{4,5\}$

$$\begin{aligned}
\sup_j \mathbb{E}(\mathcal{R}_{ds}(j)^2|\widetilde{\mathcal{F}}_\infty) &\leq K\sum_{i=1}^{n_d}\sum_{k=1}^{n_d}\sum_{i'=1}^{n_d}\sum_{k'=1}^{n_d}|\widetilde{\Delta}\Theta_{i,k}||\widetilde{\Delta}\Theta_{i',k'}|\sup_j|\mathbb{E}(\mathcal{R}_{es}(j)_{i,k}\mathcal{R}_{es}(j)_{i',k'}|\widetilde{\mathcal{F}}_\infty)| \\
&\leq Kn^2\Delta_n^2(\iota^{(n)}+\Delta_n^{1/2})^{-4}(\Delta_n^{-1/2}\iota^{(n)}+q_n).
\end{aligned} \tag{B.54}$$

Using (B.53) instead of (B.52), we obtain for $s \in \{4,5\}$ and $l \geq 2$ that

$$\sup_j |\mathbb{E}(\mathcal{R}_{ds}(j)\mathcal{R}_{ds}(j+l)|\widetilde{\mathcal{F}}_\infty)| \leq Kn^2\Delta_n^2(\iota^{(n)} + \Delta_n^{1/2})^{-4}. \tag{B.55}$$

Using (B.54) and (B.55) and applying Cauchy-Schwarz inequality, plus using (B.2), we obtain (B.44) for $s \in \{4,5\}$. Following the same reasoning, and using (B.27) and (B.28) instead of (B.29), we have $s \in \{6,7,\ldots,11\}$,

$$\frac{1}{n}\sum_{j=1}^{J_d}\mathcal{R}_{ds}(j) = o_{\mathrm{P}}(n^{-1/2}(q_n+1)^{1/2} + n^{-1/4}(\iota^{(n)})^{1/2}). \tag{B.56}$$

Given (B.44) and (B.56), the equation (B.17) immediately leads to that (B.12) holds for $s = 3$. Since we show (B.12) for $s \in \{1,2\}$ in Steps 5 and 6, plus the decomposition (B.11), the lemma is proved. ∎

Lemma B3. *Suppose Assumptions 1 - 4 hold and q_n is deterministic. Then it holds that*

$$\widehat{\sigma}^2(q_n) - \sigma^{(n)}(q_n)^2 = \bar{\eta}^\intercal \Xi_{D,n}(\beta^{(n)}) + o_{\mathrm{P}}\big(n^{-1/4}(\iota^{(n)})^{1/2} + a_n n^{-1/2} + \sqrt{\widehat{q}_n} n^{-1/2}\big) \text{ for all } a_n \to \infty, \tag{B.57}$$

if either of the two conditions holds, with $\alpha_n = n^{1/6} \wedge (n^{1/3}(\iota^{(n)} \vee n^{-1/2})^{4/9})$:

(i) We have $n^{1/2}\iota^{(n)} \to \infty$, $\widehat{\mathcal{R}}_n(q_n,b) = o_{\mathrm{P}}(1)$, $\mathcal{R}^{(n)}(q_n,b) = o_{\mathrm{P}}(1)$, and $q_n\alpha_n^{-1} \to 0$.

(ii) We have $n^{1/2}\iota^{(n)} \leq K$, $\widehat{\mathcal{R}}_n(q_n,s) = o_{\mathrm{P}}(1)$, $\mathcal{R}^{(n)}(q_n,s) = o_{\mathrm{P}}(1)$, and $q_n^{-1} \vee \big(q_n\alpha_n^{-1}\big) \to 0$.

Proof. Step 1. (Technical preparation) Throughout the proof we omit the dependence of $\beta^{(n)}$ on q_n. We impose the restriction that $\partial f(\lambda;\beta,\Delta_n)/\partial\beta$ does not depend on β. We start by introducing $(q_n+2)\times(q_n+2)$ matrices $\partial\Xi_n(\beta_n,\beta_n',k)$ with $k \in \{1,2\}$ and $\beta_n,\beta_n' \in \Pi_n^\beta(q_n)$, defined by that for $0 \leq i,j \leq q_n+1$,

$$\partial\Xi_n(\beta_n,\beta_n';1)_{i,j} = \frac{1}{2n}\mathrm{tr}\Big(\frac{\partial\log\Omega_n(\beta_n)}{\partial\beta_i}\frac{\partial\log\Omega_n(\beta_n')}{\partial\beta_j}\Big), \tag{B.58}$$

$$\partial\Xi_n(\beta_n,\beta_n';2)_{i,j} = \frac{1}{4n}\mathrm{tr}\Big(\frac{\partial\log\Omega_n(\beta_n)}{\partial\beta_i}\frac{\partial\log\Omega_n(\beta_n')}{\partial\beta_j}(\Omega_n(\beta_n)^{-1} + \Omega_n(\beta_n')^{-1})Y_nY_n^\intercal\Big). \tag{B.59}$$

We further denote $\partial\Xi_n(\beta_n;j) \coloneqq \partial\Xi_n(\beta_n,\beta_n;j)$. In addition, since generally $\bar{\beta}^{(n)}$ is an ∞-dimensional vector, we use $\partial\Xi_n(\bar{\beta}^{(n)},q_n;j)$ and $\partial\bar{\Xi}_n^\star(\bar{\beta}^{(n)},q_n)$, respectively, to denote the $(q_n+2)\times(q_n+2)$ matrices with entries defined by (B.58) and (B.59), and with entries defined by (A.1). On the other hand, we let $\{\check{\beta}_n \in \Pi_n^\beta(q_n) : n \geq 1\}$ be a sequence of (q_n+2)-dimensional random vectors which satisfies the equation $\Xi_n(\check{\beta}_n) = 0_{q_n+2}$, and the condition that $\sup_\lambda |f(\lambda;\check{\beta}_n,\Delta_n)f(\lambda;\bar{\beta}^{(n)},\Delta_n)^{-1} - 1| = o_{\mathrm{P}}(1)$ holds if either $n^{1/2}\iota^{(n)} \to \infty$ or $n^{1/2}\iota^{(n)} \leq K$. In view of the definition of $\partial\Xi_n(\beta_n,\beta_n';j)$ introduced in (B.58) and (B.59), plus applying rules of matrix differentiation, in particular that $\Omega_n(\beta)$ and $\Omega_n(\beta')$ commute for all (β,β'), we observe

$$\check{\beta}_n - \beta^{(n)} = (2\partial\Xi_n(\check{\beta}_n,\beta^{(n)};2) - \partial\Xi_n(\check{\beta}_n,\beta^{(n)};1))^{-1}(\Xi_{A,n}(\check{\beta}_n) - \Xi_{A,n}(\beta^{(n)})). \tag{B.60}$$

On the other hand, using $\mathbb{D}_m^j = O_m \mathbb{F}_m^j O_m$ and the connection between matrix V_m and spectral density $f(\lambda;\beta,\Delta_n)$ and the positivity of both following the reasoning of step 1 of the proof of Lemma A2 of Da and Xiu (2021), plus the imposed restriction that $\partial f(\lambda;\beta,\Delta_n)/\partial\beta$ does not depend on β, we have, for all $q_n \le Kn^{1/3}$, $\alpha_n \to 0$, and $j \in \{1,2\}$, and under that $\sup_\lambda |f(\lambda; b_n,\Delta_n) f(\lambda;\bar\beta^{(n)},\Delta_n)^{-1} - 1| \to 0$ for $b_n \in \{\check\beta_n, \beta^{(n)}\}$,

$$\begin{cases} (1-\alpha_n)\partial\Xi_n(\bar\beta^{(n)}, q_n; j) \le \partial\Xi_n(\check\beta_n, \beta^{(n)}; j) \le (1+\alpha_n)\partial\Xi_n(\bar\beta^{(n)}, q_n; j) \\ (1-\alpha_n)\partial\bar\Xi_n^\star(\bar\beta^{(n)}, q_n) \le \partial\Xi_n(\bar\beta^{(n)}, q_n; 1) \le (1+\alpha_n)\partial\bar\Xi_n^\star(\bar\beta^{(n)}, q_n) \end{cases}. \tag{B.61}$$

Furthermore, using Lemma A2 of Da and Xiu (2021), we can derive $\mathbb{E}\big|\mathbb{1}_{\Omega_n'}\big(\mathrm{tr}(\Omega_n(\bar\beta^{(n)})^{-1} Y_n Y_n^\intercal - \mathbb{I}_n)\big)^2\big| \le Kn$ (Ω_n' is introduced above (B.2)), which, combined with (B.3) and (B.2), leads to that for all $q_n \le Kn^{1/3}$ and some $\alpha_n \to 0$,

$$\lim_{n\to\infty} \mathbb{P}((1-\alpha_n)\partial\Xi_n(\bar\beta^{(n)}, q_n; 1) \le \partial\Xi_n(\bar\beta^{(n)}, q_n; 2) \le (1+\alpha_n)\partial\Xi_n(\bar\beta^{(n)}, q_n; 1)) = 1. \tag{B.62}$$

Step 2. (Main proof) Now we prove that (B.57) holds if we have either of conditions (i) and (ii). We introduce some more notation. We define $\eta^{(\gamma,i)} = -(2\partial\Xi_n(\check\beta_n,\beta^{(n)};2) - \partial\Xi_n(\check\beta_n,\beta^{(n)};1))^{-1}\big(\frac{\partial\gamma_i}{\partial\beta}\big)^\intercal$, define $\Theta^{(\gamma,i)}$ as the $n_d \times n_d$ matrix with entries $\Theta^{(\gamma,i)}_{j,k} = \frac{\partial\Omega_{n_d}(\beta^{(n)})^{-1}_{j,k}}{\partial\beta}\eta^{(\gamma,i)}$, and introduce

$$\begin{aligned} \widetilde\Theta^{(\gamma,i)}(m,p) &= \frac{1}{4\bar n_d(m)} \sum_{i=1}^{\bar n_d(m)} (O_{n_d}\Theta^{(\gamma,i)} O_{n_d})_{\widetilde n_d(m)+i, \widetilde n_d(m)+i} \left(e^{\mathbb{i}\pi \frac{(i-1/2)p}{\bar n_d(m)}} + e^{-\mathbb{i}\pi\frac{(i-1/2)p}{\bar n_d(m)}} \right), \\ \overline{\mathcal{R}}_g^{(\gamma,i)} &= \sum_{m=0}^{\widetilde J_d - 1} \sum_{p=1-\bar n_d(m)}^{\bar n_d(m)} \widetilde\Theta^{(\gamma,i)}(m,p) \sum_{j=1}^{J_d} \overline{\mathcal{R}}_f^{(\gamma)}(j,m,p). \end{aligned}$$

Here we define $\overline{\mathcal{R}}_f^{(\gamma)}(j,m,p)$ as $\sum_{1\le k,l\le n_d} \widetilde O(m,p)_{k,l}(Y_n(j)_k Y_n(j)_l - \Omega_n^Y(j)_{k,l})$, with $\widetilde O(m,p)$ introduced in (A.7). The notation $\bar n_d(m)$, $\widetilde n_d(m)$, $\widetilde J_d$, and $\bar J_d$ below is introduced above (A.7). Following the same reasoning of step 3 of the proof of Lemma B2 and in view of (B.61), (B.62), and the definition of $\check\beta_n$, we obtain that in restriction to $\omega \in \Omega_n'$ for which $\sup_\lambda |f(\lambda;\beta^{(n)},\Delta_n) f(\lambda;\bar\beta^{(n)},\Delta_n)^{-1} - 1| \to 0$ holds, under $n^{1/2}\iota^{(n)} \le K$, and uniformly over $1 \le m \le \widetilde J_d - 1$,

$$\sup_{0\le i\le q_n} \sum_{p=1-\bar n_d}^{\bar n_d} |\widetilde\Theta^{(\gamma,i)}(m,p)| \le \big(K(m+1)2^{-m}\bar J_d^2\big) \wedge \big(K(m+1)2^{-5m}\bar J_d^6/(n^{1/2}\iota^{(n)})^4\big). \tag{B.63}$$

On the other hand, using $|(O_{n_d})_{\widetilde n_d(m)+i,k+1} - (O_{n_d})_{\widetilde n_d(m)+i,k}| \le K n_d^{-3/2}\bar n_d(m)$ and following the analysis of step 2 of the proof of Lemma A1, we obtain that for all $1 \le m, m' \le \widetilde J_d - 1$,

$$\sup_{p,p'} \mathbb{E}\Big|\mathbb{1}_{\Omega_n'} \sum_{1\le j,j'\le J_d+1} \overline{\mathcal{R}}_f^{(\gamma)}(j,m,p)\overline{\mathcal{R}}_f^{(\gamma)}(j',m',p')\Big| \le K2^{(m+m')/2} n\bar J_d^{-1}(n^{-2} + 2^{2(m+m')}\bar J_d^{-4}(\iota^{(n)})^4). \tag{B.64}$$

The range of (p, p') over which the supremum is taken is $1 - \bar{n}_d(m) \le p, p' \le \bar{n}_d(m)$. From the definitions we have $-(2n)^{-1}\overline{\mathcal{R}}_g^{(\gamma,i)}$ is the same as $(\eta^{(\gamma,i)})^\intercal \Xi_{D,n}(\beta^{(n)})$ except that it does not include the last block $\Omega_{n'_d}$ of the matrix $\Omega_{D,n}$, accommodating which is only a matter of notation. Therefore, in view of the proofs of Lemmas A7 and A8 of Da and Xiu (2021), the equation (B.60), and the definition of $\breve{\beta}_n$, plus using the convergence in probability of $\mathcal{R}^{(n)}$ under respective drifting sequences of $n^{1/2}\iota^{(n)}$, we obtain that, for all $\varepsilon > 0$, there exists a M^* that for all $M > M^*$,

$$\limsup_{n\to\infty} \mathbb{P}\bigg(\sum_{i=0}^{q_n} \Big| \frac{\partial \gamma_i}{\partial \beta}(\breve{\beta}_n - \beta^{(n)}) \Big|^2 \ge M(q_n+1)^4 n^{-3} + M(q_n+1)n^{-1}(\iota^{(n)})^4 \bigg) < \varepsilon, \tag{B.65}$$

which comes from (B.63) and (B.64) and Hölder's inequality for the case $n^{1/2}\iota^{(n)} \le K$ and additionally using the properties of of $\Omega_{n_d}^{-1}$ characterized above (B.26) for the case $n^{1/2}\iota^{(n)} \to \infty$. We can then obtain that, under either condition (i) or condition (ii) and for all fixed $M > 0$,

$$\limsup_{n\to\infty} \mathbb{P}\bigg(\sum_{i=0}^{q_n} (i+1)^2 \Big| \frac{\partial \gamma_i}{\partial \beta}(\breve{\beta}_n - \beta^{(n)}) \Big| \ge M \bigg) = 0. \tag{B.66}$$

In view of the proofs of Lemmas A2 and B2, we obtain that (B.61) and (B.62) jointly indicate that, under either of conditions (i) and (ii),

$$(\partial\sigma_n^2)(\breve{\beta}_n - \beta^{(n)}) = \bar{\eta}^\intercal \Xi_{D,n}(\beta^{(n)}) + o_{\mathrm{P}}\big(n^{-1/4}(\iota^{(n)})^{1/2} + a_n n^{-1/2} + \sqrt{\widehat{q}_n} n^{-1/2}\big) \text{ for all } a_n \to \infty. \tag{B.67}$$

Here we also use Lemmas A7 and A8 of Da and Xiu (2021) and the relation (B.60). At this stage, in view of the fact that by definition $(\widehat{\sigma}_n^2(q_n), \widehat{\gamma}_n(q_n))$ maximizes $L_n(\sigma^2, \gamma)$ over $\Pi_n^{(\sigma,\gamma^2)}(q_n)$ and the definition of $\breve{\beta}_n$ plus conditions (i) and (ii), we conclude that $\widehat{\sigma}_n^2(q_n)$ satisfies (B.57) and finish the proof. ∎

Lemma B4. *Suppose Assumptions 1 - 5 hold. Then, if either $n^{1/2}\iota^{(n)} \to \infty$ or $n^{1/2}\iota^{(n)} \le K$, it holds that*

$$\sigma^{(n)}(\widehat{q}_n)^2 = C_T + o_{\mathrm{P}}\big(n^{-1/4}(\iota^{(n)})^{1/2} + a_n n^{-1/2} + \sqrt{\widehat{q}_n} n^{-1/2}\big) \textit{ for all } a_n \to \infty,$$

that $\widehat{q}_{n,\mathrm{AIC}} = o_{\mathrm{P}}(n^{1/6})$, and that there exists some $0 < k < K$ such that $|\widehat{q}_{n,\mathrm{AIC}} - q_n^\star(k)| = o_{\mathrm{P}}(q_n^\star(k) + n^{1/2}\iota^{(n)} + a_n)$ for all $a_n \to \infty$.

Proof. With mean value theorem and for any sequence q_n, we write that

$$-n^{-1}L_{A,n}(\widehat{\beta}_n) + n^{-1}L_{A,n}(\beta^{(n)}) = (\widehat{\beta}_n - \beta^{(n)})^\intercal \Xi_{A,n}(\beta^{(n)}) + \frac{1}{2}(\widehat{\beta}_n - \beta^{(n)})^\intercal \partial\Xi_{A,n}(\widetilde{\beta}_n)(\widehat{\beta}_n - \beta^{(n)}), \tag{B.68}$$

where $\widetilde{\beta}_n = \lambda_n \widehat{\beta}_n + (1-\lambda_n)\beta^{(n)}$ for some $\lambda_n \in (0,1)$ and we omit the argument q_n of $(\widehat{\beta}_n, \beta^{(n)}, \widetilde{\beta}_n)$. With notation introduced in and after (B.58) and (B.59), we observe $\partial\Xi_{A,n}(\widetilde{\beta}_n) = 2\partial\Xi_n(\widetilde{\beta}_n; 2) -$

$\partial\Xi_n(\widetilde{\beta}_n;1)$. On the other hand, in view of (B.61), (B.62) and Lemma B1, the definitions of $\widehat{q}_{n,\mathrm{AIC}}$ and $q_n^\star(k)$ and (B.2), we conclude that, with either $n^{1/2}\iota^{(n)}\to\infty$ or $n^{1/2}\iota^{(n)}\le K$, for $q_n\in\{\widehat{q}_{n,\mathrm{AIC}},q_n^\star(k)\}$ with any fixed $0<k<K$ and for some $a_n\to 0$,

$$\lim_{n\to\infty}\mathbb{P}\big((1-a_n)\partial\bar{\Xi}_n^\star(\bar{\beta}^{(n)},q_n)\le\partial\Xi_{A,n}(\widetilde{\beta}_n(q_n))\le(1+a_n)\partial\bar{\Xi}_n^\star(\bar{\beta}^{(n)},q_n)\big)=1.$$

Here the notation $\partial\bar{\Xi}_n^\star(\bar{\beta}^{(n)},q_n)$ is also introduced after (B.59) and we need the condition $\widehat{\mathcal{R}}_n(q_n,j)=o_{\mathrm{P}}(1)$ and $\mathcal{R}^{(n)}(q_n,j)=o_{\mathrm{P}}(1)$ for $j\in\{b,s\}$ and $q_n\in\{\widehat{q}_{n,\mathrm{AIC}},q_n^\star(k)\}$. Close scrutiny of (B.1) reveals that the result of Lemma B1 for $\widehat{q}_n$ also holds for $\widehat{q}_{n,\mathrm{AIC}}$ and leads to the convergence of $\widehat{\mathcal{R}}_n$ and $\mathcal{R}^{(n)}$, and we can easily verify this for $q_n^\star(k)$ as well. On the other hand, we note that the randomness of $\widehat{q}_{n,\mathrm{AIC}}$ does not affect (B.61) and (B.62) by observing that $\partial\Xi_n(\bar{\beta}^{(n)},q_n;j)$ is the top-left submatrix of $\partial\Xi_n(\bar{\beta}^{(n)},q_n';j)$ for all $q_n<q_n'$. Hence, in view of (B.68) and with the shorthand notation $A(q_n)=\frac{1}{2}n\Xi_{A,n}(\beta^{(n)}(q_n))^\intercal\partial\bar{\Xi}_n^\star(\bar{\beta}^{(n)},q_n)^{-1}\Xi_{A,n}(\beta^{(n)}(q_n))$, we can write that, with either $n^{1/2}\iota^{(n)}\to\infty$ or $n^{1/2}\iota^{(n)}\le K$, and for all fixed $0<k<K$ and some $a_n\to 0$,

$$\begin{cases}\lim_{n\to\infty}\mathbb{P}\big(L_{A,n}(\widehat{\beta}_n(\widehat{q}_{n,\mathrm{AIC}}))-L_{A,n}(\beta^{(n)}(\widehat{q}_{n,\mathrm{AIC}}))\le(1+a_n)A(\widehat{q}_{n,\mathrm{AIC}})\big)=1\\ \lim_{n\to\infty}\mathbb{P}\big((1-a_n)A(q_n^\star)\le L_{A,n}(\widehat{\beta}_n(q_n^\star))-L_{A,n}(\beta^{(n)}(q_n^\star))\le(1+a_n)A(q_n^\star)\big)=1\end{cases}. \tag{B.69}$$

Here for the second line we omit the argument k of $q_n^\star$. We additionally use Assumptions 4 and 5, (B.66), (B.67), and that $\widehat{\beta}_n(q_n)$ maximizes the quasi-log likelihood $L_n(\beta)$ over $\Pi_n^\beta(q_n)$, and the proof of Lemma A7 of Da and Xiu (2021). Now we define $\bar{\Lambda}(q_n,i)=(O_nY_nY_n^\intercal O_n)_{i,i}-V_n(\beta^{(n)}(q_n))_{i,i}$ and

$$\Lambda(q_n,\lambda,\lambda')=\frac{1}{4n_T}\frac{\partial f(\lambda;\beta^{(n)}(q_n),\Delta_n)^{-1}}{\partial\beta}C(\bar{\beta}^{(n)},q_n)^{-1}\Big(\frac{\partial f(\lambda';\beta^{(n)}(q_n),\Delta_n)^{-1}}{\partial\beta}\Big)^\intercal,$$

with $C(\beta,q_n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(\frac{\partial\log f(\lambda,\beta,\Delta_n)}{\partial\beta_i}\big)^\intercal\frac{\partial\log f(\lambda,\beta,\Delta_n)}{\partial\beta_j}d\lambda$; we then obtain

$$A(q_n)=\sum_{1\le i,j\le n_T}\Lambda(q_n,i(n_T+1)^{-1}\pi,j(n_T+1)^{-1}\pi)\bar{\Lambda}(q_n,i)\bar{\Lambda}(q_n,j), \tag{B.70}$$

where we use $\mathbb{D}_m^j=O_m\mathbb{F}_m^jO_m$ and the connection between matrix V_m and spectral density $f(\lambda;\beta,\Delta_n)$. We note that the right-hand side of (B.70) is invariant over choices of bijection $\beta_n(\sigma^2,\gamma)$. Then following the proof of Lemma A1 we derive that, with either $n^{1/2}\iota^{(n)}\to\infty$ or $n^{1/2}\iota^{(n)}\le K$, for all fixed $0<k<K$ and some $a_n\to 0$,

$$\begin{cases}\lim_{n\to\infty}\mathbb{P}\big(2L_n(\widehat{\beta}_n(\widehat{q}_{n,\mathrm{AIC}}))-2L_n(\beta^{(n)}(\widehat{q}_{n,\mathrm{AIC}}))-\widehat{q}_{n,\mathrm{AIC}}\le a_n(\mathcal{R}_a(n)+\widehat{q}_{n,\mathrm{AIC}})\big)=1\\ \lim_{n\to\infty}\mathbb{P}\big(\big|2L_n(\widehat{\beta}_n(q_n^\star(k)))-2L_n\left(\beta^{(n)}\left(q_n^\star(k)\right)\right)-q_n^\star(k)\big|\le a_n(\mathcal{R}_b(n,k)+q_n^\star(k))\big)=1\end{cases}, \tag{B.71}$$

where we utilize (B.69), (B.70), and the shorthand notation $\mathcal{R}_a(n)=\bar{L}_n^\star(\bar{\beta}^{(n)})-\bar{L}_n^\star(\beta^{(n)}(\widehat{q}_{n,\mathrm{AIC}}))$ and $\mathcal{R}_b(n,k)=\bar{L}_n^\star(\bar{\beta}^{(n)})-\bar{L}_n^\star(\beta^{(n)}(q_n^\star(k)))$. Now we define $\mathcal{R}_c(n,q)=L_n(\widehat{\beta}_n(q))-L_n(\beta^{(n)}(q))$

and $\mathcal{R}_d(n,q) = L_n(\beta^{(n)}(q)) - \bar{L}_n^\star(\beta^{(n)}(q))$. From the definition of AIC that $\widehat{q}_{n,\mathrm{AIC}} = \arg\min_q\{q - L_n(\widehat{\beta}_n(q))\}$, we can write

$$\mathcal{R}_a(n) - \mathcal{R}_b(n,k) \le (q_n^\star(k) - \widehat{q}_{n,\mathrm{AIC}}) - \mathcal{R}_c(n, q_n^\star(k)) + \mathcal{R}_c(n, \widehat{q}_{n,\mathrm{AIC}}) - \mathcal{R}_d(n, q_n^\star(k)) + \mathcal{R}_d(n, \widehat{q}_{n,\mathrm{AIC}}).$$

On the other hand, we have that for some $a_n \to 0$ and all $a_n' \to \infty$ and fixed $0 < k < K$,

$$\lim_{n\to\infty} \mathbb{P}\big(|\mathcal{R}_d(n, \widehat{q}_{n,\mathrm{AIC}}) - \mathcal{R}_d(n, q_n^\star(k))| \le a_n(|\mathcal{R}_a(n) - \mathcal{R}_b(n,k)| + |\widehat{q}_{n,\mathrm{AIC}} - q_n^\star(k)| + q_n^\star(k)) + a_n'\big) = 1,$$

which can be shown by the same reasoning for (B.71). Therefore, using the properties of $\mathcal{R}_c(n, q_n^\star(k))$ and $\mathcal{R}_c(n, \widehat{q}_{n,\mathrm{AIC}})$ characterized by (B.71), we have that under either $n^{1/2}\iota^{(n)} \to \infty$ or $n^{1/2}\iota^{(n)} \le K$, and for all $a_n \to \infty$ and all fixed $0 < k < K$,

$$\mathcal{R}_a(n) - \mathcal{R}_b(n,k) \le \frac{1}{2}\left(q_n^\star(k) - \widehat{q}_{n,\mathrm{AIC}}\right) + o_{\mathrm{P}}\big(|\widehat{q}_{n,\mathrm{AIC}} - q_n^\star(k)| + q_n^\star(k) + a_n\big). \tag{B.72}$$

Using the bound on $\psi_n^2 \sum_{j=q^\star(k)+1}^{\infty} |\kappa_j^n|$ from Assumption 5, we obtain the bound on $|\widehat{q}_{n,\mathrm{AIC}} - q_n^\star(k)|$ stated in the lemma. Now we prove the bounds on $\sigma^{(n)}(\widehat{q}_n)^2 - C_T$. The definition of $q_n^\star(k)$, combined with (B.5) and (B.72), indicates that there exists a fixed k such that $q_n^\star(k) - \widehat{q}_{n,\mathrm{AIC}} \le \alpha_n + 1$ for all $\alpha_n \to \infty$ with probability approaching one. Combining this inequality with (B.72) again immediately leads to $\mathcal{R}_a(n) - \mathcal{R}_b(n,k) \le o_{\mathrm{P}}(q_n^\star(k) + a_n)$ for a fixed k and all $\alpha_n \to \infty$. Therefore, applying (B.5) and Cauchy-Schwarz, plus using the bound on $\psi_n^2 \sum_{j=q^\star(k)+1}^{\infty} |\kappa_j^n|$, we prove the bound on $\sigma^{(n)}(\widehat{q}_n)^2 - C_T$ and on $\widehat{q}_n$ in view of the definition of $\widehat{q}_n$ and Lemmas A5 and A6 of Da and Xiu (2021). Following the same reasoning, $\widehat{q}_{n,\mathrm{AIC}} = o_{\mathrm{P}}(n^{1/6})$ comes from Assumption 4. ∎

Appendix C Proofs of Corollary 1 and Proposition 1

Proof of Corollary 1. Given Theorem 1 and Lemma B1, we only need show that under either $\widehat{q}_n \to \infty$ or $n^{1/2}\iota^{(n)} \to \infty$,

$$\frac{4\widehat{q}_n\widehat{E}_n(4)_T + \Delta_n^{-1/2}\zeta^{(n)}(5\widehat{E}_n(4)_T\widehat{\sigma}_n^2(\widehat{q}_n)^{-1/2} + \widehat{\sigma}_n^2(\widehat{q}_n)^{3/2}\widehat{B}_n(\widehat{q}_n)_T)}{4\widehat{q}_nE(4,\xi)_T + \Delta_n^{-1/2}\zeta^{(n)}(5E(4,\xi)_TC_T^{-1/2} + C_T^{3/2}B(\xi)_T)} = \frac{n_T}{n} + o_{\mathrm{P}}(1). \tag{C.73}$$

In view of the subsequence argument, as in the proof of Theorem 1, we only need focus on all the DGP sequences that satisfy either $n^{1/2}\iota^{(n)} \to \infty$ or $n^{1/2}\iota^{(n)} \le K$. Under $n^{1/2}\iota^{(n)} \to \infty$, (C.73) follows from

$$\widehat{E}_n(4)_T = \frac{n_T}{n}E(4,\xi)_T + o_{\mathrm{P}}(1), \quad \widehat{\sigma}_n^2(\widehat{q}_n) = C_T + o_{\mathrm{P}}(1), \quad \text{and} \quad \widehat{B}_n(\widehat{q}_n)_T = \frac{n_T}{n}B(\xi)_T + o_{\mathrm{P}}(1).$$

Under $n^{1/2}\iota^{(n)} \le K$, because of the requirement that $\widehat{q}_n \to \infty$, (C.73) is a direct result of

$$\widehat{E}_n(4)_T = \frac{n_T}{n}E(4,\xi)_T + o_{\mathrm{P}}(1).$$

The convergence of $\widehat{E}_n(4)_T$ under either $n^{1/2}\iota^{(n)} \to \infty$ or $n^{1/2}\iota^{(n)} \le K$ holds by extending Theorem 16.4.2 and Theorem 16.5.4 in Jacod and Protter (2011) to the case of serially correlated noise and random sampling interval, which can be shown by repeating arguments of Theorem 3.1 of Jacod, Li, and Zheng (2019). Note that Jacod and Protter (2011) allow for arbitrary noise magnitude, and Jacod, Li, and Zheng (2019) consider general noise dependence structure. Given that our focus is not on the pre-averaging estimation, we omit the details of this proof, which is available upon request. The consistency of $\widehat{\sigma}^2_n(\widehat{q}_n)$ comes from Theorem 1. We hence only need $\widehat{B}_n(\widehat{q}_n)_T = \frac{n_T}{n}B(\xi)_T + o_{\mathrm{P}}(1)$ under $n^{1/2}\iota^{(n)} \to \infty$. This comes from

$$(\iota^{(n)})^{-2}\widehat{B}'_n(1) = \frac{2A_n}{T}\int_0^T \eta_s^2\sigma_s^2 ds + o_{\mathrm{P}}(1), \quad (\iota^{(n)})^{-2}\widehat{B}'_n(2) = \frac{A_n}{T}\sum_{s\le T}(\eta_s^2+\eta_{s-}^2)(\Delta X_s)^2 + o_{\mathrm{P}}(1),$$

$$(\iota^{(n)})^{-4}\widehat{B}'_n(3) = \frac{nA_n^2}{n_T T}\int_0^T \eta_s^4\xi_s^{-1}ds + o_{\mathrm{P}}(1), \quad (\iota^{(n)})^{-2}\big(\widehat{\gamma}_n(\widehat{q})_0 - \widehat{\gamma}_n(\widehat{q})_1\big) = \frac{nA_n}{n_T T}\int_0^T \eta_s^2\xi_s^{-1}ds + o_{\mathrm{P}}(1),$$

where $A_n = \sum_{j=0}^{\infty}(\theta_j^{(n)})^2 - \sum_{j=0}^{\infty}\theta_j^{(n)}\theta_{j+1}^{(n)}$. These four results follow from Lemma B1, Assumption 4, and extensions of Theorem 16.5.1 and Theorem 16.5.4 in Jacod and Protter (2011). ∎

Proof of Proposition 1. Step 1. (Limit of $G_n(x)$) We can always find a probability measure $\mathbb{P}$ which satisfies Assumptions 1 - 3 with $t_i - t_{i-1} = T/n$ for all i, $\eta_t = 1$, and the distribution of ε being Gaussian; and parameter sequence $((\iota^{(n)})^2, \theta^{(n)}) = \big(bC_T\Delta_n n^{-1/2}, \frac{1}{2}\big)$ clearly satisfies Assumptions 4 and 5 for each fixed $b \ge 0$. We denote such DGP sequence by $\{\mathbb{P}_b^{(n)}\}_{n\ge1}$. For $(\sigma^2,\gamma) \in \Pi_n^{(\sigma^2,\gamma)}(q)$, we choose $\beta_n(\sigma^2,\gamma)$ as $\beta_n(\sigma^2,\gamma)_j = \frac{1}{2\pi\Delta_n}\int_{-\pi}^{\pi} f(\lambda;\sigma^2,\gamma,\Delta_n)e^{\mathrm{i}(j-1)\lambda}d\lambda$ with $1 \le j \le q+2$. Then following the standard asymptotic analysis of MLE for classic time series, we obtain

$$n^{1/2}\begin{pmatrix}\widehat{\beta}_n(0) - \beta^{(n)}(0)\\ \widehat{\beta}_n(1) - \beta^{(n)}(1)\end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 2C_T^2\left(\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\otimes\begin{pmatrix}1 & & \\ & \frac{1}{2} & \\ & & \frac{1}{2}\end{pmatrix}\right)\right). \tag{C.74}$$

On the other hand, it holds by definition that

$$\|\beta^{(n)}(0) - C_T(1+5bn^{-1/2}/4, -bn^{-1/2}/4)\| = O_{\mathrm{P}}(n^{-1}), \tag{C.75}$$

$$\|\beta^{(n)}(1) - C_T(1+3bn^{-1/2}/2, -bn^{-1/2}/4, -bn^{-1/2}/2)\| = O_{\mathrm{P}}(n^{-1}). \tag{C.76}$$

Because we have $\widehat{\sigma}^2_n(q) = \sum_{j=1}^{\infty}(2-\delta_{j,1})\widehat{\beta}_n(q)$, it follows from (C.75) and (C.76) that

$$\widehat{\sigma}^2_n(0) - C_T = \widehat{\beta}_n(0)_1 - \beta^{(n)}(0)_1 + 2(\widehat{\beta}_n(0)_2 - \beta^{(n)}(0)_2) + \frac{3b}{4}C_T n^{-1/2} + O_{\mathrm{P}}(n^{-1}), \tag{C.77}$$

$$\widehat{\sigma}_n^2(1) - C_T = \widehat{\beta}_n(1)_1 - \beta^{(n)}(1)_1 + 2(\widehat{\beta}_n(1)_2 - 2\beta^{(n)}(1)_2) + 2(\widehat{\beta}_n(1)_3 - \beta^{(n)}(1)_3) + O_{\mathrm{P}}(n^{-1}). \quad \text{(C.78)}$$

On the other hand, in view of the definition of $\mathrm{AIC}_n(q)$, we use the mean value theorem to conclude that

$$\mathrm{AIC}_n(1) - \mathrm{AIC}_n(0) = 2 + \sum_{i,j} \frac{\partial^2 L_n(\widehat{\beta}_n(1))}{\partial\beta_i \partial\beta_j}(\widehat{\beta}_n(0)_i - \widehat{\beta}_n(1)_i)(\widehat{\beta}_n(0)_j - \widehat{\beta}_n(1)_j) + o_{\mathrm{P}}(n\|\widehat{\beta}_n(0) - \widehat{\beta}_n(1)\|^2).$$

Using $\widehat{\beta}_n(1)_j = C_T\delta_{j,1} + o_{\mathrm{P}}(1)$ from (C.76) and (C.74), we deduce

$$-\frac{2}{n}\frac{\partial^2 L_n(\widehat{\beta}_n(1))}{\partial\beta_i \partial\beta_j} = \frac{1}{C_T^2}\delta_{i,j}(2 - \delta_{j,1}) + o_{\mathrm{P}}(1). \quad \text{(C.79)}$$

Further, combination of (C.74), (C.75), and (C.76) lead to $(\widehat{\beta}_n(1)_1 - \widehat{\beta}_n(0)_1)^2 = \frac{1}{16}C_T^2 b^2 n^{-1} + o_{\mathrm{P}}(n^{-1})$ and $(\widehat{\beta}_n(1)_2 - \widehat{\beta}_n(0)_2)^2 = o_{\mathrm{P}}(n^{-1})$. Using the last row of (C.74), plus the definition of $\widehat{q}_{n,\mathrm{AIC}}$, we readily obtain that $\widehat{q}_{n,\mathrm{AIC}} \wedge 1 = 0$ if and only if

$$2 - b^2/32 - (-b/2 + n^{1/2}C_T^{-1}(\widehat{\beta}_n(1)_3 - \beta^{(n)}(1)_3))^2 + o_{\mathrm{P}}(1) \geq 0. \quad \text{(C.80)}$$

In other words, (C.80) indicates that asymptotically the selected order is determined by the realization of $n^{1/2}(\widehat{\beta}_n(1)_3 - \beta^{(n)}(1)_3)$. Meanwhile, from (C.74) we observe that $n^{1/2}(\widehat{\beta}_n(1)_3 - \beta^{(n)}(1)_3)$ is asymptotically independent of $n^{1/2}(\widehat{\beta}_n(q)_j - \beta^{(n)}(q)_j)$ for every $(q, j) \in \{0, 1\} \times \{1, 2\}$. Moreover, (C.74) implies that $n^{1/2}(\widehat{\beta}_n(0)_j - \beta^{(n)}(0)_j)$ and $n^{1/2}(\widehat{\beta}_n(1)_j - \beta^{(n)}(1)_j)$ are asymptotically perfectly correlated for both $j \in \{1, 2\}$. This implication, plus (C.77) and (C.78), means that the first two terms in $\widehat{\sigma}_n^2(0) - C_T$ and $\widehat{\sigma}_n^2(1) - C_T$ are asymptotically the same. We therefore conclude that for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}_b^{(n)}(|G_n(x) - G_\infty(x, b)| > \epsilon) = 0, \quad \text{(C.81)}$$

where $G_\infty(x, b)$ is defined by

$$\begin{aligned} G_\infty(x, b) &= \mathbb{P}\left(\sqrt{6}\mathcal{U} + 3b/4 \leq xC_T^{-1}\right)\mathbb{P}\left(|\bar{\mathcal{U}} - b/2| < \sqrt{2 - b^2/32}\right) \\ &\quad + \mathbb{P}\left(\sqrt{6}\mathcal{U} + 2\bar{\mathcal{U}} \leq xC_T^{-1}\right)\mathbb{P}\left(|\mathcal{U} - b/2| \geq \sqrt{2 - b^2/32}\right), \end{aligned}$$

with $\mathcal{U}$ and $\bar{\mathcal{U}}$ being two mutually independent standard Gaussian random variables.

Step 2. (Contiguity) In this step we prove that the sequence $\mathbb{P}_b^{(n)}$ is contiguous with respect to the sequence $\mathbb{P}_0^{(n)}$ for every $b \geq 0$. In view of Le Cam's first lemma (see, e.g., Lemma 6.4 in van der Vaart (2000)), and using $\log(d\mathbb{P}_0^{(n)}/d\mathbb{P}_b^{(n)}) = L_n(C_T, 0, 1/2) - L_n(C_T, bC_T\Delta_n n^{-1/2}, 1/2) =: \mathcal{U}_n$, it suffices to show that $\exp(\mathcal{U}_n)$ converges in distribution under $\mathbb{P}_b^{(n)}$ to a random variable that is almost surely positive. We introduce shorthand notation $\beta^{(n,b)} = C_T\left(1 + \frac{3b}{2\sqrt{n}}, -\frac{b}{4\sqrt{n}}, -\frac{b}{2\sqrt{n}}\right)$. It follows that

$$L_n\big(C_T, bC_T\Delta_n n^{-1/2}, 1/2\big) = L_n(\widehat{\beta}_n(1)) + \frac{1}{2}(\beta^{(n,b)} - \widehat{\beta}_n(1))^{\intercal}\frac{\partial^2 L_n(\widehat{\beta}_n(1))}{\partial\beta\partial\beta}(\beta^{(n,b)} - \widehat{\beta}_n(1)) + o_{\mathrm{P}}(1),$$

which holds by $L_n\left(C_T, bC_T\Delta_n n^{-1/2}, 1/2\right) = L_n(\beta^{(n,b)}) + o_{\mathrm{P}}(1)$ from the construction of $L_n(\beta)$ and (C.76), the mean value theorem, and (C.74). Making use of (C.79) and (C.74), it follows that under $\mathbb{P}_b^{(n)}$, $\mathcal{U}_n \xrightarrow{\mathcal{L}} \mathcal{N}\left(-23b^2/32, 23b^2/16\right)$, which proves the contiguity.

Step 3. (Conclusion) Now we have proved two facts. First, for each $x \in \mathbb{R}$ and under $\mathbb{P}_b^{(n)}$, $G_n(x)$ converges to $G_\infty(x,b)$ as $n \to \infty$ as in (C.81). Second, the sequence $\mathbb{P}_b^{(n)}$ is contiguous with respect to the sequence $\mathbb{P}_0^{(n)}$ for every $b \geq 0$. Because $G_\infty(x,b)$ as a function of b is nonconstant for all $x \in \mathbb{R}$, according to Lemma 3.1 in Leeb and Pötscher (2006), we have that $\liminf_{n\to\infty}\inf_{\widehat{G}_n(x)}\sup_{b\geq 0}\mathbb{P}_b^{(n)}\big(|\widehat{G}_n(x) - G_n(x)| > 1/K\big) > 0$, which concludes the proof. ∎

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