Supplement to
Bargaining Under Strategic Uncertainty: The Role of Second-Order Optimism

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Abstract

This supplementary material provides conceptual background and proofs for *Bargaining Under Strategic Uncertainty: The Role of Second-Order Optimism.*

Appendix D Bargaining in Learnable Environments

Many negotiations take place in the context of long-run relationships.[1] In a subclass of those relationships, the parties' preferences are "learnable." Arguably, in those cases, negotiation failures are not easily explained by incomplete information. Here, we expand.

Suppose the inefficiencies are a consequence of uncertainty about strategic posture—i.e., uncertainty about whether the other party is *capable* of accepting bad offers or making favorable offers. Over the course of a long-term relationship, the parties are likely to have observed past concessions. If a party has ever observed a concession, the party would have to conclude that the other was, at the time, capable of making concessions. So if, in later negotiations, there is uncertainty about strategic posture, then it must be that the parties reason that capabilities change over time and, in particular, that they diminish over time.

Of course, "capability" may be a shorthand for the preferences or incentives of a particular negotiator. For instance, a union may give its leader incentives to take particular actions, and such incentives may well vary over time. But, at times, it is possible to obtain information about those incentives: Presumably, when the parties are involved in long-term relationships, they will make it their business to gather information about the incentives of key negotiators, etc. Similarly, at the start of the relationship, there may be significant uncertainty about the preferences of the parties—e.g., a firm may not understand how union members value wages vs. benefits. But, over the course of a long-term term relationship, the firm may come to understand union members' preferences over outcomes. Fearon (2004, page 290) and Powell (2006, page 172) make this argument in the context of wars.

[1] Some examples: Labor unions (re)negotiate contracts with the same set of firms or government agencies. Divorce agreements involve marital partners that (at times) have a long history with one another. Nations have long-term negotiations, renegotiating issues of property, immigration, etc. Legislators will often negotiate policies amongst the same set of political actors.

Appendix E Proofs for Section 8

Before coming to the proof of Proposition 8.1, two important caveats are in order. First, we do not know if a degenerately complete type structure exists. Second, even if a degenerately complete type structure exists, we do not know whether $R^\infty \neq \emptyset$, let alone $R^\infty \cap C \neq \emptyset$. Nonetheless, even if these are both problems, the message of the result stands: When $N = 3$, part of the explanation of delay involves incomplete type structures.

E.1 Structure of Proof

To show Proposition 8.1, it will be convenient to define certain sets $X_1^m \times X_2^m$, which will characterize $\text{proj}_{S_1 \times S_2} R^m$ in degenerately complete type structures. Set $X_i^0 = S_i$. We now define the sets X_i^m inductively. In doing so, we write $H_i[s_i] = \{h \in H_i : s_i \in S_i(h)\}$. Given that the sets X_{-i}^m have been defined, set $H_i^m = \{h \in H_i : X_{-i}^m \cap S_{-i}(h) \neq \emptyset\}$.

Round 1: B1 Put $s_1 \in X_1^1$ if and only if, for each $h = (\cdot, x_2) \in H_1[s_1] \cap H_1^R$, the following holds: If $s_1(h) = \text{r}$, then $x_2 \geq 1 - \delta s_1(h, \text{r})$.

Round 1: B2 Put $s_2 \in X_2^1$ if and only if, for each $h \in H_2[s_2]$, the following holds:

(i) If $h = (\phi, x_1)$ and $s_2(h) = \text{r}$, then $x_1 \geq \min\{1 - \delta s_2(h, \text{r}), 1 - \delta^2\}$.

(ii) If $h = (\cdot, x_1) \in H_2^R$ is a three-period history with $x_1 < 1$, then $s_2(h) = \text{a}$.

Round 2: B1 Put $s_1 \in X_1^2$ if and only if $s_1 \in X_1^1$ and, for each $h \in H_1[s_1] \cap H_1^1$, the following holds:

(i) $s_1(\phi) \geq 1 - \delta$,

(ii) If $h \in H_1^P$ is a three-period history, then $s_1(h) = 1$.

(iii) If $h = (\cdot, x_2) \in H_1^R$ and $s_1(h) = \text{a}$, then $1 - \delta \geq x_2$.

Round 2: B2 Put $s_2 \in X_2^2$ if and only if $s_2 \in X_2^1$ and, for each $h \in H_2[s_2] \cap H_2^1$, the following holds:

(i) If $h \in H_2^P$, then $s_2(h) \geq 1 - \delta$.

(ii) If $h = (\phi, x_1) \in H_2^R$ and $s_2(h) = \text{a}$, then $1 - \delta(1 - \delta) \geq x_1$.

(iii) If $h = (\phi, x_1)$ and $s_2(h) = \text{r}$, then $x_1 \geq 1 - \delta s_2(h, \text{r})$.

Round 3: B1 Put $s_1 \in X_1^3$ if and only if $s_1 \in X_1^2$ and

- $s_1(\phi) \in [1-\delta, 1-\delta(1-\delta)]$ if $1-\delta > \delta^2$.
- $s_1(\phi) \in [1-\delta, 1]$ if $1-\delta \leq \delta^2$.

Round 3: B2 Put $s_2 \in X_2^3$ if and only if $s_2 \in X_2^2$ and, for each $h \in H_2[s_2] \cap H_2^2$, the following holds:

(i) If $h \in H_2^P$, then $s_2(h) = 1-\delta$.

(ii) If $h = (\phi, x_1)$ and $s_2(h) = \mathrm{r}$, then $x_1 \geq 1-\delta(1-\delta)$.

Round 4 and Beyond Put $s_1 \in X_1^4$ if and only if $s_1 \in X_1^3$ and $s_1(\phi) = 1-\delta(1-\delta)$.

- Set $X_2^m = X_2^3$ for each $m \geq 3$.
- Set $X_1^m = X_1^4$ for each $m \geq 4$.

Let $R^0 = S_1 \times T_1 \times S_2 \times T_2$.

Proposition E.1. *Let $N = 3$ and suppose $\mathcal{T}$ is degenerately complete. If $\bigcap_{m\geq 0} R^m \neq \emptyset$, then $\text{proj}_{S_1 \times S_2} R^m = X_1^m \times X_2^m$ for each $m \geq 0$.*

Proof of Proposition 8.1. Fix $(s_1, t_1, s_2, t_2) \in \bigcap_m R^m \cap C$. By Proposition E.1, $s_1 \in \bigcap_m X_1^m$ and so $s_1(\phi) = x_1^{\text{SPE}} = 1-\delta(1-\delta)$. We must show that $s_2(\phi, x_1^{\text{SPE}}) = \mathrm{a}$. Since $(s_1, t_1, s_2, t_2) \in C$, it suffices to show that $\beta_{1,\phi}(t_1)$ assigns probability 1 to $\{r_2 : r_2(\phi, x_1^{\text{SPE}}) = \mathrm{a}\} \times T_2$.

Suppose not. Then, by Proposition E.1, $\mathbb{E}\pi_1[s_1|t_1, \phi] = \alpha(1-\delta) + \delta^2$, for some $\alpha < 1$. Construct $r_1 \in S_1$ with $r_1(\phi) = x_1 \in (\alpha(1-\delta) + \delta^2, 1-\delta(1-\delta))$. Employing Proposition E.1, $\beta_{1,\phi}(t_1)$ must assign probability 1 to $\{r_2 : r_2(\phi, x_1) = \mathrm{a}\} \times T_2$. Thus, $\mathbb{E}\pi_1[r_1|t_1, \phi] > \mathbb{E}\pi_1[s_1|t_1, \phi]$, contradicting $(s_1, t_1) \in R_1^1$. ■

E.2 Proof of Proposition E.1

The remainder of this section is devoted to proving Proposition E.1. Throughout, we fix a type structure that is degenerately complete. We denote $\text{EFR}_i^0 = S_i$ and $\text{EFR}_i^m = \text{proj}_{S_i} R_i^m$. We loosely think of this set as the set of **extensive-form rationalizable** (**EFR**) strategies, given the relationships known in the finite game setting. It will also be convenient to define **pure EFR** (**PEFR**): Let $\text{PEFR}_i^0 = S_i$ and inductively define

PEFR_i^m. Put $s_i \in \text{PEFR}_i^{m+1}$ if and only if $s_i \in \text{PEFR}_i^m$ and, for each $h \in H_i[s_i]$ with $\text{PEFR}_{-i}^m \cap S_{-i}(h) \neq \emptyset$, there exists $s_{-i} \in \text{PEFR}_{-i}^m \cap S_{-i}(h)$ so that $\pi_i(s_i, s_{-i}) \geq \pi_i(r_i, s_{-i})$ for all $r_i \in S_i(h)$.

We will show that, for each $i = 1, 2$ and each m, the following holds:

$$R^m \neq \emptyset \qquad \Longrightarrow \qquad \text{EFR}_i^m \subseteq X_i^m \subseteq \text{PEFR}_i^m \subseteq \text{EFR}_i^m.$$

From this, the proposition follows. To show this, it will be useful to note the following:

Lemma E.1. *Suppose $\mathcal{T}$ is degenerately complete and* $\text{PEFR}_1^m \times \text{PEFR}_2^m = \text{EFR}_1^m \times \text{EFR}_2^m$. *If $R^{m+1} \neq \emptyset$, then* $\text{PEFR}_1^{m+1} \times \text{PEFR}_2^{m+1} \subseteq \text{EFR}_1^{m+1} \times \text{EFR}_2^{m+1}$.

Proof. Since $R^{m+1} \neq \emptyset$, there exists some type in T_i that strongly believes $R_{-i}^1, \ldots, R_{-i}^m$. It follows that each of $R_{-i}^1, \ldots, R_{-i}^m$ is Borel.

Fix $s_i \in \text{PEFR}_i^{m+1}$. For each $h \in H_i[s_i]$, let $k(h) = \max\{k' : \text{PEFR}_{-i}^k \cap S_{-i}(h) \neq \emptyset\}$. Then, for each $h \in H_i[s_i]$, there exists some $s_{-i,h} \in \text{PEFR}_{-i}^{k(h)} \cap S_{-i}(h)$ so that $\pi_i(s_i, s_{-i,h}) \geq \pi_i(r_i, s_{-i,h})$ for all $r_i \in S_i(h)$. Note, if h' follows h and $s_{-i,h} \in S_{-i}(h')$, we can and do take $s_{-i,h'} = s_{-i,h}$. By the induction hypothesis, for any such h, there exists $t_{-i,h}$ so that $(s_{-i,h}, t_{-i,h}) \in R_{-i}^m$. Construct $\nu_{-i,h} \in \Delta(S_{-i} \times T_{-i})$ so that $\nu_{-i,h}(\{s_{-i,h}, t_{-i,h}\}) = 1$. Since each R_{-i}^k is Borel, $\nu_{-i,h}(R_{-i}^k) = 1$ for each $k \leq k(h)$.

Given this, we can construct a degenerate CPS μ_{-i} on $(S_{-i} \times T_{-i}; \mathcal{S}_i \otimes T_{-i})$ so that (i) s_i is sequentially optimal under μ_{-i}; (ii) if $R_{-i}^k \cap (S_{-i}(h) \times T_{-i}) \neq \emptyset$ for $k \leq m$, then $\mu_{-i}(R_{-i}^k | S_{-i}(h) \times T_{-i}) = 1$. Since $\mathcal{T}$ is degenerately complete, there exists a type $t_i \in T_i$ so that $\beta_i(t_i) = \mu_{-i}$. Then, $(s_i, t_i) \in R_i^{m+1}$ as desired. ■

In light of Lemma E.1, we focus on showing the following:

$$R^m \neq \emptyset \qquad \Longrightarrow \qquad \text{EFR}_i^m \subseteq X_i^m \subseteq \text{PEFR}_i^m.$$

If we have shown the claim for m, then $R^m \neq \emptyset$ implies that $\text{EFR}_i^m = X_i^m = \text{PEFR}_i^m$. We use that fact to show the claim for $(m+1)$.

Given some $r_i \in S_i(h)$ and $\nu \in \Delta(S_{-i}(h))$, if $\pi_i(r_i, \cdot) : S_{-i} \to \mathbb{R}$ is ν-integrable, write

$$\pi_i(r_i, \nu) = \int_{S_{-i}(h)} \pi_i(r_i, s_{-i}) \mathrm{d}\nu.$$

It will be convenient to note the following:

Remark E.1. Suppose $X_1^m \times X_2^m = \text{EFR}_1^m \times \text{EFR}_2^m$. If $(s_i, t_i) \in R_i^{m+1}$, then, for each $h \in H_i[s_i]$ with $X_{-i}^m \cap S_{-i}(h) \neq \emptyset$, there exists some $\nu \in \Delta(S_{-i}(h))$ so that

(i) $\pi_i(s_i, \cdot) : S_{-i} \to \mathbb{R}$ is ν-integrable,

(ii) if $r_i \in S_i(h)$ and $\pi_i(r_i, \cdot) : S_{-i} \to \mathbb{R}$ is ν-integrable, then $\pi_i(s_i, \nu) \geq \pi_i(r_i, \nu)$, and

(iii) for some some $E_{-i} \subseteq X_{-i}^m$, $\nu(E_{-i}) = 1$.

Characterization: Round 1

Lemma E.2. $\mathrm{EFR}_1^1 \subseteq X_1^1$.

Proof. Fix $s_1 \in \mathrm{EFR}_1^1$ and $h = (\cdot, x_2) \in H_1[s_1]$ with $s_1(h) = \mathrm{r}$. Observe that $\pi_1(s_1, \nu) \leq \delta^2 s_1(h, \mathrm{r})$. Consider some r_1 with $r_1 \in S_1(h)$ and $r_1(h) = \mathrm{a}$. Note, there exists some $\nu \in \Delta(S_2(h))$ so that $\pi_1(s_1, \nu) \geq \pi_1(r_1, \nu) = \delta(1 - x_2)$. Thus, $\delta^2 s_1(h, \mathrm{r}) \geq \pi_1(s_1, \nu) \geq \delta(1 - x_2)$, from which $x_2 \geq 1 - \delta s_1(h, \mathrm{r})$. ∎

Lemma E.3. $X_1^1 \subseteq \mathrm{PEFR}_1^1$

Proof. Fix $s_1^* \in X_1^1$. We show that, for each $h \in H_1[s_1^*]$, there is some $s_2^* \in S_2(h)$ so that $\pi_1(s_1^*, s_2^*) \geq \pi_1(r_1, s_2^*)$ for each $r_1 \in S_1(h)$.

Case A: $h = (\phi)$. Write $x_1^* = s_1^*(\phi)$. Construct s_2^* so that (i) $s_2^*(\phi, x_1) = \mathrm{a}$ if and only if $x_1 = x_1^*$, (ii) $s_2^*(h) = 1$ for each $h \in H_2^P$, and (iii) $s_2^*(h) = \mathrm{r}$ for each third-period history $h = (\cdot, x_1) \in H_2^R$. Note, $\pi_1(s_1^*, s_2^*) = x_1^*$ and, for each $r_1 \in S_1$, $\pi_1(r_1, s_2^*) \in \{x_1^*, 0\}$, establishing the claim.

Case B: $h = (\phi, s_1^*(\phi), \mathrm{r}, x_2)$ and $s_1^*(h) = \mathrm{a}$. Construct $s_2^* \in S_2(h)$ so that $s_2^*(h') = \mathrm{r}$ for each third-period history $h' = (\cdot, x_1) \in H_2^R$. Then, $\pi_1(s_1^*, s_2^*) = \delta(1 - x_2)$ and, for each $r_1 \in S_1(h)$, $\pi_1(r_1, s_2^*) \in \{\delta(1 - x_2), 0\}$, establishing the claim.

Case C: $h = (\phi, s_1^*(\phi), \mathrm{r}, x_2)$ and $s_1^*(h) = \mathrm{r}$. Construct $s_2^* \in S_2(h)$ so that: (i) $s_2^*(h, \mathrm{r}, s_1^*(h, \mathrm{r})) = \mathrm{a}$, and (ii) $s_2^*(h') = \mathrm{r}$ for all third-period histories $h' \in H_2^R$ with $h' \neq (h, \mathrm{r}, s_1^*(h, \mathrm{r}))$. Then, $\pi_1(s_1^*, s_2^*) = \delta^2 s_1^*(h, \mathrm{r})$ and, for each $r_1 \in S_1(h)$, $\pi_1(r_1, s_2^*) \in \{\delta(1 - x_2), 0\}$. Using the fact that $s_1 \in X_1^1$, $\delta s_1^*(h, \mathrm{r}) \geq 1 - x_2$, establishing the claim.

Case D: $h = (\phi, s_1^*(\phi), \mathrm{r}, x_2, \mathrm{r})$. Repeat the argument in Case C to get the conclusion. ∎

Lemma E.4. $\mathrm{EFR}_2^1 \subseteq X_2^1$.

Proof. Fix $s_2 \in \mathrm{EFR}_2^1$ and some $h \in H_2[s_2]$. First, suppose $h = (\phi, x_1)$ and $s_2(h) = \mathrm{r}$. Then, for each $s_1 \in S_1(h)$, we have

- $\pi_2(s_1, s_2) = \delta s_2(h, \mathrm{r})$ if $s_1(h, \mathrm{r}, s_2(h, \mathrm{r})) = \mathrm{a}$, and
- $\pi_2(s_1, s_2) \leq \delta^2$ otherwise.

Thus, for any $\nu \in \Delta(S_1(h))$, $\pi_2(s_2, \nu) \leq \max\{\delta s_2(h, \mathrm{r}), \delta^2\}$.

Consider, instead, some $r_2 \in S_2(h)$ with $r_2(h) = \text{a}$. Then, for any $\nu \in \Delta(S_1(h))$, $\pi_2(r_2, \nu) = 1 - x_1$. Thus, for any $\nu \in \Delta(S_1(h))$,

$$\max\{\delta s_2(h, \text{r}), 1 - \delta^2\} \geq \pi_2(s_2, \nu) \geq \pi_2(r_2, \nu) = 1 - x_1/$$

As such, $x_1 \geq \min\{1 - \delta s_2(h, \text{r}), 1 - \delta^2\}$, as desired.

Second, suppose $h = (\cdot, x_1) \in H_2^R$ is a three-period history with $x_1 < 1$. Suppose, contra hypothesis, that $s_2(h) = \text{r}$. Consider an alternate strategy $r_2 \in S_2(h)$ with $r_2(h) = \text{a}$. For each $\nu \in \Delta(S_1(h))$, $\pi_2(r_2, \nu) > 0 = \pi_2(s_2, \nu)$, a contradiction. ■

Lemma E.5. $X_2^1 \subseteq \text{PEFR}_2^1$

Proof. Fix $s_2^* \in X_2^1$. We show that, for each $h \in H_2[s_2^*]$, there is some $s_1^* \in S_1(h)$ so that $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, r_2)$ for each $r_2 \in S_2(h)$.

Case A: $h = (\phi, x_1) \in H_2^R$ and $s_2^*(h) = \text{a}$. Let s_1^* be such that (i) $s_1^*(\phi) = x_1$ (ii) for each $h' \in H_1^R$, $s_1^*(h') = \text{r}$, and (iii) for each third-period $h' \in H_1^P$, $s_1^*(h') = 1$. Note $s_1^* \in S_1(h)$. Moreover, $\pi_2(s_1^*, s_2^*) = (1 - x_1)$ and, for each $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) \in \{(1 - x_1), 0\}$.

Case B: $h \in \{(\phi, x_1), (\phi, x_1, \text{r})\} \in H_2^R$ and $s_2^*((\phi, x_1)) = \text{r}$ and $x_1 \geq 1 - \delta^2$. Let s_1^* be such that (i) $s_1^*(\phi) = x_1$, (ii) for each $h' \in H_1^R$, $s_1^*(h') = \text{r}$, and (iii) for each third-period $h' \in H_1^P$, $s_1^*(h') = 0$. Note $s_1^* \in S_1(h)$. Then, $\pi_2(s_1^*, s_2^*) = \delta^2$ and, for each $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) \in \{(1 - x_1), \delta^2\}$. Since $x_1 \geq 1 - \delta^2$, $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, r_2)$ for each $r_2 \in S_2(h)$.

Case C: $h \in \{(\phi, x_1), (\phi, x_1, \text{r})\} \in H_2^R$ and $s_2^*((\phi, x_1)) = \text{r}$ and $x_1 < 1 - \delta^2$. Since $s_2^* \in X_2^1$, $x_1 \geq 1 - \delta s_2^*(h, \text{r})$. Let s_1^* be such that (i) $s_1^*(\phi) = x_1$, (ii) for each $h' = (\cdot, x_2) \in H_1^R$, $s_1^*(h') = \text{a}$ if and only if $x_2 = s_2^*(h, \text{r})$, and (iii) for each third-period $h' \in H_1^P$, $s_1^*(h') = 1$. Note $s_1^* \in S_1(h)$. Then, $\pi_2(s_1^*, s_2^*) = \delta s_2^*(h', \text{r})$ and, for each $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) \in \{(1 - x_1), \delta s_2^*(h', \text{r}), 0\}$. Since $x_1 \geq 1 - \delta s_2^*(h, \text{r})$, $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, r_2)$ for any $r_2 \in S_2(h)$.

Case D: $h = (\cdot, x_1) \in H_2^R$ is a third-period history. First, suppose $s_2^*(h) = \text{r}$. Since $s_2^* \in X_2^1$, $x_1 = 0$. Thus, for any $s_1^* \in S_1(h)$ and any $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) = 0$. Second, suppose $s_2^*(h) = \text{a}$. For any $s_1^* \in S_1(h)$, $\pi_2(s_1^*, s_2^*) = \delta^2(1 - x_1) \geq 0$. For any $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) \in \{\delta^2(1 - x_1), 0\}$, as desired. ■

Characterization: Round 2

Lemma E.6. $\text{EFR}_1^2 \subseteq X_1^2$.

Proof. Fix $s_1 \in \text{EFR}_1^2$ and some $h \in H_1[s_1] \cap H_1^1$. Then, there exists some $\nu \in \Delta(S_2(h))$ satisfying the conditions of Remark E.1—i.e., s_1 is a best response under ν given strategies in $S_1(h)$ and, for some $E_2 \subseteq X_2^1$, $\nu(E_2) = 1$.

First suppose that $h = (\phi)$ but, contra hypothesis, $s_1(h) < 1 - \delta$. Consider $r_1 \in S_1$ with $r_1(h) \in (s_1(h), 1-\delta)$. For any $s_2 \in X_2^1$, $s_2(\phi, s_1(h)) = s_2(\phi, r_1(h)) = \mathrm{a}$. Thus, $\pi_1(s_1, \nu) = s_1(h) < r_1(h) = \pi_1(r_1, \nu)$, a contradiction.

Second, suppose that $h \in H_1^P$ is a third-period history but, contra hypothesis, $s_1(h) < 1$. Consider $r_1 \in S_1$ with $r_1(h) \in (s_1(h), 1)$. For any $s_2 \in X_2^1$, $s_2(\phi, s_1(h)) = s_2(\phi, r_1(h)) = \mathrm{a}$. Thus, $\pi_1(s_1, \nu) = \delta^2 s_1(h) < \delta^2 r_1(h) = \pi_1(r_1, \nu)$, a contradiction.

Finally, suppose that $h = (\cdot, x_2) \in H_1^R$, $s_1(h) = \mathrm{a}$ but, contra hypothesis, that $\delta > 1 - x_2$. Let y_1 be such that $\delta y_1 \in (1 - x_2, \delta)$ and construct $r_1 \in S_1(h)$ so that $r_1(h) = \mathrm{r}$, $r_1(h, \mathrm{r}) = y_1$. Since $y_1 < 1$, for each $s_2 \in X_2^1 \cap S_2(h)$, $s_2(h, \mathrm{r}, y_1) = \mathrm{a}$. So, $\pi_1(r_1, \nu) = \delta^2 y_1 > \delta(1 - x_2) = \pi_1(s_1, \nu)$, a contradiction. ■

Lemma E.7. $X_1^2 \subseteq \mathrm{PEFR}_1^2$.

Proof. Fix $s_1^* \in X_1^2 \subseteq \mathrm{PEFR}_1^1$. We show that, for each $h \in H_1[s_1^*] \cap H_1^1$, there is some $s_2^* \in X_2^1 \cap S_2(h) = \mathrm{PEFR}_2^1 \cap S_2(h)$ so that $\pi_1(s_1^*, s_2^*) \geq \pi_1(r_1, s_2^*)$ for each $r_1 \in S_1(h)$.

Case A: $h = (\phi)$ with $s_1^*(\phi) = x_1^* < \delta^2$. Note, since $s_1^* \in X_1^2$, $x_1^* \in [1 - \delta, \delta^2)$. Construct s_2^* so that (i) $s_2^*(\phi, x_1) = \mathrm{a}$ if and only if $x_1 \in [0, 1 - \delta)$, (ii) $s_2^*(h') = 1$ for each $h' \in H_2^P$, and (iii) $s_2^*(h') = \mathrm{a}$ for each third-period history $h' \in H_2^R$. Observe that $s_2^* \in X_2^1$. Note that, since $s_1^* \in X_1^2$, $s_1^*(\phi, x_1^*, \mathrm{r}, 1) = \mathrm{r}$ and, for any third-period history $h' \in H_1^P$ with $s_2^* \in S_2(h')$, $s_1(h') = 1$. Thus, $\pi_1(s_1^*, s_2^*) = \delta^2$. Consider an alternate strategy r_1 and note that $\pi_1(r_1, s_2^*) \in [0, 1 - \delta) \cup [0, \delta^2]$. Thus, $\pi_1(s_1^*, s_2^*) \geq \pi_1(r_1, s_2^*)$.

Case B: $h = (\phi)$ with $s_1^*(\phi) = x_1^* \geq \delta^2$. Construct s_2^* so that (i) $s_2^*(\phi, x_1) = \mathrm{a}$ if and only if $x_1 \in [0, 1 - \delta) \cup \{x_1^*\}$, (ii) $s_2^*(h') = 1$ for each $h' \in H_2^P$, and (iii) $s_2^*(h') = \mathrm{a}$ for each third-period history $h' = (\cdot, x_1) \in H_2^R$. Observe that $s_2^* \in X_2^1$. Note that, $\pi_1(s_1^*, s_2^*) = x_1^*$. For any $r_1 \in S_1$, $\pi_1(r_1, s_2^*) \in \{x_1^*\} \cup [0, 1 - \delta) \cup [0, \delta^2]$. Using the fact that $s_1^* \in X_1^2$, $x_1^* \geq 1 - \delta$. Moreover, by assumption, $x_1^* \geq \delta^2$. Thus, $\pi_1(s_1^*, s_2^*) \geq \pi_1(r_1, s_2^*)$.

Case C: $h = (\phi, s_1^*(\phi), \mathrm{r}, x_2)$ and $s_1^*(h) = \mathrm{a}$. Since $h \in H_1^1$, there exists some $s_2^* \in X_2^1 \cap S_2(h)$. For any $s_2^* \in X_2^1 \cap S_2(h)$ and any third-period history $h' = (\cdot, x_1) \in H_2^R$ with $x_1 < 1$, $s_2^*(h') = \mathrm{a}$. Fix one such strategy s_2^* and observe that $\pi_1(s_1^*, s_2^*) = \delta(1 - x_2)$. Since $s_1^* \in X_1^2$ and $s_1^*(h) = \mathrm{a}$, $1 - x_2 \geq \delta$ and so $\pi_1(s_1^*, s_2^*) \geq \delta^2$. For any other strategy $r_1 \in S_1(h)$ either (a) $r_1(h) = \mathrm{a}$ and $\pi_1(s_1^*, s_2^*) = \pi_1(r_1, s_2^*)$ or (b) $\pi_1(r_1, s_2^*) \in [0, \delta^2]$.

Case D: $h = (\phi, s_1^*(\phi), \mathrm{r}, x_2)$ and $s_1^*(h) = \mathrm{r}$. Since $h \in H_1^1$, there exists some $s_2^* \in X_2^1 \cap S_2(h)$. We can and do choose s_2^* so that $s_2^*(h') = \mathrm{a}$ for each third-period history $h' \in H_2^R$. Since

$s_1^* \in X_1^2$, $s_1^*(h, \mathrm{r}) = 1$ and so $\pi_1(s_1^*, s_2^*) = \delta^2$. For any $r_1 \in S_1(h)$, we have $\pi_1(r_1, s_2^*) \in [0, \delta^2] \cup \{\delta(1 - x_2)\}$. Since $s_1^* \in X_1^1$, it follows that $\delta \geq 1 - x_2$, as desired.

Case E: $h = (\phi, s_1^*(\phi), \mathrm{r}, x_2, \mathrm{r})$. Repeat the argument in Case D to reach the conclusion. ■

Lemma E.8. $\mathrm{EFR}_2^2 \subseteq X_2^2$.

Proof. Fix $s_2 \in \mathrm{EFR}_2^2$ and a history $h \in H_2[s_2] \cap H_2^1$. Then, there exists some $\nu \in \Delta(S_1(h))$ satisfying the conditions of Remark E.1—i.e., s_2 is a best response under ν given strategies in $S_2(h)$ and, for some $E_1 \subseteq X_1^1$, $\nu(E_1) = 1$.

First, let $h \in H_2^P$ but, contra hypothesis, $s_2(h) < 1 - \delta$. Consider $r_2 \in S_2(h)$ so that $r_2(h) \in (s_2(h), 1 - \delta)$. For any $s_1 \in X_1^1 \cap S_1(h)$, $s_1(h, s_2(h)) = s_1(h, r_2(h)) = \mathrm{a}$. Thus, $\pi_2(\nu, s_2) = s_2(h) < r_2(h) = \pi_2(\nu, r_2)$, a contradiction.

Second, let $h = (\phi, x_1)$ and $s_2(h) = \mathrm{a}$. Suppose, contra hypothesis, $1 - x_1 < \delta(1 - \delta)$. Then, there exists y_2 so that $\delta y_2 \in (1 - x_1, \delta(1 - \delta))$. Consider an alternate strategy r_2 with $r_2(\phi, x_1) = \mathrm{r}$ and $r_2(\phi, x_1, \mathrm{r}) = y_2$. Since $1 - \delta > y_2$, for any $s_1 \in X_1^1 \cap S_1(\phi, x_1, \mathrm{r}, y_1)$, $s_1(\phi, x_1, \mathrm{r}, y_2) = \mathrm{a}$. Thus, $\pi_2(\nu, r_2) = \delta y_2 > 1 - x_1 = \pi_2(\nu, s_2)$, a contradiction.

Finally, let $h = (\phi, x_1)$ and $s_2(h) = \mathrm{r}$. Write $y_2 = s_2^*(h, \mathrm{r})$. Suppose, contra hypothesis, $1 - x_1 > \delta y_2$. Let $r_2 \in S_2(h)$ with $r_2(h) = \mathrm{a}$. Then, $\pi_2(\nu, s_2) \geq \pi_2(\nu, r_2) = 1 - x_1$. Thus, there must be some $s_1 \in X_1^1 \cap S_1(h)$ so that $\pi_2(s_1, s_2) \geq 1 - x_1$. Note, for such an s_1, it must be that $s_1(h, \mathrm{r}, y_2) = \mathrm{r}$; if not $\pi_2(s_1, s_2) = \delta y_2 < 1 - x_1$. Since $s_1 \in X_1^1$ and $s_1(h, \mathrm{r}, y_2) = \mathrm{r}$, it follows that $\delta z_1 \geq 1 - y_2$, where we write $z_1 = s_1(h, \mathrm{r}, y_2, \mathrm{r})$. Thus we must have:

(i) $\delta^2(1 - z_1) \geq 1 - x_1 > \delta y_2$, and

(ii) $\delta z_1 \geq 1 - y_2$.

Put together, these say that $\delta(1 - z_1) > y_2 \geq 1 - \delta z_1$, a contradiction. ■

Lemma E.9. $X_2^2 \subseteq \mathrm{PEFR}_2^2$

Proof. Fix $s_2^* \in X_2^2$. We show that, for each $h \in H_1[s_1^*] \cap H_2^1$, there is some $s_1^* \in X_1^1 \cap S_1(h) = \mathrm{PEFR}_1^1 \cap S_1(h)$ so that $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, r_2)$ for each $r_2 \in S_2(h)$.

Case A: $h = (\phi, x_1) \in H_2^R$ and $s_2^*(h) = \mathrm{a}$. Since $s_2^* \in X_2^2$ and $s_2^*(h) = \mathrm{a}$, $1 - x_1 \geq \delta(1 - \delta)$. Construct s_1^* so that (i) $s_1^*(\phi) = x_1$, (ii) for each $h' = (\cdot, x_2) \in H_1^R$, $s_1^*(h') = \mathrm{a}$ if and only if $x_2 \in [0, 1 - \delta)$, and (iii) $s_1^*(h') = 1$ for each third-period $h' \in H_1^P$. Observe that $s_1^* \in X_1^1 \cap S_1(h)$. Moreover, $\pi_2(s_1^*, s_2^*) = 1 - x_1$. For each $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) \in \{(1 - x_1)\} \cup [0, \delta(1 - \delta))$. Thus, $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, r_2)$, as desired.

Case B: $h \in \{(\phi, x_1), (\phi, x_1, \mathrm{r})\}$ and $s_2^*(\phi, x_1) = \mathrm{r}$. Write $s_2^*(\phi, x_1, \mathrm{r}) = x_2^*$. Since $s_2^* \in X_2^2$, $\delta x_2^* \geq 1 - x_1$ and $x_2^* \geq 1 - \delta$. Construct s_1^* so that (i) $s_1^*(\phi) = x_1$, (ii) for $h' = (\cdot, x_2) \in H_1^R$, $s_1^*(h') = \mathrm{a}$ if and only if $x_2 \in [0, x_2^*]$, and (iii) $s_1^*(h') = 1$ for each third-period $h' \in H_1^P$. Notice, for each $h' = (\cdot, x_2) \in H_1^R$ with $s_1^*(h') = \mathrm{r}$, $x_2 > x_2^* \geq 1 - \delta$. Thus, $s_1^* \in X_1^1$. Now observe that $\pi_2(s_1^*, s_2^*) = \delta x_2^*$ and, for each $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) \in \{1 - x_1\} \cup [0, \delta x_2^*]$. Since $\delta x_2^* \geq 1 - x_1$, $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, r_2)$ for each $r_2 \in S_2(h)$.

Case C: $h = (\cdot, x_1) \in H_2^R$ is a third-period history. First, suppose $s_2^*(h) = \mathrm{r}$. Since $s_2^* \in X_2^1$, $x_1 = 0$. Thus, for any $s_1^* \in S_1(h)$ and any $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) = 0$. Second, suppose $s_2^*(h) = \mathrm{a}$. For any $s_1^* \in S_1(h)$, $\pi_2(s_1^*, s_2^*) = \delta^2(1 - x_1) \geq 0$. For any $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) \in \{\delta^2(1 - x_1), 0\}$, as desired. ■

Characterization: Round 3 and Beyond

Lemma E.10. $\mathrm{EFR}_1^3 \subseteq X_1^3$

Proof. Fix $s_1 \in \mathrm{EFR}_1^3$. Then, there exists some $\nu \in \Delta(S_2)$ satisfying the conditions of Remark E.1—i.e., s_1 is a best response under ν given strategies in S_1 and, for some $E_2 \subseteq X_2^2$, $\nu(E_2) = 1$.

It suffices to show: If $1 - \delta > \delta^2$, then $s_1(\phi) \leq 1 - \delta(1 - \delta)$. Since $1 - \delta > \delta^2$, there exists some $r_1 \in S_1$ so that $r_1(\phi) \in (\delta^2, 1 - \delta)$. For any $s_2 \in X_2^2$, $s_2(\phi, r_1(\phi)) = \mathrm{a}$. Thus, $\pi_1(r_1, \nu) > \delta^2$.

Suppose, contra hypothesis, $s_1(\phi) = x_1 > 1 - \delta(1 - \delta)$. Note, for each $s_2 \in X_2^2$, $s_2(\phi, x_1) = \mathrm{r}$ and $s_2(\phi, x_1, \mathrm{r}) \geq 1 - \delta$. Thus, for each $s_2 \in X_2^2$, $\pi_1(s_1, s_2) \leq \max\{\delta(1 - s_2(\phi, x_1, \mathrm{r})), \delta^2\} \leq \delta^2$. As such, $\pi_1(s_1, \nu) \leq \delta^2 < \pi_1(r_1, \nu)$, a contradiction. ■

Lemma E.11. $X_1^3 \subseteq \mathrm{PEFR}_1^3$

Proof. Fix $s_1^* \in X_1^3 \subseteq \mathrm{PEFR}_1^2$. We show that, for each $h \in H_1[s_1^*] \cap H_1^2$, there is some $s_2^* \in X_2^2 \cap S_2(h) = \mathrm{PEFR}_2^2 \cap S_2(h)$ so that $\pi_1(s_1^*, s_2^*) \geq \pi_1(r_1, s_2^*)$ for each $r_1 \in S_1(h)$.

Case A: $h = (\phi)$ and $1 - \delta \leq \delta^2$. Write $s_1^*(\phi) = x_1^*$. Construct s_2^* as in Round 2 Case A: (i) $s_2^*(\phi, x_1) = \mathrm{a}$ if and only if $x_1 \in [0, 1 - \delta)$, (ii) $s_2^*(h) = 1$ for each $h \in H_2^P$, and (iii) $s_2^*(h) = \mathrm{a}$ for each third-period history $h = (\cdot, x_1) \in H_2^R$ with $x_1 < 1$. Observe that $s_2^* \in X_2^2$. Since $s_1^* \in X_1^2$, $x_1^* \geq 1 - \delta$, $s_1^*(\phi, x_1^*, \mathrm{r}, 1) = \mathrm{r}$ and, for any third-period history $h \in H_1^P$ with $s_2^* \in S_2(h)$, $s_1(h) = 1$. Thus, $\pi_1(s_1^*, s_2^*) = \delta^2$ and, for any $r_1 \in S_1$, $\pi_1(r_1, s_2^*) \in [0, 1 - \delta) \cup [0, \delta^2]$. Since $1 - \delta \leq \delta^2$, $\pi_1(s_1^*, s_2^*) \geq \pi_1(r_1, s_2^*)$ for each $r_1 \in S_1$.

Case B: $h = (\phi)$ with $1 - \delta > \delta^2$. Since $s_1^* \in X_1^3$, $s_1(\phi) = x_1^* \in [1 - \delta, 1 - \delta(1 - \delta)]$. Construct s_2^* so that (i) $s_2^*(\phi, x_1) = \mathrm{a}$ if and only if $x_1 \in [0, 1 - \delta) \cup \{x_1^*\}$, (ii) $s_2^*(h) = 1$

for each $h \in H_2^P$, and (iii) $s_2^*(h) = \text{a}$ for each third-period history $h = (\cdot, x_1) \in H_2^R$. Since $1 - \delta(1-\delta) \geq x_1^*$, $s_2^* \in X_2^2$. Note that, $\pi_1(s_1^*, s_2^*) = x_1^*$. Moreover, for any $r_1 \in S_1$, $\pi_1(r_1, s_2^*) \in \{x_1^*\} \cup [0, 1-\delta) \cup [0, \delta^2]$. Since $\pi_1(s_1^*, s_2^*) = x_1^* \geq 1 - \delta > \delta^2$, it follows that $\pi_1(s_1^*, s_2^*) \geq \pi_1(r_1, s_2^*)$, for each $r_1 \in S_1$.

Case C: $h = (\phi, s_1^*(\phi), \text{r}, x_2)$ *and* $s_1^*(h) = \text{a}$. Since $h \in H_1^2$, there exists some $s_2^* \in X_2^2 \cap S_2(h^*)$ and, for any such s_2^*, $s_2^*(h') = \text{a}$ for each third-period history $h' = (\cdot, x_1) \in H_2^R$ with $x_1 < 1$. Fix one such strategy s_2^* and observe that $\pi_1(s_1^*, s_2^*) = \delta(1 - x_2)$. Since $s_1^* \in X_1^2$ and $s_1^*(h) = \text{a}$, $1 - x_2 \geq \delta$ and so $\pi_1(s_1^*, s_2^*) \geq \delta^2$. For any $r_1 \in S_1(h)$ either (a) $r_1(h) = \text{a}$ and $\pi_1(s_1^*, s_2^*) = \pi_1(r_1, s_2^*)$ or (b) $\pi_1(r_1, s_2^*) = [0, \delta^2]$.

Case D: $h = (\phi, s_1^*(\phi), \text{r}, x_2)$ *and* $s_1^*(h) = \text{r}$. Since $h \in H_1^2$, there exists some $s_2^* \in X_2^2 \cap S_2(h)$. We can choose s_2^* so that $s_2^*(h') = \text{a}$ for each third-period history $h' \in H_2^R$. Notice that $s_1^*(h, \text{r}) = 1$, since $s_1^* \in X_1^2$. As such, $\pi_1(s_1^*, s_2^*) = \delta^2$. For any $r_1 \in S_1(h)$, $\pi_1(r_1, s_2^*) \in \{\delta(1 - x_2)\} \cup [0, \delta^2]$. Since $s_1^* \in X_1^1$, it follows that $\delta \geq 1 - x_2$, as desired.

Case E: $h = (\phi, s_1^*(\phi), \text{r}, x_2, \text{r})$. Repeat the argument in Case D to reach the conclusion. ■

Lemma E.12. $\text{EFR}_2^3 \subseteq X_2^3$.

Proof. Fix $s_2 \in \text{EFR}_2^3$ and a history $h \in H_2[s_2] \cap H_2^2$. Then, there exists some $\nu \in \Delta(S_1(h))$ satisfying the conditions of Remark E.1—i.e., s_2 is a best response under ν given strategies in $S_2(h)$ and, for some $E_1 \subseteq X_1^2$, $\nu(E_1) = 1$.

First, let $h \in H_2^P$. Since $s_2 \in \text{EFR}_2^2 = X_2^2$, $s_2(h) \geq 1 - \delta$. Suppose, contra hypothesis, $s_2(h) > 1 - \delta$. Then, for each $s_1 \in X_1^2 \cap S_1(h)$, $s_1(h, s_2(h)) = \text{r}$ and $s_1(h, s_2(h), \text{r}) = 1$. Thus, $\pi_2(\nu, s_2) = 0$. By contrast, consider some $r_2 \in S_2(h)$ with $r_2(h) \in (0, 1-\delta)$. For each $s_1 \in X_1^1 \cap S_1(h)$, $s_1(h, r_2(h)) = \text{a}$. So, $\pi_2(\nu, r_2) = \delta r_2(h) > 0$, a contradiction.

Second, suppose $h = (\phi, x_1)$, $s_2(h) = \text{r}$, but $1 - x_1 > \delta(1 - \delta)$. Observe that, for each $s_1 \in X_1^2 \cap S_1(h)$, $\pi_2(s_1, s_2) \in \{\delta s_2(h, \text{r}), 0\}$. Thus, $\delta s_2(h, \text{r}) \geq \pi_2(\nu, s_2)$. Moreover, there exists some $r_2 \in S_2(h)$ with $r_2(h) = \text{a}$ and $\pi_2(\nu, r_2) = 1 - x_1$. As such,

$$\delta s_2(h, \text{r}) \geq \pi_2(\nu, s_2) \geq \pi_2(\nu, r_2) = 1 - x_1 > \delta(1 - \delta).$$

From this, (a) $s_2(h, \text{r}) > 1 - \delta$ and (b) $\pi_2(\nu, s_2) = \delta s_2(h, \text{r})$. Note (b) implies that there is some $s_1 \in X_1^2 \cap S_1(h)$ with $s_1(h, \text{r}, s_2(h, \text{r})) = \text{a}$. Using the fact that $s_1 \in X_1^2$, $1 - \delta \geq s_2(h, \text{r})$, contradicting (a). ■

Lemma E.13. $X_2^3 \subseteq \text{PEFR}_2^4$, $X_2^3 = X_2^4 \subseteq \text{PEFR}_2^4$, *and* $X_2^3 = X_2^5 \subseteq \text{PEFR}_2^5$.

Proof. Fix $s_2^* \in X_2^3 = X_2^4$. We show that, for each $h \in H_1[s_1^*] \cap H_2^2$ (resp. $h \in H_1[s_1^*] \cap H_2^3$, resp. $h \in H_1[s_1^*] \cap H_2^4$), there is some $s_1^* \in X_1^2 \cap S_1(h) = \text{PEFR}_1^2 \cap S_1(h)$ (resp.

$s_1^* \in X_1^3 \cap S_1(h) = \text{PEFR}_1^3 \cap S_1(h)$, resp. $s_1^* \in X_1^4 \cap S_1(h) = \text{PEFR}_1^4 \cap S_1(h)$), so that $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, r_2)$ for each $r_2 \in S_2(h)$.

Case A: $h = (\phi, x_1) \in H_2^R$ and $s_2^*(h) = \text{a}$. Construct a strategy s_1^* so that (i) $s_1^*(\phi) = x_1$, (ii) for each $h' = (\cdot, x_2) \in H_1^R$, $s_1^*(h') = \text{a}$ if and only if $x_2 \in [0, 1-\delta)$, and (iii) $s_1^*(h') = 1$ for each third-period $h' \in H_1^P$. Observe that $s_1^* \in X_1^2 \cap S_1(h)$ and, if $h \in H_2^3$, $s_1^* \in X_1^3$ (resp. $h \in H_2^4$, $s_1^* \in X_1^4$). Moreover, $\pi_2(s_1^*, s_2^*) = (1 - x_1)$ and, for each $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) \in \{1 - x_1\} \cup [0, \delta(1-\delta))$. Since $s_2^* \in X_2^2$ and $s_2^*(h) = \text{a}$, $1 - x_1 \geq \delta(1-\delta)$. As such, $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, r_2)$ for each $r_2 \in S_2(h)$.

Case B: $h \in \{(\phi, x_1), (\phi, x_1, \text{r})\} \in H_2^R$ and $s_2^*(\phi, x_1) = \text{r}$. Since $s_2^* \in X_2^3$, $\delta(1-\delta) \geq 1 - x_1$ and $s_2(\phi, x_1, \text{r}) = 1 - \delta$. Construct a strategy s_1^* so that (i) $s_1^*(\phi) = x_1$, (ii) for $h' = (\cdot, x_2) \in H_1^R$, $s_1^*(h') = \text{a}$ if and only if $x_2 \in [0, 1-\delta]$, and (iii) $s_1^*(h') = 1$ for each third-period $h' \in H_1^P$. Observe that $s_1^* \in X_1^2 \cap S_1(h)$ and, if $h \in H_2^3$, $s_1^* \in X_1^3$ (resp. $h \in H_2^4$, $s_1^* \in X_1^4$). Moreover, $\pi_2(s_1^*, s_2^*) = \delta(1-\delta)$ and, for each $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) \in \{1 - x_1\} \cup [0, \delta(1-\delta)]$. Since $\delta(1-\delta) \geq 1 - x_1$, $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, r_2)$ for each $r_2 \in S_2(h)$.

Case C: $h = (\cdot, x_1) \in H_2^R$ is a third-period history. First, suppose $s_2^*(h) = \text{r}$. Since $s_2^* \in X_2^1$, $x_1 = 0$. Thus, for any $s_1^* \in S_1(h)$ and any $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) = 0$. Second, suppose $s_2^*(h) = \text{a}$. For any $s_1^* \in S_1(h)$, $\pi_2(s_1^*, s_2^*) = \delta^2(1 - x_1) \geq 0$. For any $r_2 \in S_2(h)$, $\pi_2(s_1^*, r_2) \in \{\delta^2(1 - x_1), 0\}$, as desired. ■

Lemma E.14. $\text{EFR}_1^4 \subseteq X_1^4$.

Proof. Fix $s_1 \in \text{EFR}_1^4$. Then, there exists some $\nu \in \Delta(S_2)$ satisfying the conditions of Remark E.1—i.e., s_1 is a best response under ν given strategies in S_1 and, for some $E_2 \subseteq X_2^3$, $\nu(E_2) = 1$.

First, we show that $s_1(\phi) \geq 1 - \delta(1-\delta)$. Suppose, contra hypothesis, $s_1(\phi) < 1 - \delta(1-\delta)$. Then, there exists some $r_1 \in S_1$ with $r_1(\phi) \in (s_1(\phi), 1 - \delta(1-\delta))$. For each $s_2 \in X_2^3$, $s_2(\phi, s_1(\phi)) = s_2(\phi, r_1(\phi)) = \text{a}$. Thus, $\pi_1(s_1, \nu) = s_1(\phi) < r_1(\phi) = \pi_1(r_1, \nu)$, a contradiction.

Next, we show that $s_1(\phi) \leq 1 - \delta(1-\delta)$. Suppose, contra hypothesis, $s_1(\phi) > 1 - \delta(1-\delta)$. Then, for any $s_2 \in X_2^3$ $s_2(\phi, s_1(\phi)) = \text{r}$ and $s_2(\phi, s_1(\phi, \text{r}) = 1 - \delta$. Thus, $\pi_1(s_1, \nu) \in [0, \delta^2]$. Consider, instead, $r_1 \in S_1$ with $r_1(\phi) = x_1 \in (\delta^2, 1 - \delta(1-\delta))$. For any $s_2 \in X_2^3$, $s_2(\phi, x_1) = \text{a}$ and so $\pi_1(r_1, \nu) = x_1 > \delta^2 \geq \pi_1(s_1, \nu)$. ■

Lemma E.15. $X_1^4 \subseteq \text{PEFR}_1^4$.

Proof. Fix $s_1^* \in X_1^4$. We show that, for each $h \in H_1[s_1^*] \cap H_1^3$, there is some $s_2^* \in S_2(h) \cap X_2^3$ so that $\pi_1(s_1^*, s_2^*) \geq \pi_1(r_1, s_2^*)$ for each $r_1 \in S_1(h)$.

Case A: $h = (\phi)$. Since $s_1^* \in X_1^4$, $s_1^*(\phi) = 1 - \delta(1-\delta)$. Construct s_2^* as in Round 2 Case A: (i) $s_2^*(\phi, x_1) = \text{a}$ if and only if $x_1 \in [0, 1-\delta(1-\delta)]$, (ii) $s_2^*(h) = 1-\delta$ for each $h \in H_2^P$, and (iii) $s_2^*(h) = \text{a}$ for each third-period history $h = (\cdot, x_1) \in H_2^R$. Observe that $s_2^* \in X_2^3$. Note, $\pi_1(s_1^*, s_2^*) = 1-\delta(1-\delta)$. If $r_1(\phi) \leq 1-\delta(1-\delta)$, then $\pi_1(r_1, s_2^*) \leq \pi_1(s_1^*, s_2^*)$. If $r_1(\phi) > 1-\delta(1-\delta)$, then $s_2^*(\phi, r_1(\phi)) = \text{r}$ and, in that case, $\pi_1(r_1, s_2^*) \in [0, \delta^2]$. Thus, $\pi_1(s_1^*, s_2^*) \geq \pi_1(r_1, s_2^*)$ for each $r_1 \in S_1$.

Case B: $h = (\phi, s_1^*(\phi), \text{r}, x_2)$. Since $h \in H_1^3$, $x_2 = 1-\delta$. Construct $s_2^* \in X_2^3 \cap S_2(h)$ with $s_2^*(h') = \text{a}$ for each third-period history $h' \in H_2^R$. Observe that $\pi_1(s_1^*, s_2^*) = \delta^2$ and, for any $r_1 \in S_1(h)$, $\pi_1(r_1, s_2^*) \in [0, \delta^2]$.

Case B: $h = (\phi, s_1^*(\phi), \text{r}, x_2)$ *and* $s_1^*(h) = \text{a}$. Since $h \in H_1^3$, $x_2 = 1-\delta$. Construct $s_2^* \in X_2^3 \cap S_2(h)$ with $s_2^*(h') = \text{a}$ for each third-period history $h' \in H_2^R$. Observe that $\pi_1(s_1^*, s_2^*) = \delta^2$. For any $r_1 \in S_1(h)$, $\pi_1(r_1, s_2^*) \in [0, \delta^2]$, as desired.

Case C: $h = (\phi, s_1^*(\phi), \text{r}, x_2, \text{r})$. Repeat the argument in Case B to reach the conclusion. ■

Appendix F   Proofs for Section 9

This Appendix proves Proposition 9.1. It then provides an example that illustrates the role of "no uncertainty about breaking indifferences." The proof of Proposition 9.1 will follow from the following results:

Proposition F.1. *Let $N < \infty$ and assume that Bi proposes in period N. Fix (s_1^*, s_2^*) so that $\xi(\zeta((s_1^*, s_2^*))) = (x_1^*, x_2^*, n) \neq (0, 0, N)$. If there exists some $k \geq 0$ so that either*

(i) $n = N - 2k - 1$ and $(s_i^*, t_i^*, s_{-i}^*, t_{-i}^*) \in R_i^2 \times R_{-i}^1$, or

(ii) $n = N - 2k$ and $(s_i^*, t_i^*, s_{-i}^*, t_{-i}^*) \in R_i^{2k+2} \times R_{-i}^{2k+1}$,

then $x_i^* \geq \delta^{N-n}$.

Definition F.1. Call a set $Q_1 \times Q_2 \subseteq S_1 \times S_2$ **constant** if, for any $(s_1, s_2), (r_1, r_2) \in Q_1 \times Q_2$, $\pi_1(s_1, s_2) = \pi_1(r_1, r_2)$ and $\pi_2(s_1, s_2) = \pi_2(r_1, r_2)$.

Remark F.1. If $\pi_1(s_1, s_2) = \pi_1(r_1, r_2)$ and $\pi_2(s_1, s_2) = \pi_2(r_1, r_2)$, then $\xi(\zeta(s_1, s_2)) = \xi(\zeta(r_1, r_2))$.

Proposition F.2. *Let $N < \infty$. Suppose $R^\infty \neq \emptyset$ and, at each state in R^∞, no Bi is uncertain about how $B(-i)$ breaks indifferences. Then,* $\text{proj}_{S_1} R_1^\infty \times \text{proj}_{S_2} R_2^\infty$ *is constant.*

Proposition F.3. *Suppose* $\text{proj}_{S_1} R_1^\infty \times \text{proj}_{S_2} R_2^\infty$ *is constant and* (x_1^*, x_2^*, n) *is an outcome induced by some* $(s_1^*, s_2^*) \in \text{proj}_{S_1} R_1^\infty \times \text{proj}_{S_2} R_2^\infty$. *Then,*

$$\delta^{n-1} x_1^* \geq 1 - \delta \qquad \textit{and} \qquad \delta^{n-1} x_2^* \geq \delta(1-\delta).$$

Proof of Proposition 9.1. Immediate from Propositions F.1-F.2-F.3. ■

To prove these results, it will be useful to note the following:

Remark F.2. For any $h, h' \in H_i$ and any Borel $E_h \subseteq C_h$, $\{s_i \in S_i(h') : s_i(h) \in E_h\} = (\text{proj}_h)^{-1}(E_h) \cap S_i(h')$ is Borel. (Use Lemma A.1.)

F.1 Proof of Proposition F.1

Lemma F.1. *Let* $N < \infty$ *and assume that Bi proposes in period* N. *Suppose* $\xi(\zeta(s_1^*, t_1^*, s_2^*, t_2^*)) = (x_1^*, x_2^*, N-2k)$ *for* $\frac{N-1}{2} \geq k \geq 1$. *If* $(s_i^*, t_i^*, s_{-i}^*, t_{-i}^*) \in R_i^{2k+2} \times R_{-i}^{2k+1}$, *then* $x_i^* \geq \delta^{2k}$.

Proof of Lemma F.1. We suppose the result is true for all j with $k > j \geq 1$ and show that it is also true for k.[2] Throughout, we fix a state $(s_i^*, t_i^*, s_{-i}^*, t_{-i}^*) \in R_i^{2k+2} \times R_{-i}^{2k+1}$ with $\xi(\zeta(s_1^*, t_1^*, s_2^*, t_2^*)) = (x_1^*, x_2^*, N-2k)$. Along the path induced by (s_1^*, s_2^*), there is a $(N-2k)$-period history $h^* \in H_i^P$ with $s_i^*(h^*) = x_i^*$ and $s_{-i}^*(h^*, x_i^*) = \text{a}$. (Here we use the fact that $N - 2k < N$.) We will show that $x_i^* \geq \delta^{2k}$.

Case A: Suppose $\beta_{i,h^*}(t_i^*)$ assigns probability 1 to

$$\mathsf{A}_{-i}[h^*, x_i^*] := \{r_{-i} \in S_{-i}(h^*) : r_{-i}(h^*, x_i^*) = \text{a}\} \times T_{-i}.$$

(Remark F.2 gives that the set is Borel.) Then, $\mathbb{E}\pi_i[s_i^* | t_i^*, h^*] = \delta^{N-2k-1} x_i^*$.

Next note that t_i^* strongly believes R_{-i}^1 and $R_{-i}^1 \cap [S_{-i}(h^*) \times T_{-i}] \neq \emptyset$ (since $(s_{-i}^*, t_{-i}^*) \in R_{-i}^1 \cap [S_{-i}(h^*) \times T_{-i}]$). It follows that, for each $x \in [0,1)$, t_i^* can secure $\delta^{N-1} x$ at h^* (Lemma B.5). Since (s_i^*, t_i^*) is rational, for each $x \in [0,1)$, $\delta^{N-2k-1} x_i^* \geq \delta^{N-1} x$ or $x_i^* \geq \delta^{2k}$.

Case B: Suppose $\beta_{i,h^*}(t_i^*)$ assigns strictly positive probability to

$$\mathsf{R}_{-i}[h^*, x_i^*] := \{r_{-i} \in S_{-i}(h^*) : r_{-i}(h^*, x_i^*) = \text{r}\} \times T_{-i}.$$

(Remark F.2 gives that the set is Borel.) Note, t_i^* strongly believes R_{-i}^{2k+1} and $R_{-i}^{2k+1} \cap [S_{-i}(h^*) \times T_{-i}] \neq \emptyset$ (since $(s_{-i}^*, t_{-i}^*) \in R_{-i}^{2k+1} \cap [S_{-i}(h^*) \times T_{-i}]$). So, $\beta_{i,h^*}(t_i^*)(\mathsf{R}_{-i}[h^*, x_i^*] \cap R_{-i}^{2k+1}) > 0$, which implies $\mathsf{R}_{-i}[h^*, x_i^*] \cap R_{-i}^{2k+1} \neq \emptyset$.

Fix some $(r_{-i}, u_{-i}) \in \mathsf{R}_{-i}[h^*, x_i^*] \cap R_{-i}^{2k+1}$. We will show that

$$\delta^{N-2k}(1 - \delta^{2k-1}) \geq \mathbb{E}\pi_{-i}[r_{-i} | u_{-i}, (h^*, x_i^*, \text{r})]. \tag{2}$$

[2] The base case of $k = 1$ follows the same proof with the following two amendments: In Case B.2, take $j = 0$, and Case B.3 does not obtain.

From this, the claim follows: Since (r_{-i}, u_{-i}) is rational,

$$\mathbb{E}\pi_{-i}[r_{-i}|u_{-i}, (h^*, x_i^*, \mathrm{r})] = \mathbb{E}\pi_{-i}[r_{-i}|u_{-i}, (h^*, x_i^*)] \geq \mathbb{E}\pi_{-i}[q_{-i}|u_{-i}, (h^*, x_i^*)]$$

for $q_{-i} \in S_{-i}(h^*, x_i^*)$ with $q_{-i}(h^*, x_i^*) = \mathrm{a}$. Thus,

$$\delta^{N-2k}(1-\delta^{2k-1}) \geq \mathbb{E}\pi_{-i}[r_{-i}|u_{-i}, (h^*, x_i^*, \mathrm{r})] \geq \delta^{N-2k-1}(1-x_i^*)$$

or $x_i^* \geq 1 - \delta(1-\delta^{2k-1}) > \delta^{2k}$, as desired.

The remainder of the proof is devoted to showing Equation (2). For this, note that u_{-i} strongly believes R_i^{2k} and $R_i^{2k} \cap [S_i(h^*, x_i^*, \mathrm{r}) \times T_i] \neq \emptyset$. (Use the fact that $(s_i^*, t_i^*) \in R_i^{2k} \cap [S_i(h^*, x_i^*, \mathrm{r}) \times T_i]$.) With this, $\beta_{-i,(h^*,x_i^*,\mathrm{r})}(u_{-i})(R_i^{2k}) = 1$. As such, to show Equation (2) it suffices to show the following:

> **Claim:** If $(r_i, u_i) \in R_i^{2k} \cap [S_i(h^*, x_i^*, \mathrm{r}) \times T_i]$ with $\xi(\zeta(r_i, r_{-i})) = (x_1, x_2, n)$, then $x_{-i} \leq 1 - \delta^{2k-1}$ and $n \geq N - 2k + 1$.

To show this claim: Fix $(r_i, u_i) \in R_i^{2k} \cap [S_i(h^*, x_i^*, \mathrm{r}) \times T_i]$ with $\xi(\zeta(r_i, r_{-i})) = (x_1, x_2, n)$. Certainly, $n \geq N - 2k + 1$. We show $x_{-i} \leq 1 - \delta^{2k-1}$. Write $h[n]$ for the n-period history in $H_1^P \cup H_2^P$ along the path induced by (r_1, r_2). There will be three subcases.

Subcase 1. Suppose $n = N$. Note that $(r_i, u_i) \in R_i^{2k} \subseteq R_i^2$ and $(r_{-i}, u_{-i}) \in R_{-i}^1 \cap (S_{-i}(h[n]) \times T_{-i})$. So, $\beta_{i,h[n]}(u_i)$ assigns probability 1 to

$$\{s_{-i} \in S_{-i}(h[n]) : s_{-i}(h[n], x) = \mathrm{a}, \text{ for all } x \in [0, 1)\} \times T_{-i}.$$

(Remark F.2 gives that the set is Borel.) Since $(r_i, u_i) \in R_i^1$, $r_i(h[n]) = 1$ and so $x_{-i} = 0$.

Subcase 2. Suppose $n = N - 2j + 1$ for some j with $k \geq j \geq 1$, so that $h[n] \in H_{-i}^P$. Since $(r_i, u_i, r_{-i}, u_{-i}) \in R^{2k} \subseteq R^2$. It then follows from Lemma B.6(i) that $x_i \geq \delta^{N-(N-2j+1)} \geq \delta^{2k-1}$. Thus, $x_{-i} \leq 1 - \delta^{2k-1}$, as desired.

Subcase 3. Suppose $n = N - 2j$ for some j with $k > j \geq 1$, so that $h[n] \in H_i^P$. Note, $(r_i, u_i, r_{-i}, u_{-i}) \in R^{2k} \subseteq R^{2(j+1)} \subseteq R_i^{2j+2} \times R_{-i}^{2j+1}$. It follows from the assumption that the claim holds for all $j < k$ that $x_i \geq \delta^{2j} \geq \delta^{2k-1}$. Thus, $x_{-i} \leq 1 - \delta^{2k-1}$, as desired. ■

Proof of Proposition F.1. Immediate from Lemmata B.1(ii), B.6(i), and F.1. ■

F.2 Proof of Proposition F.2

It will be convenient to have the following:

Lemma F.2. *R_i^∞ is Borel.*

Proof. This is immediate if $R_i^\infty = \emptyset$. Suppose $R_i^\infty \neq \emptyset$. It suffices to show that, for each m, R_i^{m-1} is Borel: Since $R_i^\infty \neq \emptyset$, there is a t_i that strongly believes $R_{-i}^1, R_{-i}^2, \ldots$. Thus, each of the sets R_{-i}^m are non-empty—i.e., for each m, there exists $(s_{-i}^m, t_{-i}^m) \in R_{-i}^m$. As such, for each m, t_{-i}^m strongly believes R_i^{m-1}, which implies that each R_i^{m-1} is Borel. ■

It will be convenient to introduction notation/terminology. First, write $R_i^\infty(h) = R_i^\infty \cap [S_i(h) \times T_i]$ and $R^\infty(h) = R^\infty \cap [S(h) \times T]$. Second, say R^∞ is **bounded** if there exists some $\overline{n} < \infty$ so that the following holds: If $(s_1, s_2) \in \text{proj}_{S_1} R_1^\infty \times \text{proj}_{S_2} R_2^\infty$ with $\xi(\zeta(s_1, s_2)) = (x_1, x_2, n)$ then $n \leq \overline{n}$. Note, if the bargaining game has a deadline, then R^∞ is bounded.

Proof of Proposition F.2. Suppose R^∞ is non-empty and bounded, but $\text{proj}_{S_1} R_1^\infty \times \text{proj}_{S_2} R_2^\infty$ is not constant. Then, we can find some $\text{B}(-i)$ and some history $h_{-i} \in H_{-i}$, so that the following holds:

(a) $\xi(\zeta(\text{proj}_S R^\infty(h_{-i})))$ contains at least two outcomes, but

(b) for any history $h' \in H$ that strictly follows h_{-i}, $\xi(\zeta(\text{proj}_S R^\infty(h')))$ contains, at most, one outcome.

Write $h_i \in H_i$ for the last history in H_i that precedes h_{-i}. So, if $h_{-i} \in H_{-i}^R$, then $h_{-i} = (h_i, x_i)$ for some $x_i \in [0,1]$. If $h_{-i} \in H_{-i}^P$, then $h_{-i} = (h_i, x_i, \text{r})$ for some $x_i \in [0,1]$.

We will show that, for any $(s_i, t_i) \in R_i^\infty(h_{-i})$, there is some $(s_{-i}, t_{-i}) \in R_{-i}^\infty$ so that, at $(s_i, t_i, s_{-i}, t_{-i})$, B$i$ faces uncertainty about how $\text{B}(-i)$ breaks indifferences.

Step A: This step shows that any two outcomes in $\xi(\zeta(\text{proj}_S R^\infty(h)))$ are $\text{B}(-i)$ equivalent. Fix $(s_i, t_i, s_{-i}, t_{-i}), (r_i, u_i, r_{-i}, u_{-i}) \in R^\infty(h_{-i})$. Then, t_{-i} and u_{-i} strongly believe $R_i^1, R_i^2, \ldots$. It follows from the conjunction property of strong belief that t_{-i} and u_{-i} strongly believe R_i^∞. Using the fact that $(s_{-i}, t_{-i}), (r_{-i}, u_{-i}) \in R_{-i}^1$ plus condition (b):

(i) $\mathbb{E}\pi_{-i}[s_{-i}|t_{-i}, h_{-i}] = \Pi_{-i}(\xi(\zeta(s_i, s_{-i}))) \geq \Pi_{-i}(\xi(\zeta(r_i, r_{-i}))) = \mathbb{E}\pi_{-i}[r_{-i}|t_{-i}, h_{-i}]$, and

(ii) $\mathbb{E}\pi_{-i}[r_{-i}|u_{-i}, h_{-i}] = \Pi_{-i}(\xi(\zeta(r_i, r_{-i}))) \geq \Pi_{-i}(\xi(\zeta(s_i, s_{-i}))) = \mathbb{E}\pi_{-i}[s_{-i}|u_{-i}, h_{-i}]$.

Thus, $\Pi_{-i}(\xi(\zeta(s_i, s_{-i}))) = \Pi_{-i}(\xi(\zeta(r_i, r_{-i})))$, as required.

Step B: First, suppose $h_{-i} = (h_i, x) \in H_{-i}^R$. Fix some $(s_i, t_i) \in R_i^\infty(h_{-i})$. Define the sets

- $\mathsf{A}_{-i} := \{q_{-i} \in S_{-i}(h_{-i}) : q_{-i}(h_{-i}) = \text{a}\} \times T_{-i}$, and
- $\mathsf{R}_{-i} := \{q_{-i} \in S_{-i}(h_{-i}) : q_{-i}(h_{-i}) = \text{r}\} \times T_{-i}$.

Notice, by construction of h_{-i}, both $R_{-i}^{\infty}(h_{-i}) \cap \mathsf{A}_{-i}$ and $R_{-i}^{\infty}(h_{-i}) \cap \mathsf{R}_{-i}$ are non-empty and Borel (Remark F.2-Lemma A.1). Their union is $R_{-i}^{\infty}(h_{-i})$. Thus, we must have either $\beta_{i,h_i}(t_i)(R_{-i}^{\infty}(h_{-i}) \cap \mathsf{A}_{-i}) > 0$ or $\beta_{i,h_i}(t_i)(R_{-i}^{\infty}(h_{-i}) \cap \mathsf{R}_{-i}) > 0$ (or both). Suppose $\beta_{i,h_i}(t_i)(R_{-i}^{\infty}(h_{-i}) \cap \mathsf{A}_{-i}) > 0$ (resp. $\beta_{i,h_i}(t_i)(R_{-i}^{\infty}(h_{-i}) \cap \mathsf{R}_{-i}) > 0$) and fix $(s_{-i}, t_{-i}) \in R_{-i}^{\infty}(h_{-i}) \cap \mathsf{R}_{-i}$ (resp. $(s_{-i}, t_{-i}) \in R_{-i}^{\infty}(h_{-i}) \cap \mathsf{A}_{-i}$). Then, at the state $(s_i, t_i, s_{-i}, t_{-i}) \in R^{\infty}$, B$i$ is uncertain about how B$(-i)$ breaks indifferences.

Step C: Second, suppose $h_{-i} = (h_i, x, \mathrm{r}) \in H_{-i}^P$. Fix some $(s_i, t_i) \in R_i^{\infty}(h_{-i})$ and suppose $(s_i, \beta_{i,h}(t_i))$ has a distinguished outcome. Then, there exists some $E_{-i} \subseteq S_{-i}(h) \times T_{-i}$ so that $\beta_{i,h}(t_i)(E_{-i}) > 0$ and $\xi(\zeta(\{s_i\} \times \mathrm{proj}_{S_{-i}} E_{-i})) = \{(x_1^*, x_2^*, n)\}$. Note that, by (a), there exists some $(s_{-i}, t_{-i}) \in R_{-i}^{\infty}(h_{-i})$ with $\xi(\zeta(s_i, s_{-i})) \neq (x_1^*, x_2^*, n)$. Then, at $(s_i, t_i, s_{-i}, t_{-i})$, B$i$ is uncertain about how B$(-i)$ breaks indifferences. ■

F.3 Proof of Proposition F.3

Proof of Proposition F.3. Throughout, fix some $(s_1^*, t_1^*, s_2^*, t_2^*) \in R^{\infty}$ with $\xi(\zeta(s_1^*, s_2^*)) = (x_1^*, x_2^*, n)$.

Since t_1^* strongly believes each R_2^m, $\beta_{1,\phi}(t_1^*)(R_2^m) = 1$ for each $m \geq 1$. From this, $\beta_{1,\phi}(t_1^*)(R_2^{\infty}) = 1$. Since $\mathrm{proj}_S R^{\infty}$ is constant and $(s_1^*, s_2^*) \in \mathrm{proj}_S R^{\infty}$, it follows from Remark F.1 that

$$R_2^{\infty} \subseteq \{r_2 : \xi(\zeta(s_1^*, r_2)) = \xi(\zeta(s_1^*, s_2^*))\} \times T_2.$$

Thus, by Lemma B.2, $x_1^* \geq \frac{1-\delta}{\delta^{n-1}}$.

If $n = 1$, then it follows from Lemma B.1(ii) that $x_2^* \geq \frac{\delta(1-\delta)}{\delta^{n-1}}$. So, we focus on the case of $n \geq 2$. Note that, along the path of play, there is a two-period history $h^* \in H_2^P$. Since t_2^* strongly believes each R_1^m and $(s_1^*, t_1^*) \in R_1^m \cap [S_1(h^*) \times T_1] \neq \emptyset$, $\beta_{2,h^*)}(t_2^*)(R_1^m) = 1$ for each $m \geq 1$. From this, $\beta_{2,h^*}(t_2^*)(R_1^{\infty}) = 1$. Since $\mathrm{proj}_S R^{\infty}$ is constant and $(s_1^*, s_2^*) \in \mathrm{proj}_S R^{\infty}$, it follows from Remark F.1 that

$$R_1^{\infty} \subseteq \{r_1 : \xi(\zeta(r_1, s_2^*)) = \xi(\zeta(s_1^*, s_2^*))\} \times T_1.$$

Thus, by Lemma B.3, $x_2^* \geq \frac{\delta(1-\delta)}{\delta^{n-1}}$. ■

F.4 Uncertainty About Breaking Indifferences

Example F.1 (Three-Period Deadline)**.** Let $N = 3$. We will show that there is a type structure and a state $(s_1^*, t_1^*, s_2^*, t_2^*) \in R^{\infty}$, where (s_1^*, s_2^*) induces the outcome $(0, 0, 3)$.

Define s_1^* as follows: (i) $s_1^*(\phi) = 1 - \delta(1-\delta)$; (ii) for each second-period history $(h_2, x_2) \in H_1^R$, $s_1^*(h_2, x_2) = \mathrm{a}$ if and only if $x_2 < 1 - \delta$; and (iii) for each third-period history $h_1 \in H_1^P$, $s_1^*(h_1) = 1$. Define s_2^* as follows: (i) $s_2^*(\phi, x_1) = \mathrm{a}$ if and only if $x_1 < 1 - \delta(1-\delta)$;

(ii) for each second-period history $h_2 \in H_2^P$, $s_2^*(h_2) = 1 - \delta$; and (iii) for each third-period $(h_1, x_1) \in H_2^R$, $s_2^*(h_1, x_1) = \mathrm{a}$ if and only if $x_1 < 1$.

To define the belief maps, it will be convenient to define strategies r_1 and r_2. Let r_1 be a strategy with (i) $r_1(h_2, 1-\delta) = \mathrm{a}$ for any second-period history $h_2 \in H_2^P$, and (ii) $r_1(h) = s_1^*(h)$ otherwise. Let r_2 be a strategy with (i) $r_2(\phi, 1 - \delta(1-\delta)) = \mathrm{a}$, (ii) $r_2(h_1, x_1) = \mathrm{a}$ for any third-period period history $(h_1, x_1) \in H_2^R$, and (iii) $r_2(h) = s_2^*(h)$ otherwise. Write r_i^h for a strategy in $S_i(h)$ that otherwise agrees with r_i.

Now, the type structure $\mathcal{T}$ is defined as follows: For each i, $T_i = \{t_i^*\}$. The belief maps each have $\beta_{i,\phi}(t_i^*)(r_{-i}, t_{-i}^*) = 1$. At each history $h \in H_i$ with $r_{-i} \notin S_{-i}(h)$, set $\beta_{i,\phi}(t_i^*)(r_{-i}^h, t_{-i}^*) = 1$. It is readily verified that this defines a countable CPS.

Write $h_2^* = (\phi, 1 - \delta(1-\delta), \mathrm{r}) \in H_2^P$. It is readily verified that, for each m,

$$\{s_1^*, r_1\} \times T_1 \times \{s_2^*, r_2, r_2^{h_2^*}\} \times T_2 \subseteq R^m.$$

But, the strategy profile (s_1^*, s_2^*) induces the outcome $(0, 0, 3)$. □

Appendix G Implications for Delay

This Appendix proves Proposition B.1 and provides the proof for Section 10.C.

G.1 Proof of Proposition B.1

It will be convenient to define functions corresponding to the B1-B2 UCs and the DC. Specifically, define $\mathrm{U}_i : (0,1) \times \mathbb{N}^+ \to \mathbb{R}$, and $\mathrm{D}_i : (0,1) \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{R}$ so that

$$\mathrm{U}_1(\delta, n) = \frac{1-\delta}{\delta^{n-1}} \qquad \mathrm{U}_2(\delta, n) = 1 - \frac{\delta(1-\delta)}{\delta^{n-1}} \qquad \mathrm{D}_1(\delta, N, n) = \delta^{N-n}$$

and $\mathrm{D}_2(\delta, N, n) = 1 - \delta^{N-n}$.

For given parameters N and δ, $\underline{x}^n = \max\{\mathrm{U}_1(\delta, n), \mathrm{D}_1(\delta, N, n)\}$ if $N < \infty$ is odd and $\underline{x}^n = \mathrm{U}_1(\delta, n)$ otherwise. Likewise, for given parameters N and δ, $\overline{x}^n = \min\{\mathrm{U}_2(\delta, n), \mathrm{D}_2(\delta, N, n)\}$ if $N < \infty$ is even and $\overline{x}^n = \mathrm{U}_2(\delta, n)$ otherwise.

Lemma G.1. *Fix $n \geq 2$. There exists $\overline{\delta}[n] \in (\frac{1}{2}, 1)$ so that $\mathrm{U}_2(\delta, n) \geq \mathrm{U}_1(\delta, n)$ if and only if $\delta \geq \overline{\delta}[n]$.*

Lemma G.2. *Fix some n with $N - 2 \geq n \geq 2$.*

(i) There exists $\tilde{\delta}[N, n] \in (0, 1)$ so that $\mathrm{U}_2(\delta, n) \geq \mathrm{D}_1(\delta, N, n)$ if and only if $\delta \geq \tilde{\delta}[N, n]$.

(ii) There exists $\hat{\delta}[N, n] \in (0, 1)$ so that $\mathrm{D}_2(\delta, N, n) \geq \mathrm{U}_1(\delta, n)$ if and only if $\delta \geq \hat{\delta}[N, n]$.

Proof of Proposition B.1. Immediate from Lemmata G.1-G.2. ■

We begin with the proof of Lemma G.1. For this, it will be useful to observe the following:

Remark G.1. Fix $n \geq 2$.

(i) $U_1(\cdot, n)$ is a strictly decreasing continuous function.

(ii) $U_2(\cdot, n)$ is a strictly increasing continuous function.

Proof of Lemma G.1. Note that $\mathrm{U}_2(\delta, n) \geq \mathrm{U}_1(\delta, n)$ if and only if $f(\delta, n) := 1 - \delta^2 - \delta^{n-1} \leq 0$. For any given n observe that (i) $f(\delta, n)$ is strictly decreasing and continuous in δ, (ii) $\lim_{\delta \to 0} f(\delta, n) = 1$, and (iii) $\lim_{\delta \to 1} f(\delta, n) = -1$. Thus, for any given n, there exists $\overline{\delta}[n] \in (0, 1)$ so that $f(\overline{\delta}[n], n) = 0$. It follows that $\mathrm{U}_2(\delta, n) \geq \mathrm{U}_1(\delta, n)$ if and only if $\delta \geq \overline{\delta}[n]$. We show that $\overline{\delta}[n] > \frac{1}{2}$.

First we show that $\overline{\delta}[2] > \frac{1}{2}$: Note, $U_1(\frac{1}{2}, 2) = 1 > \frac{1}{2} = U_2(\frac{1}{2}, 2)$. Since $U_1(\cdot, 2)$ is a strictly decreasing continuous function and $U_2(\cdot, 2)$ is a strictly increasing continuous function, it follows that, $U_1(\overline{\delta}[2], 2) = U_2(\overline{\delta}[2], 2)$ implies $\overline{\delta}[2] > \frac{1}{2}$.

Second we show that $\overline{\delta}[n]$ is strictly increasing in n: For any given δ, the function $f(\delta, \cdot)$ is strictly increasing in n. Thus, if $f(\overline{\delta}[n], n) = 0$, then $f(\overline{\delta}[n], n + 1) > 0$. Since $f(\cdot, n + 1)$ is strictly decreasing in δ, it follows that $\overline{\delta}[n + 1] > \overline{\delta}[n]$. ■

Now we will turn to the proof of Lemma G.2. It will be convenient to define functions $g : [0, 1] \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{R}$ and $h : [0, 1] \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{R}$, so that

$$g(\delta, N, n) = (1 - \delta) - \delta^{n-2}(1 - \delta^{N-n}) \qquad \text{and} \qquad h(\delta, N, n) = (1 - \delta) - \delta^{n-1}(1 - \delta^{N-n}).$$

Lemma G.3. *Fix $n \geq 2$.*

(i) $0 \geq g(\delta, N, n)$ if and only if $\mathrm{U}_2(\delta, n) \geq \mathrm{D}_1(\delta, N, n)$.

(ii) $0 \geq h(\delta, N, n)$ if and only if $\mathrm{D}_2(\delta, n) \geq \mathrm{U}_1(\delta, N, n)$.

The proof follows immediately from algebra.

Lemma G.4. *Fix $n = N - 2 \geq 2$.*

(i) There exists $\tilde{\delta}[N, N-2] \in (0, 1)$ so that $g(\delta, N, N-2) \leq 0$ if and only if $\delta \geq \tilde{\delta}[N, N-2]$.

(ii) There exists $\hat{\delta}[N, N-2] \in (0, 1)$ so that $h(\delta, N, N-2) \leq 0$ if and only if $\delta \geq \hat{\delta}[N, N-2]$.

Proof. First note that

$$g(\delta, N, N - 2) = g(\delta, n + 2, n) = (1 - \delta) - \delta^{n-2}(1 - \delta^2) = (1 - \delta) - \delta^{n-2}(1 - \delta)(1 + \delta).$$

Then, $g(\delta, N, N - 2) \leq 0$ if and only if $1 - \delta^{n-2}(1 + \delta) \leq 0$. Since $n \geq 2$, the function $k(\delta, n) = 1 - \delta^{n-2}(1 + \delta)$ is strictly decreasing and continuous in δ, with $\lim_{\delta \to 0} k(\delta, n) = 1$ and $\lim_{\delta \to 1} k(\delta, n) = -1$. From this, we can find $\tilde{\delta}[n+2, n] \in (0, 1)$ so that $1-\delta^{n-2}(1+\delta) \leq 0$ if and only if $\delta \geq \tilde{\delta}[n + 2, n]$.

Next, note that

$$h(\delta, N, N-2) = h(\delta, n+2, n) = (1-\delta) - \delta^{n-1}(1-\delta^2) = (1-\delta) - \delta^{n-1}(1-\delta)(1+\delta).$$

Then, $h(\delta, N, N-2) \leq 0$ if and only if $1 - \delta^{n-1}(1+\delta) \leq 0$. Note, the function $k(\delta, n) = 1 - \delta^{n-1}(1+\delta)$ is strictly decreasing and continuous in δ with $\lim_{\delta \to 0} k(\delta, n) = 1$ and $\lim_{\delta \to 1} k(\delta, n) = -1$. From this, we can find $\hat{\delta}[n+2, n] \in (0,1)$ so that $1 - \delta^{n-1}(1+\delta) \leq 0$ if and only if $\delta \geq \hat{\delta}[n+2, n]$. ■

Corollary G.1. *Let $N - 2 \geq n \geq 2$. There exists $\tilde{\delta}[n], \hat{\delta}[n] \in (0,1)$ so that:*

(i) For all $\delta \geq \tilde{\delta}[n]$, $g(\delta, N, n) \leq 0$.

(ii) For all $\delta \geq \hat{\delta}[n]$, $h(\delta, N, n) \leq 0$.

To see this, apply Lemma G.4 taking $\tilde{\delta}[n] = \tilde{\delta}[N, N-2]$ and $\hat{\delta}[n] = \hat{\delta}[N, N-2]$. The claim follows since $g(\delta, N, \cdot)$ and $h(\delta, N, \cdot)$ are increasing in n.

Lemma G.5. *Fix some n with $N - 2 \geq n \geq 2$.*

(i) There exists $\tilde{\delta}[N, n] \in (0,1)$ so that $g(\delta, N, n) \leq 0$ if and only if $\delta \geq \tilde{\delta}[N, n]$.

(ii) There exists $\hat{\delta}[N, n] \in (0,1)$ so that $h(\delta, N, n) \leq 0$ if and only if $\delta \geq \hat{\delta}[N, n]$.

Proof. Begin with part (i) and note that $g(\cdot, N, n) : [0,1] \to \mathbb{R}$ is a continuous function with $\lim_{\delta \to 0} g(\delta, N, n) = 1$ and $\lim_{\delta \to 1} g(\delta, N, n) = 0$. Moreover, by Corollary G.1(i), there is some $\tilde{\delta}[n] \in (0,1)$ so that $g(\delta, N, n) \leq 0$ if $\delta \geq \tilde{\delta}[n]$. Thus, to show the claim, it suffices to show that the function $g(\cdot, N, n)$ does not achieve a local maximum in $(0,1)$.

To show that the function $g(\cdot, N, n)$ does not achieve a local maximum in $(0,1)$, note:

$$\frac{\mathrm{d}g(\cdot, N, n)}{\mathrm{d}\delta} = -1 - (n-2)\delta^{n-3} + (N-2)\delta^{N-3}.$$

So, if $\delta_* \in (0,1)$ is a local minimum or local maximum, then

$$(N-2)\delta_*^{N-3} = 1 + (n-2)\delta_*^{n-3}. \tag{3}$$

Moreover,

$$\frac{\mathrm{d}^2 g(\cdot, N, n)}{\mathrm{d}\delta^2} = -(n-2)(n-3)\delta^{n-4} + (N-2)(N-3)\delta^{N-4}.$$

We show that if $\delta_* \in (0,1)$ satisfies Equation 3, then $\frac{\mathrm{d}^2 g(\cdot, N, n)}{\mathrm{d}\delta^2}$ is strictly positive at δ_*. This implies that there is no local maximum in $(0,1)$.

Notice that the sign of $\frac{\mathrm{d}^2 g(\cdot, N, n)}{\mathrm{d}\delta^2}$ is the same as the sign of

$$-(n-2)(n-3)\delta^{n-3} + (N-2)(N-3)\delta^{N-3}.$$

Thus, if δ_* satisfies Equation 3, then the sign of $\frac{\mathrm{d}^2 g(\cdot, N, n)}{\mathrm{d}\delta^2}$ at δ_* is the same as the sign of

$$-(n-2)(n-3)\delta_*^{n-3} + (N-3)[1 + (n-2)\delta_*^{n-3}] = \delta_*^{n-3}(n-2)[N-n] + (N-3).$$

The fact that $\delta_*^{n-3}(n-2)[N-n]+(N-3)>0$ follows from the fact that $N-2\geq n\geq 2$.

Turn to part (ii) and note that $h(\cdot,N,n):[0,1]\to\mathbb{R}$ is a continuous function with $\lim_{\delta\to 0}h(\delta,N,n)=1$ and $\lim_{\delta\to 1}h(\delta,N,n)=0$. Moreover, by Corollary G.1(ii), there is some $\hat{\delta}[n]\in(0,1)$ so that $h(\delta,N,n)\leq 0$ if $\delta\geq\hat{\delta}[n]$. Thus, to show the claim, it suffices to show that the function $h(\cdot,N,n)$ does not achieve a local maximum in $(0,1)$.

To show that the function $h(\cdot,N,n)$ does not achieve a local maximum in $(0,1)$, note:

$$\frac{\mathrm{d}h(\cdot,N,n)}{\mathrm{d}\delta}=-1-(n-1)\delta^{n-2}+(N-1)\delta^{N-2}.$$

So, if $\delta\in(0,1)$ is a local minimum or local maximum, then

$$(N-1)\delta_*^{N-2}=1+(n-1)\delta_*^{n-2}. \tag{4}$$

Moreover,

$$\frac{\mathrm{d}^2h(\cdot,N,n)}{\mathrm{d}\delta^2}=-(n-1)(n-2)\delta^{n-3}+(N-1)(N-2)\delta^{N-3}.$$

We show that if $\delta_*\in(0,1)$ satisfies Equation 4, then $\frac{\mathrm{d}^2h(\cdot,N,n)}{\mathrm{d}\delta^2}$ is strictly positive at δ_*. This implies that there is no local maximum in $(0,1)$.

Notice that the sign of $\frac{\mathrm{d}^2h(\cdot,N,n)}{\mathrm{d}\delta^2}$ is the same as the sign of

$$-(n-1)(n-2)\delta^{n-2}+(N-1)(N-2)\delta^{N-2}.$$

Thus, if δ_* satisfies Equation 4, then the sign of $\frac{\mathrm{d}^2h(\cdot,N,n)}{\mathrm{d}\delta^2}$ at δ_* is the same as the sign of

$$-(n-1)(n-2)\delta_*^{n-2}+(N-2)[1+(n-1)\delta_*^{n-2}]=\delta_*^{n-2}(n-1)[N-n]+(N-2).$$

Since $N-2\geq n\geq 2$, $\delta_*^{n-2}(n-1)[N-n]+(N-2)>0$s. ■

Proof of Lemma G.2. Immediate from Lemma G.3 and Lemma G.5. ■

G.2 Section 10.C

Lemma G.6. *Let $N=\infty$.*

(i) $\overline{n}(\delta,\infty)=\lfloor 1+\frac{\ln(1-\delta^2)}{\ln(\delta)}\rfloor$.

(ii) For each $n\leq\overline{n}(\delta,\infty)$, $x_1^{SPE}\in[\underline{x}^n,\overline{x}^n]$.

Proof. Observe that $1-\delta({(1-\delta)}/{\delta^{n-1}})\geq{(1-\delta)}/{\delta^{n-1}}$ if and only if $(n-1)\ln(\delta)\geq\ln(1-\delta^2)$, or equivalently, if and only if $n-1\leq{\ln(1-\delta^2)}/{\ln(\delta)}$. Thus, $\overline{n}(\delta,\infty)=\lfloor 1+\frac{\ln(1-\delta^2)}{\ln(\delta)}\rfloor$.

Now note $n\leq\overline{n}(\delta,\infty)$ if and only if $\delta^{n-1}\geq 1-\delta^2$. Algebra shows that $\mathrm{U}_2(\delta,n)\geq x_1^{\mathrm{SPE}}\geq\mathrm{U}_1(\delta,n)$ whenever $\delta^{n-1}\geq 1-\delta^2$. ■

Appendix H Model Extensions

This appendix studies three extensions of the model.

H.1 Frequent Offers

Consider a continuous time variant of the model, where there is no deadline and the bargainers are restricted to making offers at intervals of length $\Delta > 0$. The original model can be embedded into this one: Taking $\delta = e^{-r\Delta}$, where r is a common discount rate. If the bargainers agree to an allocation in period $n \in \mathbb{N}^+$, then the length of time until agreement is $(n-1)\Delta$. In this case, there is **delay of length** $(n-1)\Delta$.

Note, B1's UC requires $e^{-r(n-1)\Delta}x_1^* \geq 1 - e^{-r\Delta}$ and B2's UC requires that $e^{-r(n-1)\Delta}(1 - x_1^*) \geq e^{-r\Delta}(1 - e^{-r\Delta})$. For any given (n, Δ), the gap between B2's and B1's upfront constraints is given by

$$\text{gap}(n, \Delta) = 1 - \frac{(1 + e^{-r\Delta})(1 - e^{-r\Delta})}{e^{-r(n-1)\Delta}}.$$

Let $\overline{n} : (0, \infty) \to \mathbb{R}_+$ be defined by $\overline{n}(\Delta) = 1 - \frac{\ln(1-e^{-2r\Delta})}{r\Delta} > 1$.

Lemma H.1.

(i) $n \leq \overline{n}(\Delta)$ *if and only if* $\text{gap}(n, \Delta) \geq 0$.

(ii) $n \geq \overline{n}(\Delta)$ *if and only if* $\text{gap}(n, \Delta) \leq 0$

Proof. Observe that $\text{gap}(n, \Delta) \geq 0$ if and only if

$$\ln(1) \geq \ln(e^{r(n-1)\Delta}) + \ln(1 - e^{-2r\Delta})$$

or if and only if $-\ln(1 - e^{-2r\Delta}) \geq r(n-1)\Delta$. Thus, $n \leq \overline{n}(\Delta)$ if and only if $\text{gap}(n, \Delta) \geq 0$. Reversing the inequalities gives that $n \geq \overline{n}(\Delta)$ if and only if $\text{gap}(n, \Delta) \leq 0$. ■

In light of Lemma H.1, there can be delay of length $(n-1)\Delta$ if and only if $n \leq \overline{n}(\Delta)$. So, the **maximum length of delay** is given by $(\lfloor \overline{n}(\Delta) \rfloor - 1)\Delta$. Lemma H.2 will show that the maximum length of delay is essentially decreasing in Δ. Lemma H.3 will show that, when the length of time between intervals gets small, the length of delay gets large.

Define $\text{del} : (0, \infty) \to \mathbb{R}_+$ so that $\text{del}(\Delta) = (\lfloor \overline{n}(\Delta) \rfloor - 1)\Delta$. Also define functions $\overline{\text{del}} : (0, \infty) \to \mathbb{R}_+$ and $\underline{\text{del}} : (0, \infty) \to \mathbb{R}_+$, so that $\overline{\text{del}}(\Delta) = (\overline{n}(\Delta) - 1)\Delta$ and $\underline{\text{del}}(\Delta) = (\overline{n}(\Delta) - 2)\Delta$. Observe that, for each Δ, $\overline{\text{del}}(\Delta) \geq \text{del}(\Delta) > \underline{\text{del}}(\Delta)$.

Lemma H.2. $\overline{\text{del}}(\Delta)$ *is strictly decreasing in* Δ *and convex.*

Proof. Notice that $\overline{\text{del}}(\Delta) = -\frac{1}{r}\ln(1-e^{-2r\Delta})$. So,

$$\frac{\partial\overline{\text{del}}}{\partial\Delta} = -\frac{2e^{-2r\Delta}}{(1-e^{-2r\Delta})} < 0$$

Since $e^{-2r\Delta} < 1$, $\partial\overline{\text{del}}/\partial\Delta < 0$. Moreover,

$$\frac{\partial^2\overline{\text{del}}}{\partial\Delta^2} = 4re^{-2r\Delta}\frac{(1-e^{-2r\Delta})+e^{-2r\Delta}}{(1-e^{-2r\Delta})^2}.$$

Again using the fact that $e^{-2r\Delta} \in (0,1)$, $\partial^2\overline{\text{del}}/\partial\Delta^2 > 0$. ■

Lemma H.3. $\lim_{\Delta\to 0^+} \text{del} = \infty$.

Proof. Fix $\varepsilon > 0$. Since $\lim_{\Delta\to 0^+} \underline{\text{del}} = \infty$, there exists some $\rho > 0$ so that $\text{del}(\Delta) \geq \underline{\text{del}}(\Delta) > \varepsilon$ whenever $\Delta \in (0,\rho)$. ■

H.2 Outside Options

We begin by describing the delayed allocations consistent with the OOCs and the GUCs. From this, our description of behavior follows. Then, we turn to argue that the constraints characterize RCSBR and on-path strategic certainty.

Constraint For each $n \geq 2$, set

$$\underline{\underline{x}}^n = \max\left\{\frac{\delta w_1}{\delta^{n-1}}, \frac{1-\delta}{\delta^{n-1}}\right\}.$$

If $n = 2$, set

$$\overline{\overline{x}}^n = \min\left\{1 - \frac{w_2}{\delta}, \max\{\delta, w_1\}\right\}$$

and, if $n \geq 3$, set

$$\overline{\overline{x}}^n = \min\left\{1 - \frac{w_2}{\delta^{n-1}}, 1 - \frac{\delta(1-\delta)}{\delta^{n-1}}\right\}.$$

Lemma H.4. *Let $n \geq 2$. An outcome (x_1^*, x_2^*, n) satisfies the OOCs and the GUCs if and only if $x_1^* \in [\underline{\underline{x}}^n, \overline{\overline{x}}^n]$.*

Proof. First, the OOCs and GUCs imply that $\delta^{n-1}x_1^* \geq \max\{\delta w_1, 1-\max\{\delta, w_2\}\}$. Thus, to show that $x_1^* \geq \underline{\underline{x}}^n$, it suffices to show that $\delta \geq w_2$. However, this follows from the OOC, since $\delta \geq \delta^{n-1}x_2^* \geq w_2$.

Second, the OOCs and GUC imply that

$$\min\left\{1 - \frac{w_2}{\delta^{n-1}}, 1 - \frac{\delta(1-\max\{\delta, w_1\})}{\delta^{n-1}}\right\} \geq x_1^*.$$

Thus, to show that $\overline{\overline{x}}^n \geq x_1^*$, it suffices to show that, when $n \geq 3$, $\delta \geq w_1$. However, this follows from the OOC since, when $n \geq 3$, $\delta^2 \geq \delta^{n-1}x_1^* \geq \delta w_1$. ■

Observe that, for each $n \geq 3$, $[\underline{\underline{x}}^n, \overline{\overline{x}}^n] \subseteq [\underline{x}^n, \overline{x}^n]$. The same is true for $n = 2$ when $\delta \geq w_1$. When $n = 2$ and $w_1 > \delta$, either $[\underline{\underline{x}}^n, \overline{\overline{x}}^n] = \emptyset$ or $[\underline{\underline{x}}^n, \overline{\overline{x}}^n] = [w_1, w_1]$. This latter situation is disjoint from $[\underline{x}^n, \overline{x}^n]$. (This can indeed occur: take $\delta = .7$, $w_1 = .8$, $w_2 = .1$.)

Characterization An argument analogous to Appendix B.1 establishes the following:

Fix an epistemic game $(\mathcal{B}, \mathcal{T})$ and a state $(s_1^*, t_1^*, s_2^*, t_2^*) \in R^2 \cap C$. If (s_1^*, s_2^*) induces an outcome (x_1^*, x_2^*, n) with $n \geq 2$, then (x_1^*, x_2^*, n) satisfies the OOCs and GUCs.

We here focus on the converse.

Fix a bargaining game $\mathcal{B}$ and an outcome (x_1^*, x_2^*, n) with $n \geq 2$ that satisfies the OOCs and GUCs. There is an epistemic game $(\mathcal{B}, \mathcal{T})$ and a state $(s_1^*, t_1^*, s_2^*, t_2^*)$ thereof, so that: (i) $(s_1^*, t_1^*, s_2^*, t_2^*) \in R^\infty \cap C$, and (ii) the strategy profile (s_1^*, s_2^*) induces the outcome (x_1^*, x_2^*, n).

The argument follows Appendix B.2. The key step is to redefine the strategy profile (s_1^*, s_2^*).

Recall from Appendix B.2 that $h^* = (1, \mathrm{r}, \ldots, 1, \mathrm{r})$; that is, it is a history in which there are $(n-1)$ offers of 1 followed by $(n-1)$ rejections. The strategy s_i^* satisfies the following properties: First, for any history $h \in H_i^P$, set (i) $s_i^*(h) = x_i^*$ if $h = h^*$, and (ii) $s_i^*(h) = 1$ if $h \neq h^*$. Second, for any history $h \in H_{-i}^P$, let $s_i^*(h, x) = \mathrm{a}$ if and only if either (i) $x \in [0, \min\{1 - w_i, 1 - \delta\})$, or (ii) $h = h^*$ and $x = x_{-i}^*$. Third, for any history $h \in H_{-i}^P$, let $s_i^*(h, x)$ take the outside option if and only if $x \in [1 - w_i, 1 - \delta)$. Fourth, for all other histories $h \in H_i^P$, let $s_i^*(h, x)$ reject the offer and continue negotiations if and only if $x \in [1 - \delta, 1]$.

The construction of the type structure is as in Appendix B.2. The proofs of Lemmata B.8-B.9 need amendment. The key change in those proofs comes in terms of Lemma B.11. Now it is this:

Lemma H.5. *Fix and n-period history $h \in H_i$ with $s_i^* \in S_i(h)$ but $s_{-i}^* \notin S_{-i}(h)$. For each $r_i \in S_i(h)$,*

(i) $\pi_i(s_i^*, \alpha_{-i}^h) \geq \pi_i(r_i, \alpha_{-i}^h)$, and

(ii) $\pi_i(s_i^*, \alpha_{-i}^h) = \pi_i(r_i, \alpha_{-i}^h)$ if and only if either

- $\zeta(r_i, \alpha_{-i}^h) = \zeta(s_i^*, \alpha_{-i}^h)$,
- $h = (\cdot, 1 - \delta) \in H_i^R$, $w_i \neq \delta$, and $r_i(h) = \mathrm{a}$,
- $h = (\cdot, 1 - w_i) \in H_i^R$, $w_i > \delta$, and $r_i(h) = \mathrm{a}$, or
- $h = (\cdot, 1 - \delta) \in H_i^R$, $w_i = \delta$, and $r_i(h)$ is either a or exercise the outside option.

The proof is analogous to the proof of Lemma B.11 and so omitted.

H.3 Discrete Grid of Feasible Allocations

Suppose the set of feasible allocations is constrained to lie in the discrete grid

$$\mathcal{A} = \left\{(x_1, x_2) \text{ is an allocation and } x_1 \in \left\{\frac{0}{K}, \frac{1}{K}, \ldots, \frac{K-1}{K}, \frac{K}{K}\right\}\right\}.$$

That is, Bi can offer an allocation (x_1, x_2) if and only if it lies in $\mathcal{A}$. Say that $\mathcal{A}$ has a grid of size K.

Recall that the UC is driven by the fact that, upfront, Bi reasons that B$(-i)$ will accept any offer (x_1, x_2) with $x_{-i} \in (\delta, 1]$. In the case where $\mathcal{A}$ is a continuum, rationality implies that Bi must offer an allocation with $x_{-i} = \delta$, expecting that offer to be accepted. Notice that this conclusion is driven by the fact that, in the continuum case, no $x_{-i} < \delta$ can maximize Bi's subjective expected utility. But, in the case where $\mathcal{A}$ is a discrete grid with some $\frac{j+1}{K} > \delta > \frac{j}{K}$, $\frac{j+1}{K}$ may well maximize Bi's subjective expected utility (if she expects B$(-i)$ to reject offers with $\delta > x_{-i}$). In light of this, write

$$\delta^+ = \min\left\{\frac{j}{K} : \frac{j}{K} \geq \delta\right\}.$$

Bi can only conclude that (x_1, x_2) will be accepted if $x_{-i} \geq \delta^+$. This relaxes Bi's UC: B1's UC is given by $\delta^{n-1}x_1^* \geq (1 - \delta^+)$ and B2's UC is given by $\delta^{n-1}(1 - x_1^*) \geq \delta(1 - \delta^+)$.

Suppose Bi makes the proposal in the last period. When she accepts an earlier allocation, she reasons that, in the last period, B$(-i)$ would accept any (x_1, x_2) with $x_{-i} > 0$. In the case where $\mathcal{A}$ is a continuum, this would allow Bi to anticipate getting the full share of the pie, if the final period were reached. But, when $\mathcal{A}$ is a discrete grid, this only allows Bi to anticipate (for sure) getting $\frac{K-1}{K}$. (She may or may not assign probability 1 to getting the full pie.) Thus, Bi's deadline constraint is given by $\delta^{n-1}x_i^* \geq \delta^{N-1}\frac{K-1}{K}$.

Thus, the B1-B2 UCs and the DC are relaxed. If the grid is coarse, this can lead to new possibilities for delay. In particular, choose K so that $\delta \in (\frac{K-1}{K}, 1)$. (If δ is small, this may require choosing $K = 1$; however, if δ is large, this may involve choosing a somewhat finer grid.) In that case, $1 - \delta^+ = 0$, and so any allocation trivially satisfies the two UCs. Moreover, for any period $n \leq N$, there is an n-period outcome (x_1^*, x_2^*, n) that would satisfy the DC; simply take $x_i^* = \frac{K-1}{K}$ for the Bi with the deadline bargaining power.

However, when the grid is sufficiently fine—that is, when K is large—the limitations and possibilities for delay correspond to those in the case of the continuum. For example, take the case of a three-period deadline. When K is large, there cannot be delay until the last period. If there is delay until the penultimate period, then the agreed upon allocation (x_1^*, x_2^*) must satisfy $x_1^* \in [\max\{\delta\frac{K-1}{K}, \frac{(1-\delta^+)}{\delta}\}, \delta^+]$. This is a weaker requirement than the case of the continuum; but, as K gets large, it converges to the requirement that x_1^* is δ.