Supplement to "Changes in the span of systematic risk exposures"

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This Appendix contains the proofs of all theoretical results in the paper.

Appendix A: Preliminary bounds

In this subsection, we collect some preliminary bounds that are used throughout the proof. They hold both under the null and alternative hypotheses. Here and in the rest of the proof, we assume that Assumptions A1–A3 hold. In fact, following a standard localization argument (see, e.g., Section 4.4.1 of Jacod and Protter (2011)), it is enough to prove the results under the stronger version of Assumption A1.

SA1. *We have Assumption A*1 *for* $s, t \in [0, T]$.

Therefore, the proof below is done under Assumptions SA1, A2, and A3 without further mention in the statements of the theorems, lemmas, and propositions. We also assume that $k_n\Delta_n < \varepsilon$ so that the discrete factor model in equation (19) in the main text holds. This is not a restriction because $k_n\Delta_n \to 0$ for all of our theoretical results in the paper. Finally, we remind the reader the sequence ζ_p from Assumption A3, the tuning parameters $K_{\max}$ and g_{np} related to the selection of the number of factors given in equation (22) in the main text, and the parameter $\widetilde{\varpi}$ from the statement of Theorem 4.1.

Lemma A.1. *Let* $p \to \infty$, $\Delta_n \to 0$, $k_n \to \infty$, *and* $k_n\Delta_n \to 0$. *Then we have for* $c = a, b$:

(i) $\|R_c\overline{F}_c\|^2 = O_P(pk_n^2\Delta_n^{2\widetilde{\varpi}})$.

(ii) $\|\overline{U}_c R_c'\|^2 = O_P(pk_n^2\Delta_n^{2\widetilde{\varpi}})$.

(iii) $\|R_c'\|^2 = O_P(pk_n\Delta_n^{2\widetilde{\varpi}})$.

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PROOF. Given the integrability conditions of Assumption SA1, we have for any constant $q > 0$:

$$\mathbb{E}\left|\int_{(i_t^c-1)\Delta_n}^{i_t^c\Delta_n} \alpha_{s,j}\,ds\right|^q \leq C_q\Delta_n, \qquad \mathbb{E}\left(\|\overline{f}_{c,t}\|^q + |\overline{\epsilon}_{c,tj}|^q\right) \leq C_q,$$
$$\mathbb{E}\left|\Delta_{i_t^c}^n J_j\right|^q \leq C_{q,\iota}\Delta_n^{1-\iota}, \tag{A.1}$$

for $j = 1, \ldots, p$, $t = 1, \ldots, k_n$, and $c = a, b$, some arbitrary small $\iota > 0$, and where C_q and $C_{q,\iota}$ are constants that depend on q and ι, only. From here, we also have for $q \geq 2$:

$$\mathbb{E}|r_{c,tj}|^q \leq C_q\Delta_n^{(\varpi-1/2)q+1}. \tag{A.2}$$

With these bounds, we can now proceed with the proof of the lemma. Applying the bounds in (A.1) and Hölder's inequality yields

$$\mathbb{E}\left(\sum_{t=1}^{k_n} r_{c,tj}^2\overline{f}_{c,tk}^2\right) \leq Ck_n\Delta_n^{2\widetilde{\varpi}} \tag{A.3}$$

and, therefore,

$$\|R_c\overline{F}_c\|^2 = O_P\left(pk_n^2\Delta_n^{2\widetilde{\varpi}}\right). \tag{A.4}$$

Next, given the $\mathcal{C}$-conditional independence of $\beta_{t,j}$, $\sigma_{t,j}$, and $Y_{t,j}$ across j from A2, we have

$$\mathbb{E}(r_{c,tj}\overline{\epsilon}_{c,tk}r_{c,sj}\overline{\epsilon}_{c,sk}) = 0, \quad \text{for } j \neq k \text{ and } s \neq t. \tag{A.5}$$

Using conditioning on $\mathcal{C}$, the bounds in (A.1) and (A.2), Hölder's inequality as well as Assumption SA1, we have

$$\left|\mathbb{E}(r_{c,tj}\overline{\epsilon}_{c,tj}r_{c,sj}\overline{\epsilon}_{c,sj})\right| \leq C\Delta_n^{2\widetilde{\varpi}}, \quad \text{for } s \neq t, \tag{A.6}$$

$$\left|\mathbb{E}\left(r_{c,tj}^2\overline{\epsilon}_{c,tj}^2\right)\right| + \mathbb{E}\left(r_{c,tj}^2\right) \leq C\Delta_n^{2\widetilde{\varpi}}. \tag{A.7}$$

Combining the above three bounds, we get

$$\left\|\overline{U}_cR_c'\right\|^2 = O_P\left(pk_n^2\Delta_n^{2\widetilde{\varpi}}\right) \quad \text{and} \quad \left\|R_c'\right\|^2 = O_P\left(pk_n\Delta_n^{2\widetilde{\varpi}}\right). \tag{A.8}$$

□

LEMMA A.2. *Let $p \to \infty$, $\Delta_n \to 0$, $k_n \to \infty$, and $k_n\Delta_n \to 0$. We have for $c, d \in \{a, b\}$:*

(i) $\max_{it}\sum_{j=1}^p |\mathbb{E}(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}|\mathcal{C})| \leq C$, *for some positive constant $C > 0$.*

(ii) $\|\overline{U}_c\| = O_P(\sqrt{(k_n+p)\zeta_p} + \sqrt{(k_n+p)\sqrt{\frac{pk_n}{n}}})$, *for the matrix operator norm.*

(iii) $\frac{1}{k_np^2}\sum_{ijt}(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj} - \mathbb{E}(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}|\mathcal{C})) = O_P(\frac{1}{p\sqrt{k_n}})$.

(iv) $\|\frac{1}{k_n}\overline{U}_c\overline{F}_c\| = O_P(\sqrt{\frac{p}{k_n}})$, $\|\frac{1}{k_n p}\beta_d'\overline{U}_c\overline{F}_c\| = O_P(\frac{1}{\sqrt{k_n p}} + \frac{\sqrt{\Delta_n}}{\sqrt{k_n}})$ *and* $\frac{1}{pk_n}\|\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{U}_c'\beta_c\| = O_P(\frac{\sqrt{k_n}}{\sqrt{p}} + 1)$.

(v) $\frac{1}{p}\sum_{j=1}^p |\frac{1}{pk_n}\sum_{it}\beta_{d,i}(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj} - \mathbb{E}(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}|\mathcal{C}))|^2 = O_P(\frac{1}{pk_n} + \frac{\sqrt{\Delta_n}}{k_n})$.

(vi) $\|\frac{1}{p^2k_n}\beta_d'(\overline{U}_c\overline{U}_c' - \mathbb{E}(\overline{U}_c\overline{U}_c'|\mathcal{C}))\beta_c\| = O_P(\frac{1}{p\sqrt{k_n}} + \frac{\sqrt{\Delta_n}}{\sqrt{k_n}})$.

Proof. We start with (i). We have $\sum_{j=1}^p|\mathbb{E}(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}|\mathcal{C})| = \mathbb{E}(\overline{\epsilon}_{c,ti}^2|\mathcal{C})$ and since $\sup_{i\geq 1}\mathbb{E}(\overline{\epsilon}_{c,ti}^2) < \infty$ by Assumption SA1, the result follows.

To proceed further for (ii), we introduce the following notation:

$$\widetilde{\epsilon}_{c,tj} = \frac{1}{\sqrt{\Delta_n}}\widetilde{\sigma}_{cj}\Delta_{i_t^c}^n W_j, \quad \text{for } c = a, b,\ j = 1, \dots, p,\ t = 1, \dots, k_n, \tag{A.9}$$

with $\widetilde{\sigma}_{c,j} = \sigma_{\lfloor c/\Delta_n \rfloor - k_n + 1, j}$. The matrix constructed from $\widetilde{\epsilon}_{c,tj}$ is denoted with $\widetilde{U}_c$. We first bound $\|\widetilde{U}_c\|$. Let $\widetilde{\Sigma}_{u,c} = \frac{1}{k_n}\mathbb{E}(\widetilde{U}_c\widetilde{U}_c'|\mathcal{F}_{(\lfloor c/\Delta_n \rfloor - k_n)\Delta_n})$, which is a diagonal matrix with entries $\widetilde{\sigma}_{c,j}^2$, and denote its counterpart in which $\widetilde{\sigma}_{c,j}^2$ is replaced with $\sigma_{c,j}^2$ with $\Sigma_{u,c}$. Theorem 4.6.1 of Vershynin (2018) implies

$$\left\|\widetilde{\Sigma}_{u,c}^{-1/2}\frac{1}{k_n}\widetilde{U}_c\widetilde{U}_c'\widetilde{\Sigma}_{u,c}^{-1/2} - I\right\| = O_P\left(\frac{p}{k_n} + \sqrt{\frac{p}{k_n}}\right),$$

so we need a bound for $\|\widetilde{\Sigma}_{u,c}\|$. For this, we can use triangular inequality, Assumptions A2 and A3, and the fact that $\|\cdot\| \leq \|\cdot\|_F$ to get

$$\|\widetilde{\Sigma}_{u,c}\| \leq \|\Sigma_{u,c}\| + \|\widetilde{\Sigma}_{u,c} - \Sigma_{u,c}\|_F = O_P\left(\zeta_p + \sqrt{p}\sqrt{\frac{k_n}{n}}\right). \tag{A.10}$$

As a result, $\|\widetilde{U}_c\| = O_P(\sqrt{(k_n+p)\zeta_p} + \sqrt{(k_n+p)\sqrt{\frac{pk_n}{n}}})$. Therefore, it suffices to show $\|\overline{U}_c - \widetilde{U}_c\| = O_P(\sqrt{k_n+p})$ in order to establish the bound for $\|\overline{U}_c\|$. First, note that $\mathbb{E}|\overline{\epsilon}_{c,tj} - \widetilde{\epsilon}_{c,tj}|^2 \leq C\Delta_n$ because of our assumption for $\sigma_{t,j}$. From here,

$$\|\overline{U}_c - \widetilde{U}_c\| \leq \|\overline{U}_c - \widetilde{U}_c\|_F \leq C\sqrt{pk_n}\sqrt{\Delta_n} = o_P(\sqrt{p}). \tag{A.11}$$

We continue with (iii). Using successive conditioning, we have

$$\begin{aligned}&\mathbb{E}\big[(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}\overline{\epsilon}_{c,t'i'}\overline{\epsilon}_{c,t'j'})|\mathcal{C}\big] = 0,\\ &\quad \text{if } t \neq t' \text{ or one of the indices } i, i', j, j' \text{ differs from the others,}\end{aligned} \tag{A.12}$$

and of course $\mathbb{E}[\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}]^2 \leq C$ given our integrability assumptions in SA1. From here, the result to be proved follows.

For the first of the bounds in (iv), given the definitions of $\overline{f}_{c,t}$ and $\overline{\epsilon}_{c,tj}$ as well as the integrability assumptions in SA1, we have

$$\mathbb{E}(\overline{\epsilon}_{c,tj}\overline{f}_{c,tk}) = 0, \qquad \mathbb{E}(\overline{\epsilon}_{c,tj}\overline{f}_{c,tk}\overline{\epsilon}_{c,sj}\overline{f}_{c,sk})\begin{cases} = 0 & \text{if } s \neq t,\\ \leq C & \text{if } s = t\end{cases} \tag{A.13}$$

and, therefore,

$$\|\overline{F}_c'\overline{U}_c'\|^2 = O_P(pk_n), \tag{A.14}$$

from which the first result in (iv) follows.

For the second bound in (iv), we use in addition the following result:

$$\left|\mathbb{E}\left[\beta_{d,jk'}\sum_{t=1}^{k_n}\overline{\epsilon}_{c,ti}\overline{f}_{c,tk}\sum_{t=1}^{k_n}\overline{\epsilon}_{c,tj}\overline{f}_{c,tk}\right]\right| \leq C\sqrt{k_n\Delta_n}, \quad i\neq j, k, k' = 1, \ldots, K, \tag{A.15}$$

for some constant $C > 0$. This follows from the $\mathcal{C}$-conditional independence of the processes β_i, σ_i, and $\widetilde{W}_i$ from β_j, σ_j, and $\widetilde{W}_j$, for $i \neq j$, as well as the smoothness condition for the processes σ_i and Λ in Assumption A2(i). We note that when $c = d$, the expectation in the above inequality is equal to zero.

Finally, for the third result in (iv), we apply the Cauchy–Schwarz inequality and we have

$$\|\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{U}_c'\beta_c\| \leq \|\overline{F}_d'\overline{U}_d'\|\,\|\overline{U}_c\overline{U}_c'\beta_c\|. \tag{A.16}$$

Given the above bound for $\|\overline{F}_d'\overline{U}_d'\|$, we need only a bound for $\|\overline{U}_c\overline{U}_c'\beta_c\|$. Given the independence of W_i and W_j for $i \neq j$, and the integrability conditions for the processes $\{\sigma_i\}_{i\geq 1}$, we have

$$\mathbb{E}\left(\sum_{t=1}^{k_n}\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}\sum_{t=1}^{k_n}\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj'}\middle|\mathcal{F}_{(\lfloor c/\Delta_n\rfloor-k_n)\Delta_n}\right) = 0, \quad \text{if } i\neq j \text{ and } j\neq j', \tag{A.17}$$

$$\mathbb{E}\left(\sum_{t=1}^{k_n}\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}\right)^2 \leq \begin{cases} Ck_n & \text{if } i\neq j, \\ Ck_n^2 & \text{if } i = j. \end{cases} \tag{A.18}$$

Therefore, given the integrability conditions for the processes $\beta_{c,j}$, we have

$$\mathbb{E}\big(\|\overline{U}_c\overline{U}_c'\beta_c\|^2\big) \leq C\big(k_n^2 + pk_n\big). \tag{A.19}$$

From here, the third bound in (iv) follows.

We turn next to the bound in (v). Using the $\mathcal{C}$-conditional independence of the processes β_i, W_i, and σ_i from β_j, W_j, and σ_j, for $i\neq j$, we have

$$\mathbb{E}\left(\sum_{i\neq i' \text{ or } s\neq t}\beta_{d,i}'\beta_{d,i'}\big(\widetilde{\epsilon}_{c,ti}\widetilde{\epsilon}_{c,tj} - \mathbb{E}(\widetilde{\epsilon}_{c,ti}\widetilde{\epsilon}_{c,tj}|\mathcal{C})\big)\big(\widetilde{\epsilon}_{c,si'}\widetilde{\epsilon}_{c,sj} - \mathbb{E}(\widetilde{\epsilon}_{c,si'}\widetilde{\epsilon}_{c,sj}|\mathcal{C})\big)\right) = 0, \tag{A.20}$$

where we denote $\widetilde{\epsilon}_{c,ti} = \sigma_{(i_t^c-1)\Delta_n,i}\Delta_{i_t^c}^n W_i/\sqrt{\Delta_n}$. Using the smoothness condition for the processes $\{\sigma_i\}_{i\geq 1}$ in Assumption SA1, we have

$$\sum_j\left(\sum_{i\neq i' \text{ or } s\neq t}\beta_{d,i}'\beta_{d,i'}\big(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj} - \mathbb{E}(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}|\mathcal{C})\big)\big(\overline{\epsilon}_{c,si'}\overline{\epsilon}_{c,sj} - \mathbb{E}(\widetilde{\epsilon}_{c,si'}\widetilde{\epsilon}_{c,sj}|\mathcal{C})\big)\right)$$

$$-\sum_j\Bigg(\sum_{i\neq i' \text{ or } s\neq t}\beta'_{d,i}\beta_{d,i'}\big(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}-\mathbb{E}(\overline{\epsilon}_{c,ti}\overline{\epsilon}_{c,tj}|\mathcal{C})\big)\big(\overline{\epsilon}_{c,si'}\overline{\epsilon}_{c,sj}-\mathbb{E}(\overline{\epsilon}_{c,si'}\overline{\epsilon}_{c,sj}|\mathcal{C})\big)\Bigg)$$
$$=O_P\big(p^3k_n\sqrt{\Delta_n}\big). \tag{A.21}$$

From here, the result in (v) follows after taking into account the integrability conditions for the processes β and $\{\sigma_i\_i\geq 1$. The second result in (vi) can be shown in a similar way. ☐

For stating our next result, we need some notation. For $c\in\{a,b\}$, let $\widehat{Q}_c$ be the $K\times K$ diagonal matrix consisting of the first K eigenvalues of $\overline{Y}_c\overline{Y}'_c/(pk_n)$, where K is the true number of nonredundant factors at time c.

LEMMA A.3. *We have* $\|\widehat{Q}_c\|+\|\widehat{Q}_c^{-1}\|+\frac{1}{p}\|\beta'_c\widehat{\beta}_c\widehat{Q}_c^{-1}\|=O_P(1)$.

PROOF. Using Lemma A.1 and Lemma A.2, we have

$$\frac{1}{pk_n}\|\overline{Y}_c\overline{Y}'_c-\beta_c\overline{F}'_c\overline{F}_c\beta'_c\|$$
$$\leq\frac{2}{pk_n}\|\beta_c\overline{F}'_c(\overline{U}_c+R_c)'\|+\frac{1}{pk_n}\|\overline{U}_c\|^2+\frac{2}{pk_n}\|\overline{U}_cR'_c\|+\frac{1}{pk_n}\|R_c\|^2=o_P(1).$$

Let $\bar{Q}_c$ be the $K\times K$ diagonal matrix of top K eigenvalues of $\frac{1}{p}\beta_c\Lambda'_c\Lambda_c\beta'_c$. We then have $\|\widehat{Q}_c-\bar{Q}_c\|=o_P(1)$ because, using Assumption SA1, we have $\frac{1}{k_n}\|\overline{F}'_c\overline{F}_c-\Lambda'_c\Lambda_c\|=O_P(\frac{1}{\sqrt{k_n}}+\sqrt{\frac{k_n}{n}})$. The eigenvalues of $\bar{Q}_c$ equal those of $(\Lambda'_c\Lambda_c)^{1/2}\frac{1}{p}\beta'_c\beta_c(\Lambda'_c\Lambda_c)^{1/2}$, which are bounded away from zero and infinity and, therefore, so are those of $\widehat{Q}_c$. Then $\|\widehat{Q}_c^{-1}\|=O_P(1)$ and from here $\frac{1}{p}\|\beta'_c\widehat{\beta}_c\widehat{Q}_c^{-1}\|=O_P(1)$. ☐

APPENDIX B: ESTIMATING THE NUMBER OF FACTORS

THEOREM B.1. *Let* $K_{\max}=o(\sqrt{k_n})$, *and* $g_{n,p}$ *be such that*

$$\frac{k_n+p}{k_np}g_{np}=o(1),\qquad \zeta_p+\sqrt{\frac{pk_n}{n}}=o(g_{np}),\qquad \Delta_n^{2\widetilde{\varpi}}=o\bigg(\frac{k_n+p}{k_np}g_{np}\bigg). \tag{B.1}$$

We then have

$$\mathbb{P}(\widehat{K}_a=K_a,\widehat{K}_b=K_b,\widehat{K}_{\text{mix}}=K_{\text{mix}})\to 1.$$

We note that the condition $\Delta_n^{2\widetilde{\varpi}}=o(\frac{k_n+p}{k_np}g_{np})$ in the statement of the above theorem is implied by Conditions (31)–(32) in Theorem 4.1. This is because from these conditions, we have $g_{np}\to\infty$ and $(\sqrt{p}k_n+p)\Delta_n^{2\widetilde{\varpi}}\to 0$.

PROOF. First, note that for $\widehat{F}_K$ and $\widehat{\beta}_K$ being the estimated factors and betas using K eigenvectors, we can write

$$V(K):=\frac{1}{pk_n}\|\overline{Y}_c-\widehat{\beta}_K\widehat{F}'_K\|_F^2=\sum_{m>K}v_{c,m}.$$

Therefore, the criterion (22) is equivalent to the IC criterion in Bai and Ng (2002). From here, the proof of the case $K < K_c$ is very similar to that of Bai and Ng (2002), so we omit it for brevity. However, there is a technical flaw in the published version of Bai and Ng (2002) for the case $K > K_c$, so we present a proof of this case here using random matrix theory.

Recall $S_c = \frac{1}{k_n p}\overline{Y}_c\overline{Y}_c'$, $c \in \{a, b\}$. We first bound $\max_{m>K_c} v_{c,m}$. Let us separately consider two cases: $K_c > 0$ (there are factors) and $K_c = 0$ (there are no factors).

Case I: $K_c > 0$**.** For two semipositive definition matrices A, B, the $a+b$ largest eigenvalue satisfy

$$\lambda_{a+b}(A+B) \leq \lambda_{a+1}(A) + \lambda_b(B).$$

We will use this inequality and the following decomposition:

$$\begin{aligned} S_c &= \Gamma + W, \\ \Gamma &= \frac{1}{k_n p}\Phi\overline{F}_c'\overline{F}_c\Phi', \qquad \operatorname{rank}(\Gamma) = K_c, \\ W &= \frac{1}{k_n p}(R_c + \overline{U}_c)M_{\overline{F}_c}(R_c + \overline{U}_c)', \end{aligned} \tag{B.2}$$

where $M_{\overline{F}_c} := I - P_{\overline{F}_c}$, and $\Phi = \beta_c + (\overline{U}_c + R_c)\overline{F}_c(\overline{F}_c'\overline{F}_c)^{-1}$. For $m > K_c$, there is $i = 1, 2, \ldots$, so that $m = K_c + i$. Then, by making use of Lemma A.1 and Lemma A.2, we have

$$\begin{aligned} v_{c,m} &= \lambda_m(S_c) = \lambda_{K_c+i}(W+\Gamma) \leq \lambda_{K_c+1}(\Gamma) + \lambda_i(W) = \lambda_i(W) \\ &\leq \frac{2}{k_n p}\|R_c\|^2 + \frac{2}{k_n p}\|\overline{U}_c\|^2 \leq O_P(\delta), \\ \delta &:= \left(\frac{1}{p} + \frac{1}{k_n}\right)\left(\zeta_p + \sqrt{\frac{pk_n}{n}}\right) + \Delta_n^{2\widetilde{\varpi}}. \end{aligned}$$

Let $d_{np} = (\frac{k_n+p}{k_n p})g_{np}$ denote the penalty rate. Note that $V(K_c)$ is the rescaled residual sum of squares when the true number of factors is used, which consistently estimates $\frac{1}{p}\sum_i \mathbb{E}(\sigma_{c,i}^2|\mathcal{C})$. So, $V(K_c) > c$ is bounded away from zero with probability approaching one. When $K > K_c$,

$$\begin{aligned} \Delta &:= \log V(K) + Kd_{np} - \big(\log V(K_c) + K_c d_{np}\big) = \log\frac{V(K)}{V(K_c)} + (K - K_c)d_{np} \\ &\geq \log\left(1 - \frac{\sum_{K_c<m\leq K} v_{c,m}}{V(K_c)}\right) + d_{np} \geq d_{np} - O_P\left(\sum_{K_c<m\leq K_{\max}} v_{c,m}\right) \\ &\geq d_{np} - O_P\left(\max_{m>K_c} v_{c,m}\right) \\ &\geq d_{np} - O_P(\delta) > 0, \end{aligned}$$

because of the rate condition in (B.1).

Case II: $K_c = 0$**.** We have $S_c = \frac{1}{k_n p}\overline{U}_c\overline{U}_c'$, $c \in \{a, b\}$, whose eigenvalues are bounded by $\frac{1}{k_n p}\|\overline{U}_c\|^2 \le O_P(\delta)$. In addition, $V(K_c)$ still converges to $\frac{1}{p}\sum_i \mathbb{E}(\sigma_{c,i}^2|\mathcal{C})$, which is bounded away from zero. Hence, $\Delta \ge d_{np} - O_P(\delta) > 0$. □

Appendix C: Proof of Theorem 4.1

C.1 *Outline of the proof*

Since by Theorem B.1, the number of nonredundant factors over a given period can be recovered with probability approaching one, we can conduct the proof assuming that the true number of factors is known. We do so henceforth. The proof of Theorem 4.1 is structured as follows.

Part I. PCA expansion. As discussed in Section 2, we have the following discrete factor model:

$$\overline{Y}_c = \beta_c \overline{F}_c' + \overline{U}_c + R_c, \quad c = a, b, \tag{C.1}$$

where recall R_c is a residual component containing the approximation error to the discrete factor model. We can apply PCA to $\overline{Y}_c$. Using the definition of PCA, we will make the following expansion:

$$\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 - (B_a + B_b) = \widetilde{\mu}_a + \widetilde{\mu}_b - \widehat{\mu}_{ab} + \Delta_5,$$

where B_a and B_b are certain centering terms, the first three terms on the right-hand side of the above equality are the leading terms that jointly determine the asymptotic distribution of the statistic under the null hypothesis, and Δ_5 is a higher-order term. In the above, B_a and B_b are the leading bias terms. Using the estimates $\widehat{B}_a$ and $\widehat{B}_b$ for them leads to

$$\begin{aligned} &k_n\sqrt{p}\big[\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 - (\widehat{B}_a + \widehat{B}_b)\big] \\ &\quad = k_n\sqrt{p}(\widetilde{\mu}_a + \widetilde{\mu}_b - \widehat{\mu}_{ab}) + k_n\sqrt{p}\big[\Delta_5 + B_a + B_b - (\widehat{B}_a + \widehat{B}_b)\big]. \end{aligned}$$

Finally, we also use the bias-mimicking projections that are in the term $\widehat{\mathcal{A}}_{\text{mix}}$, and hence we need to consider $P_{\widehat{\beta}_{\text{mix},o}}$ and $P_{\widehat{\beta}_{\text{mix},e}}$. These two terms are the projection matrices associated with $\widehat{\beta}_{\text{mix},o}$ and $\widehat{\beta}_{\text{mix},e}$. We can get a similar decomposition for

$$\widehat{\mathcal{A}}_{\text{mix}} = \|P_{\widehat{\beta}_{\text{mix},o}} - P_{\widehat{\beta}_{\text{mix},e}}\|_F^2 - (\widehat{B}_{\text{mix},o} + \widehat{B}_{\text{mix},e}), \tag{C.2}$$

where $\widehat{B}_{\text{mix},o} + \widehat{B}_{\text{mix},e}$ is the estimated bias term for $\|P_{\widehat{\beta}_{\text{mix},o}} - P_{\widehat{\beta}_{\text{mix},e}}\|_F^2$. Namely, we can write

$$\begin{aligned} k_n\sqrt{p}\widehat{\mathcal{A}}_{\text{mix}} &= k_n\sqrt{p}(\widetilde{\mu}_{\text{mix},o} + \widetilde{\mu}_{\text{mix},e} - \widehat{\mu}_{\text{mix}}) \\ &\quad + k_n\sqrt{p}\big[\Delta_{5,\text{mix}} + B_{\text{mix},o} + B_{\text{mix},e} - (\widehat{B}_{\text{mix},o} + \widehat{B}_{\text{mix},e})\big]. \end{aligned}$$

The terms in the above decomposition are the natural counterparts of the ones for the projection discrepancy $P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}$ above. Putting things together, this will lead to an expansion for the test statistic $\mathcal{S}$. This expansion and the definition of all the terms in the above decompositions will be given in Section C.2.

Part II. Higher-order terms. In this part of the proof, we will show that the higher-order terms are negligible, in the sense that, for $c = a, b$ and $d = o, e$, the following terms: $k_n\sqrt{p}\Delta_5$, $k_n\sqrt{p}\Delta_{5,\text{mix}}$, $k_n\sqrt{p}(B_c - \widehat{B}_c)$, and $k_n\sqrt{p}(B_{\text{mix},d} - \widehat{B}_{\text{mix},d})$ are all $o_P(1)$. As a result, under the null hypothesis,

$$\mathcal{S} = k_n\sqrt{p}(\widetilde{\mu}_a + \widetilde{\mu}_b - \widehat{\mu}_{ab}) - k_n\sqrt{p}(\widetilde{\mu}_{\text{mix},o} + \widetilde{\mu}_{\text{mix},e} - \widehat{\mu}_{\text{mix}}) + o_P(1).$$

Part III. Asymptotic null distribution. We will then derive the asymptotic distribution of the leading term. This is done in Section C.4.

Part IV. Bootstrap limit result. In the next step, we characterize the asymptotic behavior of the bootstrap statistic in Section C.5.

Part V. Asymptotic Test Size. In a final step in Section C.6, we use the results in parts I–IV to derive the result in (35) concerning the asymptotic size of the test.

C.2 *PCA expansion*

Step 1. For $c \in \{a, b\}$, let $\widehat{Q}_c$ be the $K \times K$ diagonal matrix consisting of the first K eigenvalues of $\overline{Y}_c\overline{Y}_c'/(pk_n)$. By the definition of eigenvectors, $\overline{Y}_c\overline{Y}_c'\widehat{\beta}_c/(pk_n) = \widehat{\beta}_c\widehat{Q}_c$. Expanding $\overline{Y}_c$ using (C.1), we can verify that the following identity holds:

$$\widehat{\beta}_c - \beta_c H_c = \frac{1}{k_n}\overline{U}_c\overline{F}_c\widehat{A}_c + \Delta_{1c}, \tag{C.3}$$

where

$$\begin{aligned}
\Delta_{1c} &= \frac{1}{pk_n}\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}R_c\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1},\\
\widehat{A}_c &= \frac{1}{p}\beta_c'\widehat{\beta}_c\widehat{Q}_c^{-1},\\
H_c &= \frac{1}{k_n p}\overline{F}_c'\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}.
\end{aligned} \tag{C.4}$$

Next, Lemma C.2 below shows that H_c is invertible with probability approaching one, hence $P_{\beta_c} = P_{\beta_c H_c}$. As a result,

$$P_{\widehat{\beta}_c} = P_{\beta_c} + \frac{1}{p}(\widehat{\beta}_c - \beta_c H_c)\widehat{\beta}_c' + \beta_c\Delta_{2c}\widehat{\beta}_c' + \beta_c\left(\beta_c'\beta_c\right)^{-1}H_c'^{-1}(\widehat{\beta}_c - \beta_c H_c)', \tag{C.5}$$

where

$$\Delta_{2c} = H_c\frac{1}{p}\left[H_c'\beta_c'\beta_c H_c - \widehat{\beta}_c'\widehat{\beta}_c\right]\left(H_c'\beta_c'\beta_c H_c\right)^{-1}. \tag{C.6}$$

From here and building on (C.3), we further expand (after some tedious algebra):

$$\begin{aligned}
\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 - (B_a + B_b) &= \widetilde{\mu}_a + \widetilde{\mu}_b - \widehat{\mu}_{ab} + \Delta_5, \\
\widetilde{\mu}_c &= \frac{2}{pk_n^2} \operatorname{tr} \widehat{A}_c' \big[\overline{F}_c' \overline{U}_c' \overline{U}_c \overline{F}_c - \mathrm{BIAS}_c\big] \widehat{A}_c, \quad c \in \{a, b\}, \\
\widehat{\mu}_{ab} &= \frac{2}{pk_n^2} \operatorname{tr} \widehat{A}_a' \overline{F}_a' \overline{U}_a' \overline{U}_b \overline{F}_b \widehat{A}_b \widehat{G}, \\
B_c &= \frac{2}{pk_n^2} \operatorname{tr} \widehat{A}_c' \mathrm{BIAS}_c \widehat{A}_c = \frac{2}{k_n^2} \sum_{t=1}^{k_n} \operatorname{tr} \widehat{A}_c' \bar{f}_{c,t} \bar{f}_{c,t}' \widehat{A}_c \mathbb{E}\big(\sigma_{c,1}^2 | \mathcal{C}\big), \\
\mathrm{BIAS}_c &= \sum_{i=1}^{p} \mathbb{E}\big(\sigma_{c,1}^2 | \mathcal{C}\big) \overline{F}_c' \overline{F}_c, \\
\widehat{G} &:= \frac{1}{p} \widehat{\beta}_b' \widehat{\beta}_a + H_b^{-1} \left(\frac{1}{p} \beta_b' \beta_b\right)^{-1} \beta_b' \beta_a \big(\beta_a' \beta_a\big)^{-1} H_a'^{-1},
\end{aligned} \tag{C.7}$$

with Δ_5 being a remainder term, whose lengthy decomposition will be given in Section C.3.1, and we remind the reader our notation in (16) and (17) in the main text.

We then estimate B_c by

$$\widehat{B}_c = \frac{2}{k_n^2} \operatorname{tr}\big(\widehat{Q}_c^{-1} \widehat{F}_c' \widehat{F}_c \widehat{Q}_c^{-1}\big) \mathbb{E}\widehat{\big(\sigma_{c,1}^2 | \mathcal{C}\big)}, \quad c \in \{a, b\}.$$

As a result, we can write

$$\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 - (\widehat{B}_a + \widehat{B}_b) = \widetilde{\mu}_a + \widetilde{\mu}_b - \widehat{\mu}_{ab} + \Delta_5 + B_a + B_b - (\widehat{B}_a + \widehat{B}_b). \tag{C.8}$$

Step 2. We continue with $\widehat{\mathcal{A}}_{\mathrm{mix}}$, the bias-mimicking statistic. The expansion for this term requires introducing significantly more notation. For $c \in \{a, b\}$ and $k \in \{o, e\}$, let $\overline{Y}_{c,k}$, $\overline{F}_{c,k}$, $\overline{U}_{c,k}$ denote the columns of $\overline{Y}_c$, $\overline{F}_c$, and $\overline{U}_c$ realized on k time points during period c. Recall that $\widehat{\beta}_{\mathrm{mix},k}$ is constructed as the eigenvector using data $\overline{Y}_{\mathrm{mix},k} = (\overline{Y}_{a,k}, \overline{Y}_{b,k})$. Let $S_{f,c,k} = \frac{1}{k_n} \overline{F}_{c,k}' \overline{F}_{c,k}$. Then

$$\frac{1}{k_n p} \overline{Y}_{\mathrm{mix},k} \overline{Y}_{\mathrm{mix},k}' = \frac{1}{p} \beta_a S_{f,a,k} \beta_a' + \frac{1}{p} \beta_b S_{f,b,k} \beta_b' + \Delta, \tag{C.9}$$

which holds under both null and alternatives, and

$$\Delta = \sum_{c \in \{a,b\}} \frac{1}{pk_n} \beta_c \overline{F}_{c,k}' \overline{U}_{c,k}' + \frac{1}{pk_n} \overline{U}_{c,k} \overline{U}_{c,k}' + \frac{1}{pk_n} \overline{U}_{c,k} \overline{F}_{c,k} \beta_c' + Rem_1,$$

with Rem_1 being a remainder term that depends on R_a and R_b in (C.1). Let $\widehat{Q}_{\mathrm{mix},k}$ be the $K \times K$ diagonal matrix consisting of the first K eigenvalues of $\overline{Y}_{\mathrm{mix},k} \overline{Y}_{\mathrm{mix},k}' / (pk_n)$. By the definition of the eigenvector defining $\widehat{\beta}_{\mathrm{mix},k}$, we have an identity similar to (C.3):

$$\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab} H_{\mathrm{mix},k}$$

$$
\begin{aligned}
&= \frac{1}{k_n}\overline{U}_{a,k}\overline{F}_{a,k}\frac{1}{p}\beta_a'\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k} + \frac{1}{k_n}\overline{U}_{b,k}\overline{F}_{b,k}\frac{1}{p}\beta_b'\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k} \\
&\quad + \frac{1}{pk_n}\overline{U}_{a,k}\overline{U}'_{a,k}\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k} \\
&\quad + \frac{1}{pk_n}\overline{U}_{b,k}\overline{U}'_{b,k}\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k} + Rem, \qquad \text{(C.10)}
\end{aligned}
$$

with the following notation:

$$
\beta_{ab} = (\beta_a, \beta_b), \qquad H_{c,\mathrm{mix},k} = \frac{1}{p_n k_n}\overline{F}'_{c,k}\overline{Y}'_{c,k}\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k}, \qquad H_{\mathrm{mix},k} = \begin{pmatrix} H_{a,\mathrm{mix},k} \\ H_{b,\mathrm{mix},k} \end{pmatrix},
$$

and where Rem is a remainder term depending on R_a, R_b similar to that in (C.3).

Let $\Delta_{2\mathrm{mix},k} = H_{\mathrm{mix},k}\frac{1}{p}[H'_{\mathrm{mix},k}\beta'_{ab}\beta_{ab}H_{\mathrm{mix},k} - \widehat{\beta}'_{\mathrm{mix},k}\widehat{\beta}_{\mathrm{mix},k}](H'_{\mathrm{mix},k}\beta'_{ab}\beta_{ab}H_{\mathrm{mix},k})^{-1}$. Then, similar to the identity (C.5), we have

$$
\begin{aligned}
&P_{\widehat{\beta}_{\mathrm{mix},k}} - P_{\beta_{ab}H_{\mathrm{mix},k}} \\
&\quad = \frac{1}{p}(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})\widehat{\beta}'_{\mathrm{mix},k} + \beta_{ab}\Delta_{2\mathrm{mix},k}\widehat{\beta}'_{\mathrm{mix},k} \\
&\quad\quad + \beta_{ab}H_{\mathrm{mix},k}\big(H'_{\mathrm{mix},k}\beta'_{ab}\beta_{ab}H_{\mathrm{mix},k}\big)^{-1}(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})'. \qquad \text{(C.11)}
\end{aligned}
$$

Identities (C.10) and (C.11) hold under both the null and the alternative hypotheses.

Under the null that $\beta_b = \beta_a H$ for some invertible matrix H,

$$
\beta_{ab}H_{\mathrm{mix},k} = \beta_a L_k, \qquad L_k := (H_{a,\mathrm{mix},k} + HH_{b,\mathrm{mix},k}). \qquad \text{(C.12)}
$$

Lemma C.7 below shows $\frac{1}{\sqrt{p}}\|\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k}\| = o_P(1)$. It follows that

$$
I = \frac{1}{p}\widehat{\beta}'_{\mathrm{mix},k}\widehat{\beta}_{\mathrm{mix},k} = \frac{1}{p}H'_{\mathrm{mix},k}\beta'_{ab}\beta_{ab}H_{\mathrm{mix},k} + o_P(1) = \frac{1}{p}L'_k\beta'_a\beta_a L_k + o_P(1).
$$

Also, the eigenvalues of $\frac{1}{p}\beta'_a\beta_a$ are bounded away from zero. Hence, by Lemma C.1, L_k is invertible with probability approaching one. Hence, $P_{\beta_{ab}H_{\mathrm{mix},k}} = P_{\beta_a L_k} = P_{\beta_a}$ under the null. Then the left-hand side of (C.11) can be replaced by $P_{\widehat{\beta}_{\mathrm{mix},k}} - P_{\beta_a}$.

Next, define

$$
\overline{U}_{\mathrm{mix},k} = (\overline{U}_{a,k}, \overline{U}_{b,k}), \qquad \overline{F}_{\mathrm{mix},k} = \begin{pmatrix} \overline{F}_{a,k} \\ \overline{F}_{b,k}H, \end{pmatrix},
$$

$$
\widehat{G}_{\mathrm{mix}} = \frac{2}{p}\widehat{\beta}'_{\mathrm{mix},o}\widehat{\beta}_{\mathrm{mix},e}, \qquad \widehat{A}_{\mathrm{mix},k} = \frac{1}{p}\beta'_a\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k}.
$$

Then under the null, (C.10) can be rewritten as

$$
\begin{aligned}
\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k} &= \frac{1}{k_n}\overline{U}_{\mathrm{mix},k}\overline{F}_{\mathrm{mix},k}\widehat{A}_{\mathrm{mix},k} + \Delta_{1\mathrm{mix},k}, \\
\Delta_{1\mathrm{mix},k} &= \frac{1}{pk_n}\overline{U}_{\mathrm{mix},k}\overline{U}'_{\mathrm{mix},k}\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k} + Rem,
\end{aligned} \qquad \text{(C.13)}
$$

for Rem that depends on R_a, R_b.

Combine with (C.11) to obtain an identity similar to the one in (C.7) under the null,

$$\|P_{\widehat{\beta}_{\text{mix},o}} - P_{\widehat{\beta}_{\text{mix},e}}\|_F^2 = \mu_{\text{mix},o} + \mu_{\text{mix},e} - \mu_{\text{mix},oe} + \Delta_{5,\text{mix}},$$

$$\mu_{\text{mix},k} = \frac{2}{pk_n^2} \operatorname{tr} \widehat{A}'_{\text{mix},k}\big[\overline{F}'_{\text{mix},k}\overline{U}'_{\text{mix},k}\overline{U}_{\text{mix},k}\overline{F}_{\text{mix},k}\big]\widehat{A}_{\text{mix},k},$$

$$\mu_{\text{mix},oe} = \frac{2}{pk_n^2} \operatorname{tr} \widehat{A}'_{\text{mix},o}\overline{F}'_{\text{mix},o}\overline{U}'_{\text{mix},o}\overline{U}_{\text{mix},e}\overline{F}_{\text{mix},e}\widehat{A}_{\text{mix},e}\widehat{G}_{\text{mix}},$$

where $\Delta_{5,\text{mix}}$ is a remainder term similar to Δ_5. Let

$$B_{\text{mix},k} = \frac{2}{k_n^2} \operatorname{tr} \widehat{A}'_{\text{mix},k}\big[F'_{a,k}F_{a,k}\mathbb{E}(\sigma^2_{a,1}|\mathcal{C}) + H'F'_{b,k}F_{b,k}H\mathbb{E}(\sigma^2_{b,1}|\mathcal{C})\big]\widehat{A}_{\text{mix}},$$

$$\widehat{B}_{\text{mix},k} = \frac{2}{k_n^2} \operatorname{tr} \widehat{Q}^{-1}_{\text{mix},k}\widehat{F}'_{a,k}\widehat{F}_{a,k}\widehat{Q}^{-1}_{\text{mix},k}\mathbb{E}\big(\widehat{\sigma^2_{a,1}|\mathcal{C}}\big) + \frac{2}{k_n^2} \operatorname{tr} \widehat{Q}^{-1}_{\text{mix},k}\widehat{F}'_{b,k}\widehat{F}_{b,k}\widehat{Q}^{-1}_{\text{mix},k}\mathbb{E}\big(\widehat{\sigma^2_{b,1}|\mathcal{C}}\big).$$

Then

$$\begin{aligned}\widehat{\mathcal{A}}_{\text{mix}} &= \|P_{\widehat{\beta}_{\text{mix},o}} - P_{\widehat{\beta}_{\text{mix},e}}\|_F^2 - (\widehat{B}_{\text{mix},o} + \widehat{B}_{\text{mix},e}) \\ &= (\mu_{\text{mix},o} - B_{\text{mix},o}) + (\mu_{\text{mix},e} - B_{\text{mix},e}) + \mu_{\text{mix},oe} \\ &\quad + (B_{\text{mix},o} - \widehat{B}_{\text{mix},o}) + (B_{\text{mix},e} - \widehat{B}_{\text{mix},e}) + \Delta_{5,\text{mix}}.\end{aligned}$$

Altogether, we have

$$\begin{aligned}&\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 - (\widehat{B}_a + \widehat{B}_b) - \widehat{\mathcal{A}}_{\text{mix}} \\ &\quad = \widetilde{\mu}_a + \widetilde{\mu}_b - \widehat{\mu}_{ab} - (\mu_{\text{mix},o} - B_{\text{mix},o}) - (\mu_{\text{mix},e} - B_{\text{mix},e}) - \mu_{\text{mix},oe} \\ &\quad\quad + \Delta_5 + (B_a - \widehat{B}_a) + (B_b - \widehat{B}_b) - (B_{\text{mix},o} - \widehat{B}_{\text{mix},o}) \\ &\quad\quad - (B_{\text{mix},e} - \widehat{B}_{\text{mix},e}) - \Delta_{5,\text{mix}}.\end{aligned} \tag{C.14}$$

The term in the second line of the above expression is the leading term, jointly determining the asymptotic null distribution, while the terms in the third line of the above expression are higher-order terms. We need to show that, after multiplying them by $k_n\sqrt{p}$, these terms are asymptotically negligible.

LEMMA C.1. *Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively, denote the minimum and maximum eigenvalue of a semipositive definite matrix A. Suppose Σ is semipositive definite, and*

$$\lambda_{\max}(\Sigma) < C, \qquad \lambda_{\min}\big(L'\Sigma L\big) > c$$

for constants $c, C > 0$. Then $\lambda_{\min}(L'L) \geq c/C$. If L is a square matrix, then L is invertible.

PROOF. Let v be the eigenvector of $L'L$ so that $v'L'Lv = \lambda_{\min}(L'L)$. Let $\theta = Lv$. Let I be a generic identity matrix. Then $CI - \Sigma$ is semipositive definite, implying $\theta'\Sigma\theta \leq C\|\theta\|^2$, which is

$$C\lambda_{\min}\big(L'L\big) = Cv'L'Lv \geq v'L'\Sigma Lv.$$

Because $L'\Sigma L - cI$ is semipositive definite, $v'L'\Sigma Lv \geq c$. Hence, $\lambda_{\min}(L'L) \geq c/C$. This shows the singular values of L, which are square roots of the eigenvalues of $L'L$, are nonzero. Hence, L is invertible if it is a square matrix. $\square$

C.3 *Higher-order terms*

According to (C.14), there are four higher-order terms:

$$\Delta_5, \qquad (B_c - \widehat{B}_c), \qquad (B_{\text{mix},k} - \widehat{B}_{\text{mix},k}), \qquad \Delta_{5,\text{mix}}, \quad c = a, b, k = o, e.$$

We aim to show that, after multiplication by $k_n\sqrt{p}$, these terms are all asymptotically negligible.

C.3.1 *Higher-order terms I: Δ_5 and $\Delta_{5,\text{mix}}$* In this subsection, we focus on Δ_5 and $\Delta_{5,\text{mix}}$. In particular, Δ_5 has a lengthy expression, given as follows ($\Delta_{5,\text{mix}}$ is defined similarly):

$$\begin{aligned}\Delta_5 = &-\frac{2}{p}\operatorname{tr}\frac{1}{k_n}\widehat{A}'_a\overline{F}'_a\overline{U}'_a\Delta_{1,b}\frac{1}{p}\widehat{\beta}'_b\widehat{\beta}_a - \frac{2}{p}\operatorname{tr}\frac{1}{k_n}\widehat{A}'_b\overline{F}'_b\overline{U}'_b\Delta_{1,a}\frac{1}{p}\widehat{\beta}'_a\widehat{\beta}_b \\ &-\frac{2}{p}\operatorname{tr}\Delta'_{1,a}\Delta_{1,b}\frac{1}{p}\widehat{\beta}'_b\widehat{\beta}_a - 2\left(\Delta_4 + \Delta_{3,a} + \Delta_{3,b} + \frac{2}{p}\|\Delta_{1,a}\|_F^2 + \frac{2}{p}\|\Delta_{1,b}\|_F^2\right) \\ &+\frac{4}{p}\operatorname{tr}\frac{1}{k_n}\widehat{A}'_a\overline{F}'_a\overline{U}'_a\Delta_{1,a} + \frac{4}{p}\operatorname{tr}\frac{1}{k_n}\widehat{A}'_b\overline{F}'_b\overline{U}'_b\Delta_{1,b}. \end{aligned} \tag{C.15}$$

The expression for Δ_5 depends on $\Delta_{1,c}$ and $\Delta_{2,c}$, given in (C.4) and (C.6). It also depends on $\Delta_{3,c}$, Δ_4, which are defined as

$$\begin{aligned}\Delta_{3,c} = &\,2\operatorname{tr}(\widehat{\beta}_c - \beta_c H_c)'\beta_c\Delta_{2,c} + 2\operatorname{tr}\frac{1}{p}(\widehat{\beta}_c - \beta_c H_c)'\beta_c\big(\beta'_c\beta_c\big)^{-1}H_c'^{-1}(\widehat{\beta}_c - \beta_c H_c)'\widehat{\beta}_c \\ &+ p\|\beta_c\Delta_{2,c}\|_F^2 + 2\operatorname{tr}\widehat{\beta}'_c(\widehat{\beta}_c - \beta_c H_c)H_c^{-1}\Delta_{2,c} \\ &- \operatorname{tr}(\widehat{\beta}_c - \beta_c H_c)\Delta'_{2,c}H_c'^{-1}(\widehat{\beta}_c - \beta_c H_c)', \\ \Delta_4 = &\sum_{c\neq d}\operatorname{tr}\frac{1}{p}\widehat{\beta}_c(\widehat{\beta}_c - \beta_c H_c)'\beta_d\Delta_{2,d}\widehat{\beta}'_d \\ &+\sum_{c\neq d}\operatorname{tr}\frac{1}{p}\widehat{\beta}_c(\widehat{\beta}_c - \beta_c H_c)'\beta_d\big(\beta'_d\beta_d\big)^{-1}H_d'^{-1}(\widehat{\beta}_d - \beta_d H_d)' \\ &+\sum_{c\neq d}\operatorname{tr}\widehat{\beta}_c\Delta'_{2,c}\beta'_c\beta_d\big(\beta'_d\beta_d\big)^{-1}H_d'^{-1}(\widehat{\beta}_d - \beta_d H_d)' \\ &+\sum_{c\neq d}\operatorname{tr}\frac{1}{k_n}\overline{U}_c\overline{F}_c\widehat{A}_cH_c^{-1}\big(\beta'_c\beta_c\big)^{-1}\beta'_c\beta_d\big(\beta'_d\beta_d\big)^{-1}H_d'^{-1}\Delta'_{1d} \\ &+\operatorname{tr}\Delta_{1a}H_a^{-1}\big(\beta'_a\beta_a\big)^{-1}\beta'_a\beta_b\big(\beta'_b\beta_b\big)^{-1}H_b'^{-1}\Delta'_{1b} + \operatorname{tr}\widehat{\beta}_a\Delta'_{2,a}\beta'_a\beta_b\Delta_{2,b}\widehat{\beta}'_b. \end{aligned} \tag{C.16}$$

The above expression for Δ_5 can be derived after tedious algebraic calculations. Here, we illustrate the sources of all the terms in Δ_5. From (C.5), by substituting the expression for $\widehat{\beta}_c - \beta_c H_c$, we have $P_{\widehat{\beta}_c} - P_{\beta_c} = g_{1,c} + \cdots + g_{5,c}$, where

$$
\begin{aligned}
g_{1,c} &= \frac{1}{pk_n}\overline{U}_c\overline{F}_c\widehat{A}_c\widehat{\beta}_c',\\
g_{2,c} &= \frac{1}{p}\Delta_{1c}\widehat{\beta}_c',\\
g_{3,c} &= \beta_c\Delta_{2c}\widehat{\beta}_c',\\
g_{4,c} &= \beta_c\big(\beta_c'\beta_c\big)^{-1}H_c'^{-1}\frac{1}{k_n}\widehat{A}_c'\overline{F}_c'\overline{U}_c',\\
g_{5,c} &= \beta_c\big(\beta_c'\beta_c\big)^{-1}H_c'^{-1}\Delta_{1c}'.
\end{aligned}
$$

Therefore,

$$
\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 = \sum_{d,c}\|g_{d,c}\|_F^2 + \sum_{c,d_1\neq d_2}\operatorname{tr}\big(g_{d_1,c}'g_{d_2,c}\big) - \sum_{d_1,d_2}\operatorname{tr}\big(g_{d_1,a}'g_{d_2,b}\big).
$$

(1) In $\sum_{d,c}\|g_{d,c}\|_F^2$, the leading terms are $\|g_{1,c}\|_F^2 + \|g_{4,c}\|_F^2$. The higher-order terms are: $\|g_{2,c}\|_F^2 + \|g_{5,c}\|_F^2 = O_P(\frac{1}{p}\|\Delta_{1c}\|^2)$, and $\|g_{3,c}\|_F^2 = O_P(\|\frac{1}{p}\widehat{\beta}_c'(\widehat{\beta}_c - \beta_c H_c)\|^2)$.

(2) In $\sum_{c,d_1\neq d_2}\operatorname{tr}(g_{d_1,c}'g_{d_2,c})$, all terms are of higher-order, which involves terms like $O_P(\frac{1}{p}\|\Delta_{1c}\|^2 + \|\frac{1}{pk_n}\overline{F}_c'\overline{U}_c'\beta_c\|^2 + \|\frac{1}{p}\widehat{\beta}_c'(\widehat{\beta}_c - \beta_c H_c)\|^2 + \|\frac{1}{k_n p}\overline{F}_c'\overline{U}_c'\Delta_{1c}\|)$.

(3) In $\sum_{d_1,d_2}\operatorname{tr}(g_{d_1,a}'g_{d_2,b})$, only $\operatorname{tr}(g_{1,a}'g_{1,b})$ and $\operatorname{tr}(g_{4,a}'g_{4,b})$ are the leading terms, all other terms are of higher order, involving $O_P(\frac{1}{p}\|\Delta_{1c}\|^2 + \|\frac{1}{pk_n}\overline{F}_c'\overline{U}_c'\beta_d\|^2 + \|\frac{1}{p}\widehat{\beta}_d'(\widehat{\beta}_c - \beta_c H_c)\|^2 + \|\frac{1}{k_n p}\overline{F}_c'\overline{U}_c'\Delta_{1d}\|)$ for $c, d \in \{a, b\}$.

We start with establishing some preliminary bounds in Lemmas C.2–C.5. With their help, we derive the bounds for Δ_5 and $\Delta_{5,\mathrm{mix}}$ that we need in Lemmas C.6 and C.7.

Lemma C.2. *Assume* $\zeta_p = O(\sqrt{k_n}\wedge\sqrt{p})$ *and* $pk_n\Delta_n = O_p(1)$, *as* $p, n \to \infty$. *Under both null and alternatives,* $\|\widehat{\beta}_c - \beta_c H_c\| \le O_P(\sqrt{\frac{p}{k_n}} + \frac{\zeta_p}{\sqrt{p}} + \delta_4)$ *and* $\|\Delta_{1c}\| \le O_P(\frac{\sqrt{p}}{k_n} + \frac{\zeta_p}{\sqrt{p}} + \delta_4)$, *where* $\Delta_{1c} = \frac{1}{pk_n}\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}R_c\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}$ *and*

$$
\delta_4 = \left\|\frac{1}{pk_n}\overline{U}_c R_c'\widehat{\beta}_c + \frac{1}{pk_n}R_c\overline{Y}_c'\widehat{\beta}_c\right\|. \tag{C.17}
$$

Also, $\|H_c\| + \|H_c^{-1}\| = O_P(1)$.

Proof. Recall that $\widehat{Q}_c$ is a diagonal matrix consisting of the top K eigenvalues of $Y_c'Y_c/(k_n p)$. From Lemma A.3, $\|\widehat{Q}_c\| = O_P(1) = \|\widehat{Q}_c^{-1}\|$. Also, recall that

$$
\widehat{\beta}_c - \beta_c H_c
$$

$$= \frac{1}{k_n}\overline{U}_c\overline{F}_c\widehat{A}_c$$
$$+ \underbrace{\frac{1}{pk_n}\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}R_c\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}}_{\Delta_{1c}}. \qquad \text{(C.18)}$$

The first term $\|\frac{1}{k_n}\overline{U}_c\overline{F}_c\widehat{A}_c\| \le \|\frac{1}{k_n}\overline{U}_c\overline{F}_c\|O_P(1) \le O_P(\sqrt{p/k_n})$ by Lemma A.2 (iv). The second term

$$\left\|\frac{1}{pk_n}\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}\right\| \le o_P(1)\frac{1}{pk_n}\|\overline{U}_c'\|^2\|\widehat{\beta}_c\| = O_P\left(\frac{\sqrt{p}}{k_n} + \frac{1}{\sqrt{p}}\right)\zeta_p,$$

using $\widehat{\beta}_c = O_P(\sqrt{p})$, Lemma A.2 (ii), the condition $pk_n\Delta_n = O_p(1)$ and Lemma A.3. For the third and fourth terms, we have

$$\left\|\frac{1}{pk_n}\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}R_c\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}\right\| \le O_P(1)\delta_4,$$

by making again use of Lemma A.3. Together, because $\zeta_p = O(\sqrt{k_n})$, $\|\widehat{\beta}_c - \beta_c H_c\| \le O_P(\sqrt{\frac{p}{k_n}} + \frac{\zeta_p}{\sqrt{p}} + \delta_4)$.

Finally, to show $\|H_c\| + \|H_c^{-1}\| = O_P(1)$, we have proved $\frac{1}{\sqrt{p}}\|\widehat{\beta}_c - \beta_c H_c\| = o_P(1)$. Hence,

$$I = \frac{1}{p}\widehat{\beta}_c'\widehat{\beta}_c = \frac{1}{p}H_c'\beta_c'\overline{\beta_c}H_c + o_P(1).$$

This then implies that all singular values of H_c are bounded away from zero and infinity. □

LEMMA C.3. *Under both null and alternatives,*

$$\frac{1}{p}\beta_d'(\widehat{\beta}_c - \beta_c H_c) = O_P\left(\frac{1}{p} + \frac{1}{\sqrt{pk_n}} + \frac{\delta_4}{p\sqrt{k_n}} + \frac{\delta_4\Delta_n^{1/4}}{\sqrt{pk_n}} + \frac{\Delta_n^{1/4}}{k_n} + \frac{\zeta_p\Delta_n^{1/4}}{p\sqrt{k_n}} + \frac{\sqrt{\Delta_n}}{\sqrt{k_n}} + \delta_5\right),$$

and

$$\frac{1}{p}\widehat{\beta}_d'(\widehat{\beta}_c - \beta_c H_c) = O_P\left(\frac{1}{p} + \frac{\delta_4}{p\sqrt{k_n}} + \frac{\delta_4\Delta_n^{1/4}}{\sqrt{pk_n}} + \frac{\zeta_p\Delta_n^{1/4}}{p\sqrt{k_n}} + \frac{\sqrt{\Delta_n}}{\sqrt{k_n}} + \frac{1}{k_n} + \frac{\delta_4^2}{p} + \delta_5\right),$$

for $c, d \in \{a, b\}$, and where $\delta_5 := \frac{1}{pk_n}\frac{1}{p}\beta_c'\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}\frac{1}{p}\beta_c' R_c\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}$, and δ_4 is defined in (C.17).

PROOF. Recall that, for R_c being the matrix of discretization error in the factor model,

$$\frac{1}{p}\beta_d'(\widehat{\beta}_c - \beta_c H_c) = \frac{1}{k_n p}\beta_d'\overline{U}_c\overline{F}_c\widehat{A}_c + \frac{1}{pk_n}\frac{1}{p}\beta_d'\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}$$
$$+ \frac{1}{pk_n}\frac{1}{p}\beta_d'\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}\frac{1}{p}\beta_d' R_c\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}.$$

It is easy to see for the first two terms on the right-hand side of the above equality that

$$\frac{1}{k_n p}\beta_d' \overline{U}_c \overline{F}_c \widehat{A}_c \le \left\| \frac{1}{k_n p}\beta_d' \overline{U}_c \overline{F}_c \right\| \|\widehat{A}_c\| \le O_P\Big(\frac{1}{\sqrt{k_n p}} + \frac{\sqrt{\Delta_n}}{\sqrt{k_n}}\Big) O_P(1),$$

$$\frac{1}{pk_n}\frac{1}{p}\beta_d' \overline{U}_c \overline{U}_c' \widehat{\beta}_c \widehat{Q}_c^{-1} \le O_P(1) \left\| \frac{1}{p^2 k_n}\beta_d'\big(\overline{U}_c\overline{U}_c' - \mathbb{E}(\overline{U}_c\overline{U}_c'|\mathcal{C})\big)\right\| \|\widehat{\beta}_c - \beta_c H_c\|$$
$$+ O_P(1)\left\| \frac{1}{p^2 k_n}\beta_d'\big(\overline{U}_c\overline{U}_c' - \mathbb{E}(\overline{U}_c\overline{U}_c'|\mathcal{C})\big)\beta_c\right\| + O_P(p^{-1})$$
$$\le O_P\bigg(\Big(\frac{1}{p\sqrt{k_n}} + \frac{\Delta_n^{1/4}}{\sqrt{pk_n}}\Big)\|\widehat{\beta}_c - \beta_c H_c\| + \frac{\sqrt{\Delta_n}}{\sqrt{k_n}} + p^{-1}\bigg)$$
$$\le O_P\Big(\frac{1}{p} + \frac{1}{k_n\sqrt{p}} + \frac{\delta_4}{p\sqrt{k_n}} + \frac{\delta_4\Delta_n^{1/4}}{\sqrt{pk_n}} + \frac{\Delta_n^{1/4}}{k_n} + \frac{\zeta_p\Delta_n^{1/4}}{p\sqrt{k_n}} + \frac{\sqrt{\Delta_n}}{\sqrt{k_n}}\Big),$$

because $\mathbb{E}(\overline{U}_c\overline{U}_c'|\mathcal{C})$ is a diagonal matrix with bounded elements and by application of Lemma A.2(vi), (v), and (vi) as well as Lemma C.2. Combining these bounds and using the definition of δ_p, we get the first result of the lemma.

For the second result of the lemma, we have

$$\frac{1}{p}\widehat{\beta}_d'(\widehat{\beta}_c - \beta_c H_c) \le \frac{1}{p}\beta_d'(\widehat{\beta}_c - \beta_c H_c) + \frac{1}{p}\|\widehat{\beta}_c - \beta_c H_c\|^2.$$

From here, the result to be proved follows from the bound for the first result of the lemma derived above plus application of Lemma C.2. □

LEMMA C.4. *Suppose* $\frac{\sqrt{p}}{k_n} = O(\zeta_p^3)$ *and* $pk_n\Delta_n = O_p(1)$, *as* $p, n \to \infty$. *Let*

$$\delta_6 := \frac{1}{pk_n}\overline{F}_d'\overline{U}_d'\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}\overline{F}_d'\overline{U}_d' R_c \overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}$$

and $\Delta_{1c} := \frac{1}{pk_n}\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}R_c\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}$.

Under both null and alternatives, for $c, d \in \{a, b\}$,

$$\overline{F}_d'\overline{U}_d'\Delta_{1c} = O_P\Big(1 + \sqrt{\frac{k_n}{p}} + \delta_6\Big) + O_P\Big(1 + \frac{p}{k_n}\Big)\zeta_p + O_P\Big(\sqrt{\frac{p}{k_n}} + \sqrt{\frac{k_n}{p}}\Big)\delta_4\zeta_p$$
$$+ O_P\Big(\frac{1}{\sqrt{k_n}} + \frac{\sqrt{k_n}}{p}\Big)\zeta_p^2.$$

PROOF. Recall that $\overline{F}_d'\overline{U}_d'\Delta_{1c} = \frac{1}{pk_n}\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n}\overline{F}_d'\overline{U}_d'\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{pk_n} \times \overline{F}_d'\overline{U}_d' R_c\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}$.

First, $(\frac{1}{pk_n}\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{U}_c'\beta_c)^2 = O_P(1 + \frac{k_n}{p})$, $\|\overline{U}_c\overline{F}_c\| = O_P(\sqrt{k_n p})$, and $\|\overline{U}_c\| = O_P(\sqrt{(k_n + p)\zeta_p})$, by Lemma A.2 and because $pk_n\Delta_n = O_p(1)$. Hence, by using Lemma A.2, Lemma C.2, and the the expression (C.3) for $\widehat{\beta}_c - \beta_c H_c$, we have

$$\frac{1}{pk_n}\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{U}_c'\widehat{\beta}_c = \frac{1}{pk_n}\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{U}_c'\beta_c + \frac{1}{pk_n}\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{U}_c'(\widehat{\beta}_c - \beta_c H_c)$$

$$\leq \frac{1}{pk_n}\overline{F}'_d\overline{U}'_d\overline{U}_c\overline{U}'_c\beta_c + O_P(1)\frac{1}{pk_n^2}\|\overline{F}_c\overline{U}_c\|^2\|\overline{U}_c\|^2$$

$$+ \frac{1}{pk_n}\|\overline{F}_c\overline{U}_c\|\|\overline{U}_c\|^2\|\Delta_{1c}\|$$

$$\leq O_P\left(1+\sqrt{\frac{k_n}{p}}\right) + O_P\left(1+\frac{p}{k_n}\right)\zeta_p + O_P\left(\sqrt{\frac{p}{k_n}}+\sqrt{\frac{k_n}{p}}\right)\delta_4\zeta_p$$

$$+ O_P\left(\frac{1}{\sqrt{k_n}} + \frac{\sqrt{k_n}}{p}\right)\zeta_p^2. \qquad \square$$

LEMMA C.5. *Let $p \to \infty$, $\Delta_n \to 0$, $k_n \to \infty$, and $k_n\Delta_n \to 0$. Under both null and alternatives, we have for $c = a, b$,*

$$\frac{\delta_4^2}{p} = \frac{1}{p}\left\|\frac{1}{pk_n}\overline{U}_c R'_c\widehat{\beta}_c + \frac{1}{pk_n}R_c\overline{Y}'_c\widehat{\beta}_c\right\|^2 = O_P\left(\Delta_n^{2\widetilde{\varpi}}\right), \tag{C.19}$$

$$\frac{\|\delta_6\|}{pk_n} = \left\|\frac{1}{p^2k_n^2}\overline{F}'_c\overline{U}'_c\overline{U}_c R'_c\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{p^2k_n^2}\overline{F}'_c\overline{U}'_c R_c\overline{Y}'_c\widehat{\beta}_c\widehat{Q}_c^{-1}\right\| = O_P\left(\frac{\Delta_n^{\widetilde{\varpi}}}{\sqrt{k_n}}\right), \tag{C.20}$$

$$\|\delta_5\|^2 = \left\|\frac{1}{p^2k_n}\beta'_c\overline{U}_c R'_c\widehat{\beta}_c\widehat{Q}_c^{-1} + \frac{1}{p^2k_n}\beta'_c R_c\overline{Y}'_c\widehat{\beta}_c\widehat{Q}_c^{-1}\right\|^2 = O_P\left(\Delta_n^{2\widetilde{\varpi}}\right). \tag{C.21}$$

PROOF. First, we note that

$$\|\widehat{\beta}_c\|^2 = O_P(p), \quad \text{and} \quad \|\beta_c\|^2 = O_P(p), \tag{C.22}$$

from the assumption for β_c and the fact that each column of $\widehat{\beta}_c/\sqrt{p}$ is an eigenvector (and hence has a norm of 1). From here, all results follow by application of the Cauchy–Schwarz inequality and Lemmas A.1 and A.2. $\square$

LEMMA C.6. *Suppose $k_n \to \infty$, $p\zeta_p^8 = o(k_n^2)$, $k_n^2\zeta_p^8 = o(p^3)$, and $pk_n\Delta_n^{2\widetilde{\varpi}} \to 0$ as $p, n \to \infty$. Then, under both the null and alternatives, $k_n\sqrt{p}\Delta_5 = o_P(1)$.*

PROOF. From the definition of Δ_5 and since $\|\beta_c\|^2 + \|\widehat{\beta}_c\|^2 = O_P(p)$, it is easy to see that to bound it, it suffices to derive bounds for the following terms:

$$\frac{1}{p}\|\Delta_{1c}\|^2, \qquad \left\|\frac{1}{p}\widehat{\beta}'_c(\widehat{\beta}_d - \beta_d H_d)\right\|^2, \qquad \left\|\frac{1}{pk_n}\overline{F}'_c\overline{U}'_c\beta_d\right\|^2,$$

$$\left\|\frac{1}{k_n p}\overline{F}'_c\overline{U}'_c\Delta_{1d}\right\|, \qquad c, d \in \{a, b\},$$

provided $\|H_c\| + \|H_c^{-1}\| = O_P(1)$. These terms are bounded in Lemmas A.2, C.2, C.3, C.4, and $\|H_c\| + \|H_c^{-1}\| = O_P(1)$ is shown Lemma C.2.

Applying these lemmas, for $c, d \in \{a, b\}$, we get

$$\Delta_5 \leq O_P\left(\left\|\frac{1}{pk_n}\overline{F}'_c\overline{U}'_c\beta_d\right\|^2 + \left\|\frac{1}{k_n p}\overline{F}'_c\overline{U}'_c\Delta_{1d}\right\| + \frac{1}{p}\|\Delta_{1,c}\|^2 + \left\|\frac{1}{p}\widehat{\beta}'_c(\widehat{\beta}_d - \beta_d H_d)\right\|^2\right)$$

$$\le O_P\left(\frac{1}{k_n^2}+\frac{1}{p^2}+\frac{\delta_4^2}{p}+\frac{\delta_6}{k_n p}+\delta_5^2+\frac{\delta_4^4}{p^2}\right)+O_P\left(\frac{1}{k_n p}+\frac{1}{k_n^2}\right)\zeta_p+O_P\left(\frac{1}{k_n^{3/2}p}+\frac{1}{p^2}\right)\zeta_p^2$$
$$+O_P\left(\frac{1}{p^{1/2}k_n^{3/2}}+\frac{1}{k_n^{1/2}p^{3/2}}\right)\delta_4\zeta_p+O_P\left(\frac{\Delta_n}{k_n}\right),$$

where δ_4, δ_5, δ_6 are defined in the statements of Lemmas C.2, C.3, C.4 repeated here:

$$\begin{aligned}\delta_4 &:= \left\|\frac{1}{pk_n}\overline{U}_c R_c'\widehat{\beta}_c+\frac{1}{pk_n}R_c\overline{Y}_c'\widehat{\beta}_c\right\|,\\ \delta_5 &:= \frac{1}{pk_n}\frac{1}{p}\beta_c'\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1}+\frac{1}{pk_n}\frac{1}{p}\beta_c' R_c\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1},\\ \delta_6 &:= \frac{1}{pk_n}\overline{F}_c'\overline{U}_c'\overline{U}_c R_c'\widehat{\beta}_c\widehat{Q}_c^{-1}+\frac{1}{pk_n}\overline{F}_c'\overline{U}_c' R_c\overline{Y}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}.\end{aligned} \tag{C.23}$$

Hence, to show $\sqrt{p}k_n\Delta_5=o_P(1)$, it suffices to have $\zeta_p=o(\sqrt{p})$, $p\zeta_p^2=o(k_n^2)$, $k_n^2\zeta_p^4=o(p^3)$ and $pk_n\Delta_n=O_P(1)$ (implied by the requirements of the lemma), and in addition show that $\sqrt{p}k_n(\frac{1}{p^{1/2}k_n^{3/2}}+\frac{1}{k_n^{1/2}p^{3/2}})\delta_4\zeta_p=o(1)$ and $\sqrt{p}k_n(\frac{\delta_4^2}{p}+\frac{\delta_4^4}{p^2}+\frac{\delta_6}{k_n p}+\delta_5^2)=o_P(1)$. The last results follow by application of Lemma C.5 and the conditions $k_n\to\infty$, $k_n\Delta_n\to 0$ and $pk_n\Delta_n^{2\widetilde{\varpi}}\to 0$. □

LEMMA C.7. *Under both the null and alternatives, and under the same condition as in Lemma* C.6, $k_n\sqrt{p}\Delta_{5,\mathrm{mix}}=o_P(1)$. *Also,* $\frac{1}{\sqrt{p}}\|\widehat{\beta}_{\mathrm{mix},k}-\beta_{ab}H_{\mathrm{mix},k}\|=o_P(1)$, *for* $k=o,e$.

PROOF. The proof is the same as that of Lemma C.6, as the higher-order terms of $\Delta_{5,\mathrm{mix}}$ and Δ_5 are of the same type. In addition, exactly as the proof of Lemma C.2, we have

$$\frac{1}{\sqrt{p}}\|\widehat{\beta}_{\mathrm{mix},k}-\beta_{ab}H_{\mathrm{mix},k}\|\le\frac{1}{\sqrt{p}}O_P\left(\sqrt{\frac{p}{k_n}}+\frac{\zeta_p}{\sqrt{p}}+\delta_4\right)=o_P(1),\quad k=o,e. \qquad \square$$

C.3.2 *Higher-order terms II: Bias estimation* Recall the definitions: $\widehat{B}_c=\frac{2}{k_n}\mathrm{tr}(\widehat{Q}_c^{-1})\times\widehat{\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})}$ and $B_c=\frac{2}{k_n^2}\sum_{t=1}^{k_n}\mathrm{tr}\,\widehat{A}_c'\bar{f}_{c,t}\bar{f}_{c,t}'\widehat{A}_c\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})$. Here, $\widehat{B}_c$ is an estimate of B_c, where we estimate $\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})$ by

$$\widehat{\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})}:=\frac{1}{pk_n}\|\widehat{U}_c\|_F^2(1+K_c/k_n)+\frac{1}{p^2}\mathrm{tr}\big(\widehat{\beta}_c'\widehat{D}_c\widehat{\beta}_c\big),$$

with $\widehat{D}_c=\mathrm{diag}\{\widehat{\sigma}_{c,1}^2,\ldots,\widehat{\sigma}_{c,p}^2\}$ being the $p\times p$ diagonal matrix of estimates of the idiosyncratic variances, and K_c is the number of factors in period $c\in\{a,b\}$.

The goal of this section is to show that $\sqrt{p}k_n(\widehat{B}_c-B_c)=o_P(1)$, and $\sqrt{p}k_n(\widehat{B}_{\mathrm{mix},k}-B_{\mathrm{mix},k})=o_P(1)$. This is established in Lemma C.10 below, which uses the auxiliary results in Lemmas C.8 and C.9. Before giving these results, we provide the rationale behind $\widehat{\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})}$. A naive estimator of $\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})$ is $\frac{1}{pk_n}\|\widehat{U}_c\|_F^2$, which however underestimates the volatility because of a higher-order bias in $\frac{1}{pk_n}\|\widehat{U}_c\|_F^2$ for estimating $\frac{1}{pk_n}\|U_c\|_F^2$. This bias can be derived and estimated as follows. We have

$$\begin{aligned}
\overline{U}_c - \widehat{U}_c &= (\widehat{\beta}_c - \beta_c H_c)\widehat{F}_c' + \beta_c H_c \frac{1}{p}\widehat{\beta}_c'(\beta_c H_c - \widehat{\beta}_c)H_c^{-1}\overline{F}_c' + \beta_c H_c \frac{1}{p}\widehat{\beta}_c'\overline{U}_c + \beta_c H_c \frac{1}{p}\widehat{\beta}_c' R_c \\
&= g_1 + \cdots + g_6, \\
g_1 &= \frac{1}{k_n}\overline{U}_c\overline{F}_c\widehat{A}_c\widehat{F}_c', \\
g_2 &= \beta_c H_c \frac{1}{p}\widehat{\beta}_c'\overline{U}_c, \\
g_3 &= \frac{1}{pk_n}\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}\widehat{F}_c', \\
g_4 &= -\frac{1}{p^2k_n}\beta_c H_c\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}H_c^{-1}\overline{F}_c', \\
g_5 &= -\frac{1}{pk_n}\beta_c H_c\widehat{\beta}_c'\overline{U}_c\overline{F}_c\widehat{A}_c H_c^{-1}\overline{F}_c', \\
g_6 &= Rem_3,
\end{aligned} \tag{C.24}$$

where Rem_3 means remaining terms that depend on R_c. Hence,

$$\|\widehat{U}_c\|_F^2 - \|\overline{U}_c\|_F^2 = \sum_{d=1}^{6}\|g_d\|_F^2 + \sum_{d_1,d_2=1,\ldots,6:d_1\neq d_2}\operatorname{tr}\big(g_{d_1}' g_{d_2}\big) - \sum_{d=1}^{6} 2\operatorname{tr}\big(\overline{U}_c' g_d\big).$$

Here, $\|g_1\|_F^2 + \|g_2\|_F^2 - 2\operatorname{tr}(\overline{U}_c' g_1) - 2\operatorname{tr}(\overline{U}_c' g_2)$ is the leading term. To estimate its components, note that $\overline{F}_c\widehat{A}_c$ can be estimated by $\widehat{F}_c\widehat{Q}_c^{-1}$ and note the identity $\frac{1}{k_n}\widehat{F}_c'\widehat{F}_c = \widehat{Q}_c$. Hence, $\frac{1}{k_np}[\|g_1\|_F^2 + \|g_2\|_F^2 - 2\operatorname{tr}(\overline{U}_c' g_1) - 2\operatorname{tr}(\overline{U}_c' g_2)]$ can be estimated by

$$\delta_c := -\frac{K_c}{k_n}\frac{1}{pk_n}\|\widehat{U}_c\|_F^2 - \frac{1}{p^2}\operatorname{tr}\big(\widehat{\beta}_c'\widehat{D}_c\widehat{\beta}_c\big).$$

Therefore, we can correct the bias of $\|\widehat{U}_c\|_F^2$ by

$$\widehat{\mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big)} := \frac{1}{pk_n}\|\widehat{U}_c\|_F^2 - \delta_c = \frac{1}{pk_n}\|\widehat{U}_c\|_F^2(1 + K_c/k_n) + \frac{1}{p^2}\operatorname{tr}\big(\widehat{\beta}_c'\widehat{D}_c\widehat{\beta}_c\big). \tag{C.25}$$

LEMMA C.8. *Let $p\to\infty$, $\Delta_n\to 0$, $k_n\to\infty$, and $k_n\Delta_n\to 0$. Under both null and alternatives, we have for $c = a, b$,*

$$\frac{\sqrt{p}}{k_n^2p^2}\big\|\widehat{\beta}_c' R_c\overline{U}_c'\overline{U}_c\overline{F}_c\big\| + \frac{\sqrt{p}}{k_np^2}\big\|\widehat{\beta}_c' R_c\overline{U}_c'\beta_c\big\| = O_P\big(\Delta_n^{\widetilde{\varpi}}\big), \tag{C.26}$$

$$\frac{\sqrt{p}}{k_np^2}\big\|R_c'\widehat{\beta}_c\big\|^2 = O_P\big(\sqrt{p}\Delta_n^{2\widetilde{\varpi}}\big), \tag{C.27}$$

$$\frac{\sqrt{p}}{pk_n}\overline{F}_c' R_c'\widehat{\beta}_c = O_P\big(\sqrt{p}\Delta_n^{\widetilde{\varpi}}\big), \qquad \frac{\sqrt{p}}{k_n^2p^3}\big\|\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\overline{U}_c R_c'\widehat{\beta}_c\big\| = o_P(1), \tag{C.28}$$

$$\frac{\sqrt{p}}{k_n^2 p^3}\|\widehat{\beta}_c'\overline{U}_c\overline{U}_c'R_c\overline{Y}_c'\widehat{\beta}_c\| = O_P\left(\left(1+\sqrt{\frac{p}{k_n}}\right)\Delta_n^{\widetilde{\varpi}}\right), \tag{C.29}$$

$$\frac{\sqrt{p}}{k_n p^2}\left\|\widehat{\beta}_c'R_c\overline{U}_c'\left(\frac{1}{pk_n}\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}+\frac{1}{pk_n}\overline{U}_cR_c'\widehat{\beta}_c\widehat{Q}_c^{-1}+\frac{1}{pk_n}R_c\overline{Y}_c'\widehat{\beta}_cV_c^{-1}\right)\right\|$$
$$= o_P(1). \tag{C.30}$$

Proof. The proof of all results of the lemma follows by application of the Cauchy–Schwarz inequality and the bounds derived in the proof of Lemma A.1. □

Lemma C.9. *Suppose* $\zeta_p^2 p = o(k_n^3)$, $\zeta_n = o(p^{3/4})$, $\zeta_n = o(\sqrt{k_n p})$, *and* $p\Delta_n^{2\widetilde{\varpi}} \to 0$ *and* $pk_n\Delta_n = O_p(1)$. *Then* $\frac{\sqrt{p}}{p^2 k_n}\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c = o_P(1)$.

Proof. We have $\frac{\sqrt{p}}{p^2k_n}\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c \le v_1+v_2+v_3$ where

$$v_1 = \frac{\sqrt{p}}{p^2k_n}(\widehat{\beta}_c-\beta_cH_c)'\overline{U}_c\overline{U}_c'\widehat{\beta}_c + \frac{\sqrt{p}}{p^2k_n}H_c'\beta_c'\overline{U}_c\overline{U}_c'(\widehat{\beta}_c-\beta_cH_c),$$

$$v_2 = \frac{\sqrt{p}}{p^2k_n}H_c'\beta_c'\left(\overline{U}_c\overline{U}_c' - \mathbb{E}\overline{U}_c\overline{U}_c'|\mathcal{C}\right)\beta_cH_c,$$

$$v_3 = \frac{\sqrt{p}}{p^2k_n}H_c'\beta_c'\mathbb{E}\left(\overline{U}_c\overline{U}_c'|\mathcal{C}\right)\beta_cH_c.$$

For v_1, we apply Lemma C.2 and Cauchy–Schwarz,

$$v_1 \le O_P\left(\frac{1}{pk_n}\right)\|\overline{U}_c\|^2\|\widehat{\beta}_c-\beta_cH_c\| \le O_P\left(\frac{p+k_n}{pk_n}\right)\zeta_p\left(\sqrt{\frac{p}{k_n}}+\frac{\zeta_p}{\sqrt{p}}+\delta_4\right) = o_P(1).$$

For v_2, we apply Lemma A.2(vi) $\|\frac{1}{p^2k_n}\beta_d'(\overline{U}_c\overline{U}_c' - \mathbb{E}(\overline{U}_c\overline{U}_c'|\mathcal{C}))\beta_c\| = O_P(\frac{1}{p\sqrt{k_n}}+\frac{\sqrt{\Delta_n}}{\sqrt{k_n}})$. So, $v_2 = o_P(1)$. Finally, $v_3 = O_P(p^{-1/2})$. □

Lemma C.10. *Suppose* $k_n\to\infty$, $p\zeta_p^8 = o(k_n^2)$, $k_n^2\zeta_p^8 = o(p^3)$, $p\Delta_n^{2\widetilde{\varpi}}\to 0$, *and* $pk_n\Delta_n = O_p(1)$, *as* $p,n\to\infty$. *Under both null and alternatives,* $\sqrt{p}k_n\|\widehat{B}_c - B_c\| = o_P(1)$, *for* $c\in\{a,b\}$. *Also,* $\sqrt{p}k_n(\widehat{B}_{\text{mix},k}-B_{\text{mix},k}) = o_P(1)$ *for* $k\in\{o,e\}$.

Proof. Define

$$\widetilde{B}_c = \frac{2}{k_n^2}\sum_{t=1}^{k_n}\operatorname{tr}\widehat{Q}_c^{-1}\widehat{f}_{c,t}\widehat{f}_{c,t}'\widehat{Q}_c^{-1}\frac{1}{pk_n}\sum_{i=1}^{p}\sum_{s=1}^{k_n}\widehat{\epsilon}_{c,si}^2 = \frac{2}{k_n^3p}\operatorname{tr}\left(\widehat{Q}_c^{-1}\widehat{F}_c'\widehat{F}_c\widehat{Q}_c^{-1}\right)\|\widehat{U}_c\|_F^2.$$

We first show $\sqrt{p}k_n\|\widetilde{B}_c - B_c\| = o_P(1)$, and then show $\sqrt{p}k_n\|\widehat{B}_c-\widetilde{B}_c\| = o_P(1)$.

First, because $\bar{\epsilon}_{c,ti}^2$ are $\mathcal{C}$-conditionally cross-sectionally independent and given Assumptions SA1 and A2, we have

$$\frac{\sqrt{p}}{k_n}\sum_{t=1}^{k_n}\bar{f}_{c,t}\bar{f}_{c,t}'\frac{1}{k_np}\sum_{s=1}^{k_n}\sum_{i=1}^{p}\left(\mathbb{E}\left(\sigma_{c,1}^2|\mathcal{C}\right)-\bar{\epsilon}_{c,si}^2\right) = o_P(1).$$

For $\sqrt{p}k_n\|\widetilde{B}_c - B_c\| = o_P(1)$, it remains to show $\frac{\sqrt{p}}{k_n^2 p}(\widehat{A}_c\overline{F}'_c\overline{F}_c\widehat{A}'_c)\|\overline{U}_c\|_F^2 - \frac{\sqrt{p}}{k_n^2 p} \times (\widehat{Q}_c^{-1}\widehat{F}'_c\widehat{F}_c\widehat{Q}_c^{-1})\|\widehat{U}_c\|_F^2 = o_P(1)$.

The left-hand side is bounded by the sum of the following three terms:

$$a_1 = \frac{\sqrt{p}}{k_n^2 p}\|\overline{F}_c\widehat{A}_c - \widehat{F}_c\widehat{Q}_c^{-1}\|^2\|\overline{U}_c\|_F^2 + \frac{\sqrt{p}}{k_n^2 p}\|\widehat{F}_c\widehat{Q}_c^{-1}\|^2\|\overline{U}_c - \widehat{U}_c\|_F^2,$$

$$a_2 = 2\left\|\frac{\sqrt{p}}{k_n^2 p}(\widehat{A}'_c\overline{F}'_c - \widehat{Q}_c^{-1}\widehat{F}'_c)\overline{F}_c\widehat{A}_c\right\|\|\overline{U}_c\|_F^2,$$

$$a_3 = \frac{\sqrt{p}}{k_n^2 p}\|\widehat{F}_c\widehat{Q}_c^{-1}\|^2\sum_{i=1}^{p}\sum_{t=1}^{k_n}(\widehat{\epsilon}_{c,ti} - \overline{\epsilon}_{c,ti})\overline{\epsilon}_{c,ti}.$$

To proceed, note that $\widehat{F}_c = p^{-1}\overline{Y}'_c\widehat{\beta}_c$ implies $\widehat{F}_c\widehat{Q}_c^{-1} - \overline{F}_c\widehat{A}_c = \overline{U}'_c\widehat{\beta}_c\widehat{Q}_c^{-1}/p + R'_c\widehat{\beta}_c\widehat{Q}_c^{-1}/p$. Also, recall the expansion in (C.24). Then, for a_1, by using Lemma A.2, Lemma A.3, and Lemma C.2, we have

$$\begin{aligned}\frac{1}{pk_n}\|\widehat{U}_c - \overline{U}_c\|_F^2 &\le O_P(1)\frac{1}{p}\|\widehat{\beta}_c - \beta_c H_c\|^2 + O_P(1)\frac{1}{k_n}\left\|\frac{1}{p}\widehat{\beta}'_c R_c\right\|_F^2 \\ &\quad + O_P(1)\frac{1}{k_n p^2}\|\widehat{\beta}_c\|^2\|\overline{U}_c\|^2 \\ &\le O_P\left(\frac{\zeta_p^2}{k_n} + \frac{\zeta_p^2}{p} + \frac{\delta_4^2}{p}\right) + O_P(1)\frac{1}{k_n}\left\|\frac{1}{p}\widehat{\beta}'_c R_c\right\|_F^2, \end{aligned} \tag{C.31}$$

$$\frac{1}{k_n}\|\widehat{F}_c\widehat{Q}_c^{-1} - \overline{F}_c\widehat{A}_c\|^2 \le O_P\left(\frac{\zeta_p^2}{k_n} + \frac{\zeta_p^2}{p}\right) + O_P(1)\frac{1}{k_n}\left\|\frac{1}{p}\widehat{\beta}'_c R_c\right\|_F^2. \tag{C.32}$$

Therefore, with $\zeta_p^2 = o(\sqrt{p} \wedge \frac{k_n}{\sqrt{p}})$, which is implied by the conditions in the statement of the lemma, we have $a_1 = o_P(1)$.

For a_2, we note $\frac{1}{k_n p}\|\overline{U}_c\|_F^2 = O_P(1)$. Also, $\widehat{\beta}_c - \beta_c H_c = \frac{1}{k_n}\overline{U}_c\overline{F}_c\widehat{A}_c + \Delta_{1c}$. Lemma C.4 showed $\frac{\sqrt{p}}{k_n p}\|\overline{F}'_c\overline{U}'_c\Delta_{1c}\| = o_P(1)$ under the conditions of the current lemma. Also, Lemma C.8 showed $O_P(\frac{\sqrt{p}}{k_n p})\|\overline{F}'_c R'_c\widehat{\beta}_c\| = o_P(1)$ because $\sqrt{p}\Delta_n^{\widetilde{\varpi}} \to 0$. Then combined with Lemma A.2, and under the condition that $p = o(k_n^2)$,

$$\begin{aligned} a_2 &\le O_P(\sqrt{p})\left\|\frac{1}{k_n}(\widehat{A}'_c\overline{F}'_c - \widehat{Q}_c^{-1}\widehat{F}'_c)\overline{F}_c\right\| \le O_P\left(\frac{\sqrt{p}}{k_n p}\right)\|\overline{F}'_c\overline{U}'_c\widehat{\beta}_c\| + O_P\left(\frac{\sqrt{p}}{k_n p}\right)\|\overline{F}'_c R'_c\widehat{\beta}_c\| \\ &\le O_P\left(\frac{\sqrt{p}}{k_n p}\right)\|\overline{F}'_c\overline{U}'_c\beta_c\| + O_P\left(\frac{\sqrt{p}}{k_n^2 p}\right)\|\overline{F}'_c\overline{U}'_c\|^2 + \frac{\sqrt{p}}{k_n p}\|\overline{F}'_c\overline{U}'_c\Delta_{1c}\| + o_P(1) = o_P(1). \end{aligned}$$

Finally, for a_3, we need $\frac{\sqrt{p}}{k_n p}\operatorname{tr}[(\widehat{U}_c - \overline{U}_c)'\overline{U}_c] = o_P(1)$, which is bounded using (C.24),

$$a_3 \le O_P\left(\frac{\sqrt{p}}{k_n p}\right)\operatorname{tr}\big[(\widehat{U}_c - \overline{U}_c)'\overline{U}_c\big]$$

$$\begin{aligned}
&\leq o_P(1)\frac{\sqrt{p}}{k_n^2 p}\|\overline{F}_c'\overline{U}_c'\overline{U}_c\overline{F}_c\| + o_P(1)\frac{\sqrt{p}}{k_n p}\|\overline{F}_c'\overline{U}_c'\Delta_{1c}\| \\
&\quad + O_P\left(\frac{1}{\sqrt{k_n}}\right)\left\|\frac{1}{p}\widehat{\beta}_c'(\beta_c H_c - \widehat{\beta}_c)\right\| + o_P(1)\frac{\sqrt{p}}{k_n p^2}\|\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c\| \\
&\quad + o_P(1)\frac{\sqrt{p}}{k_n p}\left\|\frac{1}{p}\widehat{\beta}_c' R_c\overline{U}_c'\beta_c\right\| \\
&\quad + o_P(1)\frac{\sqrt{p}}{k_n p^2}\|\widehat{\beta}_c' R_c\overline{U}_c'(\widehat{\beta}_c - \beta_c H_c)\| \\
&\leq o_P(1) + O_P(1)\frac{\sqrt{p}}{k_n^2 p^2}\|\widehat{\beta}_c' R_c\overline{U}_c'\overline{U}_c\overline{F}_c\| + O_P(1)\frac{\sqrt{p}}{k_n p^2}\|\widehat{\beta}_c' R_c\overline{U}_c'\beta_c\| \\
&\quad + O_P(1)\frac{\sqrt{p}}{k_n p^2}\|\widehat{\beta}_c' R_c\overline{U}_c'\Delta_{1c}\| + O_P(1)\frac{\sqrt{p}}{k_n^2 p^3}\|\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\overline{U}_c R_c'\widehat{\beta}_c\| \\
&\quad + O_P(1)\frac{\sqrt{p}}{k_n^2 p^3}\|\widehat{\beta}_c'\overline{U}_c\overline{U}_c' R_c\overline{Y}_c'\widehat{\beta}_c\|.
\end{aligned}$$

Here, we used the Lemma C.9, showing $\frac{\sqrt{p}}{p^2 k_n}\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c \leq o_P(1)$, under the conditions of the current lemma. Also, Lemma C.3 showed

$$\begin{aligned}
&\frac{1}{p\sqrt{k_n}}\widehat{\beta}_c'(\widehat{\beta}_c - \beta_c H_c) \\
&\quad \leq O_P\left(\frac{1}{p\sqrt{k_n}} + \frac{\delta_4}{pk_n} + \frac{\delta_4\Delta_n^{1/4}}{\sqrt{p}k_n} + \frac{\zeta_p\Delta_n^{1/4}}{pk_n} + \frac{\sqrt{\Delta_n}}{k_n} + \frac{1}{k_n^{3/2}} + \frac{\delta_4^2}{p\sqrt{k_n}} + \frac{\delta_5}{\sqrt{k_n}}\right) = o_P(1),
\end{aligned}$$

with the last result due to the conditions of the current lemma. The asymptotic negligibility of a_3 then follows by application of Lemma C.8 provided $p\Delta_n^{2\widetilde{\varpi}}/k_n \to 0$.

To show $\sqrt{p}k_n\|\widehat{B}_c - \widetilde{B}_c\| = o_P(1)$, note that

$$\begin{aligned}
\widetilde{B}_c &= \frac{2}{k_n^3 p}\,\mathrm{tr}\big(\widehat{Q}_c^{-1}\widehat{F}_c'\widehat{F}_c\widehat{Q}_c^{-1}\big)\|\widehat{U}_c\|_F^2, \\
\widehat{B}_c &= \frac{2}{k_n^2}\,\mathrm{tr}\big(\widehat{Q}_c^{-1}\widehat{F}_c'\widehat{F}_c\widehat{Q}_c^{-1}\big)\widehat{\mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big)}, \quad c \in \{a, b\}.
\end{aligned}$$

From Lemma A.3, $\|\widehat{Q}_c^{-1}\| = O_P(1)$ and together with the identity $\frac{1}{k_n}\widehat{F}_c'\widehat{F}_c = \widehat{Q}_c$, we have $\frac{1}{k_n}\mathrm{tr}(\widehat{Q}_c^{-1}\widehat{F}_c'\widehat{F}_c\widehat{Q}_c^{-1}) = O_P(1)$. Also, by (C.25), $\widehat{\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})} := \frac{1}{pk_n}\|\widehat{U}_c\|_F^2 - \delta_c$, where $|\delta_c| = O_P(\frac{1}{k_n} + \frac{1}{p})$. Hence, using (C.31), Lemma C.2 and Lemma C.8, we have

$$\begin{aligned}
\sqrt{p}k_n\|\widehat{B}_c - \widetilde{B}_c\| &= \frac{2k_n\sqrt{p}}{k_n^2}\,\mathrm{tr}\big(\widehat{Q}_c^{-1}\widehat{F}_c'\widehat{F}_c\widehat{Q}_c^{-1}\big)\left|\frac{1}{pk_n}\|\widehat{U}_c\|_F^2 - \widehat{\mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big)}\right| \\
&\leq O_P(\sqrt{p})\left|\frac{1}{pk_n}\|\widehat{U}_c\|_F^2 - \widehat{\mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big)}\right| = O_P(\sqrt{p}\delta_c) \\
&\leq O_P(\sqrt{p})O_P\left(\frac{1}{k_n} + \frac{1}{p}\right) = o_P(1).
\end{aligned}$$

As for $\sqrt{p}k_n\|\widehat{B}_{\mathrm{mix},k}-B_{\mathrm{mix},k}\|$, note that

$$B_{\mathrm{mix},k}=\frac{2}{k_n^2}\operatorname{tr}\widehat{A}'_{\mathrm{mix},k}\big[F'_{a,k}F_{a,k}\mathbb{E}(\sigma^2_{a,1}|\mathcal{C})+H'F'_{b,k}F_{b,k}H\mathbb{E}(\sigma^2_{b,1}|\mathcal{C})\big]\widehat{A}_{\mathrm{mix}},$$

$$\widehat{B}_{\mathrm{mix},k}=\frac{2}{k_n^2}\operatorname{tr}\widehat{Q}^{-1}_{\mathrm{mix},k}\widehat{F}'_{a,k}\widehat{F}_{a,k}\widehat{Q}^{-1}_{\mathrm{mix},k}\mathbb{E}\widehat{(\sigma^2_{a,1}|\mathcal{C})}+\frac{2}{k_n^2}\operatorname{tr}\widehat{Q}^{-1}_{\mathrm{mix},k}\widehat{F}'_{b,k}\widehat{F}_{b,k}\widehat{Q}^{-1}_{\mathrm{mix},k}\mathbb{E}\widehat{(\sigma^2_{b,1}|\mathcal{C})}.$$

Also, $A_{\mathrm{mix},k}=\frac{1}{p}\beta'_a\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k}$ and $\widehat{F}_{c,k}=p^{-1}\overline{Y}'_{c,k}\widehat{\beta}_{\mathrm{mix},k}$ imply

$$\widehat{F}_{c,k}\widehat{Q}^{-1}_{\mathrm{mix},k}-\overline{F}_{c,k}\frac{1}{p}\beta'_c\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k}=\frac{1}{p}\overline{U}'_{c,k}\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k}+\frac{1}{p}R'_{c,k}\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k},$$

where $F_{a,k}\frac{1}{p}\beta'_a\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k}=F_{a,k}\widehat{A}_{\mathrm{mix},k}$ and $F_{b,k}\frac{1}{p}\beta'_b\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k}=F_{b,k}H'\widehat{A}_{\mathrm{mix},k}$ when $\beta_b=\beta_aH$. From here, the proof of $\sqrt{p}k_n\|\widehat{B}_{\mathrm{mix},k}-B_{\mathrm{mix},k}\|$ follows from the same arguments, so we omit it for brevity. □

C.4 *Asymptotic null distribution*

Lemmas C.6, C.7, and C.10 show that $k_n\sqrt{p}\Delta_5=o_P(1)$, $k_n\sqrt{p}\Delta_{5,\mathrm{mix}}=o_P(1)$, $\sqrt{p}k_n(\widehat{B}_c-B_c)=o_P(1)$ and $\sqrt{p}k_n(\widehat{B}_{\mathrm{mix},k}-B_{\mathrm{mix},k})=o_P(1)$, for $c\in\{a,b\}$ and $k\in\{o,e\}$. It then follows from (C.14) that

$$\begin{aligned}&\sqrt{p}k_n\|P_{\widehat{\beta}_a}-P_{\widehat{\beta}_b}\|_F^2-(\widehat{B}_a+\widehat{B}_b)-\sqrt{p}k_n\widehat{A}_{\mathrm{mix}}\\&\quad=\sqrt{p}k_n\big[\widetilde{\mu}_a+\widetilde{\mu}_b-\widehat{\mu}_{ab}-(\mu_{\mathrm{mix},o}-B_{\mathrm{mix},o})-(\mu_{\mathrm{mix},e}-B_{\mathrm{mix},e})-\mu_{\mathrm{mix},oe}\big]\\&\qquad+o_P(1),\end{aligned}\tag{C.33}$$

where we recall here the definitions of these terms:

$$\widetilde{\mu}_c=\frac{2}{pk_n^2}\operatorname{tr}\widehat{A}'_c\left[\overline{F}'_c\overline{U}'_c\overline{U}_c\overline{F}_c-\sum_{i=1}^{p}\mathbb{E}(\sigma^2_{c,1}|\mathcal{C})\overline{F}'_c\overline{F}_c\right]\widehat{A}_c,\quad c\in\{a,b\},$$

$$\widehat{\mu}_{ab}=\frac{2}{pk_n^2}\operatorname{tr}\widehat{A}'_a\overline{F}'_a\overline{U}'_a\overline{U}_b\overline{F}_b\widehat{A}_b\widehat{G},$$

$$\mu_{\mathrm{mix},k}=\frac{2}{pk_n^2}\operatorname{tr}\widehat{A}'_{\mathrm{mix},k}\big[\overline{F}'_{\mathrm{mix},k}\overline{U}'_{\mathrm{mix},k}\overline{U}_{\mathrm{mix},k}\overline{F}_{\mathrm{mix},k}\big]\widehat{A}_{\mathrm{mix},k},$$

$$\mu_{\mathrm{mix},oe}=\frac{2}{pk_n^2}\operatorname{tr}\widehat{A}'_{\mathrm{mix},o}\overline{F}'_{\mathrm{mix},o}\overline{U}'_{\mathrm{mix},o}\overline{U}_{\mathrm{mix},e}\overline{F}_{\mathrm{mix},e}\widehat{A}_{\mathrm{mix},e}\widehat{G}_{\mathrm{mix}},$$

$$B_{\mathrm{mix},k}=\frac{2}{k_n^2}\operatorname{tr}\widehat{A}'_{\mathrm{mix},k}\big[F'_{a,k}F_{a,k}\mathbb{E}(\sigma^2_{a,1}|\mathcal{C})+H'F'_{b,k}F_{b,k}H\mathbb{E}(\sigma^2_{b,1}|\mathcal{C})\big]\widehat{A}_{\mathrm{mix}}.$$

We now derive the asymptotic distribution of the leading term. Using the notation in (C.41),

$$\sqrt{p}k_n\big[\widetilde{\mu}_a+\widetilde{\mu}_b-\widehat{\mu}_{ab}-(\mu_{\mathrm{mix},o}-B_{\mathrm{mix},o})-(\mu_{\mathrm{mix},e}-B_{\mathrm{mix},e})-\mu_{\mathrm{mix},oe}\big]$$

$$
\begin{aligned}
&= 2\widehat{Z}_a(\widehat{A}_a) + 2\widehat{Z}_a(\widehat{A}_a) - 2\widehat{Z}_{ab}(\widehat{A}_a, \widehat{A}_b\widehat{G}) - 2\widehat{Z}_{\mathrm{mix},o}(\widehat{A}_{\mathrm{mix},o}, H\widehat{A}_{\mathrm{mix},o}) \\
&\quad - 2\widehat{Z}_{\mathrm{mix},e}(\widehat{A}_{\mathrm{mix},e}, H\widehat{A}_{\mathrm{mix},e}) \\
&\quad + 2\widehat{Z}_{\mathrm{mix}}(\widehat{A}_{\mathrm{mix},o}, H\widehat{A}_{\mathrm{mix},o}, \widehat{A}_{\mathrm{mix},e}\widehat{G}_{\mathrm{mix}}, H\widehat{A}_{\mathrm{mix},e}\widehat{G}_{\mathrm{mix}}). \qquad \text{(C.34)}
\end{aligned}
$$

Also, recall that the left-hand side is $\sqrt{p}k_n\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 - (\widehat{B}_a + \widehat{B}_b) - \sqrt{p}k_n\widehat{\mathcal{A}}_{\mathrm{mix}} + o_P(1)$.

Lemma C.11 below shows that $\widehat{A}_c \xrightarrow{\mathbb{P}} \bar{A}_c$, $\widehat{G} \xrightarrow{\mathbb{P}} \bar{G}$, $\widehat{A}_{\mathrm{mix},k} \xrightarrow{\mathbb{P}} \bar{A}_{\mathrm{mix}}$, and $\widehat{G}_{\mathrm{mix},k} \xrightarrow{\mathbb{P}} 2I$, for some $\bar{A}_c$, $\bar{G}$, and $\bar{A}_{\mathrm{mix}}$. In particular, $\bar{A}_{\mathrm{mix}}$ does not depend on k. Hence, by Lemma C.12 below, (C.34) also holds up to $o_P(1)$ term if on the right-hand side $(\widehat{A}_c, \widehat{G}, \widehat{A}_{\mathrm{mix},k}, \widehat{G}_{\mathrm{mix}})$ is replaced by $(\bar{A}_c, \bar{G}, \bar{A}_{\mathrm{mix}}, 2I)$. That is,

$$
\begin{aligned}
&\sqrt{p}k_n\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 - (\widehat{B}_a + \widehat{B}_b) - \sqrt{p}k_n\widehat{\mathcal{A}}_{\mathrm{mix}} \\
&\quad = 2\widehat{Z}_a(\bar{A}_a) + 2\widehat{Z}_a(\bar{A}_a) - 2\widehat{Z}_{ab}(\bar{A}_a, \bar{A}_b\bar{G}) - 2\widehat{Z}_{\mathrm{mix},o}(\bar{A}_{\mathrm{mix}}, H\bar{A}_{\mathrm{mix}}) \\
&\qquad - 2\widehat{Z}_{\mathrm{mix},e}(\bar{A}_{\mathrm{mix}}, H\bar{A}_{\mathrm{mix}}) + 2\widehat{Z}_{\mathrm{mix}}(\bar{A}_{\mathrm{mix}}, H\bar{A}_{\mathrm{mix}}, 2\bar{A}_{\mathrm{mix}}, 2HA_{\mathrm{mix}}) + o_P(1) \\
&\quad = \frac{1}{\sqrt{p}}\sum_{i=1}^{p} z_{i,n} + o_P(1), \qquad \text{(C.35)}
\end{aligned}
$$

for some $z_{i,n}$. Lemma C.12 below implies

$$
\frac{1}{\sqrt{p}}\sum_{i=1}^{p} z_{i,n} \xrightarrow{\mathcal{L}|\mathcal{F}} \sqrt{\mathcal{V}}Z, \qquad \text{(C.36)}
$$

where Z is a standard normal random variable defined on an extension of the original probability space and independent of $\mathcal{F}$ and $\mathcal{V}$ is some $\mathcal{C}$-adapted strictly positive random variable.

C.4.1 *An auxiliary probability bound* We restate here some notation that will be used in showing the next lemma. We define

$$
\begin{aligned}
\widehat{A}_c &= \frac{1}{p}\beta_c'\widehat{\beta}_c\widehat{Q}_c^{-1}, \\
\widehat{G} &= \frac{1}{p}\widehat{\beta}_b'\widehat{\beta}_a + H_b^{-1}\left(\frac{1}{p}\beta_b'\beta_b\right)^{-1}\beta_b'\beta_a\left(\beta_a'\beta_a\right)^{-1}H_a'^{-1}, \\
\widehat{A}_{\mathrm{mix},k} &= \frac{1}{p}\beta_a'\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}_{\mathrm{mix},k}^{-1}, \\
\widehat{G}_{\mathrm{mix}} &= \frac{2}{p}\widehat{\beta}_{\mathrm{mix},o}'\widehat{\beta}_{\mathrm{mix},e}, \\
J_n &:= \frac{1}{p}\beta_c'\widehat{\beta}_c = \widehat{A}_c\widehat{Q}_c, \\
\Sigma_{f,c} &:= \Lambda_c'\Lambda_c, \\
D_n &= K \times K \text{ diagonal matrix of the diagonal elements of } J_n'\Lambda_c'\Lambda_c J_n,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{Q}_c &= K \times K \text{ diagonal matrix of top } K \text{ eigenvalues of } \overline{Y}_c \overline{Y}_c' / (pk_n), \\
\bar{Q}_c &= K \times K \text{ diagonal matrix of top } K \text{ eigenvalues of } \frac{1}{p} \beta_c \Lambda_c' \Lambda_c \beta_c', \\
\bar{Q}_c^* &= K \times K \text{ diagonal matrix of top } K \text{ eigenvalues of } \Sigma_{f,c}^{1/2} \Sigma_{\beta,c} \Sigma_{f,c}^{1/2}, \\
\bar{Q}_{\text{mix}} &= K \times K \text{ diagonal matrix of top } K \text{ eigenvalues of} \\
&\quad \Sigma_{\beta,a}^{1/2} (0.5 \Sigma_{f,a} + 0.5 H \Sigma_{f,b} H') \Sigma_{\beta,a}^{1/2}.
\end{aligned}
$$

Lemma C.11. *Under the null hypothesis, provided $\zeta_p / p \to 0$ and $pk_n \Delta_n = O_p(1)$ as $p, n \to \infty$, we have*

(1) $\|\widehat{A}_c - \bar{A}_c\| + \|\widehat{G} - \bar{G}\| + \|\frac{2}{p} \widehat{\beta}_b' \widehat{\beta}_a - \bar{G}\| = O_P(\widehat{T}_n + \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{p}} + \frac{\zeta_p}{p}) = o_P(1)$, *for some* $(\bar{A}_c, \bar{G})$ *adapted to* $\mathcal{C}$, *where*

$$
\begin{aligned}
\widehat{T}_n &= \left\| \frac{1}{p} \beta_c \Sigma_{f,c} \beta_c' - \frac{1}{pk_n} \overline{Y}_c \overline{Y}_c' \right\| \\
&\leq O_P\left(\frac{1}{\sqrt{k_n}} + \sqrt{\frac{k_n}{n}} + \frac{1}{\sqrt{p}} \left\| \frac{1}{k_n} R_c \overline{F}_c \right\| + \frac{1}{pk_n} \|\overline{U}_c \overline{Y}_c'\| + \frac{1}{pk_n} \|R_c \overline{Y}_c'\| \right).
\end{aligned}
$$

(2) $\widehat{A}_{\text{mix},k} \xrightarrow{\mathbb{P}} \bar{A}_{\text{mix}}$ *and* $\widehat{G}_{\text{mix},k} \xrightarrow{\mathbb{P}} 2I$, *for an* $\bar{A}_{\text{mix}}$ *adapted to* $\mathcal{C}$.

Proof. (1) Note that the top K eigenvalues of $\frac{1}{p} \beta_c \Lambda_c' \Lambda_c \beta_c'$ are also those of $\Sigma_{f,c}^{1/2} \frac{1}{p} \beta_c' \beta_c \times \Sigma_{f,c}^{1/2}$. Also from Assumptions SA1 and A2, we have $\|\frac{1}{p} \beta_c' \beta_c - \Sigma_{\beta,c}\| = O_P(p^{-1/2})$. Hence, as the proof of Lemma A.3, we have

$$
\|\widehat{Q}_c - \bar{Q}_c^*\| \leq \|\bar{Q}_c - \bar{Q}_c^*\| + \|\widehat{Q}_c - \bar{Q}_c\| \leq \|\Sigma_{f,c}\| \left\| \frac{1}{p} \beta_c' \beta_c - \Sigma_{\beta,c} \right\| + \widehat{T}_n \leq O_P\big(\widehat{T}_n + p^{-1/2}\big).
$$

Meanwhile, $\frac{1}{k_n} \overline{F}_c' \overline{F}_c = \Sigma_{f,c} + O_P(k_n^{-1/2} + \sqrt{k_n/n})$. Hence,

$$
H_c = \frac{1}{k_n p} \overline{F}_c' \overline{Y}_c' \widehat{\beta}_c \widehat{Q}_c^{-1} = \Sigma_{f,c} \widehat{A}_c + O_P\left(\frac{1}{\sqrt{k_n}} + \sqrt{\frac{k_n}{n}} + \left\| \frac{1}{k_n p} \overline{F}_c' R_c' \widehat{\beta}_c \right\| \right).
$$

This implies that singular values of $\widehat{A}_c$ are bounded away from zero and infinity.

We now show that the eigenvalues of $J_n' \Sigma_{f,c} J_n$ converge in probability. We have

$$
J_n' \Sigma_{f,c} J_n = \frac{1}{p^2 k_n} \widehat{\beta}_c' \overline{Y}_c \overline{Y}_c' \widehat{\beta}_c + o_P(\widehat{T}_n) = \widehat{Q}_c + O_P(\widehat{T}_n) = \bar{Q}_c^* + O_P\big(\widehat{T}_n + p^{-1/2}\big).
$$

Then $\|D_n - \bar{Q}_c^*\| \leq O_P(\widehat{T}_n + p^{-1/2})$. We now prove the convergence of J_n following the same argument as in Bai (2003). First, singular values of J_n are bounded away from zero, which follows from the fact that singular values of $\widehat{A}_c$ and $\widehat{Q}_c$ are bounded away from zero. From $\frac{1}{pk_n} \overline{Y}_c \overline{Y}_c' \widehat{\beta}_c = \widehat{\beta}_c \widehat{Q}_c$, left multiply $\frac{1}{p} \beta_c'$,

$$
\left[\frac{1}{p} \beta_c' \beta_c \Sigma_{f,c} + J_n^{-1} O_P(\widehat{T}_n) \right] J_n = J_n \widehat{Q}_c.
$$

Note that each column of $\Sigma_{f,c}^{1/2} J_n D_n^{-1/2}$ is a unit vector (whose Euclidean norm is one), so that they are also eigenvectors. Also, D_n^{-1} and $\widehat{Q}_c$ are commutable because both are diagonal. Thus, left multiply by $\Sigma_{f,c}^{1/2}$ and right multiply by $D_n^{-1/2}$,

$$\left[\Sigma_{f,c}^{1/2}\frac{1}{p}\beta_c'\beta_c\Sigma_{f,c}^{1/2} + J_n^{-1}O_P(\widehat{T}_n)\right]\Sigma_{f,c}^{1/2}J_nD_n^{-1/2} = \Sigma_{f,c}^{1/2}J_n\widehat{Q}_cD_n^{-1/2} = \Sigma_{f,c}^{1/2}J_nD_n^{-1/2}\widehat{Q}_c.$$

Then by the assumption that $\bar{Q}_c^*$ has distinct diagonal elements, the sin-theta theorem implies $\|\Sigma_{f,c}^{1/2}J_nD_n^{-1/2} - M_c\| = O_P(\widehat{T}_n + p^{-1/2})$ where columns of M_c are the eigenvectors of $\Sigma_{f,c}^{1/2}\Sigma_{\beta,c}\Sigma_{f,c}^{1/2}$. So, $\|J_n - \Sigma_{f,c}^{-1/2}M_c\bar{Q}_c^{1/2}\| = O_P(\widehat{T}_n)$. Recall that $\widehat{A}_c = J_n\widehat{Q}_c^{-1}$. Hence,

$$\|\widehat{A}_c - \bar{A}_c\| = O_P\big(\widehat{T}_n + p^{-1/2}\big), \qquad \bar{A}_c = \Sigma_{f,c}^{-1/2}M_c\bar{Q}_c^{*-1/2}. \tag{C.37}$$

Finally, we bound $\widehat{G}$. Lemma C.2 implies, for δ_4 defined in (C.17),

$$\left\|\frac{2}{p}\widehat{\beta}_b'\widehat{\beta}_a - 2H_b'\Sigma_{\beta,ba}H_a\right\| + \left\|\frac{2}{p}\widehat{\beta}_b'\widehat{\beta}_a - \widehat{G}\right\| = O_P\left(\frac{1}{\sqrt{k_n}} + \frac{\zeta_p}{p} + \frac{\delta_4}{\sqrt{p}} + \frac{1}{\sqrt{p}}\right),$$

where $\Sigma_{\beta,ba}$ is the probability limit of $\frac{1}{p}\beta_b'\beta_a$. Meanwhile,

$$\begin{aligned}\|H_c - \Sigma_{f,c}\bar{A}_c\| &= \|\Sigma_{f,c}\widehat{A}_c - \Sigma_{f,c}\bar{A}_c\| + \|H_c - \Sigma_{f,c}\widehat{A}_c\| \\ &= O_P\left(\frac{1}{\sqrt{k_n}} + \left\|\frac{1}{k_np}\overline{F}_c'R_c'\widehat{\beta}_c\right\| + \widehat{T}_n + \frac{1}{\sqrt{p}}\right).\end{aligned} \tag{C.38}$$

This implies

$$\left\|\frac{2}{p}\widehat{\beta}_b'\widehat{\beta}_a - \bar{G}\right\| + \|\bar{G} - \widehat{G}\| = O_P\left(\frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{p}} + \left\|\frac{1}{k_np}\overline{F}_c'R_c'\widehat{\beta}_c\right\| + \widehat{T}_n + \frac{\zeta_p}{p} + \frac{\delta_4}{\sqrt{p}}\right) = o_P(1),$$

where $\bar{G} = 2\bar{A}_b'\Sigma_{f,c}\Sigma_{\beta,ba}\Sigma_{f,c}\bar{A}_a$. The last result above follows by applying Lemma A.1 and Lemma A.2, and making use of $\zeta_p/p \to 0$, which is assumed in the statement of the lemma.

(3) Recall that $\widehat{Q}_{\text{mix},k}$ contains top eigenvalues of the sample covariance from $(Y_{a,k}, Y_{b,k})$, which are equal to the top K eigenvalues of $\frac{1}{2p}\beta_a\Sigma_{f,a}\beta_a' + \frac{1}{2p}\beta_b\Sigma_{f,b}\beta_b'$ up to $o_P(1)$. Under the null hypothesis, they also converge to the distinct eigenvalues of $\bar{Q}_{\text{mix}}$. Thus, we have proved $\widehat{Q}_{\text{mix},k} \xrightarrow{\mathbb{P}} \bar{Q}_{\text{mix}}$. These eigenvalues are also bounded away from zero and infinity so long as those of $\Sigma_{f,b}$, $\Sigma_{\beta,a}$, and H do.

Under the null, $\frac{1}{2p}\beta_a\Sigma_{f,a}\beta_a' + \frac{1}{2p}\beta_b\Sigma_{f,b}\beta_b' = \beta_a\Sigma_{f,\text{mix}}\beta_a'$ where $\Sigma_{f,\text{mix}} := 0.5\Sigma_{f,a} + 0.5H\Sigma_{f,b}H'$. Then the same argument for $\|J_n - \Sigma_{f,c}^{-1/2}M_c\bar{Q}_c^{1/2}\| = O_P(\widehat{T}_n)$ in part (1) can be repeated here to show

$$\left\|\frac{1}{p}\beta_a'\widehat{\beta}_{\text{mix},k} - \Sigma_{f,\text{mix}}^{-1/2}M_{\text{mix}}\bar{Q}_{\text{mix}}^{1/2}\right\| = o_P(1), \tag{C.39}$$

where the columns of M_{mix} are the eigenvectors of $\Sigma_{f,\text{mix}}^{1/2}\Sigma_{\beta,a}\Sigma_{f,\text{mix}}^{1/2}$. Hence, under the null,

$$\widehat{A}_{\text{mix},k} = \frac{1}{p}\beta_a'\widehat{\beta}_{\text{mix},k}\widehat{Q}_{\text{mix},k}^{-1} \xrightarrow{\mathbb{P}} \bar{A}_{\text{mix}} := \Sigma_{f,\text{mix}}^{-1/2}M_{\text{mix}}\bar{Q}_{\text{mix}}^{-1/2}. \tag{C.40}$$

To find the probability limit of $\widehat{G}_{\text{mix},k}$, we recall $H_{c,\text{mix},k} = \frac{1}{p_n k_n}\overline{F}_{c,k}'\overline{Y}_{c,k}'\widehat{\beta}_{\text{mix},k}\widehat{Q}_{\text{mix},k}^{-1}$, and $L_k := (H_{a,\text{mix},k} + HH_{b,\text{mix},k})$. Then $H_{c,\text{mix},k} = 0.5\Sigma_{f,c}\frac{1}{p_n}\beta_c'\widehat{\beta}_{\text{mix},k}\widehat{Q}_{\text{mix},k}^{-1} + o_P(1)$, which with (C.39) imply

$$H_{a,\text{mix},k} = 0.5\Sigma_{f,a}\frac{1}{p_n}\beta_a'\widehat{\beta}_{\text{mix},k}\widehat{Q}_{\text{mix},k}^{-1} + o_P(1) \xrightarrow{\mathbb{P}} 0.5\Sigma_{f,a}\bar{A}_{\text{mix}},$$

$$H_{b,\text{mix},k} \xrightarrow{\mathbb{P}} 0.5\Sigma_{f,b}H'\bar{A}_{\text{mix}}.$$

This shows that L_k converges in probability to some $\bar{L}$ that does not depend on $k \in \{o, e\}$. From (C.12),

$$o_P(1) = \frac{1}{\sqrt{p}}\|\widehat{\beta}_{\text{mix},k} - \beta_{ab}H_{\text{mix},k}\|_F = \frac{1}{\sqrt{p}}\|\widehat{\beta}_{\text{mix},k} - \beta_a\bar{L}\|_F + o_P(1).$$

Thus, $\widehat{G}_{\text{mix},k} = \frac{2}{p}\widehat{\beta}_{\text{mix},o}'\widehat{\beta}_{\text{mix},e} = \frac{2}{p}\widehat{\beta}_{\text{mix},o}'\beta_a\bar{L} + o_P(1) = \frac{2}{p}\widehat{\beta}_{\text{mix},o}'\widehat{\beta}_{\text{mix},o} + o_P(1) \xrightarrow{\mathbb{P}} 2I.$ □

C.4.2 *An auxiliary CLT result* Consider the following statistics for $c = a, b$, and $k = o, e$:

$$\begin{aligned}
\widehat{Z}_c(\zeta_1) &= \frac{1}{\sqrt{p}k_n} \\
&\quad \times \sum_{i=1}^{p}\left[\left(\sum_{t=1}^{k_n}\overline{\epsilon}_{c,ti}\overline{f}_{c,t}'\zeta_1\right)\left(\sum_{t=1}^{k_n}\zeta_1'\overline{\epsilon}_{c,ti}\overline{f}_{c,t}\right) - \text{tr}\big(\zeta_1'\overline{F}_c'\overline{F}_c\zeta_1\big)\mathbb{E}\big(\sigma_{ci}^2|\mathcal{C}\big)\right], \\
\widehat{Z}_{ab}(\zeta_1,\zeta_2) &= \frac{1}{\sqrt{p}}\sum_{i=1}^{p}\left(\frac{1}{\sqrt{k_n}}\sum_{t=1}^{k_n}\overline{\epsilon}_{a,ti}\overline{f}_{a,t}'\zeta_1\right)\left(\frac{1}{\sqrt{k_n}}\sum_{t=1}^{k_n}\zeta_2'\overline{\epsilon}_{b,ti}\overline{f}_{b,t}\right), \\
\widehat{Z}_{\text{mix},k}(\zeta_1,\zeta_2) &= \frac{1}{\sqrt{p}} \\
&\quad \times \sum_{i=1}^{p}\left[\|\gamma_{a,k,i}'\zeta_1 + \gamma_{b,k,i}'\zeta_2\|^2 - \frac{1}{k_n}\sum_{c=a,b}\text{tr}\big(\zeta_1'\overline{F}_{c,k}'\overline{F}_{c,k}\zeta_1\big)\mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big)\right], \\
\widehat{Z}_{\text{mix}}(\zeta_1,\zeta_2,\zeta_3,\zeta_4) &= \frac{1}{\sqrt{p}}\sum_{i=1}^{p}\big(\gamma_{a,o,i}'\zeta_1 + \gamma_{b,o,i}'\zeta_2\big)\big(\zeta_3'\gamma_{a,e,i} + \zeta_4'\gamma_{b,e,i}\big),
\end{aligned} \tag{C.41}$$

for some $K \times K$ matrices $\zeta_1, \zeta_2, \zeta_3, \zeta_4$, and where

$$\gamma_{a,k,i} = \frac{1}{\sqrt{k_n}}\sum_{t\in\mathcal{T}_k}\overline{\epsilon}_{a,ti}\overline{f}_{a,t}, \qquad \gamma_{b,k,i} = \frac{1}{\sqrt{k_n}}\sum_{t\in\mathcal{T}_k}\overline{\epsilon}_{b,ti}\overline{f}_{b,t}, \quad k = o, e,$$

with $\mathcal{T}_o = \{1, 3, \ldots, 2\lfloor(k_n-1)/2\rfloor + 1\}$ and $\mathcal{T}_e = \{2, 4, \ldots, 2\lfloor k_n/2\rfloor\}$.

We note that for two $K \times K$ matrices A, B, we can write

$$\operatorname{tr}\big(A'\overline{F}_c'\overline{U}_c'\overline{U}_c\overline{F}_cB\big) = \sum_{i=1}^{p}\left[\sum_{t=1}^{k_n}\overline{\epsilon}_{c,ti}\overline{f}_{c,t}'B\sum_{t=1}^{k_n}A'\overline{f}_{c,t}\overline{\epsilon}_{c,ti}\right],$$

$$\operatorname{tr} A'\big[\overline{F}_{\mathrm{mix},k}'\overline{U}_{\mathrm{mix},k}'\overline{U}_{\mathrm{mix},k}\overline{F}_{\mathrm{mix},k}\big]A = \sum_{i=1}^{p}\big\|\gamma_{a,k,i}'A + \gamma_{b,k,i}'HA\big\|^2,$$

where the matrix H in the second line arises from the definition: $\overline{F}_{\mathrm{mix},k} = (\overline{F}_{a,k}', H'\overline{F}_{b,k}')'$.

We stack together the above random variables into a vector. Let $\underline{\zeta} = (\zeta_1, \dots, \zeta_{12})$ for $\{\zeta_i\}_{i=1,\dots,12}$ being a set of $K \times K$ matrices. We then set

$$\widehat{Z}(\underline{\zeta}) = \big(\widehat{Z}_b(\zeta_1), \widehat{Z}_a(\zeta_2), \widehat{Z}_{ab}(\zeta_3, \zeta_4), \widehat{Z}_{\mathrm{mix},o}(\zeta_5, \zeta_6), \widehat{Z}_{\mathrm{mix},e}(\zeta_7, \zeta_8),$$
$$\widehat{Z}_{\mathrm{mix}}(\zeta_9, \zeta_{10}, \zeta_{11}, \zeta_{12})\big). \tag{C.42}$$

The next theorem states a CLT for $\widehat{Z}(\underline{\zeta})$.

Lemma C.12. *Let $\{\zeta_k\}_{k=1,\dots,12}$ be $\mathcal{C}$-adapted $K \times K$ matrices and set $\underline{\zeta} = (\zeta_1, \dots, \zeta_{12})$. We have the following convergence as $p \to \infty$, $\Delta_n \to 0$, and $k_n \to \infty$ with $k_n\Delta_n \to 0$:*

$$\widehat{Z}(\underline{\zeta}) \xrightarrow{\mathcal{L}|\mathcal{C}} V(\underline{\zeta})^{1/2}\mathbf{Z}, \tag{C.43}$$

where $\mathbf{Z}$ is a standard normal random vector defined on an extension of the original probability space and independent of $\mathcal{C}$, and $V(\underline{\zeta})$ is some $\mathcal{C}$-adapted positive semidefinite matrix.

In addition, if $\widehat{\underline{\zeta}} - \underline{\zeta} = o_P(1)$, we have

$$\widehat{Z}(\widehat{\underline{\zeta}}) - \widehat{Z}(\underline{\zeta}) = o_P(1). \tag{C.44}$$

Proof. In the proof, we will denote with C_n a $\mathcal{C}$-adapted random variable that can change from line to line, depends on n and k_n, and is $O_P(1)$. We can write

$$\widehat{Z}(\underline{\zeta}) = \sum_{i=1}^{p} z_i(\underline{\zeta}). \tag{C.45}$$

We will apply Theorem VIII.5.25 in Jacod and Shiryaev (2003) to establish the convergence in (C.43). It suffices to show the following three convergence results:

$$\sum_{i=1}^{p}\mathbb{E}\big(z_i(\underline{\zeta})|\mathcal{C}\big) \xrightarrow{\mathbb{P}} 0, \tag{C.46}$$

$$\sum_{i=1}^{p}\big[\mathbb{E}\big(z_i(\underline{\zeta})z_i'(\underline{\zeta})|\mathcal{C}\big) - \mathbb{E}\big(z_i(\underline{\zeta})|\mathcal{C}\big)\mathbb{E}\big(z_i'(\underline{\zeta})|\mathcal{C}\big)\big] \xrightarrow{\mathbb{P}} V(\underline{\zeta}), \tag{C.47}$$

$$\sum_{i=1}^{p}(\mathbb{E}\big(\big|z_i(\underline{\zeta})\big|^3|\mathcal{C}\big) \xrightarrow{\mathbb{P}} 0. \tag{C.48}$$

Using Assumption SA1 and the fact that $\mathbb{E}(|\overline{\epsilon}_{t,i}|^q + \|\overline{f}_t\|_F^q) < C_q$, for any $q > 1$ and $\mathcal{C}$-adapted random variable that depends on q but not on t and i, we have

$$\left|\sum_{i=1}^{p} \mathbb{E}\big(z_i(\underline{\zeta})|\mathcal{C}\big)\right| = O_P\left(\sqrt{p}\frac{k_n}{n}\right) \quad \text{and} \quad \sum_{i=1}^{p} \big(\mathbb{E}\big(\big|z_i(\underline{\zeta})\big|^3\big) \to 0. \tag{C.49}$$

Therefore, to establish the convergence result of the theorem, we need to establish the convergence of the second conditional moments above. We will show here the convergence of the top three by three block of the matrix, with the rest of the convergence results in (C.47) being established in an analogous way. Toward this end, we denote the first three elements of $z_i(\underline{\zeta})$ with $z_{b,i}$, $z_{a,i}$, and $z_{ab,i}$, and we further set

$$\begin{aligned} V_c(\zeta) &= \mathbb{E}\big(\sigma_{c,i}^4|\mathcal{C}\big)\,\|\Lambda_c'\zeta\zeta'\Lambda_c\|_F^2, \\ V_{ab}(\zeta_1,\zeta_2) &= \mathbb{E}\big(\sigma_{b,i}^2\sigma_{a,i}^2|\mathcal{C}\big)\,\mathrm{tr}\big(\zeta_1'\Lambda_a\Lambda_a'\zeta_1\zeta_2'\Lambda_b\Lambda_b'\zeta_2\big). \end{aligned} \tag{C.50}$$

With this notation, we will show $\sum_{i=1}^{p} \mathbb{E}(z_{b,i}^2|\mathcal{C}) \xrightarrow{\mathbb{P}} V_b(\zeta_1)$, $\sum_{i=1}^{p} \mathbb{E}(z_{a,i}^2|\mathcal{C}) \xrightarrow{\mathbb{P}} V_a(\zeta_2)$ as well as $\sum_{i=1}^{p} \mathbb{E}(z_{ab,i}^2|\mathcal{C}) \xrightarrow{\mathbb{P}} V_b(\zeta_3,\zeta_4)$. We start with the first of them. Using the fact that $\mathbb{E}(\overline{\epsilon}_{b,ti}\overline{f}_{b,t}|\mathcal{F}_{(i_t^b-1)\Delta_n} \cap \mathcal{C}) = \mathbf{0}_{K\times 1}$ (for $\mathbf{0}_{K\times 1}$ being $K \times 1$ vector of zeros) and the integrability conditions of Assumption SA1, we have

$$\begin{aligned} &\Bigg|\mathbb{E}\Bigg(\Bigg[\Bigg(\frac{1}{\sqrt{k_n}}\sum_{t=1}^{k_n}\overline{\epsilon}_{b,ti}\overline{f}_{b,t}'\zeta_1\Bigg)\Bigg(\frac{1}{\sqrt{k_n}}\sum_{t=1}^{k_n}\overline{\epsilon}_{b,ti}\zeta_1'\overline{f}_{b,t}\zeta_2\Bigg)\Bigg]^2\Bigg|\mathcal{C}\Bigg) \\ &\quad - \mathbb{E}\Bigg[\Bigg(\frac{1}{k_n}\sum_{t=1}^{k_n}\overline{\epsilon}_{b,ti}^2\overline{f}_{b,t}'\zeta_1\zeta_1'\overline{f}_{b,t}\Bigg)^2\Bigg|\mathcal{C}\Bigg] \\ &\quad - \frac{2}{k_n^2}\sum_{t=1}^{k_n}\sum_{s=1}^{k_n}\mathbb{E}\big(\overline{\epsilon}_{b,ti}^2\overline{\epsilon}_{b,si}^2(\overline{f}_{b,t}'\zeta_1\zeta_1'\overline{f}_{b,s})^2|\mathcal{C}\big)\Bigg| \le \frac{C_n}{\sqrt{k_n}}. \end{aligned} \tag{C.51}$$

In addition, using the smoothness conditions for the processes Λ and σ_i, we have

$$\begin{aligned} &\Bigg|\frac{1}{k_n^2}\sum_{t=1}^{k_n}\sum_{s=1}^{k_n}\mathbb{E}\big(\overline{\epsilon}_{b,ti}^2\overline{\epsilon}_{b,si}^2(\overline{f}_{b,t}'\zeta_1\zeta_1'\overline{f}_{b,s})^2\big|\mathcal{C}\big) \\ &\quad - \mathbb{E}\big(\overline{\sigma}_{b,i}^4|\mathcal{C}\big)\frac{1}{k_n^2}\,\mathrm{tr}\Bigg(\Lambda_b\sum_{t=1}^{k_n}\frac{\big[\Delta_{i_t^b}^n W\Delta_{i_t^b}^n W'\big]}{\Delta_n}\Lambda_b'\zeta_1\zeta_1'\Lambda_b\sum_{s=1}^{k_n}\frac{\big[\Delta_{i_s^b}^n W\Delta_{i_s^b}^n W'\big]}{\Delta_n^2}\Lambda_b'\zeta_1\zeta_1'\Bigg)\Bigg| \\ &\le C_n\sqrt{\frac{k_n}{n}}, \end{aligned} \tag{C.52}$$

and by CLT for i.i.d. random variables,

$$\frac{1}{k_n}\sum_{t=1}^{k_n}\frac{\big[\Delta_{i_t^b}^n W\Delta_{i_t^b}^n W'\big]}{\Delta_n^2} = I_K + \frac{C_n}{\sqrt{k_n}}. \tag{C.53}$$

Further, we have

$$\mathbb{E}\left|\mathbb{E}\left[\left(\frac{1}{k_n}\sum_{t=1}^{k_n}\overline{\epsilon}_{b,ti}^2\overline{f}_{b,t}'\zeta_1\zeta_1'\overline{f}_{b,t}\right)^2\Big|\mathcal{C}\right]-\mathbb{E}\left(\frac{1}{k_n}\sum_{t=1}^{k_n}\overline{\epsilon}_{b,ti}^2\overline{f}_{b,t}'\zeta_1\zeta_1'\overline{f}_{b,t}\Big|\mathcal{C}\right)^2\right|^2\leq\frac{C}{k_n},\qquad\text{(C.54)}$$

for some $\mathcal{C}$-adapted random variable that does not depend on i. From here, we have $\sum_{i=1}^p\mathbb{E}(z_{b,i}^2|\mathcal{C})\xrightarrow{\mathbb{P}}V_b(\zeta_1)$ and similarly $\sum_{i=1}^p\mathbb{E}(z_{a,i}^2|\mathcal{C})\xrightarrow{\mathbb{P}}V_a(\zeta_2)$. Next, following similar steps as above, we get

$$\begin{aligned}&\left|\mathbb{E}(z_{ab,i}^2|\mathcal{C})-\frac{1}{p}\mathbb{E}\left[\sigma_{b,i}^2\sigma_{a,i}^2\,\mathrm{tr}\left(\Lambda_a\Lambda_a'\zeta_2\zeta_1'\Lambda_b\frac{1}{k_n}\sum_{t=1}^{k_n}\frac{(\Delta_{i_t^b}^n\widetilde{W}_i)^2\Delta_{i_t^b}^nW\Delta_{i_t^b}^nW'}{\Delta_n^2}\Lambda_b'\zeta_3\zeta_4'\right)\Big|\mathcal{C}\right]\right|\\&\quad\leq C_n\frac{1}{p}\left(\sqrt{\frac{k_n}{n}}+\frac{1}{\sqrt{k_n}}\right).\qquad\text{(C.55)}\end{aligned}$$

Using the law of iterated expectations, we can write

$$\mathbb{E}\left[\sigma_{b,i}^2\sum_{t=1}^{k_n}\frac{(\Delta_{i_t^b}^n\widetilde{W}_i)^2\Delta_{i_t^b}^nW\Delta_{i_t^b}^nW'}{\Delta_n^2}\sigma_{a,i}^2\Big|\mathcal{C}\right]=\mathbb{E}\left[\sum_{t=1}^{k_n}\frac{(\Delta_{i_t^b}^n\widetilde{W}_i)^2\Delta_{i_t^b}^nW\Delta_{i_t^b}^nW'}{\Delta_n^2}\sigma_{ba,i,b}^2\Big|\mathcal{C}\right],\qquad\text{(C.56)}$$

where we denote $\widetilde{\mathcal{F}}_t^{(i)}=\mathcal{C}\vee\sigma(\widetilde{W}_{s,i}:s\leq t)$ and $\sigma_{ba,i,t}^2=\mathbb{E}(\sigma_{b,i}^2\sigma_{a,i}^2|\widetilde{\mathcal{F}}_t^{(i)})$ for $t\leq b$. Using a martingale representation theorem (Theorem II.4.33 of Jacod and Shiryaev (2003)), we have $\sigma_{ba,i,t}^2=\mathbb{E}(\sigma_{b,i}^2\sigma_{a,i}^2|\widetilde{\mathcal{F}}_0^{(i)})+\int_0^t\widetilde{\sigma}_{s,i}\,d\widetilde{W}_{s,i}$, for some $\widetilde{\sigma}_{s,i}$ adapted to $\widetilde{\mathcal{F}}_s^{(i)}$ and with $\mathbb{E}(\int_0^b\widetilde{\sigma}_{s,i}^2\,ds|\mathcal{C})<\infty$ almost surely. From here, by applying a law of iterated expectations, we get

$$\left\|\frac{1}{k_n}\mathbb{E}\left[\sigma_{b,i}^2\sum_{t=1}^{k_n}\frac{(\Delta_{i_t^b}^n\widetilde{W}_i)^2\Delta_{i_t^b}^nW\Delta_{i_t^b}^nW'}{\Delta_n^2}\sigma_{a,i}^2\Big|\mathcal{C}\right]-\mathbb{E}(\sigma_{b,i}^2\sigma_{a,i}^2|\mathcal{C})I_K\right\|_F\leq C_n\frac{k_n}{n}.\qquad\text{(C.57)}$$

As a result, we have $\sum_{i=1}^p\mathbb{E}(z_{ab,i}^2|\mathcal{C})\xrightarrow{\mathbb{P}}V_{ab}(\zeta_3,\zeta_4)$. Next, we have

$$\begin{aligned}&\mathbb{E}\left(\sum_{s,t,u}\overline{\epsilon}_{a,si}\overline{f}_{a,s}\zeta_1\zeta_1'\overline{\epsilon}_{a,ti}\overline{f}_{a,t}\overline{\epsilon}_{a,ui}\overline{f}_{a,u}\zeta_1|\mathcal{C}\cup\mathcal{F}_b\right)\\&=\mathbb{E}\left(\sum_{s,t:s\geq t}\overline{\epsilon}_{a,si}^2\overline{f}_{a,s}\zeta_1\zeta_1'\overline{f}_{a,s}\overline{\epsilon}_{a,ti}\overline{f}_{a,t}\zeta_1|\mathcal{C}\cup\mathcal{F}_b\right)\\&\quad+\mathbb{E}\left(\sum_{s,t:s>t}\overline{\epsilon}_{a,si}\overline{f}_{a,s}\zeta_1\zeta_1'\overline{\epsilon}_{a,ti}^2\overline{f}_{a,t}\overline{f}_{a,t}\zeta_1|\mathcal{C}\cup\mathcal{F}_b\right),\qquad\text{(C.58)}\end{aligned}$$

and from here, by using the integrability conditions of Assumption SA1 and applying the Cauchy–Schwarz inequality, we have $\sum_{i=1}^p\mathbb{E}(z_{a,i}z_{ab,i}|\mathcal{C})\xrightarrow{\mathbb{P}}0$. In a similar way, $\sum_{i=1}^p\mathbb{E}(z_{b,i}z_{ab,i}|\mathcal{C})\xrightarrow{\mathbb{P}}0$. The convergence result in (C.47) for the rest of the elements of the matrix $\sum_{i=1}^{p}[\mathbb{E}(z_i(\underline{\zeta})z_i'(\underline{\zeta})|\mathcal{C}) - \mathbb{E}(z_i(\underline{\zeta})|\mathcal{C})\mathbb{E}(z_i'(\underline{\zeta})|\mathcal{C})]$ follows the same steps as above. From here, the CLT result in (C.43) follows.

We are left with showing (C.44). Note that we can write

$$\widehat{Z}_c(\zeta_1) = \frac{1}{\sqrt{p}}\sum_{i=1}^{p}\operatorname{tr}\left\{\zeta_1'\left[\left(\frac{1}{\sqrt{k_n}}\sum_{t=1}^{k_n}\overline{\epsilon}_{c,ti}\overline{f}_{c,t}\right)\left(\frac{1}{\sqrt{k_n}}\sum_{t=1}^{k_n}\overline{\epsilon}_{c,ti}\overline{f}_{c,t}'\right) - \mathbb{E}\left(\sigma_{cj}^2|\mathcal{C}\right)\Lambda_c\Lambda_c'\right]\zeta_1\right\}. \tag{C.59}$$

By applying the CLT result in (C.43) for ζ_1 being a matrix with 1 at the (k,k) element and zeros elsewhere, for $k = 1,\dots,K$, we see that

$$\frac{1}{\sqrt{p}}\sum_{i=1}^{p}\left[\left(\frac{1}{\sqrt{k_n}}\sum_{t=1}^{k_n}\overline{\epsilon}_{c,ti}\overline{f}_{c,t}\right)\left(\frac{1}{\sqrt{k_n}}\sum_{t=1}^{k_n}\overline{\epsilon}_{c,ti}\overline{f}_{c,t}'\right) - \frac{1}{k_n}\mathbb{E}\left(\sigma_{cj}^2|\mathcal{C}\right)\overline{F}_c'\overline{F}_c\right] = O_P(1) \tag{C.60}$$

and similarly

$$\frac{1}{\sqrt{p}}\sum_{i=1}^{p}\left[\left(\frac{1}{\sqrt{k_n}}\sum_{t=1}^{k_n}\overline{\epsilon}_{b,ti}\overline{f}_{b,t}\right)\left(\frac{1}{\sqrt{k_n}}\sum_{t=1}^{k_n}\overline{\epsilon}_{a,ti}\overline{f}_{a,t}'\right)\right] = O_P(1). \tag{C.61}$$

From here, if $\widehat{\zeta}_i - \zeta_i = o_P(1)$, for $i = 1,\dots,4$, we have the asymptotic negligibility result in (C.44) for the first four elements of the vector. Similar analysis can be done for the rest as well. □

C.5 *Bootstrap limit result*

The statistic in the cross-sectional bootstrap is given by

$$\mathcal{S}^* := k_n\sqrt{p}\left[\|P_{\widehat{\beta}_a^*} - P_{\widehat{\beta}_b^*}\|_F^2 - \left(\widehat{B}_a^* + \widehat{B}_b^*\right) - \|P_{\widehat{\beta}_o^*} - P_{\widehat{\beta}_e^*}\|_F^2 + \left(\widehat{B}_{\text{mix},o}^* + \widehat{B}_{\text{mix},e}^*\right)\right].$$

The following lemma establishes the CLT result that needs to be proved.

LEMMA C.13. *Suppose Conditions* (31)–(32) *in Theorem* 4.1 *hold. Under the null,*

$$\mathcal{S}^* - \mathcal{S} \xrightarrow{\mathcal{L}|\mathcal{F}} \sqrt{\mathcal{V}}Z,$$

where $\mathcal{V}$ is defined in (C.36) *and Z is a standard normal random variable defined on an extension of $\mathcal{F}$ and independent from it.*

PROOF. The asymptotic expansion of the bootstrap statistics is very similar to the expansion of the original one. We omit the details in order to avoid repeating the same arguments. As a result, we have

$$k_n\sqrt{p}\mathcal{S}^* = \frac{1}{\sqrt{p}}\sum_{i=1}^{p} z_{i,n}^* + o_P(1), \tag{C.62}$$

where $z^*_{i,n}$ is drawn at random with replacement from $\{z_{i,n} : i \le p\}$ in (C.35). With the notation $\bar{z}_n := \frac{1}{p}\sum_{i=1}^{p} z_{i,n}$, we have

$$\mathcal{S}^* - \mathcal{S} = \sqrt{p}\left(\frac{1}{p}\sum_{i=1}^{p} z^*_{i,n} - \bar{z}_n\right) + o_P(1). \tag{C.63}$$

We note that

$$\mathbb{E}\big(z^*_{i,n}|\mathcal{F}\big) = \bar{z}_n \quad \text{and} \quad \text{Var}\left(\frac{1}{\sqrt{p}}\sum_i z^*_{i,n}|\mathcal{F}\right) = \frac{1}{p}\sum_{i=1}^{p} z^2_{i,n} - \bar{z}^2_n = \mathcal{V} + o_P(1). \tag{C.64}$$

Indeed, let W_p be a p-dim multinomial random vector that extracts p outcomes from $z_{i,n}$ with replacement, each with probability $1/p$. Let $z_n = (z_{1,n} \dots z_{p,n})$. Then $\text{Var}(\frac{1}{\sqrt{p}}\sum_i z^*_{i,n}|\mathcal{F}) = \frac{1}{p}\text{Var}(z_n' W_p) = \frac{1}{p} z_n' \text{Cov}(W_p) z_n$. From here, the second result in (C.64) follows because $\text{Cov}(W_p) = I - \frac{1}{p} 1_p 1_p'$.

In addition, suppose $\mathcal{V} > 0$ is bounded away from zero, a claim we show at the end of the proof. Then

$$\begin{aligned} a_1 &:= \frac{\text{Var}\big(z^*_{i,n}|\mathcal{F}\big)^{1/2}}{\sqrt{\mathcal{V}}} \xrightarrow{\mathbb{P}} 1, \\ a_2 &:= \text{Var}\big(z^*_{i,n}|\mathcal{F}\big)^{-1/2}\big(\mathcal{S}^* - \mathcal{S}\big) \xrightarrow{\mathcal{L}|\mathcal{F}} \mathcal{N}(0,1), \\ \mathcal{S}^* - \mathcal{S} &= \sqrt{\mathcal{V}} a_1 a_2 \xrightarrow{\mathcal{L}|\mathcal{F}} \sqrt{\mathcal{V}} Z, \end{aligned} \tag{C.65}$$

and the result to be proved follows.

We are left thus with showing that the limiting variance $\mathcal{V}$ is strictly positive almost surely. We can decompose $z_{i,n}$ in (C.36) into $z^{(1)}_{i,n}$ and $z^{(2)}_{i,n}$, corresponding to the part due to $\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2$ and $\|P_{\widehat{\beta}_o} - P_{\widehat{\beta}_e}\|_F^2$, respectively. From the above CLT result, we have

$$\frac{1}{p}\sum_{i=1}^{p}\begin{pmatrix} z^{(1)}_{i,n} \\ z^{(2)}_{i,2} \end{pmatrix} \xrightarrow{\mathcal{L}|\mathcal{C}} \begin{pmatrix} \mathcal{Z}^{(1)} \\ \mathcal{Z}^{(2)} \end{pmatrix}, \tag{C.66}$$

where $(\mathcal{Z}^{(1)}, \mathcal{Z}^{(2)})$ is $\mathcal{C}$-conditionally zero-mean bivariate normal vector. With this notation, we have $\mathcal{V} = \text{Var}(\mathcal{Z}^{(1)}|\mathcal{C}) + \text{Var}(\mathcal{Z}^{(2)}|\mathcal{C}) - 2\,\text{Cov}(\mathcal{Z}^{(1)}, \mathcal{Z}^{(2)}|\mathcal{C})$. Since $\text{Var}(\mathcal{Z}^{(1)}|\mathcal{C}) + \text{Var}(\mathcal{Z}^{(2)}|\mathcal{C}) > 0$ a.s. (because of our assumption for nonvanishing idiosyncratic volatility in A2(i)), to establish $\mathcal{V} > 0$ a.s., we need to show that $\mathcal{Z}^{(1)}$ and $\mathcal{Z}^{(2)}$ are not $\mathcal{C}$-conditionally perfectly positively correlated, that is, that there is no $\mathcal{C}$-adapted random variable ϕ such that $\mathcal{Z}^{(2)} = \phi\mathcal{Z}^{(1)}$.

To show this, we can look at terms in $z^{(1)}_{i,n}$ and $z^{(2)}_{i,n}$ of the type $\overline{\epsilon}_{b,ti}\overline{\epsilon}_{b,si}(\overline{f}'_{b,t}\zeta_1'\zeta_2\overline{f}_{b,s})$. These summands are uncorrelated with the rest of the summands in $z^{(1)}_{i,n}$ and $z^{(2)}_{i,n}$ and generate positive variance in $\mathcal{Z}^{(1)}$ and $\mathcal{Z}^{(2)}$. However, they generate dependence in $\mathcal{Z}^{(1)}$ and $\mathcal{Z}^{(2)}$ of the opposite sign depending on whether both s and t correspond to odd or even increments or whether one of them correspond to odd increment and the other one to even increment. To see this note that, these summands appear in $\widehat{Z}_b(\overline{A}_b)$ and in: (1) $\widehat{Z}_{\text{mix},o}(\overline{A}_{\text{mix}}, H\overline{A}_{\text{mix}})$ if s, t both correspond to odd increments, (2) $\widehat{Z}_{\text{mix},e}(\overline{A}_{\text{mix}}, H\overline{A}_{\text{mix}})$ if s, t both correspond to even increments, and (3) $\widehat{Z}_{\text{mix},e}(\overline{A}_{\text{mix}}, H\overline{A}_{\text{mix}}, 2\overline{A}_{\text{mix}}, 2H\overline{A}_{\text{mix}})$ is one of s, t corresponds to odd increment and the other one to an even one. Therefore, we cannot have $\mathcal{Z}^{(2)} = \phi\mathcal{Z}^{(1)}$ for $\mathcal{C}$-adapted random variable ϕ. This proves that $\mathcal{V} > 0$ a.s. □

C.6 *Asymptotic test size*

Proof. Expressions (C.35), (C.36) (C.63), and Lemma C.13 show the convergences of $\mathcal{S}$ and $\mathcal{S}^* - \mathcal{S}$ under the null. More specifically, these results imply

$$\begin{aligned}
\mathcal{S} &= X_n + y_n, \qquad X_n := \frac{1}{\sqrt{p}}\sum_{i=1}^{p} z_{i,n}, \qquad y_n = o_P(1),\\
\mathcal{S}^* - \mathcal{S} &= X_n^* + y_n^*, \qquad X_n := \frac{1}{\sqrt{p}}\sum_{i=1}^{p}(z_{i,n} - \bar{z}_n), \qquad y_n^* = o_P(1),\\
&\sqrt{\mathcal{V}}^{-1} X_n \xrightarrow{\mathcal{L}} Z,\\
&\sqrt{\mathcal{V}}^{-1} X_n^* \xrightarrow{\mathcal{L}} Z^*,
\end{aligned}$$

and Z and Z^* being standard normal random variables. Let q^* be the τth upper quantile of $\mathcal{S}^* - \mathcal{S}$ so that $\mathbb{P}(\mathcal{S}^* - \mathcal{S} > q^*) = \tau$. Since $\mathcal{V}$ is strictly positive almost surely, we have $\mathbb{P}(\mathcal{S}^* - \mathcal{S} > q^*) = \mathbb{P}(\widetilde{\mathcal{S}}^* - \widetilde{\mathcal{S}} > \widetilde{q}^*)$, where $\widetilde{\mathcal{S}} = \sqrt{\mathcal{V}}^{-1}\mathcal{S}$, $\widetilde{\mathcal{S}}^* = \sqrt{\mathcal{V}}^{-1}\mathcal{S}^*$, and $\widetilde{q}^* = \sqrt{\mathcal{V}}^{-1}q^*$. Therefore, we need to show $\mathbb{P}(\mathcal{S} > q^*) = \mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q}^*) \to \tau$.

To this end, first note that $\widetilde{\mathcal{S}}^* - \widetilde{\mathcal{S}} \xrightarrow{\mathcal{L}} Z^*$ implies $\widetilde{q}^* \xrightarrow{\mathbb{P}} \widetilde{q}$, for $\widetilde{q}$ being the τth upper quantile of Z by, for example, Lemma 21.2 of Van der Vaart (2000). For any $\delta > 0$,

$$\begin{aligned}
\mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q} + \delta) &\leq \mathbb{P}\big(\widetilde{\mathcal{S}} > \widetilde{q} + \delta, |\widetilde{q}^* - \widetilde{q}| < \delta\big) + \mathbb{P}\big(|\widetilde{q}^* - \widetilde{q}| > \delta\big) \leq \mathbb{P}\big(\widetilde{\mathcal{S}} > \widetilde{q}^*\big) + o(1),\\
\mathbb{P}\big(\widetilde{\mathcal{S}} > \widetilde{q}^*\big) &\leq \mathbb{P}\big(\widetilde{\mathcal{S}} > \widetilde{q}^*, |\widetilde{q}^* - \widetilde{q}| < \delta\big) + o(1) \leq \mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q} - \delta) + o(1).
\end{aligned}$$

Therefore, $\mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q} + \delta) + o(1) \leq \mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q}^*) \leq \mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q} - \delta) + o(1)$, which implies

$$\begin{aligned}
\big|\mathbb{P}\big(\widetilde{\mathcal{S}} > \widetilde{q}^*\big) - \tau\big| &\leq \big|\mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q} + \delta) - \tau\big| + \big|\mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q} - \delta) - \tau\big| + o(1)\\
&\leq \big|\mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q} + \delta) - \mathbb{P}(Z > \widetilde{q} + \delta)\big| + \big|\mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q} - \delta) - \mathbb{P}(Z > \widetilde{q} - \delta)\big|\\
&\quad + \big|\mathbb{P}(Z > \widetilde{q} + \delta) - \mathbb{P}(Z > \widetilde{q})\big| + \big|\mathbb{P}(Z > \widetilde{q} - \delta) - \mathbb{P}(Z > \widetilde{q})\big| + o(1)\\
&\leq o(1) + C\delta,
\end{aligned}$$

for some $C > 0$ that depends on the density of Z. Because $\delta > 0$ is arbitrarily small, $\mathbb{P}(\mathcal{S} > q^*) = \mathbb{P}(\widetilde{\mathcal{S}} > \widetilde{q}^*) \to \tau$. □

Appendix D: Proof of Theorem 4.2

We remind the reader of following notation, which is going to be used in this section:

$$
\begin{aligned}
\beta_c &= \text{the true beta}, \quad c \in \{a, b\},\\
\beta_c^{(k)} &= \text{see (4)}, \quad c \in \{a, b\}, k = 1, \ldots, 4,\\
\beta_c^r &= \text{see (5)}, \quad c \in \{a, b\},\\
\beta_{ab} &= (\beta_a, \beta_b),\\
\beta_{\text{mix}} &= \text{ unique columns of } \beta_{ab}.
\end{aligned}
$$

Recall that $q^* = q^*_\tau\{\mathcal{S}^* - \mathcal{S}\}$ is the bootstrap quantile so that $\mathbb{P}(\mathcal{S}^* - \mathcal{S} > q^*) = \tau$, for some significance level $\tau > 0$. We reject the null if $\mathcal{S} > q^*$. Let

$$
\mathcal{A} := \|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 - (\widehat{B}_a + \widehat{B}_b) - \|P_{\widehat{\beta}_{\text{mix},o}} - P_{\widehat{\beta}_e}\|_F^2 - (\widehat{B}_{\text{mix},o} + \widehat{B}_{\text{mix},e}).
$$

Also, let $\mathcal{A}^*$ be its bootstrap version. Let g^* be the bootstrap quantile so that $\mathbb{P}(\mathcal{A}^* - \mathcal{A} > g^*) = \tau$. Then $\mathcal{S} = \sqrt{p} k_n \mathcal{A}$ and $\mathcal{S}^* = \sqrt{p} k_n \mathcal{A}^*$ and $q^* = \sqrt{p} k_n g^*$. The key to the proof is to show that under the alternative, $\mathcal{A}$ is bounded away from zero and $\mathcal{A}^* - \mathcal{A} = o_P(1)$.

Specifically, from Proposition D.1 below, $\mathbb{P}(\mathcal{A} > c_0) \to 1$ for some constant $c_0 > 0$. Also, Lemma D.3 below shows $\mathbb{P}(g^* > c_0) \to 0$. Combining these two results, we get

$$
\mathbb{P}\big(\mathcal{S} < q^*\big) = \mathbb{P}\big(\mathcal{A} < g^*\big) \le \mathbb{P}\big(\mathcal{A} < g^*, g^* \le c_0\big) + \mathbb{P}\big(g^* > c_0\big) \le \mathbb{P}(\mathcal{A} < c_0) + o(1) = o(1).
$$

Hence, $\mathbb{P}(\mathcal{S} > q^*) \to 1$ under the two alternatives considered in the theorem.

D.1 *The behavior of $\mathcal{S}$ under the alternative*

We show in this section that $\mathbb{P}(\mathcal{A} > c_0) \to 1$, for some constant $c_0 > 0$. We start with an auxiliary result concerning the true factor loadings.

Lemma D.1. *Suppose either alternative hypothesis* (*i*) *or* (*ii*) *of Theorem* 4.2 *holds*:

Alternative (*i*): *there is an invertible matrix H so that $\beta_a = (\beta_b^{(1)} H, \mathbf{0}_{p\times K_3})$, and $\beta_b = (\beta_b^{(1)}, \beta_b^{(3)})$. Then there is $m > 0$ so that*

$$
\|P_{\beta_b^{(1)}} - P_{\beta_b}\|_F^2 > m.
$$

Alternative (*ii*): *$K_a = K_b$, and there are $c, C > 0$ so that $\|\beta_b\| \le C p^{1/2}$ and*

$$
\min_{H \in \mathbb{R}^{K_a \times K_a}} \frac{1}{\sqrt{p}} \|\beta_a H - \beta_b\|_F > c.
$$

Then there is $m > 0$ so that

$$
\|P_{\beta_a} - P_{\beta_b}\|_F^2 > m.
$$

PROOF. (i) We will show that $\|P_{\beta_b^{(1)}} - P_{\beta_b}\|_F^2 = K_b - K_a$.

Write $g = \beta_b^{(3)}$ and $\beta_b = (\beta_b^{(1)}, g)$. In addition, let $A = \beta_b^{(1)'}\beta_b^{(1)}$, and $B = \beta_b^{(1)'} g$, $T = g'g - B'A^{-1}B$. Because both A and $\beta_b'\beta_b$ are invertible, we have $\det(\beta_b'\beta_b) = \det(A)\det(T)$. Then $\det(T) \neq 0$, meaning that T is invertible. We then apply the matrix block inversion formula:

$$(\beta_b'\beta_b)^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BT^{-1}B'A^{-1} & -A^{-1}BT^{-1} \\ -T^{-1}B'A^{-1} & T^{-1} \end{pmatrix}.$$

Next, let $M_a = I - P_{\beta_b^{(1)}}$. Some algebra shows

$$P_{\beta_b} - P_{\beta_b^{(1)}} = M_a g T^{-1} g' M_a = LL', \quad L = M_a g T^{-1/2}.$$

Next, $T = g'g - g'P_{\beta_b^{(1)}} g = g'M_a g$. So, $L'L = T^{-1/2} g' M_a g T^{-1/2} = I$. This implies

$$P_{\beta_b} - P_{\beta_b^{(1)}} = L\big(L'L\big)^{-1}L'.$$

As such, $\|P_{\beta_b} - P_{\beta_b^{(1)}}\|_F^2 = \mathrm{tr}(L(L'L)^{-1}L') = K_b - K_a$.

(ii) Note that the result holds by taking $m = c/C$, because

$$\begin{aligned} c &< \min_{H \in \mathbb{R}^{K_a \times K_a}} \frac{1}{\sqrt{p}} \|\beta_a H - \beta_b\|_F = \frac{1}{\sqrt{p}} \|P_{\beta_a}\beta_b - \beta_b\|_F = \frac{1}{\sqrt{p}} \big\|(P_{\beta_a} - P_{\beta_b})\beta_b\big\|_F \\ &\leq \|P_{\beta_a} - P_{\beta_b}\|_F \frac{1}{\sqrt{p}} \|\beta_b\| \leq C\|P_{\beta_a} - P_{\beta_b}\|_F. \quad \square \end{aligned}$$

PROPOSITION D.1. *Suppose Conditions* (31)–(32) *in Theorem* 4.1 *hold. Under either the alternative* (*i*) *or the alternative* (*ii*), $\mathbb{P}(\mathcal{A} > c_0) \to 1$ *for some constant* $c_0 > 0$.

PROOF. The expansion (C.5) holds for $c \in \{a, b\}$ under either the null or the alternative hypotheses. Let β_c^r denote the nonzero unique columns of β_c. Thus, under either alternative hypothesis, $\|P_{\widehat{\beta}_c} - P_{\beta_c^r}\|_F = o_P(1)$. This implies

$$\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F \geq \|P_{\beta_a^r} - P_{\beta_b^r}\|_F - \sum_{c \in \{a,b\}} \|P_{\widehat{\beta}_c} - P_{\beta_c^r}\|_F \geq \|P_{\beta_a^r} - P_{\beta_b^r}\|_F - o_P(1).$$

By Lemma D.1, under either alternative (i) or alternative (ii), $\|P_{\beta_a^r} - P_{\beta_b^r}\|_F > c_1$ for some constant $c_1 > 0$. In addition, $\widehat{B}_a + \widehat{B}_b = O_P(k_n^{-1})$ because of Lemma C.10 and since $B_a = O_P(k_n^{-1})$ and $B_b = O_P(k_n^{-1})$. Hence, with probability approaching one,

$$\|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 - (\widehat{B}_a + \widehat{B}_b) > c_1/2. \tag{D.1}$$

Next, we show that $\|P_{\widehat{\beta}_{\mathrm{mix},o}} - P_{\widehat{\beta}_{\mathrm{mix},e}}\|_F^2 - (\widehat{B}_{\mathrm{mix},o} + \widehat{B}_{\mathrm{mix},e}) = o_P(1)$ under the alternative, where $\widehat{\beta}_{\mathrm{mix},k}$ is the PCA estimates for beta from the data matrix $\overline{Y}_{\mathrm{mix},k}$. As above, we have $(\widehat{B}_{\mathrm{mix},o} + \widehat{B}_{\mathrm{mix},e}) = o_P(1)$, so we focus on proving $\|P_{\widehat{\beta}_{\mathrm{mix},o}} - P_{\widehat{\beta}_{\mathrm{mix},e}}\|_F^2 = o_P(1)$.

From (C.9), which holds also under the alternatives, the eigenvalues of $\frac{1}{k_n p}\overline{Y}_{\mathrm{mix},k} \times \overline{Y}'_{\mathrm{mix},k}$ converge to those of $\frac{1}{p}\beta_a\Sigma_{f,a}\beta'_a + \frac{1}{p}\beta_b\Sigma_{f,b}\beta'_b$.

We now show that: (1) Diagonal entries of $\widehat{Q}_{\mathrm{mix},k}$ are bounded away from zero; (2) $\|\frac{1}{\sqrt{p}}(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})\| = o_P(1)$.

Alternative (i): $\beta_a = (\beta_b^{(1)}H, \mathbf{0}_{p\times K_3})$, and $\beta_b = (\beta_b^{(1)}, \beta_b^{(3)})$, so $K_2 + K_4 = 0$. Here, H is a $K_1 \times K_1$ invertible matrix. In this case, both β_a and β_b are $p \times K_b$-dimensional where $K_b = K_1 + K_3$.

Recall $S_{f,c,k} = \frac{1}{k_n}\overline{F}'_{c,k}\overline{F}_{c,k}$, which is $K_b \times K_b$-dimensional. Also, let $S^{\mathrm{sub}}_{f,a,k}$ denote the $K_1 \times K_1$ upper block submatrix of $S_{f,a,k}$. Then

$$\frac{1}{p}\beta_a S_{f,a,k}\beta'_a + \frac{1}{p}\beta_b S_{f,b,k}\beta'_b = \frac{1}{p}\beta_b\widehat{S}_{f,k}\beta'_b, \qquad \widehat{S}_{f,k} := S_{f,b,k} + \begin{pmatrix} H^{-1}S^{\mathrm{sub}}_{f,a,k}H'^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

The top K_b eigenvalues are bounded from below by those of $(\frac{1}{p}\beta'_b\beta_b)^{1/2}\widehat{S}_{f,k}(\frac{1}{p}\beta'_b\beta_b)^{1/2}$, which are bounded away from zero under the assumption that those of $\frac{1}{p}\beta'_b\beta_b$ and $S_{f,b,k}$ are bounded away from zero. Therefore, $\widehat{Q}^{-1}_{\mathrm{mix},k} = O_P(1)$.

For (2), using Lemma C.7, we have $\|\frac{1}{\sqrt{p}}(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})\| = o_P(1)$. They imply that the eigenvalues of $H'_{\mathrm{mix},k}\beta'_{ab}\beta_{ab}H_{\mathrm{mix},k}$ are bounded away from zero, so $P_{\beta_{ab}H_{\mathrm{mix},k}}$ exists. In addition, under this alternative, $K_{\mathrm{mix}} = K_b$ and $\beta_{ab}H_{\mathrm{mix},k} = \beta_b\bar{H}_b$, for some square matrix $\bar{H}_b$. The fact $\|\frac{1}{\sqrt{p}}(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})\| = o_P(1)$ implies $\bar{H}_b$ is invertible with probability approaching one. Hence, $P_{\beta_{ab}H_{\mathrm{mix},k}} = P_{\beta_b}$. Thus,

$$\begin{aligned} \|P_{\widehat{\beta}_{\mathrm{mix},o}} - P_{\widehat{\beta}_{\mathrm{mix},e}}\|_F &\le \|P_{\widehat{\beta}_{\mathrm{mix},o}} - P_{\beta_{ab}H_{\mathrm{mix},o}}\|_F + \|P_{\widehat{\beta}_{\mathrm{mix},e}} - P_{\beta_{ab}H_{\mathrm{mix},e}}\|_F \\ &\quad + \|P_{\beta_{ab}H_{\mathrm{mix},o}} - P_{\beta_{ab}H_{\mathrm{mix},e}}\|_F \\ &\le o_P(1) + \|P_{\beta_{ab}H_{\mathrm{mix},o}} - P_{\beta_{ab}H_{\mathrm{mix},e}}\|_F = o_P(1), \end{aligned} \tag{D.2}$$

where the second inequality follows from the expression in (C.11) that $\|P_{\widehat{\beta}_{\mathrm{mix},k}} - P_{\beta_{ab}H_{\mathrm{mix},k}}\|_F = o_P(1)$.

Alternative (ii). $\beta_a = \beta_a^{(2)}$, $\beta_b = \beta_b^{(2)}$, and $K_a = K_b$. Also, there is $c > 0$ so that with probability approaching one,

$$\min_{H\in\mathbb{R}^{K\times K}} \frac{1}{\sqrt{p}}\|\beta_a H - \beta_b\|_F > c.$$

Denote with β_{mix} a $p \times K_{\mathrm{mix}}$ matrix whose columns are the unique components of the factor loadings over the two periods. For this matrix, the eigenvalues of $\frac{1}{p}\beta'_{\mathrm{mix}}\beta_{\mathrm{mix}}$ are bounded away from zero. Then $\frac{1}{p}\beta_a S_{f,a,k}\beta'_a + \frac{1}{p}\beta_b S_{f,b,k}\beta'_b = \frac{1}{p}\beta_{\mathrm{mix}}M\beta'_{\mathrm{mix}}$ for some invertible matrix M whose eigenvalues are bounded away from zero. As a result, the top K_{mix} eigenvalues of $\frac{1}{p}\beta_{\mathrm{mix}}M\beta'_{\mathrm{mix}}$ are bounded from below by those of $(\frac{1}{p}\beta'_{\mathrm{mix}}\beta_{\mathrm{mix}})^{1/2}M(\frac{1}{p}\beta'_{\mathrm{mix}}\beta_{\mathrm{mix}})^{1/2}$, which in turn are bounded away from zero. Therefore, $\widehat{Q}^{-1}_{\mathrm{min},k} = O_P(1)$.

In addition, there is a $K_{\text{mix}} \times K_{\text{mix}}$ matrix $\bar{H}$ so that $\beta_{ab} H_{\text{mix},k} = \beta_a H_{a,\text{mix},k} + \beta_b H_{b,\text{mix},k} = \beta_{\text{mix}} \bar{H}$. Applying Lemma C.7, we have $\|\frac{1}{\sqrt{p}}(\widehat{\beta}_{\text{mix},k} - \beta_{\text{mix}}\bar{H})\| = \|\frac{1}{\sqrt{p}} \times (\widehat{\beta}_{\text{mix},k} - \beta_{ab} H_{\text{mix},k})\| = o_P(1)$. This implies $I = \frac{1}{p}\widehat{\beta}'_{\text{mix},k}\widehat{\beta}_{\text{mix},k} = \bar{H}'\frac{1}{p}\beta'_{\text{mix}}\beta_{\text{mix}}\bar{H} + o_P(1)$. Hence, $\bar{H}$ is invertible and, therefore,

$$P_{\beta_{ab} H_{\text{mix},k}} = P_{\beta_{\text{mix}}\bar{H}} = P_{\beta_{\text{mix}}}.$$

Thus, similar to (D.2), we have $\|P_{\widehat{\beta}_{\text{mix},o}} - P_{\widehat{\beta}_{\text{mix},e}}\|_F \le o_P(1)$.

In addition, $\widehat{B}_{\text{mix},o} + \widehat{B}_{\text{mix},e} = o_P(1)$. Hence,

$$\|P_{\widehat{\beta}_{\text{mix},o}} - P_{\widehat{\beta}_e}\|_F^2 - (\widehat{B}_{\text{mix},o} + \widehat{B}_{\text{mix},e}) = o_P(1).$$

Combining with (D.1), we have shown that under the two alternatives, there is a constant $c_0 = c_1/4$, such that $\mathbb{P}(\mathcal{A} > c_0) \to 1$. □

D.2 *The behavior of the bootstrap quantile under the alternative*

Recall that g^* is the bootstrap quantile so that $\mathbb{P}(\mathcal{A}^* - \mathcal{A} > g^*) = \tau$ for some significance level $\tau > 0$. All results in this subsection hold under either alternative (i) or alternative (ii) of Theorem 4.2.

LEMMA D.2. *Suppose Conditions* (31)–(32) *hold. We have* $\mathcal{A}^* - \mathcal{A} = o_P(1)$.

PROOF. In the proof of Proposition D.1, we have shown that $\|P_{\widehat{\beta}_{\text{mix},o}} - P_{\widehat{\beta}_{\text{mix},e}}\|_F^2 - (\widehat{B}_{\text{mix},o} + \widehat{B}_{\text{mix},e}) = o_P(1)$ under the two alternatives. Similarly, their bootstrap counterpart is $o_P(1)$. The proof of this can be established in the same way as showing $\|P_{\widehat{\beta}_{\text{mix},o}} - P_{\widehat{\beta}_{\text{mix},e}}\|_F^2 - (\widehat{B}_{\text{mix},o} + \widehat{B}_{\text{mix},e}) = o_P(1)$, and we omit this for brevity. In addition, $\widehat{B}_c = o_P(1)$ and $\widehat{B}_c^* = o_P(1)$. It remains to show the following under the two alternatives:

$$\left\|P^*_{\widehat{\beta}_a} - P^*_{\widehat{\beta}_b}\right\|_F^2 - \|P_{\widehat{\beta}_a} - P_{\widehat{\beta}_b}\|_F^2 = o_P(1).$$

Let β_a^* and β_b^* denote the bootstrap counterparts of β_a and β_b, respectively, obtained by randomly drawing from the rows of (β_a, β_b) with replacement. We have $\|P^*_{\widehat{\beta}_c} - P^*_{\beta_c^r}\|_F^2 = o_P(1)$ and $\|P_{\widehat{\beta}_c} - P_{\beta_c^r}\|_F^2 = o_P(1)$. Thus, it suffices to show

$$\left\|P^*_{\beta_a^r} - P^*_{\beta_b^r}\right\|_F^2 - \|P_{\beta_a^r} - P_{\beta_b^r}\|_F^2 = o_P(1).$$

This will be the case if we can show that both $\|P^*_{\beta_a^r} - P^*_{\beta_b^r}\|_F^2$ and $\|P_{\beta_a^r} - P_{\beta_b^r}\|_F^2$ converge in probability to the same limiting constant under either alternative.

For the convergence of $\|P_{\beta_a^r} - P_{\beta_b^r}\|_F^2$, under alternative (i), the proof of Lemma D.1 shows that $\|P_{\beta_a^r} - P_{\beta_b^r}\|_F^2 = K_b - K_a$. Under alternative (ii), we have $\beta_a^r = \beta_a$ and $\beta_b^r = \beta_b$ and

$$\|P_{\beta_a^r} - P_{\beta_b^r}\|_F^2 = K_a + K_b - 2\,\text{tr}(P_{\beta_a} P_{\beta_b}) \xrightarrow{\mathbb{P}} K_a + K_b - 2\,\text{tr}\big(\Sigma_{\beta,a}^{-1}\Sigma_{\beta,ab}\Sigma_{\beta,b}^{-1}\Sigma'_{\beta,ab}\big).$$

The proof of the bootstrap counterpart is very similar, noting that $\frac{1}{p}\beta_a^{*'}\beta_b^* = \frac{1}{p}\beta'_a\beta_b + o_P(1) = \Sigma_{\beta,ab} + o_P(1)$. □

Lemma D.3. *Suppose Conditions* (31)–(32) *in Theorem* 4.1 *hold. We have* $\mathbb{P}(g^* > c_0) \to 0$, *for the constant* $c_0 > 0$ *in Proposition* D.1.

Proof. From Lemma D.2, we have $\mathcal{A}^* - \mathcal{A} = o_P(1)$. This implies $\mathbb{P}(\mathcal{A}^* - \mathcal{A} > c_0) = o_P(1)$. Let $J := \mathbb{P}(\mathcal{A}^* - \mathcal{A} > g^*)$. Because $\tau > 0$ is the significance level,

$$\mathbb{P}(J \ge \tau) \le \mathbb{P}\big(J \ge \tau, g^* > c_0\big) + \mathbb{P}\big(g^* < c_0\big) \le \mathbb{P}\big(\mathbb{P}\big(\mathcal{A}^* - \mathcal{A} > c_0\big) \ge \tau\big) + \mathbb{P}\big(g^* < c_0\big)$$
$$= o(1) + \mathbb{P}\big(g^* < c_0\big).$$

Meanwhile, $\mathbb{P}(J \ge \tau) \to 1$ because of the definition of g^*. Thus, $\mathbb{P}(g^* > c_0) \to 0$. □

Appendix E: Proof of Theorem 4.3

In addition to Assumptions SA1, A2, and A3, we will assume throughout this section, without further mention, that Assumption A4 holds as well. Denote

$$\widehat{Z}_n = \sqrt{p}k_n\big[\widetilde{\mu}_a + \widetilde{\mu}_b - \widehat{\mu}_{ab} - (\mu_{\mathrm{mix},o} - B_{\mathrm{mix},o}) - (\mu_{\mathrm{mix},e} - B_{\mathrm{mix},e}) - \mu_{\mathrm{mix},oe}\big].$$

Then the decomposition of $\mathcal{S}$ in Section C.2 shows

$$\mathcal{S} = \widehat{Z}_n + \widehat{\mathcal{RA}} - \widehat{\mathcal{RA}}_{\mathrm{mix}},$$

where

$$\widehat{\mathcal{RA}} := \sqrt{p}k_n\Delta_5 - \sqrt{p}k_n \sum_{c\in\{a,b\}} (\widehat{B}_c - B_c),$$
$$\widehat{\mathcal{RA}}_{\mathrm{mix}} := \sqrt{p}k_n\Delta_{5,\mathrm{mix}} - \sqrt{p}k_n \sum_{k\in\{o,e\}} (\widehat{B}_{\mathrm{mix},k} - B_{\mathrm{mix},k}).$$

In Section C.4, we have shown that $\widehat{Z}_n$ converges in distribution, provided $k_n, p \to \infty$, $\zeta_p/p \to 0$, and $pk_n\Delta_n = O_p(1)$. Previously, we have also shown that both $\widehat{\mathcal{RA}}$ and $\widehat{\mathcal{RA}}_{\mathrm{mix}}$ are $o_P(1)$ under the conditions in (31)–(32) in the statement of Theorem 4.1, and in particular under the assumption $\frac{p}{k_n^2}\zeta_p^8 \to 0$.

In this section, by assuming A4, we aim to show that both $\widehat{\mathcal{RA}}$ and $\widehat{\mathcal{RA}}_{\mathrm{mix}}$ have the same higher-order expansion (proved in Lemmas E.7 and E.8):

$$\widehat{\mathcal{RA}} = \frac{\sqrt{p}}{k_n}\big[\mathrm{tr}(\mathbb{B}_3) - 2\mathbb{M}\big] + o_P(1) + o_P\bigg(\frac{\sqrt{p}}{k_n}\bigg)^{1/2},$$
$$\widehat{\mathcal{RA}}_{\mathrm{mix}} = \frac{\sqrt{p}}{k_n}\big[\mathrm{tr}(\mathbb{B}_3) - 2\mathbb{M}\big] + o_P(1) + o_P\bigg(\frac{\sqrt{p}}{k_n}\bigg)^{1/2}, \quad \text{(E.1)}$$

where $\mathbb{B}_3 = [8\mathbb{B}\bar{A}\bar{Q}^{-1}\bar{A}' + 4\mathbb{C}^2 - 6\mathbb{C}^2\bar{H}'\Sigma_\beta\bar{H}]$, $\mathbb{B} := \Lambda_c\Lambda_c'\mathbb{E}(\sigma_{1c}^2|\mathcal{C})^2$, $\mathbb{C} = \bar{A}'\Lambda_c\Lambda_c'\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})\bar{A}$ and

$$\mathbb{M} = \mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big)\,\mathrm{tr}\,\bar{Q}^{-1}\big(\mathrm{tr}\,2\Sigma_f\bar{A}\bar{Q}\bar{A}' - 2K_c\big) + 4\,\mathrm{tr}\,\Sigma_f\bar{A}\bar{Q}_c^{-1}\bar{A}'\big(\mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big)\big)^2,$$

with $\bar{A}$, $\bar{Q}$, $\bar{H}$ defined in Lemma E.2. Then, given these results, it will follow that

$$\widehat{\mathcal{RA}} - \widehat{\mathcal{RA}}_{\text{mix}} = o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right)^{1/2},$$

under the rate conditions of the current theorem (and in particular when $\frac{p}{k_n^2}\zeta_p^8 \to \kappa$ for some finite $\kappa \geq 0$). So, this weakens the condition from requiring $\kappa = 0$ to allowing $\kappa \geq 0$.

The analysis of $\sqrt{p}k_n\Delta_5$ and $\sqrt{p}k_n\Delta_{5,\text{mix}}$ is done in Section E.1 and of $\sqrt{p}k_n \times (\sum_{c\in\{a,b\}}(\widehat{B}_c - B_c) - \sum_{k\in\{o,e\}}(\widehat{B}_{\text{mix},k} - B_{\text{mix},k}))$ in Section E.2. Prior to that, we establish the following preliminary result.

Lemma E.1. *Let $p \to \infty$, $k_n \to \infty$, and $k_n = o(p^{3/2})$. We then have for $c, d \in \{a, b\}$:*

(i) $\frac{1}{\sqrt{p}}\frac{1}{p^2k_n}\beta_c'\overline{U}_c\overline{U}_c'\overline{U}_d\overline{U}_d'\beta_d = o_P(1)$,

(ii) $W - \mathbb{E}(W|\mathcal{C}) = o_P(1)$, *where* $W := \frac{1}{\sqrt{p}}\frac{1}{pk_n^2}\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{U}_c'\overline{U}_c\overline{F}_c$.

(ii) $W_2 - \mathbb{E}(W_2|\mathcal{C}) = o_P(1)$, *where* $W_2 := \frac{\sqrt{p}}{pk_n^2}\operatorname{tr}\overline{F}_c'\overline{U}_c'\overline{U}_c\overline{F}_c$.

Proof. The case $c \neq d$ is easier than the case $c = d$, so we focus on the latter case. The proof is straightforward calculation.

We focus on an arbitrary element, say $M := \frac{1}{\sqrt{p}}\frac{1}{p^2k_n}g'\overline{U}_c\overline{U}_c'\overline{U}_c\overline{U}_c'h$, where g and h are two arbitrary columns of β_c. Then it is straightforward to check that

$$\mathbb{E}(M^2|\mathcal{C}) = O_P\left(\frac{k_n^2}{p^3} + \frac{k_n}{p^2} + \frac{1}{k_n} + \frac{1}{p}\right) = o_P(1),$$

as long as $k_n = o(p^{3/2})$.

As for W and W_2, it is also straightforward to check that the $\mathcal{C}$-conditional variance of an arbitrary element of W is of the order $O_P(\frac{1}{k_n} + \frac{1}{p}) = o_P(1)$. Similarly, the variance of $W_2 = O_P(\frac{1}{k_n^2}) = o_P(1)$. □

E.1 *Asymptotic expansion of* Δ_5 *and* $\Delta_{5,\text{mix}}$

We introduce the following notation related with the higher-order terms of $\widehat{\mathcal{A}}_{\text{mix}}$:

$$\begin{aligned}
\Delta_{2\text{mix},k} &= H_{\text{mix},k}J_{\text{mix},k}, \qquad \Delta_{3\text{mix},k} = \frac{1}{k_n\sqrt{p}}(d_{\text{mix},1} + d_{\text{mix},2}),\\
\Delta_{4\text{mix}} &= \frac{1}{k_n\sqrt{p}}(g_{\text{mix},1} + g_{\text{mix},2} + g_{\text{mix},3}),\\
\Delta_{5,\text{mix}} &= \frac{1}{k_n\sqrt{p}}(c_{\text{mix},1} + c_{\text{mix},2}) - 2(\Delta_{4\text{mix}} + \Delta_{3\text{mix,even}} + \Delta_{3\text{mix,odd}}),
\end{aligned} \tag{E.2}$$

where, for k, k_1, $k_2 = o$, e, we denote

$$
\begin{aligned}
J_{\mathrm{mix},k} &:= \frac{1}{p}\big[H'_{\mathrm{mix},k}\beta'_{ab}\beta_{ab}H_{\mathrm{mix},k} - \widehat{\beta}'_{\mathrm{mix},k}\widehat{\beta}_{\mathrm{mix},k}\big]\big(H'_{\mathrm{mix},k}\beta'_{ab}\beta_{ab}H_{\mathrm{mix},k}\big)^{-1},\\
d_{\mathrm{mix},1} &= k_n\sqrt{p}\,\mathrm{tr}\left[p\Delta_{2\mathrm{mix},k}(p\Delta_{2\mathrm{mix},k})'\frac{1}{p}\beta'_{ab}\beta_{ab}\right]\\
&\quad + 2k_n\sqrt{p}\,\mathrm{tr}\,\widehat{\beta}'_{\mathrm{mix},k}(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})J_{\mathrm{mix},k}\\
&\quad - k_n\sqrt{p}\,\mathrm{tr}(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})'(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})J_{\mathrm{mix},k},\\
d_{\mathrm{mix},2} &= 2k_n\sqrt{p}\,\mathrm{tr}(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})'\beta_{ab}\Delta_{2\mathrm{mix},k}\\
&\quad + 2k_n\sqrt{p}\,\mathrm{tr}\,\frac{1}{p}(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})'\beta_a\big(\beta'_a\beta_a\big)^{-1}\\
&\quad \times L_k'^{-1}(\widehat{\beta}_{\mathrm{mix},k} - \beta_{ab}H_{\mathrm{mix},k})'\widehat{\beta}_{\mathrm{mix},k},\\
g_{\mathrm{mix},1} &= k_n\sqrt{p}\,\mathrm{tr}\,\Delta'_{2\mathrm{mix},o}\beta'_{ab}\beta_{ab}\Delta_{2\mathrm{mix},e}\widehat{\beta}'_{\mathrm{mix},e}\widehat{\beta}_{\mathrm{mix},o},\\
g_{\mathrm{mix},2} &= k_n\sqrt{p}\sum_{k_1\neq k_2}\mathrm{tr}\,\frac{1}{p}\widehat{\beta}'_{\mathrm{mix},k_2}\widehat{\beta}_{\mathrm{mix},k_1}(\widehat{\beta}_{\mathrm{mix},k_1} - \beta_{ab}H_{\mathrm{mix},k_1})'\beta_{ab}\Delta_{2\mathrm{mix},k_2}\\
&\quad + k_n\sqrt{p}\sum_{k_1\neq k_2}\mathrm{tr}\,\frac{1}{p}(\widehat{\beta}_{\mathrm{mix},k_1} - \beta_{ab}H_{\mathrm{mix},k_1})'\beta_a\big(\beta'_a\beta_a\big)^{-1}\\
&\quad \times L_{k_2}'^{-1}(\widehat{\beta}_{\mathrm{mix},k_2} - \beta_{ab}H_{\mathrm{mix},k_2})'\widehat{\beta}_{\mathrm{mix},k_1}\\
&\quad + k_n\sqrt{p}\,\mathrm{tr}\,\Delta_{1\mathrm{mix},o}L^{-1}_{\mathrm{odd}}\big(\beta'_a\beta_a\big)^{-1}L'^{-1}_{\mathrm{even}}\Delta'_{1\mathrm{mix},e},\\
g_{\mathrm{mix},3} &= k_n\sqrt{p}\sum_{k_1\neq k_2}\mathrm{tr}\,\frac{1}{k_n}\widehat{A}_{\mathrm{mix},k_1}L^{-1}_{k_1}\big(\beta'_a\beta_a\big)^{-1}L'^{-1}_{k_2}\Delta'_{1\mathrm{mix},k_2}\overline{U}_{\mathrm{mix},k_1}\overline{F}_{\mathrm{mix},k_1}\\
&\quad + k_n\sqrt{p}\sum_{k_1\neq k_2}\mathrm{tr}\,\Delta'_{2\mathrm{mix},k_1}\beta'_{ab}\beta_a\big(\beta'_a\beta_a\big)^{-1}L'^{-1}_{k_2}(\widehat{\beta}_{\mathrm{mix},k_2} - \beta_{ab}H_{\mathrm{mix},k_2})'\widehat{\beta}_{\mathrm{mix},k_1},\\
c_{\mathrm{mix},1} &= -\frac{2}{\sqrt{p}}\,\mathrm{tr}\,\widehat{A}'_{\mathrm{mix},o}\overline{F}'_{\mathrm{mix},o}\overline{U}'_{\mathrm{mix},o}\Delta_{1\mathrm{mix},e}\frac{1}{p}\widehat{\beta}'_{\mathrm{mix},e}\widehat{\beta}_{\mathrm{mix},o}\\
&\quad - \frac{2}{\sqrt{p}}\,\mathrm{tr}\,\widehat{A}'_{\mathrm{mix},e}\overline{F}'_{\mathrm{mix},e}\overline{U}'_{\mathrm{mix},e}\Delta_{1\mathrm{mix},o}\frac{1}{p}\widehat{\beta}'_{\mathrm{mix},o}\widehat{\beta}_{\mathrm{mix},e}\\
&\quad + \frac{4}{\sqrt{p}}\,\mathrm{tr}\,\widehat{A}'_{\mathrm{mix},o}\overline{F}'_{\mathrm{mix},o}\overline{U}'_{\mathrm{mix},o}\Delta_{1\mathrm{mix},o} + \frac{4}{\sqrt{p}}\,\mathrm{tr}\,\widehat{A}'_{\mathrm{mix},e}\overline{F}'_{\mathrm{mix},e}\overline{U}'_{\mathrm{mix},e}\Delta_{1\mathrm{mix},e},\\
c_{\mathrm{mix},2} &= -\frac{2}{p}\,\mathrm{tr}\,\Delta'_{1\mathrm{mix},o}\Delta_{1\mathrm{mix},e}\frac{1}{p}\widehat{\beta}'_{\mathrm{mix},k_2}\widehat{\beta}_{\mathrm{mix},k_1} - \frac{4k_n}{\sqrt{p}}\|\Delta_{1\mathrm{mix},o}\|_F^2 - \frac{4k_n}{\sqrt{p}}\|\Delta_{1\mathrm{mix},e}\|_F^2.
\end{aligned}
\tag{E.3}
$$

We start with showing that some probability limits associated with estimation based on the different sets of data considered in the construction of the test are the same.

LEMMA E.2. *There are some matrices $\bar{H}$, $\bar{A}$, $\bar{Q}$ such that $\widehat{A}_c$, $\widehat{A}_{\mathrm{mix},k} \xrightarrow{\mathbb{P}} \bar{A}$, $\widehat{G} := \frac{2}{p}\widehat{\beta}'_a\widehat{\beta}_b \xrightarrow{\mathbb{P}} 2I$, $G_{\mathrm{mix}} := \frac{2}{p}\widehat{\beta}'_{\mathrm{mix},o}\widehat{\beta}_{\mathrm{mix},e} \xrightarrow{\mathbb{P}} 2I$, $\widehat{Q}_c$, $\widehat{Q}_{\mathrm{mix},k} \xrightarrow{\mathbb{P}} \bar{Q}$, and also H_c, $2H_{c,\mathrm{mix},k} \xrightarrow{\mathbb{P}} \bar{H}$.*

PROOF. (i) Convergence of $\widehat{A}_c$, $\widehat{A}_{\mathrm{mix},k}$. From Lemma C.2, and expression (C.38), $\frac{1}{\sqrt{p}} \times \|\widehat{\beta}_c - \beta_c H_c\| = o_P(1)$, and $H_c = \Sigma_{f,c}\bar{A}_c + o_P(1)$, where $\bar{A}_c = \Sigma_{f,c}^{-1/2} M_c \bar{Q}_c^{*-1/2}$; columns of M_c are the eigenvectors of $\Sigma_{f,c}^{1/2}\Sigma_{\beta,c}\Sigma_{f,c}^{1/2}$, and $\bar{Q}_c^*$ is a $K \times K$ diagonal matrix of top K eigenvalues of $\Sigma_{f,c}^{1/2}\Sigma_{\beta,c}\Sigma_{f,c}^{1/2}$. Under Assumption A4, $\Sigma_{\beta,c}$, $\Sigma_{f,c}$ do not vary over time, and hence we can conclude that $H_c \xrightarrow{\mathbb{P}} \bar{H}$ and $\bar{A}_c = \bar{A}$, for $\bar{H}$ and $\bar{A}$ that do not depend on $c \in \{a, b\}$. Therefore, $\frac{1}{\sqrt{p}}\|\widehat{\beta}_c - \beta_c \bar{H}\| = o_P(1)$. For $\beta_b = \beta_a H$, and with the identity $\frac{1}{p}\widehat{\beta}_a'\widehat{\beta}_a = I$, we can write

$$\frac{1}{p}\widehat{\beta}_a'\widehat{\beta}_b = \frac{1}{p}\widehat{\beta}_a'\beta_b H_b + o_P(1) = \frac{1}{p}\widehat{\beta}_a'\beta_a \bar{H}\bar{H}^{-1}H\bar{H} + o_P(1) = \bar{H}^{-1}H\bar{H} + o_P(1).$$

If $H = I$ (assumed in A4), the probability limit of the above is the identity matrix. Also, (C.37) implies $\widehat{A}_c \xrightarrow{\mathbb{P}} \bar{A}$.

In addition, by (C.40), $\widehat{A}_{\mathrm{mix},k} \xrightarrow{\mathbb{P}} \bar{A}_{\mathrm{mix}} := \Sigma_{f,\mathrm{mix}}^{-1/2} M_{\mathrm{mix}} \bar{Q}_{\mathrm{mix}}^{-1/2}$ where $\bar{Q}_{\mathrm{mix}}$ is $K \times K$ diagonal matrix of top K eigenvalues of $\Sigma_{\beta,a}^{1/2}\Sigma_{f,\mathrm{mix}}\Sigma_{\beta,a}^{1/2}$; $\Sigma_{f,\mathrm{mix}} := 0.5\Sigma_{f,a} + 0.5H\Sigma_{f,b}H'$ and the columns of M_{mix} are the eigenvectors of $\Sigma_{f,\mathrm{mix}}^{1/2}\Sigma_{\beta,a}\Sigma_{f,\mathrm{mix}}^{1/2}$. When $H = I$, and $\Sigma_{\beta,a} = \Sigma_{\beta,b}$, $\Sigma_{f,c} = \Sigma_f$ (assumed in A4), we have $\Sigma_{f,\mathrm{mix}} = \Sigma_f$, $\bar{Q}_c^* = \bar{Q}_{\mathrm{mix}}$, $M_c = M_{\mathrm{mix}}$. This implies $\bar{A}_{\mathrm{mix}} = \bar{A}$.

(ii) Convergence of $\widehat{G}$, $\widehat{G}_{\mathrm{mix}}$. From Lemma C.11, $\widehat{G} = \bar{G} + o_P(1) = \frac{2}{p}\widehat{\beta}_b'\widehat{\beta}_a + o_P(1) \xrightarrow{\mathbb{P}} 2I$.

(iii) Convergence of $\widehat{Q}_c$, $\widehat{Q}_{\mathrm{mix},k}$, and H_c, $H_{\mathrm{mix},k}$. From the proof of Lemma C.11, $\widehat{Q}_c = \bar{Q}_c^* + o_P(1) = \bar{Q}_{\mathrm{mix}}$, $\widehat{Q}_{\mathrm{mix},k} = \bar{Q}_{\mathrm{mix}} + o_P(1)$, $H_{a,\mathrm{mix},k} = 0.5\Sigma_{f,a}\bar{A}_{\mathrm{mix}} + o_P(1)$, and $H_{b,\mathrm{mix},k} = 0.5\Sigma_{f,b}H'\bar{A}_{\mathrm{mix}} + o_P(1)$. Also, from (i) we showed $H_c \xrightarrow{\mathbb{P}} \bar{H} = \Sigma_f\bar{A}$. Hence, we can simply write the probability limit as $\bar{Q} := \bar{Q}_{\mathrm{mix}}$, $\bar{H} = \Sigma_f\bar{A}$, and conclude $2H_{c,\mathrm{mix},k} \xrightarrow{\mathbb{P}} \bar{H}$. □

LEMMA E.3. *Suppose $\zeta_p^2 p = o(k_n^3)$, $\zeta_p^4 k_n = o(p^3)$, $k_n = o(p^2)$, and $\zeta_p^2 = o(\sqrt{k_n p})$. Recall the definitions of Δ_{1c} and $\Delta_{1\mathrm{mix},k}$ in (C.4) and (C.13). Then, for $c, d \in \{a, b\}$,*

$$\frac{1}{\sqrt{p}}\overline{F}_d'\overline{U}_d'\Delta_{1c} = \begin{cases} o_P(1) + \frac{\sqrt{p}}{k_n}\mathbb{B}\big[\bar{A}\bar{Q}^{-1} + o_P(1)\big], & c = d, \\ o_P(1), & c \neq d, \end{cases}$$

where $\mathbb{B} := \Lambda_c\Lambda_c'\mathbb{E}(\sigma_{1c}^2|\mathcal{C})^2$. And for $k_1, k_2 \in \{o, e\}$,

$$\frac{1}{\sqrt{p}}\overline{F}_{\mathrm{mix},k_1}'\overline{U}_{\mathrm{mix},k_1}'\Delta_{1\mathrm{mix},k_2} = \begin{cases} o_P(1) + \frac{\sqrt{p}}{k_n}\mathbb{B}\big[\bar{A}\bar{Q}^{-1} + o_P(1)\big], & k_1 = k_2, \\ o_P(1), & k_1 \neq k_2. \end{cases}$$

PROOF. We have the following identity:

$$\frac{1}{\sqrt{p}}\overline{F}_d'\overline{U}_d'\Delta_{1c} = W_1 H_c \widehat{Q}_c^{-1} + W_2\widehat{A}_c\widehat{Q}_c^{-1} + Rem_1 + Rem_2,$$

$$
\begin{aligned}
W_1 &= \frac{1}{\sqrt{p}} \frac{1}{pk_n} \overline{F}'_d \overline{U}'_d \overline{U}_c \overline{U}'_c \beta_c, \\
W_2 &= \frac{1}{\sqrt{p}} \frac{1}{pk_n^2} \overline{F}'_d \overline{U}'_d \overline{U}_c \overline{U}'_c \overline{U}_c \overline{F}_c, \\
Rem_1 &= \frac{1}{\sqrt{p}} \frac{1}{pk_n} \overline{F}'_d \overline{U}'_d \overline{U}_c \overline{U}'_c \Delta_{1c} \widehat{Q}_c^{-1}, \\
Rem_2 &= \frac{1}{\sqrt{p}} \left(\frac{1}{pk_n} \overline{F}'_d \overline{U}'_d \overline{U}_c R'_c \widehat{\beta}_c \widehat{Q}_c^{-1} + \frac{1}{pk_n} \overline{F}'_d \overline{U}'_d R_c \overline{Y}'_c \widehat{\beta}_c \widehat{Q}_c^{-1} \right).
\end{aligned}
$$

Using Cauchy–Schwarz and Lemma C.4, $Rem_1 = o_P(1)$ because $\zeta_p^2 p = o(k_n^3)$, $\zeta_p^4 k_n = o(p^3)$, and $\zeta_p^2 = o(\sqrt{k_n p})$, and $\zeta_p \delta_4(\frac{1}{\sqrt{k_n}} + \frac{\sqrt{k_n}}{p}) = o_P(1)$. In addition, Lemma A.2 implies that $W_1 = o_P(1)$ under the condition $k_n = o(p^2)$, which is needed for the convergence of its variance.

Using the $\mathcal{C}$-conditional independence of $\overline{U}_c$ from $\overline{U}_d$, for $c \neq d$, we have $\mathbb{E}(W_2) = 0$. On the other hand, if $d = c$, then $\mathbb{E}(W_2|\mathcal{C}) = \frac{\sqrt{p}}{k_n} \mathbb{B}_c + O_P(p^{-1/2})$. Also, Lemma E.1 shows that the $\mathcal{C}$-conditional variance of each element of W_2 is $o_P(1)$. Then by Lemma E.2 and Assumption A4, $W_2 \widehat{A}_c \widehat{Q}_c^{-1} = \frac{\sqrt{p}}{k_n} \mathbb{B}[\bar{A}\bar{Q}^{-1} + o_P(1)] + O_P(p^{-1/2})$.

We can bound $\frac{1}{\sqrt{p}} \overline{F}'_{\text{mix},k_1} \overline{U}'_{\text{mix},k_1} \Delta_{1\text{mix},k_2}$ in a similar way:

$$
\frac{1}{\sqrt{p}} \overline{F}'_{\text{mix},k_1} \overline{U}'_{\text{mix},k_1} \Delta_{1\text{mix},k_2} = o_P(1) + \mathbb{E}(W_{2,\text{mix}}|\mathcal{C}) \widehat{A}_{\text{mix},k_1} \widehat{Q}^{-1}_{\text{mix},k_2},
$$

where $W_{2,\text{mix}} = \frac{1}{\sqrt{p}} \frac{1}{pk_n^2} \overline{F}'_{\text{mix},k_1} \overline{U}'_{\text{mix},k_1} \overline{U}_{\text{mix},k_2} \overline{U}'_{\text{mix},k_2} \overline{U}_{\text{mix},k_2} \overline{F}_{\text{mix},k_2}$. If $k_1 = k_2 = e$,

$$
\mathbb{E}(W_{2,\text{mix}}|\mathcal{C}) = \frac{\sqrt{p}}{k_n} \frac{1}{p^2 k_n} \sum_{t \text{ is even}} \sum_{c \in \{a,b\}} \sum_{i,j \le p} \mathbb{E}\big(\overline{f}_{c,t} \overline{f}'_{c,t} \overline{\epsilon}^2_{c,ti} \overline{\epsilon}^2_{c,tj} |\mathcal{C}\big).
$$

If k_n is also an even number, then $\sum_{t \text{ is even}} \sum_{c \in \{a,b\}} 1 = k_n$, so $\mathbb{E}(W_{2,\text{mix}}|\mathcal{C}) = \frac{\sqrt{p}}{k_n} \mathbb{B} + O_P(\frac{1}{\sqrt{p} k_n})$. If k_n is an odd number, then $\sum_{t \text{ is even}} \sum_{c \in \{a,b\}} 1 = k_n - 2$, so that $\mathbb{E}(W_{2,\text{mix}}|\mathcal{C}) = \frac{k_n - 2}{k_n} \frac{\sqrt{p}}{k_n} \mathbb{B} + O_P(\frac{1}{\sqrt{p} k_n}) = \frac{\sqrt{p}}{k_n} \mathbb{B} + O(\frac{\sqrt{p}}{k_n^2} + \frac{1}{\sqrt{p} k_n})$. The same proof also carries over to the case $k_1 = k_2 = $ "o." Altogether, we have proved $\mathbb{E}(W_{2,\text{mix}}|\mathcal{C}) = \frac{\sqrt{p}}{k_n} \mathbb{B} + O_P(\frac{\sqrt{p}}{k_n^2} + \frac{1}{\sqrt{p} k_n})$, if $k_1 = k_2$. Therefore, by Lemma E.2,

$$
\begin{aligned}
k_n \sqrt{p} \left[\frac{1}{p} \overline{F}'_{\text{mix},k_1} \overline{U}'_{\text{mix},k_1} \Delta_{1\text{mix},k_2} \right] &= o_P(1) + \frac{\sqrt{p}}{k_n} \mathbb{B} \widehat{A}_{\text{mix},k_1} \widehat{Q}^{-1}_{\text{mix},k_2} \\
&= o_P(1) + \frac{\sqrt{p}}{k_n} \mathbb{B}\big(\bar{A}\bar{Q}^{-1} + o_P(1)\big). \qquad \text{(E.4)}
\end{aligned}
$$

Finally, if $k_1 \neq k_2$, we trivially have $\mathbb{E}(W_{2,\text{mix}}) = 0$. ☐

LEMMA E.4. *Recall* $\Delta_{1c} = \frac{1}{pk_n} \overline{U}_c \overline{U}'_c \widehat{\beta}_c \widehat{Q}_c^{-1} + \frac{1}{pk_n} \overline{U}_c R'_c \widehat{\beta}_c \widehat{Q}_c^{-1} + \frac{1}{pk_n} R_c \overline{Y}'_c \widehat{\beta}_c \widehat{Q}_c^{-1}$. *Assume* $\zeta_p^3 = O(k_n \sqrt{p})$, $\zeta_p^4 p = o(k_n^3)$, *and* $k_n = o(p^{3/2})$. *Then* $\frac{k_n}{\sqrt{p}} \|\Delta_{1,c}\|^2 = o_P(1)$.

Proof. We write $\Delta_{1c} = \frac{1}{pk_n}\overline{U}_c\overline{U}'_c\widehat{\beta}_c\widehat{Q}_c^{-1} + Rem$, where Rem denotes a term that depends on R_c. Then

$$\begin{aligned}
\frac{k_n}{\sqrt{p}}\|\Delta_{1c}\|^2 &\leq \frac{2k_n}{\sqrt{p}}\left\|\frac{1}{pk_n}\overline{U}_c\overline{U}'_c\widehat{\beta}_c\widehat{Q}_c^{-1}\right\|^2 + Rem\\
&\leq B + 2\operatorname{tr}\widehat{Q}_c^{-1}H'_c v_c H_c\widehat{Q}_c^{-1} + Rem,\\
B &= \frac{2k_n}{\sqrt{p}}\left\|\frac{1}{pk_n}\overline{U}_c\overline{U}'_c(\widehat{\beta}_c - \beta_c H_c)\widehat{Q}_c^{-1}\right\|^2 \leq O_P\left(\frac{1}{\sqrt{p}p^2k_n}\right)\|\overline{U}_c\|^2\|\widehat{\beta}_c - \beta_c H_c\|^2,\\
v_c &= \frac{1}{\sqrt{p}}\frac{1}{p^2k_n}\beta'_c\overline{U}_c\overline{U}'_c\overline{U}_c\overline{U}'_c\beta_c.
\end{aligned}$$

Lemma A.3 showed $\|\widehat{Q}_c^{-1}\| = O_P(1)$. Using Lemma C.2, we have that $\|\widehat{\beta}_c - \beta_c H_c\| \leq O_P(\sqrt{\frac{p}{k_n}} + \frac{\zeta_p}{\sqrt{p}} + \delta_4)$. Lemma A.2 also bounds $\|\overline{U}_c\|$. The assumption that $\zeta_p^3 = O(k_n\sqrt{p})$, $\zeta_p^4 p = o(k_n^3)$, and $k_n\zeta^3 = o(p^{5/2})$ then imply that $B = o_P(1)$. In addition, Lemma E.1 showed $v_c = o_P(1)$. Combining these two results, we have the result of the lemma.

Finally, the term Rem depends on the remainder term R_c, whose effect is also negligible. In fact, the effect of R_c is given in δ_4, defined in (C.23). By Lemma C.5, Rem is negligible under the conditions of the current lemma. □

Lemma E.5. *Suppose $\zeta_p^2 = o(p^{3/4}k_n)$, $\zeta_p^2 = o(p^{3/2})$, $p\zeta_p^4 = o(k_n^6)$, $p = o(k_n^4)$, and $k_n = o(p^{3/2})$.*

(i) *Recall $\Delta_{2c} = H_c\frac{1}{p}[H'_c\beta'_c\beta_c H_c - \widehat{\beta}'_c\widehat{\beta}_c](H'_c\beta'_c\beta_c H_c)^{-1}$. Then*

$$\frac{1}{p}(\widehat{\beta}_c - \beta_c H_c)'(\widehat{\beta}_c - \beta_c H_c) = C_n + \frac{1}{k_n}\big[\mathbb{C} + o_P(1)\big],$$

$$\frac{1}{p}\widehat{\beta}'_d(\widehat{\beta}_c - \beta_c H_c) = A_n + \begin{cases}\frac{1}{k_n}\big[\mathbb{C} + o_P(1)\big] & \text{if } c = d,\\ 0 & \text{if } c \neq d,\end{cases}$$

$$p\Delta_{2c} = B_n - \frac{1}{k_n}\big[\bar{H}\mathbb{C} + o_P(1)\big],$$

where A_n, B_n, and C_n are such that $[\|A_n\|_F^2 + \|B_n\|_F^2 + \|C_n\|_F^2]k_n\sqrt{p} = o_P(1)$, and $\mathbb{C} = \bar{A}'\Lambda_c\Lambda'_c\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})\bar{A}$.

(ii) *Recall $\Delta_{2\mathrm{mix},k} = H_{\mathrm{mix},k}J_{\mathrm{mix},k}$ in (E.2). Then*

$$\frac{1}{p}(\widehat{\beta}_{\mathrm{mix},k_1} - \beta_{ab}H_{\mathrm{mix},k_1})'(\widehat{\beta}_{\mathrm{mix},k_1} - \beta_{ab}H_{\mathrm{mix},k_1}) = C_n^* + \frac{1}{k_n}\big[\mathbb{C} + o_P(1)\big],$$

$$\frac{1}{p}\widehat{\beta}'_{\mathrm{mix},k_2}(\widehat{\beta}_{\mathrm{mix},k_1} - \beta_{ab}H_{\mathrm{mix},k_1}) = A_n^* + \begin{cases}\frac{1}{k_n}\big[\mathbb{C} + o_P(1)\big] & \text{if } k_1 = k_2,\\ 0 & \text{if } k_1 \neq k_2,\end{cases}$$

$$pJ_{\mathrm{mix},k} = B_n^* - \frac{1}{k_n}\big[\mathbb{C} + o_P(1)\big],$$

where A_n^*, B_n^*, and C_n^* are such that $[\|A_n^*\|_F^2 + \|B_n^*\|_F^2 + \|C_n^*\|_F^2]k_n\sqrt{p} = o_P(1)$.

PROOF. (i) By (C.3),

$$\frac{1}{p}(\widehat{\beta}_d-\beta_d H_d)'(\widehat{\beta}_c-\beta_c H_c)=\frac{1}{pk_n^2}\widehat{A}_d'\big(\mathbb{E}\big(\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{F}_c|\mathcal{C}\big)\big)\widehat{A}_c+C_{1,cd}+C_{2,cd},$$

$$\frac{1}{p}\widehat{\beta}_d'(\widehat{\beta}_c-\beta_c H_c)=\frac{1}{pk_n^2}\widehat{A}_d'\big(\mathbb{E}\big(\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{F}_c|\mathcal{C}\big)\big)\widehat{A}_c+C_{1,cd}+C_{2,cd}+C_{3,cd},$$

$$C_{1,cd}=\frac{1}{pk_n}\widehat{A}_d'\overline{F}_d'\overline{U}_d'\Delta_{1c}+\frac{1}{p}\Delta_{1d}'(\widehat{\beta}_c-\beta_c H_c),$$

$$C_{2,cd}=\frac{1}{pk_n^2}\widehat{A}_d'\big(\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{F}_c-\mathbb{E}\overline{F}_d'\overline{U}_d'\overline{U}_c\overline{F}_c\big)\widehat{A}_c,$$

$$C_{3,cd}=\frac{1}{p}H_d'\beta_d'(\widehat{\beta}_c-\beta_c H_c).$$

From Lemmas C.2, C.3, and C.4, we have $[\|C_{1,cd}\|_F^2+\|C_{2,cd}\|_F^2+\|C_{3,cd}\|_F^2]k_n\sqrt{p}=o_P(1)$, provided $\zeta_p^2=o(p^{3/4}k_n)$, $\zeta_p^2=o(p^{3/2})$, $p\zeta_p^4=o(k_n^6)$, $p=o(k_n^4)$, $k_n=o(p^{3/2})$. Hence, $A_n=C_{1,cd}+C_{2,cd}+C_{3,cd}$ and $C_n=C_{1,cd}+C_{2,cd}$ satisfy $[\|C_n\|_F^2+\|A_n\|_F^2]k_n\sqrt{p}=o_P(1)$.

The first term in the above expansion of $\frac{1}{p}(\widehat{\beta}_d-\beta_d H_d)'(\widehat{\beta}_c-\beta_c H_c)$ is zero, if $d\neq c$. If $d=c$, then by making use of Assumption A4,

$$\frac{1}{pk_n^2}\widehat{A}_c'\big(\mathbb{E}\big(\overline{F}_c'\overline{U}_c'\overline{U}_c\overline{F}_c|\mathcal{C}\big)\big)\widehat{A}_c=\frac{1}{k_n}\widehat{A}_c'\frac{1}{pk_n}\sum_{i\le p}\sum_{t=1}^{k_n}\mathbb{E}\big(\overline{f}c,t\overline{f}_{c,t}'\overline{\epsilon}_{c,ti}^2|\mathcal{C}\big)\widehat{A}_c$$

$$=\frac{1}{k_n}\big[\bar{A}'\Lambda_c\Lambda_c'\mathbb{E}\big(\sigma_{c,i}^2|\mathcal{C}\big)\bar{A}+o_P(1)\big].$$

This implies the expansion result for $\frac{1}{p}(\widehat{\beta}_c-\beta_c H_c)'(\widehat{\beta}_c-\beta_c H_c)$ in the lemma. We can show the one for $\frac{1}{p}\widehat{\beta}_d'(\widehat{\beta}_c-\beta_c H_c)$ in a similar way.

Next, by Lemma C.3, for $G_1=\frac{1}{p}H_c'\beta_c'(\beta_c H_c-\widehat{\beta}_c)$, we have $\|G_1\|_F^2k_n\sqrt{p}=o_P(1)$. Then

$$M_c:=\frac{1}{p}\big(H_c'\beta_c'\beta_c H_c-\widehat{\beta}_c'\widehat{\beta}_c\big)=G_1+\frac{1}{p}\big(H_c'\beta_c'-\widehat{\beta}_c'\big)\widehat{\beta}_c=G_1-A_n-\frac{1}{k_n}\big[\mathbb{C}+o_P(1)\big],$$

$$p\Delta_{2c}\;=\;H_cM_c\bigg(\frac{1}{p}H_c'\beta_c'\beta_c H_c\bigg)^{-1}=H_c\bigg[G_1-A_n-\frac{1}{k_n}\big[\mathbb{C}+o_P(1)\big]\bigg]\bigg(\frac{1}{p}H_c'\beta_c'\beta_c H_c\bigg)^{-1}.$$

Also, $\frac{1}{\sqrt{p}}\|\widehat{\beta}_c-\beta_c H_c\|=o_P(1)$ implies $(\frac{1}{p}H_c'\beta_c'\beta_c H_c)^{-1}\xrightarrow{\mathbb{P}}I$, and by Lemma E.2, $H_c\xrightarrow{\mathbb{P}}\bar{H}$. Hence, for $B_n:=H_c(G_1-A_n)$, we have $\|B_n\|_F^2k_n\sqrt{p}=o_P(1)$, and

$$p\Delta_{2c}=B_n-\frac{1}{k_n}\big[\bar{H}\mathbb{C}+o_P(1)\big].$$

(ii) By (C.13),

$$\frac{1}{p}(\widehat{\beta}_{\text{mix},k_2}-\beta_{ab}H_{\text{mix},k_2})'(\widehat{\beta}_{\text{mix},k_1}-\beta_{ab}H_{\text{mix},k_1})$$
$$=\frac{1}{pk_n^2}\widehat{A}'_{\text{mix},k_2}\big(\mathbb{E}\big(\overline{F}'_{\text{mix},k_2}\overline{U}'_{\text{mix},k_2}\overline{U}_{\text{mix},k_1}\overline{F}_{\text{mix},k_1}|\mathcal{C}\big)\big)\widehat{A}_{\text{mix},k_1}+C_1^*+C_2^*,$$
$$\frac{1}{p}\widehat{\beta}'_{\text{mix},k_2}(\widehat{\beta}_{\text{mix},k_1}-\beta_{ab}H_{\text{mix},k_1})$$
$$=\frac{1}{pk_n^2}\widehat{A}'_{\text{mix},k_2}\big(\mathbb{E}\big(\overline{F}'_{\text{mix},k_2}\overline{U}'_{\text{mix},k_2}\overline{U}_{\text{mix},k_1}\overline{F}_{\text{mix},k_1}|\mathcal{C}\big)\big)\widehat{A}_{\text{mix},k_1}+C_1^*+C_2^*+C_3^*, \tag{E.5}$$
$$C_1^*=\frac{1}{pk_n}\widehat{A}'_{\text{mix},k_2}\overline{F}'_{\text{mix},k_2}\overline{U}'_{\text{mix},k_2}\Delta_{1\text{mix},k_1}+\frac{1}{p}\Delta'_{1\text{mix},k_2}(\widehat{\beta}_{\text{mix},k_1}-\beta_{ab}H_{\text{mix},k_1}),$$
$$C_2^*=\frac{1}{pk_n^2}\widehat{A}'_{\text{mix},k_2}\big(\overline{F}'_{\text{mix},k_2}\overline{U}'_{\text{mix},k_2}\overline{U}_{\text{mix},k_1}\overline{F}_{\text{mix},k_1}$$
$$-\mathbb{E}\big(\overline{F}'_{\text{mix},k_2}\overline{U}'_{\text{mix},k_2}\overline{U}_{\text{mix},k_1}\overline{F}_{\text{mix},k_1}|\mathcal{C}\big)\big)\widehat{A}_{\text{mix},k_1},$$
$$C_3^*=\frac{1}{p}H'_{\text{mix},k_2}\beta'_{ab}(\widehat{\beta}_{\text{mix},k_1}-\beta_{ab}H_{\text{mix},k_1}).$$

Exactly as the proof of Lemmas C.2, C.3, and C.4, we can show that $[\|C_1^*\|_F^2+\|C_2^*\|_F^2+\|C_3^*\|_F^2]k_n\sqrt{p}=o_P(1)$.

Next, if $k_1\neq k_2$, using successive conditioning and Assumption A4, we have

$$\mathbb{E}\big(\overline{F}'_{\text{mix},k_2}\overline{U}'_{\text{mix},k_2}\overline{U}_{\text{mix},k_1}\overline{F}_{\text{mix},k_1}\big)=0,\quad \text{if } k_1\neq k_2.$$

We turn to the case $k_1=k_2$. If $k_1=e$, then

$$\frac{1}{pk_n^2}\mathbb{E}\big(\overline{F}'_{\text{mix},k_2}\overline{U}'_{\text{mix},k_2}\overline{U}_{\text{mix},k_1}\overline{F}_{\text{mix},k_1}|\mathcal{C}\big)=\frac{1}{k_n}\frac{1}{pk_n}\sum_{i\leq p}\sum_{c\in\{a,b\}}\sum_{t\text{ is even}}\mathbb{E}\big(\overline{f}_{c,t}\overline{f}'_{c,t}\overline{\epsilon}^2_{c,ti}|\mathcal{C}\big).$$

If k_n is also even, then the above equals $\frac{1}{k_n}\Lambda_c\Lambda_c'\mathbb{E}(\sigma^2_{c,1}|\mathcal{C})$ due to Assumption A4. If k_n is odd, then the above equals $\frac{k_n-2}{k_n^2}\Lambda_c\Lambda_c'\mathbb{E}(\sigma^2_{c,1}|\mathcal{C})$, again by Assumption A4. Also, Lemma E.2 shows $\widehat{A}_{\text{mix},k}=\bar{A}+o_P(1)$. Thus,

$$\frac{1}{pk_n^2}\widehat{A}'_{\text{mix},k_2}\big(\mathbb{E}\big(\overline{F}'_{\text{mix},k_2}\overline{U}'_{\text{mix},k_2}\overline{U}_{\text{mix},k_1}\overline{F}_{\text{mix},k_1}|\mathcal{C}\big)\big)\widehat{A}_{\text{mix},k_1}=\frac{1}{k_n}\big[\mathbb{C}+o_P(1)\big].$$

The case $k_1=k_2=o$ follows by the same argument. This yields the expression for $\frac{1}{p}\widehat{\beta}'_{\text{mix},k_2}(\widehat{\beta}_{\text{mix},k_1}-\beta_{ab}H_{\text{mix},k_1})$ and $\frac{1}{p}(\widehat{\beta}_{\text{mix},k_1}-\beta_{ab}H_{\text{mix},k_1})'(\widehat{\beta}_{\text{mix},k_1}-\beta_{ab}H_{\text{mix},k_1})$ in the lemma.

Finally, the expansion for $pJ_{\text{mix},k}$ follows by similar arguments. More specifically, an expansion for $\frac{1}{p}\|\widehat{\beta}_{\text{mix},k}-\beta_{ab}H_{\text{mix},k}\|$ would imply

$$\left(\frac{1}{p}H'_{\text{mix},k}\beta'_{ab}\beta_{ab}H_{\text{mix},k}\right)^{-1}=\frac{1}{p}\widehat{\beta}'_{\text{mix},k}\widehat{\beta}_{\text{mix},k}+o_P(1)=I+o_P(1).$$

Let $A_n^* = C_1^* + C_2^* + C_3^*$ and $G_1^* = \frac{1}{p} H'_{\mathrm{mix},k} \beta'_{ab}(\beta_{ab} H_{\mathrm{mix},k} - \widehat{\beta}_{\mathrm{mix},k})$. Then

$$M_{\mathrm{mix}} := \frac{1}{p}\big[H'_{\mathrm{mix},k}\beta'_{ab}\beta_{ab}H_{\mathrm{mix},k} - \widehat{\beta}'_{\mathrm{mix},k}\widehat{\beta}_{\mathrm{mix},k}\big] = G_1^* - A_n^* - \frac{1}{k_n}\big[\mathbb{C} + o_P(1)\big],$$

$$pJ_{\mathrm{mix},k} = M_{\mathrm{mix}}\left(\frac{1}{p}H'_{\mathrm{mix},k}\beta'_{ab}\beta_{ab}H_{\mathrm{mix},k}\right)^{-1} = \left[G_1^* - A_n^* - \frac{1}{k_n}\big[\mathbb{C} + o_P(1)\big]\right]\big[I + o_P(1)\big].$$

We can write $B_n^* := (G_1^* - A_n^*)[I + o_P(1)]$ satisfying $\|B_n^*\|_F^2 k_n\sqrt{p} = o_P(1)$, and from here the result in the lemma for $pJ_{\mathrm{mix},k}$ follows:

$$pJ_{\mathrm{mix},k} = B_n^* - \frac{1}{k_n}\big[\mathbb{C} + o_P(1)\big]\big[I + o_P(1)\big] = B_n^* - \frac{1}{k_n}\big[\mathbb{C} + o_P(1)\big]. \qquad \square$$

LEMMA E.6. *Recall the definitions of* $\Delta_{3,c}$, Δ_4 *in* (C.16), *and of* $\Delta_{3\mathrm{mix},k}$, $\Delta_{4\mathrm{mix}}$ *in* (E.2). *Then*

$$k_n\sqrt{p}\Delta_{3,c} = o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right)^{1/2} + \frac{\sqrt{p}}{k_n}\operatorname{tr}\mathbb{C}^2\big(\bar{H}'\Sigma_\beta\bar{H} - I\big),$$

$$k_n\sqrt{p}\Delta_4 = o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right)^{1/2} + \frac{\sqrt{p}}{k_n}\operatorname{tr}\mathbb{C}^2\bar{H}'\Sigma_\beta\bar{H},$$

$$k_n\sqrt{p}\Delta_{3\mathrm{mix},k} = o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right)^{1/2} + \frac{\sqrt{p}}{k_n}\operatorname{tr}\mathbb{C}^2\big(\bar{H}'\Sigma_\beta\bar{H} - I\big),$$

$$k_n\sqrt{p}\Delta_{4\mathrm{mix}} = o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right)^{1/2} + \frac{\sqrt{p}}{k_n}\operatorname{tr}\mathbb{C}^2\bar{H}'\Sigma_\beta\bar{H},$$

where $\mathbb{C} = \bar{A}'\Lambda_c\Lambda'_c\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})\bar{A}$.

PROOF. (i) Bound for $\Delta_{3,c}$ and $\Delta_{3\mathrm{mix},k}$. Recall

$$\begin{aligned}
k_n\sqrt{p}\Delta_{3,c} &= d_1 + d_2,\\
d_1 &= k_n\sqrt{p}\operatorname{tr}\left[p\Delta_{2c}(p\Delta_{2c})'\frac{1}{p}\beta'_c\beta_c\right] + 2k_n\sqrt{p}\operatorname{tr}\widehat{\beta}'_c(\widehat{\beta}_c - \beta_c H_c)H_c^{-1}\Delta_{2,c}\\
&\quad - k_n\sqrt{p}\operatorname{tr}(\widehat{\beta}_c - \beta_c H_c)'(\widehat{\beta}_c - \beta_c H_c)H_c^{-1}\Delta_{2,c},\\
d_2 &= 2k_n\sqrt{p}\operatorname{tr}(\widehat{\beta}_c - \beta_c H_c)'\beta_c\Delta_{2,c}\\
&\quad + 2k_n\sqrt{p}\operatorname{tr}\frac{1}{p}(\widehat{\beta}_c - \beta_c H_c)'\beta_c\big(\beta'_c\beta_c\big)^{-1}H_c'^{-1}(\widehat{\beta}_c - \beta_c H_c)'\widehat{\beta}_c.
\end{aligned}$$

Lemma C.3 gives bounds for $\frac{1}{p}\beta'_c(\widehat{\beta}_c - \beta_c H_c)$ and $\frac{1}{p}\widehat{\beta}'_c(\widehat{\beta}_c - \beta_c H_c)$. Hence, using Cauchy–Schwarz, we have $d_2 = o_P(1) + o_P(\frac{\sqrt{p}}{k_n})^{1/2}$. The term d_1 is the leading one. From Lemma E.5, $p\Delta_{2c} = B_n - \frac{1}{k_n}[\bar{H}\mathbb{C} + o_P(1)]$ and $\frac{1}{p}\widehat{\beta}'_c(\widehat{\beta}_c - \beta_c H_c) = A_n + \frac{1}{k_n}[\mathbb{C} + o_P(1)]$, with $(\|A_n\|_F^2 + \|B_n\|_F^2)k_n\sqrt{p} = o_P(1)$. Also, by Lemma E.2, $H_c^{-1} \xrightarrow{\mathbb{P}} \bar{H}^{-1}$. Altogether,

$$d_1 = o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right)^{1/2} + \frac{\sqrt{p}}{k_n}\big[\operatorname{tr}\mathbb{C}^2\big(H'\Sigma_\beta\bar{H} - I\big) + o_P(1)\big].$$

The bound for $\Delta_{3\text{mix},k}$ can be shown in a similar way. Recall the definitions of $d_{\text{mix},1}$ and $d_{\text{mix},2}$ in (E.3). The term $d_{2,\text{mix}} = o_P(1) + o_P(\frac{\sqrt{p}}{k_n})^{1/2}$, and the term $d_{\text{mix},1}$ is the leading one. By Lemma E.5, $p\Delta_{2\text{mix},k} = H_{\text{mix},k}[B_n^* - \frac{1}{k_n}(\mathbb{C} + o_P(1))]$, where $\|B_n^*\|_F^2 k_n\sqrt{p} = o_P(1)$. Also, by Lemma E.2, $\beta_{ab}H_{\text{mix},k} = \beta_a[\bar{H} + o_P(1)]$. Hence,

$$\begin{aligned} d_{\text{mix},1} = {} & k_n\sqrt{p}\,\text{tr}\Big[p\Delta_{2\text{mix},k}(p\Delta_{2\text{mix},k})'\frac{1}{p}\beta_{ab}'\beta_{ab}\Big] \\ & + 2k_n\sqrt{p}\,\text{tr}\,\widehat{\beta}_{\text{mix},k}'(\widehat{\beta}_{\text{mix},k} - \beta_{ab}H_{\text{mix},k})J_{\text{mix},k} \\ & - k_n\sqrt{p}\,\text{tr}(\widehat{\beta}_{\text{mix},k} - \beta_{ab}H_{\text{mix},k})'(\widehat{\beta}_{\text{mix},k} - \beta_{ab}H_{\text{mix},k})J_{\text{mix},k} \\ = {} & o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right)^{1/2} + \frac{\sqrt{p}}{k_n}\big[\text{tr}\,\mathbb{C}^2\big(H'\Sigma_\beta\bar{H} - I\big) + o_P(1)\big]. \end{aligned}$$

This implies the bound for $k_n\sqrt{p}\Delta_{3\text{mix},k} = d_{\text{mix},1} + d_{\text{mix},2}$.

(ii) Bound for Δ_4 and $\Delta_{4\text{mix}}$. We have

$$\begin{aligned} k_n\sqrt{p}\Delta_4 &= g_1 + g_2 + g_3, \\ g_1 &= k_n\sqrt{p}\,\text{tr}\,\Delta_{2,a}'\beta_a'\beta_b\Delta_{2,b}\widehat{\beta}_b'\widehat{\beta}_a, \\ g_2 &= k_n\sqrt{p}\sum_{c\neq d}\text{tr}\,\frac{1}{p}\widehat{\beta}_d'\widehat{\beta}_c(\widehat{\beta}_c - \beta_cH_c)'\beta_d\Delta_{2,d} \\ &\quad + k_n\sqrt{p}\sum_{c\neq d}\text{tr}\,\frac{1}{p}(\widehat{\beta}_c - \beta_cH_c)'\beta_d\big(\beta_d'\beta_d\big)^{-1}H_d'^{-1}(\widehat{\beta}_d - \beta_dH_d)'\widehat{\beta}_c \\ &\quad + k_n\sqrt{p}\,\text{tr}\,\Delta_{1a}H_a^{-1}\big(\beta_a'\beta_a\big)^{-1}\beta_a'\beta_b\big(\beta_b'\beta_b\big)^{-1}H_b'^{-1}\Delta_{1b}', \\ g_3 &= k_n\sqrt{p}\sum_{c\neq d}\text{tr}\,\frac{1}{k_n}\widehat{A}_cH_c^{-1}\big(\beta_c'\beta_c\big)^{-1}\beta_c'\beta_d\big(\beta_d'\beta_d\big)^{-1}H_d'^{-1}\Delta_{1d}'\overline{U}_c\overline{F}_c \\ &\quad + k_n\sqrt{p}\sum_{c\neq d}\text{tr}\,\Delta_{2,c}'\beta_c'\beta_d\big(\beta_d'\beta_d\big)^{-1}H_d'^{-1}(\widehat{\beta}_d - \beta_dH_d)'\widehat{\beta}_c. \end{aligned} \tag{E.6}$$

Lemma C.3 provides a bound for $\frac{1}{p}\beta_d'(\widehat{\beta}_c - \beta_cH_c)$ and $\frac{1}{p}\widehat{\beta}_d'(\widehat{\beta}_c - \beta_cH_c)$. Also, Lemma C.2 derived bound for Δ_{1a}. We can then apply the Cauchy–Schwarz inequality and Lemma C.3 to verify that $g_2 = o_P(1) + o_P(\frac{\sqrt{p}}{k_n})^{1/2}$. As for g_3, it follows from Lemmas E.3 and E.5 that, when $c \neq d$, $\frac{1}{\sqrt{p}}\overline{F}_d'\overline{U}_d'\Delta_{1c} = o_P(1)$ and $\frac{1}{p}\widehat{\beta}_d'(\widehat{\beta}_c - \beta_cH_c) = A_n + o_P(1)$, where $\|A_n\|_F^2]k_n\sqrt{p} = o_P(1)$. Thus, $g_3 = o_P(1)$.

We are left with the term g_1, which is the leading one in the expansion of $k_n\sqrt{p}\Delta_4$. It follows from Lemma E.5 that since $\beta_b = \beta_a$,

$$g_1 = o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right)^{1/2} + \frac{\sqrt{p}}{k_n}\,\text{tr}\,\mathbb{C}^2\bar{H}'\Sigma_\beta\bar{H}.$$

This leads to the bound for Δ_4.

We can proceed in an analogous way for $\Delta_{4\text{mix}}$. Recall the definitions of $g_{\text{mix},1}\ldots g_{\text{mix},3}$ in (E.3). As above, we have $g_{\text{mix},2} + g_{\text{mix},3} = o_P(1) + o_P(\frac{\sqrt{p}}{k_n})^{1/2}$ and $g_{\text{mix},1}$ is the leading term in the expansion. From Lemma E.2, $\frac{1}{p}\widehat{\beta}'_{\text{mix},e}\widehat{\beta}_{\text{mix},o} = I + o_P(1)$, and $H_{c,\text{mix},k} = 0.5\bar{H} + o_P(1)$. From Lemma E.5, since $\beta_b = \beta_a$ (from Assumption A4),

$$p\beta_{ab}\Delta_{2\text{mix},k} = \beta_a(H_{a,\text{mix},k} + HH_{b,\text{mix},k})pJ_{\text{mix},k} = \beta_a\big(\bar{H} + o_P(1)\big)\bigg(B_n^* - \frac{1}{k_n}\big(\mathbb{C} + o_P(1)\big)\bigg).$$

Thus,

$$\begin{aligned} g_{\text{mix},1} &= k_n\sqrt{p}\operatorname{tr}\Delta'_{2\text{mix},o}\beta'_{ab}\beta_{ab}\Delta_{2\text{mix},e}\widehat{\beta}'_{\text{mix},e}\widehat{\beta}_{\text{mix},o} \\ &= o_P(1) + o_P\bigg(\frac{\sqrt{p}}{k_n}\bigg)^{1/2} + \frac{\sqrt{p}}{k_n}\operatorname{tr}\mathbb{C}^2\bar{H}'\Sigma_\beta H\bar{H}. \end{aligned}$$

This leads to the expansion result for $\Delta_{4\text{mix}}$ in the lemma. □

Lemma E.7. *Recall $\Delta_{5,\text{mix}}$ defined in (E.2). Suppose $\zeta_p^2 p = o(k_n^3)$, $\zeta_p^4 k_n = o(p^3)$, $k_n = o(p^{3/2})$, and $\zeta_p^2 = o(\sqrt{k_n p})$. Then*

$$\begin{aligned} k_n\sqrt{p}\Delta_5 &= \frac{\sqrt{p}}{k_n}\operatorname{tr}(\mathbb{B}_3) + o_P(1) + o_P\bigg(\frac{\sqrt{p}}{k_n}\bigg)^{1/2}, \\ k_n\sqrt{p}\Delta_{5,\text{mix}} &= \frac{\sqrt{p}}{k_n}\operatorname{tr}(\mathbb{B}_3) + o_P(1) + o_P\bigg(\frac{\sqrt{p}}{k_n}\bigg)^{1/2}, \\ k_n\sqrt{p}(\Delta_5 - \Delta_{5,\text{mix}}) &= o_P(1) + o_P\bigg(\frac{\sqrt{p}}{k_n}\bigg)^{1/2}, \end{aligned}$$

where $\mathbb{B}_3 = [8\mathbb{B}\bar{A}\bar{Q}^{-1}\bar{A}' + 4\mathbb{C}^2 - 6\mathbb{C}^2\bar{H}'\Sigma_\beta\bar{H}]$.

Proof. We use the expression for Δ_5 in (C.15) and write

$$\begin{aligned} k_n\sqrt{p}\Delta_5 &= c_1 + c_2 - 2k_n\sqrt{p}(\Delta_4 + \Delta_{3,a} + \Delta_{3,b}), \\ c_1 &= -\frac{2}{\sqrt{p}}\operatorname{tr}\widehat{A}'_a\overline{F}'_a\overline{U}'_a\Delta_{1,b}\frac{1}{p}\widehat{\beta}'_b\widehat{\beta}_a - \frac{2}{\sqrt{p}}\operatorname{tr}\widehat{A}'_b\overline{F}'_b\overline{U}'_b\Delta_{1,a}\frac{1}{p}\widehat{\beta}'_a\widehat{\beta}_b, \\ &\quad + \frac{4}{\sqrt{p}}\operatorname{tr}\widehat{A}'_a\overline{F}'_a\overline{U}'_a\Delta_{1,a} + \frac{4}{\sqrt{p}}\operatorname{tr}\widehat{A}'_b\overline{F}'_b\overline{U}'_b\Delta_{1,b}, \\ c_2 &= -\frac{2}{p}\operatorname{tr}\Delta'_{1,a}\Delta_{1,b}\frac{1}{p}\widehat{\beta}'_b\widehat{\beta}_a - \frac{4k_n}{\sqrt{p}}\|\Delta_{1,a}\|_F^2 - \frac{4k_n}{\sqrt{p}}\|\Delta_{1,b}\|_F^2. \end{aligned}$$

By Lemmas E.2 and E.3, $c_1 = \frac{8\sqrt{p}}{k_n}[\operatorname{tr}\mathbb{B}\bar{A}\bar{Q}^{-1}\bar{A}' + o_P(1)] + o_P(1)$. By Lemma C.2, $c_2 = o_P(1)$. Also, Lemma E.6 bounds $\Delta_{3,c}$ and Δ_4. Together, we obtain the desired expansion for $k_n\sqrt{p}\Delta_5$:

$$k_n\sqrt{p}\Delta_5 = \frac{\sqrt{p}}{k_n}\big[8\operatorname{tr}\mathbb{B}\bar{A}\bar{Q}^{-1}\bar{A}' + 4\operatorname{tr}\mathbb{C}^2 - 6\operatorname{tr}\mathbb{C}^2\bar{H}'\Sigma_\beta\bar{H}\big] + o_P(1) + o_P\bigg(\frac{\sqrt{p}}{k_n}\bigg)^{1/2}.$$

The expansion of $\Delta_{5,\text{mix}}$ follows analogously from Lemmas C.2, E.2, E.3, E.5, and E.6. □

E.2 *Asymptotic expansion of* $\widehat{B}_c$ *and* $\widehat{B}_{\mathrm{mix},k}$

Recall

$$
\begin{aligned}
B_c &= \frac{2}{k_n^2}\sum_{t=1}^{k_n} \operatorname{tr} \widehat{A}_c' \bar{f}_{c,t}\bar{f}_{c,t}' \widehat{A}_c \mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big),\\
\widetilde{B}_c &= \frac{2}{k_n^3 p}\operatorname{tr}\big(\widehat{Q}_c^{-1}\widehat{F}_c'\widehat{F}_c\widehat{Q}_c^{-1}\big)\|\widehat{U}_c\|_F^2,\\
\widehat{B}_c &= \frac{2}{k_n^2}\operatorname{tr}\big(\widehat{Q}_c^{-1}\widehat{F}_c'\widehat{F}_c\widehat{Q}_c^{-1}\big)\mathbb{E}\widehat{\big(\sigma_{c,1}^2|\mathcal{C}\big)},\\
B_{\mathrm{mix},k} &= \frac{2}{k_n^2}\operatorname{tr}\widehat{A}_{\mathrm{mix},k}'\big[F_{a,k}'F_{a,k}\mathbb{E}\big(\sigma_{a,1}^2|\mathcal{C}\big)+H'F_{b,k}'F_{b,k}H\mathbb{E}\big(\sigma_{b,1}^2|\mathcal{C}\big)\big]\widehat{A}_{\mathrm{mix},k},\\
\widehat{B}_{\mathrm{mix},k} &= \frac{2}{k_n^2}\operatorname{tr}\widehat{Q}_{\mathrm{mix},k}^{-1}\widehat{F}_{a,k}'\widehat{F}_{a,k}\widehat{Q}_{\mathrm{mix},k}^{-1}\mathbb{E}\widehat{\big(\sigma_{a,1}^2|\mathcal{C}\big)}+\frac{2}{k_n^2}\operatorname{tr}\widehat{Q}_{\mathrm{mix},k}^{-1}\widehat{F}_{b,k}'\widehat{F}_{b,k}\widehat{Q}_{\mathrm{mix},k}^{-1}\mathbb{E}\widehat{\big(\sigma_{b,1}^2|\mathcal{C}\big)},\\
\widetilde{B}_{\mathrm{mix},k} &= \frac{2}{k_n^3 p}\operatorname{tr}\widehat{Q}_{\mathrm{mix},k}^{-1}\widehat{F}_{a,k}'\widehat{F}_{a,k}\widehat{Q}_{\mathrm{mix},k}^{-1}\|\widehat{U}_a\|_F^2+\frac{2}{k_n^3 p}\operatorname{tr}\widehat{Q}_{\mathrm{mix},k}^{-1}\widehat{F}_{b,k}'\widehat{F}_{b,k}\widehat{Q}_{\mathrm{mix},k}^{-1}\|\widehat{U}_b\|_F^2.
\end{aligned}
$$

LEMMA E.8. *We have*

$$
\begin{aligned}
k_n\sqrt{p}(\widehat{B}_c-B_c) &= \frac{\sqrt{p}}{k_n}\mathbb{M}+o_P(1)+o_P\left(\frac{\sqrt{p}}{k_n}\right),\\
k_n\sqrt{p}(\widehat{B}_{\mathrm{mix},k}-B_{\mathrm{mix},k}) &= \frac{\sqrt{p}}{k_n}\mathbb{M}+o_P(1)+o_P\left(\frac{\sqrt{p}}{k_n}\right),
\end{aligned}
$$

where

$$
\mathbb{M}=\mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big)\operatorname{tr}\bar{Q}^{-1}\big(\operatorname{tr}2\Sigma_f\bar{A}\bar{Q}\bar{A}'-2K_c\big)+4\operatorname{tr}\Sigma_f\bar{A}\bar{Q}_c^{-1}\bar{A}'\big(\mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big)\big)^2.
$$

PROOF. We will analyze separately $k_n\sqrt{p}(\widehat{B}_c-\widetilde{B}_c)$ and $k_n\sqrt{p}(\widetilde{B}_c-B_c)$. We will denote with *Rem* terms that depend on R_c.

(i) Bound for $k_n\sqrt{p}(\widehat{B}_c-\widetilde{B}_c)$ and $\sqrt{p}k_n(\widehat{B}_{\mathrm{mix},k}-\widetilde{B}_{\mathrm{mix},k})$. We have

$$
\mathbb{E}\widehat{\big(\sigma_{c,1}^2|\mathcal{C}\big)}:=\frac{1}{pk_n}\|\widehat{U}_c\|_F^2-\delta_c,\qquad \delta_c:=-\frac{K_c}{k_n}\frac{1}{pk_n}\|\widehat{U}_c\|_F^2-\frac{1}{p^2}\operatorname{tr}\big(\widehat{\beta}_c'\widehat{D}_c\widehat{\beta}_c\big).
$$

From Lemma E.2, $\frac{1}{k_n}\widehat{F}_c'\widehat{F}_c=\widehat{Q}_c\xrightarrow{\mathbb{P}}\bar{Q}$, $\frac{1}{pk_n}\|\widehat{U}_c\|_F^2\xrightarrow{\mathbb{P}}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})$. Therefore,

$$
\begin{aligned}
k_n\sqrt{p}(\widehat{B}_c-\widetilde{B}_c) &= -\sqrt{p}\frac{2}{k_n}\operatorname{tr}\big(\widehat{Q}_c^{-1}\widehat{F}_c'\widehat{F}_c\widehat{Q}_c^{-1}\big)\delta_c=\frac{\sqrt{p}}{k_n}2K_c\operatorname{tr}\big(\widehat{Q}_c^{-1}\big)\frac{1}{pk_n}\|\widehat{U}_c\|_F^2+O_P\big(p^{-1/2}\big)\\
&= \frac{\sqrt{p}}{k_n}2K_c\operatorname{tr}\big(\bar{Q}^{-1}\big)\mathbb{E}\big(\sigma_{c,1}^2|\mathcal{C}\big)+O_P\big(p^{-1/2}\big)+o_P\left(\frac{\sqrt{p}}{k_n}\right).
\end{aligned}
$$

As for $\sqrt{p}k_n(\widehat{B}_{\text{mix},k} - \widetilde{B}_{\text{mix},k})$, we use the identity $\frac{1}{k_n}\widehat{F}'_{\text{mix},k}\widehat{F}_{\text{mix},k} = \widehat{Q}_{\text{mix},k}$ and note that from Lemma E.2, $\widehat{Q}_{\text{mix},k} \xrightarrow{\mathbb{P}} \bar{Q}$. As a result,

$$\begin{aligned}\sqrt{p}k_n(\widehat{B}_{\text{mix},k} - \widetilde{B}_{\text{mix},k}) &= -\sum_{c=a,b} \frac{2\sqrt{p}k_n}{k_n^2} \operatorname{tr} \widehat{Q}^{-1}_{\text{mix},k}\widehat{F}'_{c,k}\widehat{F}_{c,k}\widehat{Q}^{-1}_{\text{mix},k}\delta_c \\ &= \frac{2K_c\sqrt{p}}{k_n^2} \operatorname{tr} \widehat{Q}^{-1}_{\text{mix},k}\widehat{F}'_{\text{mix},k}\widehat{F}_{\text{mix},k}\widehat{Q}^{-1}_{\text{mix},k}\mathbb{E}(\sigma_c^2|\mathcal{C}) \\ &\quad + o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right) \\ &= \frac{2K_c\sqrt{p}}{k_n} \operatorname{tr} \bar{Q}^{-1}\mathbb{E}(\sigma_c^2|\mathcal{C}) + o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right).\end{aligned}$$

(ii) Bound for $k_n\sqrt{p}(\widetilde{B}_c - B_c)$ and $k_n\sqrt{p}(\widetilde{B}_{\text{mix},k} - B_{\text{mix},k})$. Using the identity $\frac{1}{k_n}\widehat{F}'_c\widehat{F}_c = \widehat{Q}_c$, we have

$$\begin{aligned}k_n\sqrt{p}(\widetilde{B}_c - B_c) &= \sqrt{p}\left(\frac{2}{k_n} \operatorname{tr} \widehat{Q}_c^{-1}\widehat{F}'_c\widehat{F}_c\widehat{Q}_c^{-1}\frac{1}{pk_n}\|\widehat{U}_c\|_F^2 - \frac{2}{k_n} \operatorname{tr} \widehat{A}'_c\overline{F}'_c\overline{F}_c\widehat{A}_c\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})\right) \\ &= a_1 + a_2 + a_3, \\ a_1 &= 2\sqrt{p} \operatorname{tr} \widehat{Q}_c^{-1}\left(\frac{1}{pk_n}\|U_c\|_F^2 - \mathbb{E}(\sigma_{c,1}^2|\mathcal{C})\right) = O_P(k_n^{-1/2}), \\ a_2 &= 2\sqrt{p} \operatorname{tr} \widehat{Q}_c^{-1}\frac{1}{pk_n}(\|\widehat{U}_c\|_F^2 - \|U_c\|_F^2), \\ a_3 &= \sqrt{p}\left(\frac{2}{k_n} \operatorname{tr} \widehat{Q}_c^{-1}\widehat{F}'_c\widehat{F}_c\widehat{Q}_c^{-1} - \frac{2}{k_n} \operatorname{tr} \widehat{A}'_c\overline{F}'_c\overline{F}_c\widehat{A}_c\right)\mathbb{E}(\sigma_{c,1}^2|\mathcal{C}).\end{aligned}$$

We start with a_2 and a_3. Recall (C.24) for the expansion of $\overline{U}_c - \widehat{U}_c = \sum_{j=1}^{6} g_j$. We can write

$$\begin{aligned}a_2 &= 2\sqrt{p} \operatorname{tr} \widehat{Q}_c^{-1}\frac{1}{pk_n}\|\widehat{U}_c - \overline{U}_c\|_F^2 + 4\sqrt{p} \operatorname{tr} \widehat{Q}_c^{-1}\frac{1}{pk_n} \operatorname{tr}(\widehat{U}_c - \overline{U}_c)'\overline{U}_c \\ &= 2\sqrt{p} \operatorname{tr} \widehat{Q}_c^{-1}\frac{1}{pk_n}\sum_j \|g_j\|_F^2 + 4\sqrt{p} \operatorname{tr} \widehat{Q}_c^{-1}\frac{1}{pk_n}\sum_{j<k} \operatorname{tr} g'_j g_k - 4\sqrt{p} \operatorname{tr} \widehat{Q}_c^{-1}\frac{1}{pk_n} \operatorname{tr}\sum_d U'_c g_d.\end{aligned}$$

In Lemma C.9, we showed $\frac{\sqrt{p}}{p^2k_n}\widehat{\beta}'_c\overline{U}_c\overline{U}'_c\widehat{\beta}_c \le o_P(1)$, under the conditions of the current lemma. Then, using Lemmas A.1, A.2, and A.3, we have

$$\begin{aligned}\frac{2\sqrt{p}}{pk_n} \operatorname{tr}(\widehat{Q}_c^{-1})\|g_1\|_F^2 &= 2\sqrt{p} \operatorname{tr} \widehat{Q}_c^{-1}\frac{1}{pk_n^2} \operatorname{tr} \overline{F}'_c\overline{U}'_c\overline{U}_c\overline{F}_c\widehat{A}_c\widehat{Q}_c\widehat{A}'_c, \\ &= \frac{2\sqrt{p}}{k_n}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C}) \operatorname{tr} \Sigma_f \bar{A}\bar{Q}\bar{A}' \operatorname{tr} \bar{Q}^{-1} + o_P\left(\frac{\sqrt{p}}{k_n}\right) + o_P(1), \\ \frac{2\sqrt{p}}{pk_n} \operatorname{tr}(\widehat{Q}_c^{-1})\|g_2\|_F^2 &= O_P(1)\left\|\frac{\sqrt{p}}{p^2k_n}\widehat{\beta}'_c\overline{U}_c\overline{U}'_c\widehat{\beta}_c\right\| = o_P(1),\end{aligned}$$

$$\frac{2\sqrt{p}}{pk_n}\operatorname{tr}(\widehat{Q}_c^{-1})\|g_3\|_F^2 = \frac{2\sqrt{p}}{p^3k_n^2}\operatorname{tr}(\widehat{Q}_c^{-1})\operatorname{tr}\widehat{Q}_c^{-1}\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c \le O_P\left(\frac{2\sqrt{p}}{p^2k_n^2}\right)\|\overline{U}_c\|^4$$
$$\le O_P\left(\frac{\sqrt{p}}{k_n}+\frac{1}{p^{3/2}}\right)\zeta_p^2 = o_P(1),$$
$$\frac{2\sqrt{p}}{pk_n}\operatorname{tr}(\widehat{Q}_c^{-1})\|g_4\|_F^2 = O_P(p^{-1/2})\left\|\frac{\sqrt{p}}{p^2k_n}\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c\right\|^2 = o_P(1),$$
$$\frac{2\sqrt{p}}{pk_n}\operatorname{tr}(\widehat{Q}_c^{-1})\|g_5\|_F^2 = \frac{2\sqrt{p}}{pk_n}\operatorname{tr}(\widehat{Q}_c^{-1})\left\|\frac{1}{pk_n}\beta_cH_cH_c'\beta_c'\overline{U}_c\overline{F}_c\widehat{A}_cH_c^{-1}\overline{F}_c'\right\|_F^2$$
$$+o_P(1)+O_P\left(\frac{\sqrt{p}}{k_n^{3/2}}\right)$$
$$= o_P(1),$$
$$\frac{2\sqrt{p}}{pk_n}\operatorname{tr}(\widehat{Q}_c^{-1})\|g_6\|_F^2 = o_P(1).$$

As for terms that involve $\sum_{j<k}\operatorname{tr} g_j'g_k$, we apply the Cauchy–Schwarz inequality:

$$4\sqrt{p}\operatorname{tr}\widehat{Q}_c^{-1}\frac{1}{pk_n}\sum_{j<k}\operatorname{tr} g_j'g_k \le \sum_{j<k}O_P\left(\sqrt{\frac{\sqrt{p}}{pk_n}\|g_j\|_F^2}\right)O_P\left(\sqrt{\frac{\sqrt{p}}{pk_n}\|g_k\|_F^2}\right)$$
$$\le\left[O_P\left(\sqrt{\frac{\sqrt{p}}{k_n}}\right)+o_P(1)\right]\left[o_P\left(\sqrt{\frac{\sqrt{p}}{k_n}}\right)+o_P(1)\right] = o_P(1),$$

under the condition $p = O(k_n^2)$,

In the proof of Lemma C.9, we showed $\|\frac{\sqrt{p}}{p^2k_n}\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\beta_cH_c\| + \|\frac{\sqrt{p}}{p^2k_n}\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c\| \le o_P(1)$, under the conditions of the current lemma. In addition, Lemma A.2(ii)(iv) bounded $\|\overline{U}_c\|$ and $\|\overline{U}_c\overline{F}_c\|$. Lemma E.1 showed the variance of each element of $\frac{\sqrt{p}}{pk_n^2}\overline{F}_c'\overline{U}_c'\overline{U}_c\overline{F}_c$ is $o(1)$. Then the identity $\bar{H} = \Sigma_f\bar{A}$ yields, when $\zeta_p = o(p^{2/3})$ and $\zeta_p^3 p = o(k_n^3)$,

$$-\frac{4\sqrt{p}}{pk_n}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\overline{U}_c'g_1 = -\frac{4\sqrt{p}}{pk_n^2}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\overline{F}_c'\overline{U}_c'\overline{U}_c\overline{F}_c\widehat{A}_cH_c^{-1}+o_P(1)+o_P\left(\frac{\sqrt{p}}{k_n}\right)$$
$$= -\frac{4\sqrt{p}}{k_n}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})\operatorname{tr}\bar{Q}^{-1}K_c+o_P(1)+o_P\left(\frac{\sqrt{p}}{k_n}\right),$$
$$-\frac{4\sqrt{p}}{pk_n}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\overline{U}_c'g_2 = -\frac{4\sqrt{p}}{p^2k_n}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\beta_cH_c = o_P(1),$$
$$-\frac{4\sqrt{p}}{pk_n}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\overline{U}_c'g_3 = -\frac{4\sqrt{p}}{pk_n}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\overline{U}_c'\frac{1}{pk_n}\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}\widehat{F}_c'$$
$$\le O_P\left(\frac{1}{pk_n^{3/2}}\right)\|\overline{U}_c\|^3 \le \zeta_p^{3/2}\left(\frac{1}{p}+\frac{p^{1/2}}{k_n^{3/2}}\right) = o_P(1),$$

$$
\begin{aligned}
-\frac{4\sqrt{p}}{pk_n}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\overline{U}_c'g_4 &= \frac{4\sqrt{p}}{pk_n}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\overline{U}_c'\frac{1}{p^2k_n}\beta_cH_c\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}H_c^{-1}\overline{F}_c' \\
&\le O_P\left(\frac{\sqrt{k}}{p^2k_n^2}\right)\|\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c\| + o_P(1) = o_P(1), \\
-\frac{4\sqrt{p}}{pk_n}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\overline{U}_c'g_5 &= \frac{4\sqrt{p}}{pk_n}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\frac{1}{pk_n}H_c\widehat{\beta}_c'\overline{U}_c\overline{F}_c\widehat{A}_cH_c^{-1}\overline{F}_c'\overline{U}_c'\beta_c \\
&\le O_P\left(\frac{\sqrt{k_n}}{pk_n^2}\right)\|\widehat{\beta}_c\|\|\overline{U}_c\overline{F}_c\| = o_P(1), \\
-\frac{4\sqrt{p}}{pk_n}\operatorname{tr}\widehat{Q}_c^{-1}\operatorname{tr}\overline{U}_c'g_6 &= o_P(1).
\end{aligned}
$$

Here, term g_6 depends on Rem, which is negligible.

Together, $a_2 = \frac{\sqrt{p}}{k_n}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})\operatorname{tr}\bar{Q}^{-1}(\operatorname{tr}2\Sigma_f\bar{A}\bar{Q}\bar{A}' - 4K_c) + o_P(1) + o_P(\frac{\sqrt{p}}{k_n})$.

Next, we have $\widehat{F}_c\widehat{Q}_c^{-1} - \overline{F}_c\widehat{A}_c = \overline{U}_c'\beta_c\widehat{Q}_c^{-1}/p + Rem$ and $\widehat{\beta}_c - \beta_cH = \frac{1}{k_n}\overline{U}_c\overline{F}_c\widehat{A}_c + \Delta_{1c}$. Also, Lemma E.3 showed $\frac{1}{\sqrt{p}}\overline{F}_c'\overline{U}_c'\Delta_{1c} = O_P(\frac{\sqrt{p}}{k_n}) + o_P(1)$. By Lemma C.9, $\frac{\sqrt{p}}{p^2k_n}\widehat{\beta}_c'\overline{U}_c\overline{U}_c' \times \widehat{\beta}_c \le o_P(1)$. And by Lemma A.2, $\overline{F}_c'\overline{U}_c'\beta_c = O_P(\sqrt{k_np})$. Hence,

$$
\begin{aligned}
a_3 &= \sqrt{p}\frac{2}{k_np}\operatorname{tr}\widehat{Q}_c^{-1}\widehat{\beta}_c'\overline{U}_c\widehat{F}_c\widehat{Q}_c^{-1}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C}) + \sqrt{p}\frac{2}{k_np}\operatorname{tr}\widehat{A}_c'\overline{F}_c'\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C}) + Rem \\
&= \sqrt{p}\frac{4}{k_np}\operatorname{tr}\widehat{A}_c'\overline{F}_c'\overline{U}_c'(\widehat{\beta}_c - \beta_cH_c)\widehat{Q}_c^{-1}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C}) + \sqrt{p}\frac{4}{k_np}\operatorname{tr}\widehat{A}_c'\overline{F}_c'\overline{U}_c'\beta_cH_c\widehat{Q}_c^{-1}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C}) \\
&\quad + \sqrt{p}\frac{2}{k_np^2}\operatorname{tr}\widehat{Q}_c^{-1}\widehat{\beta}_c'\overline{U}_c\overline{U}_c'\widehat{\beta}_c\widehat{Q}_c^{-1}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C}) + Rem \\
&= \sqrt{p}\frac{4}{k_n^2p}\operatorname{tr}\widehat{A}_c'\overline{F}_c'\overline{U}_c'\overline{U}_c\overline{F}_c\widehat{A}_c\widehat{Q}_c^{-1}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C}) + o_P\left(\frac{\sqrt{p}}{k_n}\right) + o_P(1) \\
&= \frac{4\sqrt{p}}{k_n}\operatorname{tr}\Sigma_f\bar{A}\bar{Q}_c^{-1}\bar{A}'\left(\mathbb{E}(\sigma_{c,1}^2|\mathcal{C})\right)^2 + o_P\left(\frac{\sqrt{p}}{k_n}\right) + o_P(1).
\end{aligned}
$$

Together,

$$
k_n\sqrt{p}(\widetilde{B}_c - B_c) = \frac{\sqrt{p}}{k_n}M + o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right),
$$

where $M = \mathbb{E}(\sigma_{c,1}^2|\mathcal{C})\operatorname{tr}\bar{Q}^{-1}(\operatorname{tr}2\Sigma_f\bar{A}\bar{Q}\bar{A}' - 4K_c) + 4\operatorname{tr}\Sigma_f\bar{A}\bar{Q}_c^{-1}\bar{A}'(\mathbb{E}(\sigma_{c,1}^2|\mathcal{C}))^2$.

We are left with $\sqrt{p}k_n(\widehat{B}_{\text{mix},k} - B_{\text{mix},k})$. Its proof is analogous to the one for $\sqrt{p}k_n(\widehat{B}_c - B_c)$, so we only sketch the leading terms to avoid repetition:

$$
\begin{aligned}
B_{\text{mix},k} &= \frac{2}{k_n^2}\operatorname{tr}\sum_{c=a,b}\widehat{A}_{\text{mix},k}'\overline{F}_{c,k}'\overline{F}_{c,k}\widehat{A}_{\text{mix}}\mathbb{E}(\sigma_{c,1}^2|\mathcal{C}), \\
\widetilde{B}_{\text{mix},k} &= \frac{2}{k_n^3p}\operatorname{tr}\sum_{c=a,b}\widehat{Q}_{\text{mix},k}^{-1}\widehat{F}_{c,k}'\widehat{F}_{c,k}\widehat{Q}_{\text{mix},k}^{-1}\|\widehat{U}_c\|_F^2,
\end{aligned}
$$

$$k_n\sqrt{p}(\widehat{B}_{\mathrm{mix},k} - B_{\mathrm{mix},k}) = a_{1\mathrm{mix}} + a_{2\mathrm{mix}} + a_{3\mathrm{mix}},$$

$$\begin{aligned} a_{1\mathrm{mix}} &= \sqrt{p}\frac{2}{k_n}\operatorname{tr}\sum_{c=a,b}\widehat{Q}^{-1}_{\mathrm{mix},k}\widehat{F}'_{c,k}\widehat{F}_{c,k}\widehat{Q}^{-1}_{\mathrm{mix},k}\left(\frac{1}{k_n p}\|U_c\|_F^2 - \mathbb{E}(\sigma_c^2|\mathcal{C})\right) \\ &= o_P(1), \\ a_{2\mathrm{mix}} &= \sqrt{p}\frac{2}{k_n}\operatorname{tr}\sum_{c=a,b}\widehat{Q}^{-1}_{\mathrm{mix},k}\widehat{F}'_{c,k}\widehat{F}_{c,k}\widehat{Q}^{-1}_{\mathrm{mix},k}\left(\frac{1}{k_n p}\|\widehat{U}_c\|_F^2 - \frac{1}{k_n p}\|U_c\|_F^2\right), \\ a_{3\mathrm{mix}} &= \sqrt{p}\frac{2}{k_n}\operatorname{tr}\sum_{c=a,b}(\widehat{Q}^{-1}_{\mathrm{mix},k}\widehat{F}'_{c,k}\widehat{F}_{c,k}\widehat{Q}^{-1}_{\mathrm{mix},k} - \widehat{A}'_{\mathrm{mix},k}\overline{F}'_{c,k}\overline{F}_{c,k}\widehat{A}_{\mathrm{mix}}) \\ &\qquad \times \mathbb{E}(\sigma^2_{c,1}|\mathcal{C}). \end{aligned}$$

Recall (C.24) for the expansion of $\overline{U}_c - \widehat{U}_c = \sum_{j=1}^6 g_j$. Also, recall the identity,

$$\widehat{F}_{c,k}\widehat{Q}^{-1}_{\mathrm{mix},k} - \overline{F}_{c,k}\frac{1}{p}\beta'_c\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k} = \frac{1}{p}\overline{U}'_{c,k}\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k} + \frac{1}{p}R'_{c,k}\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k},$$

where $F_{a,k}\frac{1}{p}\beta'_a\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k} = F_{a,k}\widehat{A}_{\mathrm{mix},k}$ and $F_{b,k}\frac{1}{p}\beta'_b\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k} = F_{b,k}\widehat{A}_{\mathrm{mix},k}$ when $\beta_b = \beta_a$. Hence,

$$\begin{aligned} a_{2\mathrm{mix}} &= \sqrt{p}\frac{2}{k_n}\operatorname{tr}\sum_{c=a,b}\widehat{Q}^{-1}_{\mathrm{mix},k}\widehat{F}'_{c,k}\widehat{F}_{c,k}\widehat{Q}^{-1}_{\mathrm{mix},k}\frac{1}{k_n p}(\|g_1\|_F^2 - 2U'_c g_1) + o_P(1) \\ &= \frac{2\sqrt{p}}{k_n}\operatorname{tr}\bar{Q}^{-1}\operatorname{tr}\Sigma_f\bar{A}\bar{Q}\bar{A}'E(\sigma^2_{ic}|\mathcal{C}) - \frac{4\sqrt{p}}{k_n}\mathbb{E}(\sigma^2_{ic}|\mathcal{C})\operatorname{tr}\bar{Q}^{-1}K_c + o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right), \\ a_{3\mathrm{mix}} &= \sqrt{p}\frac{4}{k_n p}\operatorname{tr}\widehat{A}'_{\mathrm{mix},k}\overline{F}'_{\mathrm{mix},k}\overline{U}'_{\mathrm{mix},k}\widehat{\beta}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k}\mathbb{E}(\sigma_c^2|\mathcal{C}) + o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right) \\ &= \sqrt{p}\frac{4}{k_n^2 p}\operatorname{tr}\widehat{A}'_{\mathrm{mix},k}\overline{F}'_{\mathrm{mix},k}\overline{U}'_{\mathrm{mix},k}\overline{U}_{\mathrm{mix},k}\overline{F}_{\mathrm{mix},k}\widehat{A}_{\mathrm{mix},k}\widehat{Q}^{-1}_{\mathrm{mix},k}\mathbb{E}(\sigma_c^2|\mathcal{C}) \\ &\quad + o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right) \\ &= \sqrt{p}\frac{4}{k_n^2 p}\operatorname{tr}\Sigma_f\bar{A}\bar{Q}^{-1}\bar{A}'\mathbb{E}(\sigma_c^2|\mathcal{C})^2 + o_P(1) + o_P\left(\frac{\sqrt{p}}{k_n}\right). \end{aligned}$$

Putting together all of these results, we get the expansion for $k_n\sqrt{p}(\widehat{B}_{\mathrm{mix},k} - B_{\mathrm{mix},k})$ in the lemma. □

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