

# Supplement to “The extended perturbation method: With applications to the New Keynesian model and the zero lower bound”

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## APPENDIX SA: THE CERTAINTY EQUIVALENT SOLUTION

This section describes the certainty equivalent solution to DSGE models and how it is obtained numerically by the Extended Path. Section SA.1 introduces our notation and describes the computational problem. We describe how to solve for the certainty equivalent solution in Section SA.2. Good starting values to compute the certainty equivalent solution and how to set the termination values for the control variables are presented in Section SA.3.

### SA.1 *The setup*

The setup is similar to the one given in the paper but included for completeness. That is, we consider the broad class of DSGE models, which can be expressed as

$$E_t[\mathbf{f}(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1})] = \mathbf{0} \quad \text{for all } t = 1, 2, \dots, \quad (\text{S1})$$

where  $E_t$  denotes the conditional expectation given information available in time period  $t$ . The state vector  $\mathbf{x}_t$  with dimension  $n_x \times 1$  belongs to the set  $\mathcal{X}_x$ , denoting the Borel subset of  $\mathbb{R}^{n_x}$ . The control variables are stored in  $\mathbf{y}_t$  with dimension  $n_y \times 1$  and  $\mathbf{y}_t \in \mathcal{X}_y$ , where  $\mathcal{X}_y$  refers to the Borel subset of  $\mathbb{R}^{n_y}$ . We further let  $n_x + n_y = n$ . The function  $\mathbf{f}$  takes elements from  $\mathcal{X}_x \times \mathcal{X}_x \times \mathcal{X}_y \times \mathcal{X}_y$  into  $\mathbb{R}^n$ .

It is useful to consider the partitioning  $\mathbf{x}_t \equiv [\mathbf{x}'_{1,t} \ \mathbf{x}'_{2,t}]'$ , where  $\mathbf{x}_{1,t}$  contains endogenous state variables and  $\mathbf{x}_{2,t}$  denotes exogenous states. The dimensions of these vectors are  $n_{x_1} \times 1$  and  $n_{x_2} \times 1$ , respectively, with  $n_{x_1} + n_{x_2} = n_x$ . We further assume that the dynamics of the exogenous state variables belong to the general class

$$\mathbf{x}_{2,t+1} = \mathbf{\Gamma}(\mathbf{x}_{2,t}) + \sigma \bar{\eta} \boldsymbol{\epsilon}_{t+1}, \quad (\text{S2})$$

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where  $\boldsymbol{\epsilon}_{t+1} \in \mathcal{X}_\epsilon$  and has dimension  $n_\epsilon \times 1$ . We also assume  $\boldsymbol{\epsilon}_{t+1}$  to be independent and identically distributed with zero and an unit covariance matrix, that is,  $\boldsymbol{\epsilon}_{t+1} \sim IID(\mathbf{0}, \mathbf{I})$ . The function  $\Gamma$  maps elements from  $\mathcal{X}_{x_2}$  into  $\mathcal{X}_{x_2}$ . We further assume that  $\Gamma$  generates a stable process for  $\mathbf{x}_{2,t}$ .<sup>1</sup> In linear systems, this corresponds to requiring that all eigenvalues of the  $n_{x_2} \times n_{x_2}$  Jacobian  $\partial\Gamma/\partial\mathbf{x}'_{2,t}$  lie inside the unit circle. For nonlinear mappings,  $\Gamma$  must satisfy the general stability condition for nonlinear first-order Markov systems provided in the paper.

In the certainty equivalent solution, uncertainty about future shocks is absence by definition. This corresponds to omitting the conditional expectation in (S1), which reduces to a deterministic system, that is,

$$\mathbf{f}(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = \mathbf{0} \quad \text{for all } t = 1, 2, \dots$$

The system contains an infinite number of equations and cannot be solved without any simplifying assumptions. The approach taken in the Extended Path of Fair and Taylor (1983) is to truncate the problem at some finite horizon  $N$  after which the variables are assumed to be constant, for instance, given by their deterministic steady-state values. Although this is an asymptotic model property it implies that the approximation errors from the truncation decreases in  $N$  and can be made arbitrary small for an appropriately chosen horizon  $N$  (see Fair and Taylor (1983) and Boucekkinne (1995)). Hence, the key assumption in the Extended Path is to close the infinite system in (S1) by considering a terminal value for  $\mathbf{y}_{t+N}$ . This gives rise to the finite dimensional system

$$\begin{aligned} \mathbf{f}(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) &= \mathbf{0}_{n \times 1}, \\ \mathbf{f}(\mathbf{x}_{t+1}, \mathbf{x}_{t+2}, \mathbf{y}_{t+1}, \mathbf{y}_{t+2}) &= \mathbf{0}_{n \times 1}, \\ \mathbf{f}(\mathbf{x}_{t+2}, \mathbf{x}_{t+3}, \mathbf{y}_{t+2}, \mathbf{y}_{t+3}) &= \mathbf{0}_{n \times 1}, \\ &\dots \\ \mathbf{f}(\mathbf{x}_{t+N-1}, \mathbf{x}_{t+N}, \mathbf{y}_{t+N-1}, \mathbf{y}_{t+N}) &= \mathbf{0}_{n \times 1}. \end{aligned} \tag{S3}$$

Given that  $\mathbf{f}(\cdot)$  constraints  $n$  equilibrium conditions, we thus have a total of  $n * N$  conditions. The initial state  $\mathbf{x}_t$  and  $\mathbf{y}_{t+N}$  are known by assumption, whereas we must solve for  $(\mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}, \mathbf{x}_{t+2}, \mathbf{y}_{t+2}, \dots, \mathbf{x}_{t+N-1}, \mathbf{y}_{t+N-1}, \mathbf{x}_{t+N})$ , constituting  $n * N$  unknowns. Thus, the certainty equivalent solution as given by the Extended Path is obtained by solving the fixed-point problem implied by (S3).

**SA.1.1 System reduction I** To reduce the computational burden when solving the fixed-point problem in (S3), the exogenous states  $\{\mathbf{x}_{2,t+j}\}_{j=1}^N$  can be concentrated out of the system as they can be computed directly by iterating on (S2). Hence, the concen-

<sup>1</sup>This implies that trends may only be included in the class of DSGE models considered, if a given model after rescaling has an equivalent representation without trending variables. The procedure is carefully described in King and Rebelo (1999).

trated fixed-point problem is given by

$$\begin{aligned}
 \mathbf{f}_1(\mathbf{x}_{1,t}, \mathbf{x}_{1,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) &= \mathbf{0}_{n_1 \times 1}, \\
 \mathbf{f}_1(\mathbf{x}_{1,t+1}, \mathbf{x}_{1,t+2}, \mathbf{y}_{t+1}, \mathbf{y}_{t+2}) &= \mathbf{0}_{n_1 \times 1}, \\
 \mathbf{f}_1(\mathbf{x}_{1,t+2}, \mathbf{x}_{1,t+3}, \mathbf{y}_{t+2}, \mathbf{y}_{t+3}) &= \mathbf{0}_{n_1 \times 1}, \\
 &\dots \\
 \mathbf{f}_1(\mathbf{x}_{1,t+N-1}, \mathbf{x}_{1,t+N}, \mathbf{y}_{t+N-1}, \mathbf{y}_{t+N}) &= \mathbf{0}_{n_1 \times 1},
 \end{aligned} \tag{S4}$$

where we have introduced the partitioning

$$\mathbf{f}(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \equiv \begin{bmatrix} \mathbf{f}_1(\mathbf{x}_{1,t}, \mathbf{x}_{1,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \mathbf{f}_2(\mathbf{x}_{2,t}, \mathbf{x}_{2,t+1}) \end{bmatrix}.$$

The function  $\mathbf{f}_2(\cdot)$  has dimensions  $n_2 \times 1$  where  $n_2 \equiv n_{x_2}$  and contains the law of motions for the exogenous variables, whereas the function  $\mathbf{f}_1(\cdot)$  has dimensions  $n_1 \times 1$  where  $n_1 \equiv n_y + n_{x_1}$  and contains all the remaining equilibrium conditions. In  $\mathbf{f}_1(\cdot)$ , we suppress the dependence of  $\mathbf{x}_{2,t}$  to reduce the notational burden. Hence, the concentrated system in (S4) has  $n_1 * N$  equations with the same number of unknowns and constitutes the certainty equivalent solution, provided that it is unaffected by increasing  $N$  (see Fair and Taylor (1983)). For numerical stability of the proposed routines below, we recommend expressing the residuals of  $\mathbf{f}$  in terms of unit-free errors.

**SA.1.2 System reduction II** Some of the most popular DSGE models are characterized by having lagged control variables in the states. For instance, lagged consumption  $c_{t-1}$  to capture the effect of consumption habits in the RBC model or the lagged interest rate  $r_{t-1}$  to capture interest rate smoothing by the central bank in a New Keynesian model. This link between some of the controls and the states can be exploited to further eliminate some of the states, as we do not need to separately solve for a path of  $\{c_t\}_{t=1}^N$  and  $\{r_{t-1}\}_{t=1}^N$ , say. The same observation is used in Binning (2013) to reduce the number of states in the standard perturbation approximation. To see how we concentrate these lagged control variables out of the system, consider the following partition:

$$\mathbf{x}_t = \begin{bmatrix} \mathbf{x}_{11,t} \\ \mathbf{x}_{12,t} \\ \mathbf{x}_{2,t} \end{bmatrix}.$$

Here,  $\mathbf{x}_{11,t}$  contains the true endogenous state variables (such as capital), whereas  $\mathbf{x}_{12,t}$  contains the lagged control variables (which also are endogenous states). As in Binning (2013), we say that  $\mathbf{x}_{11,t}$  has dimension  $m_x \times 1$  and  $\mathbf{x}_{12,t}$  has dimension  $m_{yx} \times 1$ . As before,  $\mathbf{x}_{2,t}$  refers to the exogenous states. Hence, the concentrated fixed-point problem is given

by

$$\begin{aligned}
 \mathbf{f}_1(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{x}_{11,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) &= \mathbf{0}_{n_1 \times 1}, \\
 \mathbf{f}_1(\mathbf{y}_t, \mathbf{x}_{11,t+1}, \mathbf{x}_{11,t+2}, \mathbf{y}_{t+1}, \mathbf{y}_{t+2}) &= \mathbf{0}_{n_1 \times 1}, \\
 \mathbf{f}_1(\mathbf{y}_{t+1}, \mathbf{x}_{11,t+2}, \mathbf{x}_{11,t+3}, \mathbf{y}_{t+2}, \mathbf{y}_{t+3}) &= \mathbf{0}_{n_1 \times 1}, \\
 &\dots \\
 \mathbf{f}_1(\mathbf{y}_{t+N-2}, \mathbf{x}_{11,t+N-1}, \mathbf{x}_{11,t+N}, \mathbf{y}_{t+N-1}, \mathbf{y}_{t+N}) &= \mathbf{0}_{n_1 \times 1},
 \end{aligned} \tag{S5}$$

where we have introduced the partitioning

$$\mathbf{f}(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \equiv \begin{bmatrix} \mathbf{f}_1(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{x}_{11,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \mathbf{f}_{12}(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{x}_{11,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \mathbf{f}_2(\mathbf{x}_{2,t}, \mathbf{x}_{2,t+1}) \end{bmatrix}.$$

Note that we here explicitly account for the fact that  $\mathbf{y}_{t-1}$  enters in the  $\mathbf{f}$ -function, as this effect is not captured by  $\mathbf{x}_{11,t}$ . This is what makes this additional concentration “trick” different from the one presented in Section SA.1.1. Furthermore,  $\mathbf{f}_1(\mathbf{y}_{t-1}, \mathbf{x}_{1,t}, \mathbf{x}_{1,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1})$  contains the first  $n_y + m_x$  equations in the model,  $\mathbf{f}_{12}(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{x}_{11,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1})$  contains the  $m_{yx}$  link-equations for defining the lagged control variables, and finally  $\mathbf{f}_2(\cdot)$  contains the exogenous shocks with dimension  $n_2 \times 1$  where  $n_2 \equiv n_{x_2}$ .

### SA.2 Solving the fixed-point problem

This subsection describes how we solve for the fixed-point problem in (S4) when using the system reduction described in Section SA.1.1. To do so, we let the unknowns be denoted by

$$\mathbf{Z} \equiv \begin{bmatrix} \mathbf{y}_t & \mathbf{y}_{t+1} & \cdots & \mathbf{y}_{t+N-2} & \mathbf{y}_{t+N-1} \\ \mathbf{x}_{1,t+1} & \mathbf{x}_{1,t+2} & \cdots & \mathbf{x}_{1,t+N-1} & \mathbf{x}_{1,t+N} \end{bmatrix},$$

meaning that the system in (S4) can be condensely expressed as

$$\mathbf{F}(\mathbf{Z}) \equiv \begin{bmatrix} \mathbf{f}_1(\mathbf{x}_{1,t}, \mathbf{x}_{1,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \mathbf{f}_1(\mathbf{x}_{1,t+1}, \mathbf{x}_{1,t+2}, \mathbf{y}_{t+1}, \mathbf{y}_{t+2}) \\ \mathbf{f}_1(\mathbf{x}_{1,t+2}, \mathbf{x}_{1,t+3}, \mathbf{y}_{t+2}, \mathbf{y}_{t+3}) \\ \cdots \\ \mathbf{f}_1(\mathbf{x}_{1,t+N-1}, \mathbf{x}_{1,t+N}, \mathbf{y}_{t+N-1}, \mathbf{y}_{t+N}) \end{bmatrix} = \mathbf{0}.$$

Linearizing this system around the point  $\mathbf{Z}^*$  gives

$$\mathbf{F}(\mathbf{Z}) \approx \mathbf{F}(\mathbf{Z}^*) + \mathbf{J}(\mathbf{Z}^*)(\text{vec}(\mathbf{Z}) - \text{vec}(\mathbf{Z}^*)), \tag{S6}$$

where  $\mathbf{J}(\mathbf{Z}^*)$  denotes the Jacobian evaluated at  $\mathbf{Z}^*$ . That is,

$$\mathbf{J}(\mathbf{Z}^*) \equiv \left. \frac{\partial \mathbf{F}(\mathbf{Z})}{\partial \text{vec}(\mathbf{Z})'} \right|_{\mathbf{Z}=\mathbf{Z}^*},$$

which has dimensions  $(n_1 * N) \times (n_1 * N)$ .

When also using the system reduction in Section SA.1.2, we define

$$\mathbf{Z} \equiv \begin{bmatrix} \mathbf{y}_t & \mathbf{y}_{t+1} & \cdots & \mathbf{y}_{t+N-2} & \mathbf{y}_{t+N-1} \\ \mathbf{x}_{11,t+1} & \mathbf{x}_{11,t+2} & \cdots & \mathbf{x}_{11,t+N-1} & \mathbf{x}_{11,t+N} \end{bmatrix}$$

and the system in (S5) can be condensely expressed as

$$\mathbf{F}(\mathbf{Z}) \equiv \begin{bmatrix} \mathbf{f}_1(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{x}_{11,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \mathbf{f}_1(\mathbf{y}_t, \mathbf{x}_{11,t+1}, \mathbf{x}_{11,t+2}, \mathbf{y}_{t+1}, \mathbf{y}_{t+2}) \\ \mathbf{f}_1(\mathbf{y}_{t+1}, \mathbf{x}_{11,t+2}, \mathbf{x}_{11,t+3}, \mathbf{y}_{t+2}, \mathbf{y}_{t+3}) \\ \cdots \\ \mathbf{f}_1(\mathbf{y}_{t+N-2}, \mathbf{x}_{11,t+N-1}, \mathbf{x}_{11,t+N}, \mathbf{y}_{t+N-1}, \mathbf{y}_{t+N}) \end{bmatrix} = \mathbf{0}.$$

SA.2.1 *The standard Newton–Raphson routine* The most efficient way to solve (S4) (if we can obtain convergence) is to use the Newton–Raphson routine. To describe this routine, let  $\mathbf{Z}^i$  denote the value of  $\mathbf{Z}$  at iteration  $i$ , and use (S6) for  $\mathbf{Z}^i = \mathbf{Z}^*$  to obtain

$$\mathbf{F}(\mathbf{Z}) \approx \mathbf{F}(\mathbf{Z}^i) + \mathbf{J}(\mathbf{Z}^i)(\text{vec}(\mathbf{Z}) - \text{vec}(\mathbf{Z}^i)).$$

Let  $\mathbf{F}(\mathbf{Z}) = \mathbf{0}$  and isolate for  $\text{vec}(\mathbf{Z})$ , that is,

$$\text{vec}(\mathbf{Z}) = \text{vec}(\mathbf{Z}^i) - \mathbf{J}(\mathbf{Z}^i)^{-1} \mathbf{F}(\mathbf{Z}^i).$$

Hence, the Newton–Raphson routine is then

$$\text{vec}(\mathbf{Z}^{i+1}) = \text{vec}(\mathbf{Z}^i) - \mathbf{J}(\mathbf{Z}^i)^{-1} \mathbf{F}(\mathbf{Z}^i). \quad (\text{S7})$$

We iterate until the change in  $\max |\text{vec}(\mathbf{Z}^{i+1}) - \text{vec}(\mathbf{Z}^i)|$  is smaller than some tolerance. Note that this routine is sometimes referred to as a Newton–Raphson relaxation algorithm (see Boucekine (1995)). When computing the inversion of the Jacobian we use the efficient method of Boucekine (1995) (see Section SB of this technical Appendix for a presentation of this method using our notation).

SA.2.2 *The extended Newton–Raphson routine* When the problem is very nonlinear, the standard Newton–Raphson may fail to converge. If this is the case, then we consider an extended Newton–Raphson routine of the form

$$\text{vec}(\mathbf{Z}^{i+1}(\delta)) = \text{vec}(\mathbf{Z}^i) - \delta(\mathbf{J}(\mathbf{Z}^i))^{-1} \mathbf{F}(\mathbf{Z}^i), \quad (\text{S8})$$

where the scaling parameter  $\delta \in \mathbb{R}$  is determined by a rough grid search to the problem

$$\delta = \arg \min_{\delta} \mathbf{F}(\mathbf{Z}^{i+1}(\delta))' \mathbf{F}(\mathbf{Z}^{i+1}(\delta)).$$

That is,  $\delta$  accounts for the possibility that a linear approximation to  $\mathbf{F}(\mathbf{Z})$  may be insufficiently accurate. We iterate on (S8) until the change in  $\max |\text{vec}(\mathbf{Z}^{i+1}) - \text{vec}(\mathbf{Z}^i)|$  is smaller than some tolerance, and we refer to this algorithm as an extended Newton–Raphson relaxation routine.

SA.2.3 *Minimizing the squared model-residuals by the LM optimizer* If these routines are unsuccessful in solving the fixed-point problem in (S4), then we use the Levenberg-Marquardt (LM) optimizer to minimize  $\mathbf{F}(\mathbf{Z})'\mathbf{F}(\mathbf{Z})$  across  $\mathbf{Z}$ .

### SA.3 *Determining starting values and terminal values for the controls*

To obtain fast convergence when solving the fixed-point problem in (S4), it is essential to have good starting values. They are typically computed based on a first-order approximation, that is, by iterating on

$$\begin{aligned}\mathbf{y}_{t+i}^{1st} - \mathbf{y}_{ss} &= \mathbf{g}_{\mathbf{x}}(\mathbf{x}_{t+i}^{1st} - \mathbf{x}_{ss}), \\ \mathbf{x}_{t+i}^{1st} - \mathbf{x}_{ss} &= \mathbf{h}_{\mathbf{x}}(\mathbf{x}_{t+i-1}^{1st} - \mathbf{x}_{ss})\end{aligned}\tag{S9}$$

for  $i = 1, 2, \dots, N$  where  $\mathbf{x}_t^{1st} = \mathbf{x}_t$ . Adjemian and Juillard (2010) further notes that it is possible to reduce the value of  $N$ , and hence increase the speed of the certainty equivalent solver by replacing the standard terminal condition of  $\mathbf{y}_{t+N} = \mathbf{y}_{ss}$  by  $\mathbf{y}_{t+N} = \mathbf{y}_{t+N}^{1st}$ . Of course, for  $N \rightarrow \infty$ , we clearly have  $\mathbf{x}_{t+i}^{1st} - \mathbf{x}_{ss} \rightarrow \mathbf{0}$  and  $\mathbf{y}_{t+i}^{1st} - \mathbf{y}_{ss} \rightarrow \mathbf{0}$ , given the stability of  $\mathbf{h}_{\mathbf{x}}$ , implying that this procedure reproduces the standard terminal condition in the limit. This section generalizes these ideas by using a fourth-order perturbation approximation to compute starting values and the terminal value of  $\mathbf{y}_{t+N}$ . To reduce the computational burden, we use the perturbation on perturbation (POP) method suggested by Andreasen and Zabczyk (2015) and applied to compute conditional expectations in the technical Appendix of Andreasen (2012). Below, we provide the key results from this technical Appendix for terms under certainty equivalence (i.e., we leave out the expressions for derivatives involving the perturbation parameter) and extend these results to fourth order.

SA.3.1 *Using the POP-method to compute conditional expectations in DSGE models* Consider the case where the DSGE model reports the endogenous variable  $r_t$  and we want to compute conditional expectations of this variable, that is,  $r_{1,t} \equiv E_t[r_{t+1}]$ ,  $r_{2,t} \equiv E_t[r_{t+2}]$ ,  $r_{3,t} \equiv E_t[r_{t+3}]$ , etc.<sup>2</sup> The law of iterated expectations implies  $r_{2,t} \equiv E_t[r_{t+2}] = E_t[E_{t+1}[r_{t+2}]] = E_t[r_{1,t+1}]$  and so on. Hence, we only need to derive a formula for computing  $p_t \equiv E_t[r_{t+1}]$  because all other expectations can be found by iterating this formula. We therefore consider the problem

$$p(\mathbf{x}_t, \sigma) = E_t[r(\mathbf{x}_{t+1}, \sigma)],\tag{S10}$$

where  $\sigma$  is the perturbation parameter. We then observe that

$$F(\mathbf{x}_t, \sigma) \equiv E_t[-p(\mathbf{x}_t, \sigma) + r(\mathbf{h}(\mathbf{x}_t, \sigma) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}, \sigma)] = 0,\tag{S11}$$

because

$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t, \sigma) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}.\tag{S12}$$

<sup>2</sup>If the variable of interest is a control variable, then the function  $r(\mathbf{x}_{t+1}, \sigma)$  follows from the  $\mathbf{g}$ -function. If the variable of interest is a state variable, then we let  $r_t \equiv \mathbf{i}'\mathbf{x}_t$  to obtain moments for the  $i$ 'th state variable with  $i(k, 1) = 1$  for  $k = i$ , otherwise  $i(k, 1) = 0$ .

Note then that (S11) must hold for all values of  $(\mathbf{x}_t, \sigma)$ . This allow us to compute all derivatives of  $p$  with respect to  $(\mathbf{x}_t, \sigma)$  around the deterministic steady state, that is,  $\mathbf{x}_t = \mathbf{x}_{\text{ss}}$  and  $\sigma = 0$ , given derivatives of  $\mathbf{h}(\mathbf{x}_t, \sigma)$  and  $r(\mathbf{x}_{t+1}, \sigma)$  around the same point.

For the indices, we adopt the convention that the subscript indicates the order of differentiation, that is, a subscript 1 is for the first time we take derivatives and so on. Thus,

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 1, 2, \dots, n_x,$$

$$\gamma_1, \gamma_2, \gamma_3, \gamma_4 = 1, 2, \dots, n_x,$$

where the  $\alpha$ 's and  $\gamma$ 's will be used to index the states. We will also use superscript  $t$  and  $t + 1$  on  $p, r$ , and  $\mathbf{h}$  and their derivatives. This is done to indicate that the functions are functions of  $\mathbf{x}_t$  and  $\mathbf{x}_{t+1}$ , respectively. This distinction is relevant when we compute the derivatives, but not when these derivatives are evaluated in the deterministic steady state where  $\mathbf{x}_t = \mathbf{x}_{t+1} = \mathbf{x}_{\text{ss}}$ .

SA.3.2 *The first-order terms: For  $(\mathbf{x}_t)$ ,*

$$[F_{\mathbf{x}}(\mathbf{x}_{\text{ss}}, 0)]_{\alpha_1} = E_t[-[p'_{\mathbf{x}}]_{\alpha_1} + [r'_{\mathbf{x}^{t+1}}]_{\gamma_1} [\mathbf{h}'_{\mathbf{x}}]_{\alpha_1}^{\gamma_1}] = 0$$

⇕

$$[p'_{\mathbf{x}}]_{\alpha_1} = E_t[[r'_{\mathbf{x}^{t+1}}]_{\gamma_1} [\mathbf{h}'_{\mathbf{x}}]_{\alpha_1}^{\gamma_1}].$$

Hence, in the steady state we have

$$[p_{\mathbf{x}}]_{\alpha_1} = [r_{\mathbf{x}}]_{\gamma_1} [\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1},$$

or in standard matrix notation,

$$\mathbf{p}_{\mathbf{x}}(1; :) = \mathbf{r}_{\mathbf{x}}(1, :) \mathbf{h}_{\mathbf{x}},$$

which is identical to the expression computed in (S9).

SA.3.3 *The second-order terms: For  $(\mathbf{x}_t, \mathbf{x}_t)$ ,*

$$[F_{\mathbf{xx}}(\mathbf{x}_{\text{ss}}, 0)]_{\alpha_1 \alpha_2} = E_t[-[p'_{\mathbf{xx}}]_{\alpha_1 \alpha_2} + [r'_{\mathbf{xx}^{t+1}}]_{\gamma_1 \gamma_2} [\mathbf{h}'_{\mathbf{x}}]_{\alpha_2}^{\gamma_2} [\mathbf{h}'_{\mathbf{x}}]_{\alpha_1}^{\gamma_1} + [r'_{\mathbf{x}^{t+1}}]_{\gamma_1} [\mathbf{h}'_{\mathbf{xx}}]_{\alpha_1 \alpha_2}^{\gamma_1}] = 0$$

⇕

$$[p'_{\mathbf{xx}}]_{\alpha_1 \alpha_2} = E_t[[r'_{\mathbf{xx}^{t+1}}]_{\gamma_1 \gamma_2} [\mathbf{h}'_{\mathbf{x}}]_{\alpha_2}^{\gamma_2} [\mathbf{h}'_{\mathbf{x}}]_{\alpha_1}^{\gamma_1} + [r'_{\mathbf{x}^{t+1}}]_{\gamma_1} [\mathbf{h}'_{\mathbf{xx}}]_{\alpha_1 \alpha_2}^{\gamma_1}].$$

Hence, in the steady state we have

$$[p_{\mathbf{xx}}]_{\alpha_1 \alpha_2} = [r_{\mathbf{xx}}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1} + [r_{\mathbf{x}}]_{\gamma_1} [\mathbf{h}_{\mathbf{xx}}]_{\alpha_1 \alpha_2}^{\gamma_1},$$

or in the standard matrix notation,

$$\mathbf{p}_{\mathbf{xx}} = \mathbf{h}'_{\mathbf{x}} \mathbf{r}_{\mathbf{xx}} \mathbf{h}_{\mathbf{x}} + \sum_{\gamma_1=1}^{n_x} \mathbf{r}_{\mathbf{x}}(1, \gamma_1) \mathbf{h}_{\mathbf{xx}}(\gamma_1, :, :).$$

SA.3.4 *Third-order terms: For  $(\mathbf{x}_t, \mathbf{x}_t, \mathbf{x}_t)$ ,*

$$\begin{aligned}
& [F_{\mathbf{xxx}}(\mathbf{x}_{\text{ss}}, 0)]_{\alpha_1\alpha_2\alpha_3} \\
&= E_t[-[p_{\mathbf{xxx}}^t]_{\alpha_1\alpha_2\alpha_3} \\
&\quad + [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2\alpha_3}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1\alpha_3}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1\alpha_2}^{\gamma_1} \\
&\quad + [r_{\mathbf{x}}^{t+1}]_{\gamma_1} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_1\alpha_2\alpha_3}^{\gamma_1}] \\
&= 0 \\
&\Downarrow \\
& [p_{\mathbf{xxx}}^t]_{\alpha_1\alpha_2\alpha_3} \\
&= E_t[[r_{\mathbf{xxx}}^{t+1}]_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2\alpha_3}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1\alpha_3}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1\alpha_2}^{\gamma_1} \\
&\quad + [r_{\mathbf{x}}^{t+1}]_{\gamma_1} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_1\alpha_2\alpha_3}^{\gamma_1}].
\end{aligned}$$

Hence, in the steady state we have

$$\begin{aligned}
[p_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3} &= [r_{\mathbf{xxx}}]_{\gamma_1\gamma_2\gamma_3} [\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}]_{\gamma_1\gamma_2} [\mathbf{h}_{\mathbf{xx}}]_{\alpha_2\alpha_3}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}]_{\gamma_1\gamma_2} [\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}]_{\alpha_1\alpha_3}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}]_{\gamma_1\gamma_3} [\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}]_{\alpha_1\alpha_2}^{\gamma_1} \\
&\quad + [r_{\mathbf{x}}]_{\gamma_1} [\mathbf{h}_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3}^{\gamma_1},
\end{aligned}$$

or in the standard matrix notation,

$$\begin{aligned}
\mathbf{p}_{\mathbf{xxx}}(\alpha_1, \alpha_2, \alpha_3) &= \sum_{\gamma_3=1}^{n_x} \mathbf{h}_{\mathbf{x}}(:, \alpha_1)' \mathbf{r}_{\mathbf{xxx}}(:, :, \gamma_3) \mathbf{h}_{\mathbf{x}}(:, \alpha_2) \mathbf{h}_{\mathbf{x}}(\gamma_3, \alpha_3) \\
&\quad + \mathbf{h}_{\mathbf{x}}(:, \alpha_1)' \mathbf{r}_{\mathbf{xx}} \mathbf{h}_{\mathbf{xx}}(:, \alpha_2, \alpha_3) \\
&\quad + \sum_{\gamma_1=1}^{n_x} \mathbf{r}_{\mathbf{xx}}(\gamma_1, :) \mathbf{h}_{\mathbf{x}}(:, \alpha_2) \mathbf{h}_{\mathbf{xx}}(\gamma_1, \alpha_1, \alpha_3)
\end{aligned}$$



$$\begin{aligned}
& + \sum_{\gamma_1=1}^{n_x} \mathbf{r}_{\mathbf{xx}}(\gamma_1, :) \mathbf{h}_{\mathbf{x}}(:, \alpha_3) \mathbf{h}_{\mathbf{xx}}(\gamma_1, \alpha_1, \alpha_2) \\
& + \mathbf{r}_{\mathbf{x}}(1, :) \mathbf{h}_{\mathbf{xxx}}(:, \alpha_1, \alpha_2, \alpha_3).
\end{aligned}$$

SA.3.5 *Fourth order terms: For*  $(\mathbf{x}_t, \mathbf{x}_t, \mathbf{x}_t, \mathbf{x}_t)$  Recall that

$$\begin{aligned}
& [F_{\mathbf{xxx}}(\mathbf{x}_{\text{ss}}, 0)]_{\alpha_1 \alpha_2 \alpha_3} \\
& = E_t[-[p_{\mathbf{xxx}}^t]_{\alpha_1 \alpha_2 \alpha_3} \\
& \quad + [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
& \quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2 \alpha_3}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
& \quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_3}^{\gamma_1} \\
& \quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_2}^{\gamma_1} \\
& \quad + [r_{\mathbf{x}}^{t+1}]_{\gamma_1} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_1 \alpha_2 \alpha_3}^{\gamma_1} \\
& = 0 \\
& [F_{\mathbf{xxx}}(\mathbf{x}_{\text{ss}}, 0)]_{\alpha_1 \alpha_2 \alpha_3} = E_t[-[p_{\mathbf{xxx}}^t]_{\alpha_1 \alpha_2 \alpha_3} + Q_1 + Q_2 + Q_3 + Q_4 + Q_5],
\end{aligned}$$

where

$$\begin{aligned}
Q_1 & = [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1}, \\
Q_2 & = [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2 \alpha_3}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1}, \\
Q_3 & = [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_3}^{\gamma_1}, \\
Q_4 & = [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_2}^{\gamma_1}, \\
Q_5 & = [r_{\mathbf{x}}^{t+1}]_{\gamma_1} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_1 \alpha_2 \alpha_3}^{\gamma_1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
[Q_{1,\mathbf{x}}]_{\alpha_4} & = [r_{4\mathbf{x}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_3 \gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_4}^{\gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
& \quad + [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_3 \alpha_4}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
& \quad + [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2 \alpha_4}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
& \quad + [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_4}^{\gamma_1}, \\
[Q_{2,\mathbf{x}}]_{\alpha_4} & = [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_4}^{\gamma_4} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2 \alpha_3}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
& \quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_2 \alpha_3 \alpha_4}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
& \quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2 \alpha_3}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_4}^{\gamma_1},
\end{aligned}$$

$$\begin{aligned}
[Q_{3,\mathbf{x}}]_{\alpha_4} &= [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_4}^{\gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_3}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2 \alpha_4}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_3}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_1 \alpha_3 \alpha_4}^{\gamma_1}, \\
[Q_{4,\mathbf{x}}]_{\alpha_4} &= [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_3 \gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_4}^{\gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_2}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_3 \alpha_4}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_2}^{\gamma_1} \\
&\quad + [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_1 \alpha_2 \alpha_4}^{\gamma_1}, \\
[Q_{5,\mathbf{x}}]_{\alpha_4} &= [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_4}^{\gamma_4} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_1 \alpha_2 \alpha_3}^{\gamma_1} + [r_{\mathbf{x}}^{t+1}]_{\gamma_1} [\mathbf{h}_{\mathbf{xxxx}}^t]_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{\gamma_1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
[F_{4\mathbf{x}}(\mathbf{x}_{\text{SS}}, 0)]_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} &= E_t[-[p_{4\mathbf{x}}^t]_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \\
(1) \quad &+ [r_{4\mathbf{x}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_3 \gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_4}^{\gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
(2) \quad &+ [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_3 \alpha_4}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
(3) \quad &+ [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2 \alpha_4}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
(4) \quad &+ [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_4}^{\gamma_1} \\
(5) \quad &+ [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_4}^{\gamma_4} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2 \alpha_3}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
(6) \quad &+ [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_2 \alpha_3 \alpha_4}^{\gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_1}^{\gamma_1} \\
(7) \quad &+ [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2 \alpha_3}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_4}^{\gamma_1} \\
(8) \quad &+ [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_2 \gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_4}^{\gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_3}^{\gamma_1} \\
(9) \quad &+ [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_2 \alpha_4}^{\gamma_2} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_3}^{\gamma_1} \\
(10) \quad &+ [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_2} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_2}^{\gamma_2} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_1 \alpha_3 \alpha_4}^{\gamma_1} \\
(11) \quad &+ [r_{\mathbf{xxx}}^{t+1}]_{\gamma_1 \gamma_3 \gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_4}^{\gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_2}^{\gamma_1} \\
(12) \quad &+ [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_3 \alpha_4}^{\gamma_3} [\mathbf{h}_{\mathbf{xx}}^t]_{\alpha_1 \alpha_2}^{\gamma_1} \\
(13) \quad &+ [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_3} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_3}^{\gamma_3} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_1 \alpha_2 \alpha_4}^{\gamma_1} \\
(14) \quad &+ [r_{\mathbf{xx}}^{t+1}]_{\gamma_1 \gamma_4} [\mathbf{h}_{\mathbf{x}}^t]_{\alpha_4}^{\gamma_4} [\mathbf{h}_{\mathbf{xxx}}^t]_{\alpha_1 \alpha_2 \alpha_3}^{\gamma_1} \\
(15) \quad &+ [r_{\mathbf{x}}^{t+1}]_{\gamma_1} [\mathbf{h}_{\mathbf{xxxx}}^t]_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{\gamma_1} \\
&= 0.
\end{aligned}$$

## APPENDIX SB: EFFICIENT INVERSION OF THE JACOBIAN: SYSTEM REDUCTION I

This section presents the efficient solution algorithm of Boucekine (1995) to compute  $\mathbf{J}(\mathbf{Z})^{-1}\mathbf{F}(\mathbf{Z})$ . In this section, we consider

$$\mathbf{f}(\mathbf{x}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}) = \mathbf{0}$$

or the condensed version of this system, that is,  $\mathbf{f}_1(\mathbf{x}_{1,t}, \mathbf{y}_t, \mathbf{x}_{1,t+1}, \mathbf{y}_{t+1}) = \mathbf{0}$ . Whether we use the normal or condensed system in Section SA.1.1, we will express it as

$$\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{x}_p, \mathbf{y}_p) = \mathbf{0}.$$

To reduce the notational burden, let

$$\mathbf{r}_t \equiv [\mathbf{x}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}],$$

or

$$\mathbf{r}_t \equiv [\mathbf{x}_{1,t}, \mathbf{y}_t, \mathbf{x}_{1,t+1}, \mathbf{y}_{t+1}],$$

if using the condensed system. We then consider the following notation for the partial derivatives:

$$\mathbf{f}_x(\mathbf{r}_t) = \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{x}},$$

$$\mathbf{f}_y(\mathbf{r}_t) = \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{y}},$$

$$\mathbf{f}_{xp}(\mathbf{r}_t) = \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{x}_p},$$

$$\mathbf{f}_{yp}(\mathbf{r}_t) = \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{y}_p}.$$

To understand the logic of the solution method of Boucekine (1995), it is useful to gradually extend the horizon  $N$  when solving the certainty equivalent solution.

## SB.1 2 time periods

Let  $N = 2$  and consider the system

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{f}_x(\mathbf{r}_t) & \mathbf{f}_y(\mathbf{r}_t) & \mathbf{f}_{xp}(\mathbf{r}_t) & \mathbf{f}_{yp}(\mathbf{r}_t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix}$$



$$= \begin{bmatrix} \mathbf{x}_t \\ -\mathbf{f}_x(\mathbf{r}_t)\mathbf{x}_t - \mathbf{f}(\mathbf{r}_t) \\ \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

⇕

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{I}_{n, n_y} & \mathbf{I}_{n, n_x} & \mathbf{W}_t^{-1} \mathbf{f}_{yp}(\mathbf{r}_t) & \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{x}_t \\ \mathbf{W}_t^{-1}(-\mathbf{f}_x(\mathbf{r}_t)\mathbf{x}_t - \mathbf{f}(\mathbf{r}_t)) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix},$$

where  $\mathbf{W}_t \equiv [\mathbf{f}_y(\mathbf{r}_t) \ \mathbf{f}_{xp}(\mathbf{r}_t)]$  and we use the partitioning  $\mathbf{I}_n \equiv [\mathbf{I}_{n, n_y} \ \mathbf{I}_{n, n_x}]$  where  $\mathbf{I}_{n, n_y}$  are the first  $n_y$  columns of  $\mathbf{I}_n$  and  $\mathbf{I}_{n, n_x}$  denotes the last columns of  $\mathbf{I}_n$ .

⇕

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix},$$

where  $[\mathbf{Q}_{n_y}(t)] \equiv \mathbf{W}_t^{-1} \mathbf{f}_{yp}(\mathbf{r}_t)$  and  $[\mathbf{R}_{n_x}(t)] \equiv -\mathbf{W}_t^{-1}(\mathbf{f}_x(\mathbf{r}_t)\mathbf{x}_t + \mathbf{f}(\mathbf{r}_t))$

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$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & -\mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{I}_n & \mathbf{0}_{n \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n} & \mathbf{I}_{n_y} \end{bmatrix}$$

$$\times \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & -\mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{I}_n & \mathbf{0}_{n \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

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$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{f}_x(\mathbf{r}_{t+1}) - \mathbf{f}_x(\mathbf{r}_{t+1}) & -\mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t) + \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix}$$

$$\times \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t) - \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

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$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t+1) \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ \mathbf{y}_{t+N} \end{bmatrix},$$

where we have defined  $\mathbf{W}_{t+1} \equiv [-\mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t) + \mathbf{f}_y(\mathbf{r}_{t+1}) \quad \mathbf{f}_{xp}(\mathbf{r}_{t+1})]$  and  $[\mathbf{Q}_{n_x}^{(t+1)}] \equiv \mathbf{W}_{t+1}^{-1} \mathbf{f}_{yp}(\mathbf{r}_{t+1})$  and  $[\mathbf{R}_{n_x}^{(t+1)}] \equiv -\mathbf{W}_{t+1}^{-1} (\mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t) + \mathbf{f}(\mathbf{r}_{t+1}))$

This system can then be solved by backward substitution as follows:

- $\mathbf{y}_{t+2} = \mathbf{y}_{t+N}$
- $\mathbf{x}_{t+2} = \mathbf{R}_{n_x}(t+1) - \mathbf{Q}_{n_x}(t+1)\mathbf{y}_{t+2}$
- $\mathbf{y}_{t+1} = \mathbf{R}_{n_y}(t+1) - \mathbf{Q}_{n_y}(t+1)\mathbf{y}_{t+2}$
- $\mathbf{x}_{t+1} = \mathbf{R}_{n_x}(t) - \mathbf{Q}_{n_x}(t)\mathbf{y}_{t+1}$
- $\mathbf{y}_t = \mathbf{R}_{n_y}(t) - \mathbf{Q}_{n_y}(t)\mathbf{y}_{t+1}$

Recall that for period  $t$

$$\begin{aligned} \mathbf{W}_t &\equiv \left[ \mathbf{f}_y(\mathbf{r}_t) \mathbf{f}_{\mathbf{x}p}(\mathbf{r}_t) \right], \\ \begin{bmatrix} \mathbf{Q}_{n_y}(t) \\ \mathbf{Q}_{n_x}(t) \end{bmatrix} &\equiv \mathbf{W}_t^{-1} \mathbf{f}_{yp}(\mathbf{r}_t), \\ \begin{bmatrix} \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \end{bmatrix} &\equiv -\mathbf{W}_t^{-1} (\mathbf{f}_x(\mathbf{r}_t) \mathbf{x}_t + \mathbf{f}(\mathbf{r}_t)) \end{aligned}$$

and for period  $t + 1$ ,

$$\begin{aligned} \mathbf{W}_{t+1} &\equiv \left[ -\mathbf{f}_x(\mathbf{r}_{t+1}) \mathbf{Q}_{n_x}(t) + \mathbf{f}_y(\mathbf{r}_{t+1}) \quad \mathbf{f}_{\mathbf{x}p}(\mathbf{r}_{t+1}) \right], \\ \begin{bmatrix} \mathbf{Q}_{n_y}(t+1) \\ \mathbf{Q}_{n_x}(t+1) \end{bmatrix} &\equiv \mathbf{W}_{t+1}^{-1} \mathbf{f}_{yp}(\mathbf{r}_{t+1}), \\ \begin{bmatrix} \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \end{bmatrix} &\equiv -\mathbf{W}_{t+1}^{-1} (\mathbf{f}_x(\mathbf{r}_{t+1}) \mathbf{R}_{n_x}(t) + \mathbf{f}(\mathbf{r}_{t+1})). \end{aligned}$$

### SB.2 3 time periods

Let  $N = 3$  and consider the system:

$$\underbrace{\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{f}_x(\mathbf{r}_t) & \mathbf{f}_y(\mathbf{r}_t) & \mathbf{f}_{\mathbf{x}p}(\mathbf{r}_t) & \mathbf{f}_{yp}(\mathbf{r}_t) & \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{x}p}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) & \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{f}_x(\mathbf{r}_{t+2}) & \mathbf{f}_y(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{x}p}(\mathbf{r}_{t+2}) & \mathbf{f}_{yp}(\mathbf{r}_{t+2}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix}}_{4n \times 4n}$$

$$\times \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \\ \mathbf{x}_{t+3} \\ \mathbf{y}_{t+3} \end{bmatrix} = - \begin{bmatrix} -\mathbf{x}_t \\ \mathbf{f}(\mathbf{r}_t) \\ \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{f}(\mathbf{r}_{t+2}) \\ -\mathbf{y}_{t+N} \end{bmatrix}.$$







$$\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t+2) \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+2) \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}
= \begin{bmatrix}
\mathbf{x}_t \\
\mathbf{R}_{n_y}(t) \\
\mathbf{R}_{n_x}(t) \\
\mathbf{R}_{n_y}(t+1) \\
\mathbf{R}_{n_x}(t+1) \\
\mathbf{R}_{n_y}(t+2) \\
\mathbf{R}_{n_x}(t+2) \\
\mathbf{y}_{t+N}
\end{bmatrix},$$

where  $\begin{bmatrix} \mathbf{Q}_{n_y}(t+2) \\ \mathbf{Q}_{n_x}(t+2) \end{bmatrix} \equiv \mathbf{W}_{t+2}^{-1} \mathbf{f}_{yp}(\mathbf{r}_{t+2})$  and  $\begin{bmatrix} \mathbf{R}_{n_y}(t+2) \\ \mathbf{R}_{n_x}(t+2) \end{bmatrix} \equiv -\mathbf{W}_{t+2}^{-1} (\mathbf{f}_x(\mathbf{r}_{t+2}) \mathbf{R}_{n_x}(t+1) + \mathbf{f}(\mathbf{r}_{t+2}))$ .

Thus, this system is easily solved by backwards substitution.

### SB.3 Summarizing: For $N$ time periods

Thus, the general solution is given by

- $\mathbf{y}_{t+N} = \mathbf{y}_{t+N}$  is given
- $\mathbf{y}_{t+k} = -\mathbf{R}_{n_y}(t+k) - \mathbf{Q}_{n_y}(t+k) \mathbf{y}_{t+k+1}$  for  $k = 0, 1, 2, \dots, N-1$
- $\mathbf{x}_{t+k+1} = -\mathbf{R}_{n_x}(t+k) - \mathbf{Q}_{n_x}(t+k) \mathbf{y}_{t+k+1}$  for  $k = 0, 1, 2, \dots, N-1$

The relevant matrices are given by

$$\mathbf{W}_t \equiv \begin{bmatrix} \mathbf{f}_y(\mathbf{r}_t) & \mathbf{f}_{xp}(\mathbf{r}_t) \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{Q}_{n_y}(t) \\ \mathbf{Q}_{n_x}(t) \end{bmatrix} \equiv \mathbf{W}_t^{-1} \mathbf{f}_{yp}(\mathbf{r}_t),$$

$$\begin{bmatrix} \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \end{bmatrix} \equiv -\mathbf{W}_t^{-1} (\mathbf{f}_x(\mathbf{r}_t) \mathbf{x}_t + \mathbf{f}(\mathbf{r}_t))$$

and for period  $t+k$ ,

$$\mathbf{W}_{t+k} \equiv \begin{bmatrix} -\mathbf{f}_x(\mathbf{r}_{t+k}) \mathbf{Q}_{n_x}(t+k-1) + \mathbf{f}_y(\mathbf{r}_{t+k}) & \mathbf{f}_{xp}(\mathbf{r}_{t+k}) \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{Q}_{n_y}(t+k) \\ \mathbf{Q}_{n_x}(t+k) \end{bmatrix} \equiv \mathbf{W}_{t+k}^{-1} \mathbf{f}_{yp}(\mathbf{r}_{t+k}),$$

$$\begin{bmatrix} \mathbf{R}_{n_y}(t+k) \\ \mathbf{R}_{n_x}(t+k) \end{bmatrix} \equiv -\mathbf{W}_{t+k}^{-1} (\mathbf{f}_x(\mathbf{r}_{t+k})\mathbf{R}_{n_x}(t+k-1) + \mathbf{f}(\mathbf{r}_{t+k}))$$

for  $k = 1, 2, \dots, N$

## APPENDIX SC: EFFICIENT INVERSION OF THE JACOBIAN: SYSTEM REDUCTION II

This section presents the efficient solution algorithm of Boucekkine (1995) to compute  $\mathbf{J}(\mathbf{Z})^{-1}\mathbf{F}(\mathbf{Z})$ . In this section, we consider

$$\mathbf{f}_1(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{y}_t, \mathbf{x}_{11,t+1}, \mathbf{y}_{t+1}) = \mathbf{0}.$$

To condensely represent this system of equations, we use the notation

$$\mathbf{f}(\mathbf{z}, \mathbf{x}, \mathbf{y}, \mathbf{xp}, \mathbf{yp}) = \mathbf{0},$$

meaning that  $\mathbf{z}$  refers to  $\mathbf{y}_{t-1}$ ,  $\mathbf{x}$  to  $\mathbf{x}_{11,t}$ ,  $\mathbf{xp}$  to  $\mathbf{x}_{11,t+1}$ ,  $\mathbf{y}$  to  $\mathbf{y}_t$ , and  $\mathbf{yp}$  to  $\mathbf{y}_{t+1}$ . Also, to reduce the notational burden, let

$$\mathbf{r}_t \equiv [\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{y}_t, \mathbf{x}_{11,t+1}, \mathbf{y}_{t+1}].$$

We then consider the following notation for the partial derivatives:

$$\mathbf{f}_x(\mathbf{r}_t) = \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{x}},$$

$$\mathbf{f}_y(\mathbf{r}_t) = \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{y}},$$

$$\mathbf{f}_{xp}(\mathbf{r}_t) = \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{xp}},$$

$$\mathbf{f}_{yp}(\mathbf{r}_t) = \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{yp}},$$

$$\mathbf{f}_z(\mathbf{r}_t) = \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{z}}.$$

To understand the logic of the solution method of Boucekkine (1995), it is useful to gradually extend the horizon  $N$  when solving the certainty equivalent solution.

### SC.1 2 time periods

Let  $N = 2$  and consider the system

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{f}_x(\mathbf{r}_t) & \mathbf{f}_y(\mathbf{r}_t) & \mathbf{f}_{xp}(\mathbf{r}_t) & \mathbf{f}_{yp}(\mathbf{r}_t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{f}_z(\mathbf{r}_{t+1}) & \mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = - \begin{bmatrix} -\mathbf{x}_t \\ \mathbf{f}(\mathbf{r}_t) \\ \mathbf{f}(\mathbf{r}_{t+1}) \\ -\mathbf{y}_{t+N} \end{bmatrix}$$



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$$\begin{aligned}
 & \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{I}_{n, n_x} & \mathbf{W}_t^{-1} \mathbf{f}_{yp}(\mathbf{r}_t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{f}_z(\mathbf{r}_{t+1}) & \mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{x}_t \\ \mathbf{W}_t^{-1}(-\mathbf{f}_x(\mathbf{r}_t)\mathbf{x}_t - \mathbf{f}(\mathbf{r}_t)) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix},
 \end{aligned}$$

where  $\mathbf{W}_t \equiv [\mathbf{f}_y(\mathbf{r}_t) \ \mathbf{f}_{xp}(\mathbf{r}_t)]$  and we use the partitioning  $\mathbf{I}_n \equiv [\mathbf{I}_{n, n_y} \ \mathbf{I}_{n, n_x}]$  where  $\mathbf{I}_{n, n_y}$  are the first  $n_y$  columns of  $\mathbf{I}_n$  and  $\mathbf{I}_{n, n_x}$  denotes the last columns of  $\mathbf{I}_n$ .

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$$\begin{aligned}
 & \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{f}_z(\mathbf{r}_{t+1}) & \mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix},
 \end{aligned}$$

where  $[\mathbf{Q}_{n_y}(t)] \equiv \mathbf{W}_t^{-1} \mathbf{f}_{yp}(\mathbf{r}_t)$  and  $[\mathbf{R}_{n_x}(t)] \equiv -\mathbf{W}_t^{-1}(\mathbf{f}_x(\mathbf{r}_t)\mathbf{x}_t + \mathbf{f}(\mathbf{r}_t))$

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$$\begin{aligned}
 & \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & -\mathbf{f}_z(\mathbf{r}_{t+1}) & -\mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{I}_n & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n} & \mathbf{I}_{n_y} \end{bmatrix} \\
 & \times \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{f}_z(\mathbf{r}_{t+1}) & \mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & -\mathbf{f}_z(\mathbf{r}_{t+1}) & -\mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{I}_n & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}
 \end{aligned}$$

$\Downarrow$ 

$$\begin{aligned}
 & \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{f}_z(\mathbf{r}_{t+1}) - \mathbf{f}_z(\mathbf{r}_{t+1}) & \mathbf{f}_x(\mathbf{r}_{t+1}) - \mathbf{f}_x(\mathbf{r}_{t+1}) & -\mathbf{f}_z(\mathbf{r}_{t+1})\mathbf{Q}_{n_y}(t) - \mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t) + \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \\
 & \times \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} \\
 & = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}_z(\mathbf{r}_{t+1})\mathbf{R}_{n_y}(t) - \mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t) - \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}
 \end{aligned}$$

 $\Downarrow$ 

$$\begin{aligned}
 & \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{0}_{n \times n_x} & -\mathbf{f}_z(\mathbf{r}_{t+1})\mathbf{Q}_{n_y}(t) - \mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t) + \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{xp}(\mathbf{r}_{t+1}) & \mathbf{f}_{yp}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \\
 & \times \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}_z(\mathbf{r}_{t+1})\mathbf{R}_{n_y}(t) - \mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t) - \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}
 \end{aligned}$$

Now let  $\mathbf{W}_{t+1} \equiv [-\mathbf{f}_z(\mathbf{r}_{t+1})\mathbf{Q}_{n_y}(t) - \mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t) + \mathbf{f}_y(\mathbf{r}_{t+1}) \quad \mathbf{f}_{xp}(\mathbf{r}_{t+1})]$  and  $[\mathbf{Q}_{n_x}^{-1}(t+1)] \equiv \mathbf{W}_{t+1}^{-1} \mathbf{f}_{yp}(\mathbf{r}_{t+1})$  and  $[\mathbf{R}_{n_x}^{(t+1)}] \equiv -\mathbf{W}_{t+1}^{-1} (\mathbf{f}_z(\mathbf{r}_{t+1})\mathbf{R}_{n_y}(t) + \mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t) + \mathbf{f}(\mathbf{r}_{t+1}))$ . Using this, we get

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t+1) \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ \mathbf{y}_{t+N} \end{bmatrix}.$$

This system can then be solved by backward substitution as follows:

- $\mathbf{y}_{t+2} = \mathbf{y}_{t+N}$

- $\mathbf{x}_{t+2} = \mathbf{R}_{n_x}(t+1) - \mathbf{Q}_{n_x}(t+1)\mathbf{y}_{t+2}$
- $\mathbf{y}_{t+1} = \mathbf{R}_{n_y}(t+1) - \mathbf{Q}_{n_y}(t+1)\mathbf{y}_{t+2}$
- $\mathbf{x}_{t+1} = \mathbf{R}_{n_x}(t) - \mathbf{Q}_{n_x}(t)\mathbf{y}_{t+1}$
- $\mathbf{y}_t = \mathbf{R}_{n_y}(t) - \mathbf{Q}_{n_y}(t)\mathbf{y}_{t+1}$

Recall that for period  $t$ ,

$$\mathbf{W}_t \equiv \begin{bmatrix} \mathbf{f}_y(\mathbf{r}_t) & \mathbf{f}_{\mathbf{x}\mathbf{p}}(\mathbf{r}_t) \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{Q}_{n_y}(t) \\ \mathbf{Q}_{n_x}(t) \end{bmatrix} \equiv \mathbf{W}_t^{-1} \mathbf{f}_{\mathbf{y}\mathbf{p}}(\mathbf{r}_t),$$

$$\begin{bmatrix} \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \end{bmatrix} \equiv -\mathbf{W}_t^{-1} (\mathbf{f}_x(\mathbf{r}_t)\mathbf{x}_t + \mathbf{f}(\mathbf{r}_t))$$

and for period  $t+1$ ,

$$\mathbf{W}_{t+1} \equiv \begin{bmatrix} -\mathbf{f}_z(\mathbf{r}_{t+1})\mathbf{Q}_{n_y}(t) - \mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t) + \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{x}\mathbf{p}}(\mathbf{r}_{t+1}) \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{Q}_{n_y}(t+1) \\ \mathbf{Q}_{n_x}(t+1) \end{bmatrix} \equiv \mathbf{W}_{t+1}^{-1} \mathbf{f}_{\mathbf{y}\mathbf{p}}(\mathbf{r}_{t+1}),$$

$$\begin{bmatrix} \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \end{bmatrix} \equiv -\mathbf{W}_{t+1}^{-1} (\mathbf{f}_z(\mathbf{r}_{t+1})\mathbf{R}_{n_y}(t) + \mathbf{f}_x(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t) + \mathbf{f}(\mathbf{r}_{t+1})).$$

### SC.2 3 time periods

Let  $N = 3$  and consider the system:

$$\underbrace{\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{f}_x(\mathbf{r}_t) & \mathbf{f}_y(\mathbf{r}_t) & \mathbf{f}_{\mathbf{x}\mathbf{p}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{y}\mathbf{p}}(\mathbf{r}_t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{f}_z(\mathbf{r}_{t+1}) & \mathbf{f}_x(\mathbf{r}_{t+1}) & \mathbf{f}_y(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{x}\mathbf{p}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}\mathbf{p}}(\mathbf{r}_{t+1}) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{f}_z(\mathbf{r}_{t+2}) & \mathbf{f}_x(\mathbf{r}_{t+2}) & \mathbf{f}_y(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{x}\mathbf{p}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{y}\mathbf{p}}(\mathbf{r}_{t+2}) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix}}_{4n \times 4n}$$

$$\times \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \\ \mathbf{x}_{t+3} \\ \mathbf{y}_{t+3} \end{bmatrix} = - \begin{bmatrix} -\mathbf{x}_t \\ \mathbf{f}(\mathbf{r}_t) \\ \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{f}(\mathbf{r}_{t+2}) \\ -\mathbf{y}_{t+N} \end{bmatrix}.$$







$$= \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ -\mathbf{f}_z(\mathbf{r}_{t+2})\mathbf{R}_{n_y}(t+1) - \mathbf{f}_x(\mathbf{r}_{t+2})\mathbf{R}_{n_x}(t+1) - \mathbf{f}(\mathbf{r}_{t+2}) \\ \mathbf{y}_{t+N} \end{bmatrix}.$$

$$\text{Let } \mathbf{W}_{t+2} \equiv \begin{bmatrix} -\mathbf{f}_z(\mathbf{r}_{t+2})\mathbf{Q}_{n_y}(t+1) - \mathbf{f}_x(\mathbf{r}_{t+2})\mathbf{Q}_{n_x}(t+1) + \mathbf{f}_y(\mathbf{r}_{t+2}) & \mathbf{f}_{xp}(\mathbf{r}_{t+2}) \end{bmatrix}$$

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$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t+2) \\ \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+2) \\ \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ \mathbf{R}_{n_y}(t+2) \\ \mathbf{R}_{n_x}(t+2) \\ \mathbf{y}_{t+N} \end{bmatrix},$$

where

$$\begin{bmatrix} \mathbf{Q}_{n_y}(t+2) \\ \mathbf{Q}_{n_x}(t+2) \end{bmatrix} \equiv \mathbf{W}_{t+2}^{-1} \mathbf{f}_{yp}(\mathbf{r}_{t+2})$$

and

$$\begin{bmatrix} \mathbf{R}_{n_y}(t+2) \\ \mathbf{R}_{n_x}(t+2) \end{bmatrix} \equiv -\mathbf{W}_{t+2}^{-1} (\mathbf{f}_z(\mathbf{r}_{t+2})\mathbf{R}_{n_y}(t+1) + \mathbf{f}_x(\mathbf{r}_{t+2})\mathbf{R}_{n_x}(t+1) + \mathbf{f}(\mathbf{r}_{t+2})).$$

Thus, this system is easily solved by backwards substitution.

### SC.3 Summarizing: For $N$ time periods

Thus, the general solution is given by

- $\mathbf{y}_{t+N} = \mathbf{y}_{t+N}$  is given

- $\mathbf{y}_{t+k} = -\mathbf{R}_{n_y}(t+k) - \mathbf{Q}_{n_y}(t+k)\mathbf{y}_{t+k+1}$  for  $k = 0, 1, 2, \dots, N-1$
- $\mathbf{x}_{t+k+1} = -\mathbf{R}_{n_x}(t+k) - \mathbf{Q}_{n_x}(t+k)\mathbf{y}_{t+k+1}$  for  $k = 0, 1, 2, \dots, N-1$

The relevant matrices are given by

$$\mathbf{W}_t \equiv \begin{bmatrix} \mathbf{f}_y(\mathbf{r}_t) & \mathbf{f}_{xp}(\mathbf{r}_t) \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{Q}_{n_y}(t) \\ \mathbf{Q}_{n_x}(t) \end{bmatrix} \equiv \mathbf{W}_t^{-1} \mathbf{f}_{yp}(\mathbf{r}_t),$$

$$\begin{bmatrix} \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \end{bmatrix} \equiv -\mathbf{W}_t^{-1} (\mathbf{f}_x(\mathbf{r}_t)\mathbf{x}_t + \mathbf{f}(\mathbf{r}_t))$$

and for period  $t+k$

$$\mathbf{W}_{t+k} \equiv \begin{bmatrix} -\mathbf{f}_z(\mathbf{r}_{t+k})\mathbf{Q}_{n_y}(t+k-1) - \mathbf{f}_x(\mathbf{r}_{t+k})\mathbf{Q}_{n_x}(t+k-1) + \mathbf{f}_y(\mathbf{r}_{t+k}) & \mathbf{f}_{xp}(\mathbf{r}_{t+k}) \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{Q}_{n_y}(t+k) \\ \mathbf{Q}_{n_x}(t+k) \end{bmatrix} \equiv \mathbf{W}_{t+k}^{-1} \mathbf{f}_{yp}(\mathbf{r}_{t+k}),$$

$$\begin{bmatrix} \mathbf{R}_{n_y}(t+k) \\ \mathbf{R}_{n_x}(t+k) \end{bmatrix} \equiv -\mathbf{W}_{t+k}^{-1} (\mathbf{f}_z(\mathbf{r}_{t+k})\mathbf{R}_{n_y}(t+k-1) + \mathbf{f}_x(\mathbf{r}_{t+k})\mathbf{R}_{n_x}(t+k-1) + \mathbf{f}(\mathbf{r}_{t+k}))$$

for  $k = 1, 2, \dots, N$

## APPENDIX SD: A NEW KEYNESIAN MODEL

This section presents a standard New Keynesian model with Calvo pricing.

### SD.1 Households

The dynamic optimization problem faced by the representative household is of the form:

$$\text{Max}_{c_t, b_t, h_t, k_{t+1}, i_t \forall t \geq 0} V_t = u(c_t, 1 - h_t)$$

$$\text{St. } k_{t+1} = (1 - \delta)k_t + i_t - k_t \frac{\kappa}{2} \left( \frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}} \right)^2$$

$$b_t + c_t + i_t = \frac{R_{t-1}b_{t-1}}{\pi_t} + h_t w_t + r_t^k k_t + \text{div}_t$$

a no-Ponzi-game condition

$$c_t, h_t, k_{t+1}, i_t \geq 0 \forall t \geq 0.$$

As for the notation:

- $c_t$  = consumption

- $h_t =$  hours
- $k_t =$  the capital stock
- $i_t =$  investments
- $R_t =$  the gross one-period risk-free rate, that is,  $R_t = 1 + r_t$
- $b_t =$  nominal one-period bonds
- $\pi_t =$  gross inflation of consumption good prices
- $w_t =$  the wage level measured in consumption good units
- $r_t^k =$  the rental rate for capital services sold to firms as measured in consumption good units
- $\text{div}_t =$  net transfers of profit from firms to households as measured in consumption good units

The Lagrangian is therefore given by

$$\begin{aligned} \mathcal{L} = & \sum_{l=0}^{\infty} E_t \beta^l u(c_{t+l}, 1 - h_{t+l}) \\ & + E_t \sum_{l=0}^{\infty} \beta^l \lambda_{t+l} \left[ \frac{R_{t-1+l}}{\pi_{t+l}} b_{t+l-1} + h_{t+l} w_{t+l} + r_{t+l}^k k_{t+l} + \text{div}_{t+l} - b_{t+l} - c_{t+l} - i_{t+l} \right] \\ & + E_t \sum_{l=0}^{\infty} \beta^l q_{t+l} \lambda_{t+l} \left[ (1 - \delta) k_{t+l} + i_{t+l} - k_{t+l} \frac{\kappa}{2} \left( \frac{i_{t+l}}{k_{t+l}} - \frac{i_{ss}}{k_{ss}} \right)^2 - k_{t+1+l} \right]. \end{aligned}$$

That is, we introduce two Lagrange multipliers:

- $\lambda_t =$  for the budget constraint
- $q_t \lambda_t =$  for the capital accumulation equation

FOC (Kuhn–Tucker conditions)

We look for a solution in the interior, that is,  $c_t, b_t, h_t, k_{t+1}, i_t > 0 \forall t \geq 0$

(1) *Consumption,  $c_t$ :*

$$\frac{\partial \mathcal{L}}{\partial c_t} = u_c(c_t, 1 - h_t) - \lambda_t = 0$$

⇕

$$\lambda_t = u_c(c_t, 1 - h_t).$$

(2) *Bonds,  $b_t$ :*

$$\frac{\partial \mathcal{L}}{\partial b_t} = -\lambda_t + E_t \left[ \beta \lambda_{t+1} \frac{R_t}{\pi_{t+1}} \right] = 0$$

⇕

$$1 = E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{R_t}{\pi_{t+1}} \right].$$

Note that  $D_{t,t+1} \equiv \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}}$  is the nominal stochastic discount factor.

(3) *The labor supply,  $h_t$*

$$\frac{\partial \mathcal{L}}{\partial h_t} = -u_{1-h}(c_t, 1 - h_t) + \lambda_t w_t = 0$$

⇕

$$u_{1-h}(c_t, 1 - h_t) = \lambda_t w_t.$$

(4) *The physical capital stock,  $k_{t+1}$  :*

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial k_{t+1}} = & -\lambda_t q_t + E_t \beta \lambda_{t+1} \left[ r_{t+1}^k + q_{t+1}(1 - \delta) \right. \\ & \left. - q_{t+1} \frac{\kappa}{2} \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right)^2 + q_{t+1} \kappa \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right) \frac{i_{t+1}}{k_{t+1}^2} k_{t+1} \right] = 0 \end{aligned}$$

⇕

$$q_t \lambda_t = E_t \beta \lambda_{t+1} \left[ r_{t+1}^k + q_{t+1}(1 - \delta) - q_{t+1} \frac{\kappa}{2} \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right)^2 + q_{t+1} \kappa \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right) \frac{i_{t+1}}{k_{t+1}} \right].$$

(5) *Investments,  $i_t$ :*

$$\frac{\partial \mathcal{L}}{\partial i_t} = -\lambda_t + q_t \lambda_t \left( 1 - \kappa \left( \frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}} \right) \right) = 0$$

⇕

$$1 = q_t \left( 1 - \kappa \left( \frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}} \right) \right).$$

## SD.2 Firms

SD.2.1 *Final good producers* The representative competitive consumption good producer chooses  $y_{i,t}$  for  $i \in [0, 1]$  to solve

$$\max_{y_{i,t}} P_t y_t - \int_0^1 P_{i,t} y_{i,t} di$$

s.t.

$$y_t = \left( \int_0^1 (y_{i,t})^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}}.$$

As for the notation,

- $y_{i,t}$  = denotes the output from firm  $i$  at time  $t$
- $P_{t,i}$  = the price of  $y_{i,t}$
- $y_t$  = output from the final good producers
- $P_t$  = price of  $y_t$

The first-order condition to the problem is given by

$$y_{i,t} = \left( \frac{P_{i,t}}{P_t} \right)^{-\eta} y_t.$$

To find the expression for the aggregate price level, we use the zero-profit condition, that is,

$$\begin{aligned} P_t y_t &= \int_0^1 P_{i,t} y_{i,t} di \\ &= \int_0^1 P_{i,t} \left( \frac{P_{i,t}}{P_t} \right)^{-\eta} y_t di \\ &= y_t P_t^\eta \int_0^1 (P_{i,t})^{1-\eta} di \\ &\Downarrow \\ P_t^{1-\eta} &= \int_0^1 (P_{i,t})^{1-\eta} di \\ &\Downarrow \\ P_t &= \left[ \int_0^1 (P_{i,t})^{1-\eta} di \right]^{\frac{1}{1-\eta}}. \end{aligned}$$

SD.2.2 *Intermediate good producer* This section derives the first-order conditions for the  $i$ th firm's optimization problem. We start by deriving the equation for the real dividend payments:

$$\text{div}_{i,t} \equiv \left[ \left( \frac{P_{i,t}}{P_t} \right) y_{i,t} - r_t^k k_{i,t} - w_t h_{i,t} \right].$$

So the problem is

$$\begin{aligned} \text{Max}_{h_{i,t}, k_{i,t}, P_{i,t}, \forall t \geq 0} & E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \text{div}_{i,t+l} \\ \text{St. } \text{div}_{i,t+l} &\equiv \left( \frac{P_{i,t+l}}{P_{t+l}} \right)^{1-\eta} y_{t+l} - r_{t+l}^k k_{i,t+l} - w_{t+l} h_{i,t+l} \\ a_t k_{i,t}^\theta (z_t h_{i,t})^{1-\theta} &= \left( \frac{P_{i,t}}{P_t} \right)^{-\eta} y_t \end{aligned}$$

a no-Ponzi-game condition.

Here, we use the following notation:

- $r_t^k$  = the rental rate for capital services sold to firms as measured in consumption good units
- $k_{i,t}$  = the capital stock used by firm  $i$  at time  $t$
- $w_t$  = the wage level measured in consumption good units
- $h_{i,t}$  = hours used by firm  $i$  at time  $t$
- $a_t$  = stationary technology shocks

The law of motion for  $a_t$  is given by

$$\log a_{t+1} = \rho_a \log a_t + \sigma_a \epsilon_{a,t+1},$$

where  $\epsilon_{a,t+1} \sim \mathcal{NID}(0, 1)$ . The Lagrangian is

$$\begin{aligned} \mathcal{L} = & E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \left[ \left( \frac{P_{i,t+l}}{P_{t+l}} \right)^{1-\eta} y_{t+l} - r_{t+l}^k k_{i,t+l} - w_{t+l} h_{i,t+l} \right] \\ & + E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} mc_{i,t+l} \left[ a_{t+l} k_{i,t+l}^\theta (h_{i,t+l})^{1-\theta} - \left( \frac{P_{i,t+l}}{P_{t+l}} \right)^{-\eta} y_{t+l} \right]. \end{aligned}$$

FOC (Kuhn–Tucker conditions)

We look for a solution in the interior, that is,  $h_{i,t}, k_{i,t}, P_{i,t} > 0 \forall t \geq 0$ .

(1) Demand for labor,  $h_{i,t}$ :

$$\frac{\partial \mathcal{L}}{\partial h_{i,t}} = P_t (-w_t + mc_{i,t} a_t (1 - \theta) k_{i,t}^\theta (h_{i,t})^{-\theta})$$

Since  $P_t > 0$ , we get

$$mc_{i,t} a_t (1 - \theta) k_{i,t}^\theta (h_{i,t})^{-\theta} = w_t.$$

(2) Demand for capital,  $k_{i,t}$ :

$$\frac{\partial \mathcal{L}}{\partial k_{i,t}} = P_t (-r_t^k + mc_{i,t} \theta a_t k_{i,t}^{\theta-1} (h_{i,t})^{1-\theta})$$

Since  $P_t > 0$ , we get

$$r_t^k = mc_{i,t} \theta a_t k_{i,t}^{\theta-1} (h_{i,t})^{1-\theta}.$$

(3) The optimal price,  $P_{i,t}$ :

We assume Calvo pricing determined by  $\alpha$ , giving the probability of a firm not being allowed to change its price in a given period. Notice that all the reoptimizing firms face the same problem, hence they all set the same price. We denote this price by  $\tilde{P}_t$ . Hence,

we have

$$\frac{P_{i,t+l}}{P_{t+l}} = \frac{\tilde{P}_t}{P_t} \frac{P_{t+l}}{P_t} = \frac{\tilde{P}_t \prod_{i=1}^l \frac{1}{\pi_{t+i}}}{P_t},$$

where  $\pi_{t+l} \equiv \frac{P_{t+l}}{P_{t+l-1}}$ .

If we only write the probability that the new price last forever, the Lagrangian reads

$$\begin{aligned} \mathcal{L} = & E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left[ \left( \frac{\tilde{P}_t}{P_t} \right)^{1-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{1-\eta} y_{t+l} - r_{t+l}^k k_{i,t+l} - w_{t+l} h_{i,t+l} \right] \\ & + E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l mc_{i,t+l} \left[ a_{t+l} k_{i,t+l}^\theta (h_{i,t+l})^{1-\theta} - \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{-\eta} y_{t+l} \right]. \end{aligned}$$

The first-order-condition is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \tilde{P}_t} = & E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left[ (1-\eta) \tilde{P}_t^{-\eta} P_t^{\eta-1} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{1-\eta} y_{t+l} \right. \\ & \left. + mc_{i,t+l} \eta \tilde{P}_t^{-\eta-1} P_t^\eta \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{-\eta} y_{t+l} \right] \\ = & E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{-\eta} y_{t+l} \left[ (1-\eta) P_t^{-1} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right) + \eta \frac{mc_{i,t+l}}{\tilde{P}_t} \right] \\ = & E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{-\eta} y_{t+l} \left[ \frac{(\eta-1) \tilde{P}_t}{\eta P_t} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right) - mc_{i,t+l} \right] \frac{-\eta}{\tilde{P}_t} \end{aligned}$$

since  $\eta > 1$  and  $\tilde{P}_t > 0$ ,  $\frac{\partial \mathcal{L}}{\partial \tilde{P}_t} = 0$  implies

$$E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{-\eta} y_{t+l} \left[ \frac{(\eta-1) \tilde{P}_t}{\eta P_t} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right) - mc_{i,t+l} \right] = 0.$$

**SD.2.3 Marginal costs** We next show that marginal costs are identical across firms, that is,  $mc_{i,t} = mc_t$  for all  $i$ . Note from the first-order conditions for  $k_{i,t}$  and  $h_{i,t}$  that

$$\frac{mc_{i,t} a_t z_t (1-\theta) k_{i,t}^\theta (h_{i,t})^{-\theta}}{mc_{i,t} a_t \theta k_{i,t}^{\theta-1} (h_{i,t})^{1-\theta}} = \frac{w_t}{r_t^k}$$

⇔

$$\frac{(1-\theta)}{\theta k_{i,t}^{-1} (h_{i,t})} = \frac{w_t}{r_t^k}$$



$$\Downarrow$$

$$\frac{1 - \theta}{\theta} \frac{k_{i,t}}{h_{i,t}} = \frac{w_t}{r_t^k}$$

implying that  $\frac{k_{i,t}}{h_{i,t}}$  must be constant with respect to  $i$ . That is,

$$\frac{k_{i,t}}{h_{i,t}} = \text{cons}_t$$

$$\Downarrow$$

$$k_{i,t} = h_{i,t} \text{cons}_t$$

$$\Downarrow$$

$$\int_0^1 k_{i,t} di = \int_0^1 h_{i,t} \text{cons}_t di$$

$$\Downarrow$$

$$k_t = h_t \text{cons}_t$$

$$\Downarrow$$

$$\text{cons}_t = k_t / h_t.$$

Hence,

$$mc_{i,t} a_t \theta k_{i,t}^{\theta-1} (h_{i,t})^{1-\theta} = r_t^k$$

$$\Downarrow$$

$$mc_{i,t} a_t \theta \left( \frac{k_{i,t}}{h_{i,t}} \right)^{\theta-1} = r_t^k$$

$$\Downarrow$$

$$mc_{i,t} a_t \theta \left( \frac{k_t}{h_t} \right)^{\theta-1} = r_t^k.$$

This shows that  $mc_{i,t} = mc_t$ .

Hence, we can write the first-order condition for  $h_{i,t}$  and  $k_{i,t}$  as

$$a_t (1 - \theta) \left( \frac{h_t}{k_t} \right)^{-\theta} mc_t = w_t,$$

$$r_t^k = \theta a_t \left( \frac{h_t}{k_t} \right)^{1-\theta} mc_t.$$

SD.2.4 *The recursive representation of the price relation* We start by defining

$$x_t^1 \equiv E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l y_{t+l} mc_{t+l} \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta-1} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta},$$

$$x_t^2 \equiv E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l y_{t+l} \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta}.$$

This implies that the first-order condition for the price relation can be expressed as

$$E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{-\eta} y_{t+l} \left[ \frac{\eta-1}{\eta} \frac{\tilde{P}_t}{P_t} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right) - mc_{t+l} \right] = 0$$

⇔

$$E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{-\eta} y_{t+l} \frac{(\eta-1)}{\eta} \frac{\tilde{P}_t}{P_t} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right) - E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{-\eta} y_{t+l} mc_{t+l} = 0$$

⇔

$$E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l \tilde{P}_t \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta} y_{t+l} \frac{(\eta-1)}{\eta} - E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l \tilde{P}_t \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta-1} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta} y_{t+l} mc_{t+l} = 0 \quad \text{see below}$$

⇔

since  $\tilde{P}_t > 0$

$$\underbrace{E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l y_{t+l} \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta} \frac{(\eta-1)}{\eta}}_{x_t^2} - \underbrace{E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l y_{t+l} mc_{t+l} \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta-1} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta}}_{x_t^1} = 0$$

⇔

$$x_t^2 \frac{(\eta-1)}{\eta} - x_t^1 = 0$$

⇔

$$\eta x_t^1 + (1-\eta)x_t^2 = 0.$$

We used the fact that

$$P_{t+l} \left( \frac{\tilde{P}_t}{P_t} \right)^{1-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{1-\eta} = \tilde{P}_t^{1-\eta} P_{t+l}^\eta = \tilde{P}_t \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta}$$

and

$$P_{t+l} \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}} \right)^{-\eta} = \tilde{P}_t^{-\eta} P_{t+l}^{1+\eta} = \tilde{P}_t \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta-1} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta}.$$

*The dynamic process for  $x_t^1$*

We define  $\frac{\tilde{P}_t}{P_t} = \tilde{p}_t$ . Therefore,

$$\begin{aligned} x_t^1 &= E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l y_{t+l} m c_{t+l} \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta-1} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta} \\ &= y_t m c_{t,i} \tilde{p}_t^{-\eta-1} + E_t \sum_{l=1}^{\infty} D_{t,t+l} \alpha^l y_{t+l} m c_{t+l} \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta-1} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta} \\ &= y_t m c_t \tilde{p}_t^{-\eta-1} + \tilde{p}_t^{-\eta-1} E_t \left[ \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \tilde{p}_{t+1}^{\eta+1} \left( \frac{1}{\pi_{t+1}} \right)^{-\eta} x_{t+1}^1 \right], \quad \text{see below} \\ &= y_t m c_t \tilde{p}_t^{-\eta-1} + E_t \left[ \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \left( \frac{\tilde{p}_t}{\tilde{p}_{t+1}} \right)^{-\eta-1} \left( \frac{1}{\pi_{t+1}} \right)^{-\eta} x_{t+1}^1 \right]. \end{aligned}$$

At the third equality sign, we use the fact that

$$x_{t+1}^1 = E_{t+1} \sum_{l=0}^{\infty} D_{t+1,t+1+l} \alpha^l y_{t+1+l} m c_{t+1+l} \left( \frac{\tilde{P}_{t+1}}{P_{t+1}} \right)^{-\eta-1} \prod_{i=1}^l \left( \frac{1}{\pi_{t+1+i}^{(1+\eta)/\eta}} \right)^{-\eta}$$

⇕ change of index,  $j = l + 1$

$$x_{t+1}^1 = \tilde{p}_{t+1}^{-\eta-1} E_{t+1} \sum_{j=1}^{\infty} D_{t+1,t+j} \alpha^{j-1} y_{t+j} m c_{t+j} \prod_{i=1}^{j-1} \left( \frac{1}{\pi_{t+1+i}^{(1+\eta)/\eta}} \right)^{-\eta}$$

⇕

$$x_{t+1}^1 \tilde{p}_{t+1}^{1+\eta} \alpha D_{t,t+1} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j} \alpha^j y_{t+j} m c_{t+j} \prod_{i=2}^j \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta}$$

⇕

$$x_{t+1}^1 \tilde{p}_{t+1}^{1+\eta} \alpha D_{t,t+1} \left( \frac{1}{\pi_{t+1}^{(1+\eta)/\eta}} \right)^{-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j} \alpha^j y_{t+j} m c_{t+j} \prod_{i=1}^j \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta}$$

⇕

$$x_{t+1}^1 \tilde{p}_{t+1}^{1+\eta} \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}} \left( \frac{1}{\pi_{t+1}^{(1+\eta)/\eta}} \right)^{-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j} \alpha^j y_{t+j} m c_{t+j} \prod_{i=1}^j \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta}$$

$$\text{since } D_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}}$$

⇕

$$x_{t+1}^1 \tilde{p}_{t+1}^{1+\eta} \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \left( \frac{1}{\pi_{t+1}} \right)^{-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j} \alpha^j y_{t+j} m c_{t+j} \prod_{i=1}^j \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta}$$

⇓

$$E_t \left[ x_{t+1}^1 \tilde{p}_{t+1}^{1+\eta} \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \left( \frac{1}{\pi_{t+1}} \right)^{-\eta} \right] = E_t \sum_{j=1}^{\infty} D_{t,t+j} \alpha^j y_{t+j} m c_{t+j} \prod_{i=1}^j \left( \frac{1}{\pi_{t+i}^{(1+\eta)/\eta}} \right)^{-\eta}$$

due to the law of iterated expectations,  $E_t E_{t+1}[\cdot] = E_t[\cdot]$ .

*The dynamic process for  $x_t^2$*

$$\begin{aligned} x_t^2 &= E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l y_{t+l} \left( \frac{\tilde{P}_t}{P_t} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta} \\ &= y_t \tilde{p}_t^{-\eta} + \tilde{p}_t^{-\eta} E_t \sum_{l=1}^{\infty} D_{t,t+l} \alpha^l y_{t+l} \prod_{i=1}^l \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta} \quad \text{since } \tilde{p}_t = \frac{\tilde{P}_t}{P_t} \\ &= y_t \tilde{p}_t^{-\eta} + \tilde{p}_t^{-\eta} E_t \left[ \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \tilde{p}_{t+1}^{\eta} \left( \frac{1}{\pi_{t+1}} \right)^{1-\eta} x_{t+1}^2 \right], \quad \text{see below} \\ &= y_t \tilde{p}_t^{-\eta} + E_t \left[ \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \left( \frac{\tilde{p}_t}{\tilde{p}_{t+1}} \right)^{-\eta} \left( \frac{1}{\pi_{t+1}} \right)^{1-\eta} x_{t+1}^2 \right]. \end{aligned}$$

At the third equality sign, we use the fact that

$$x_{t+1}^2 = E_{t+1} \sum_{l=0}^{\infty} D_{t+1,t+1+l} \alpha^l y_{t+1+l} \left( \frac{\tilde{P}_{t+1}}{P_{t+1}} \right)^{-\eta} \prod_{i=1}^l \left( \frac{1}{\pi_{t+1+i}^{\eta/(\eta-1)}} \right)^{1-\eta}$$

⇕ change of index  $j = 1 + l$

$$x_{t+1}^2 = \tilde{p}_{t+1}^{-\eta} E_{t+1} \sum_{j=1}^{\infty} D_{t+1,t+j} \alpha^{j-1} y_{t+j} \prod_{i=1}^{j-1} \left( \frac{1}{\pi_{t+1+i}^{\eta/(\eta-1)}} \right)^{1-\eta}$$

⇕

$$x_{t+1}^2 \alpha D_{t,t+1} \tilde{p}_{t+1}^{\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j} \alpha^j y_{t+j} \prod_{i=2}^j \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta}$$

⇕

$$x_{t+1}^2 \alpha D_{t,t+1} \tilde{p}_{t+1}^\eta \left( \frac{1}{\pi_{t+1}^{\eta/(\eta-1)}} \right)^{1-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j} \alpha^j y_{t+j} \prod_{i=1}^j \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta}$$

⇕

$$x_{t+1}^2 \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}} \tilde{p}_{t+1}^\eta \left( \frac{1}{\pi_{t+1}^{\eta/(\eta-1)}} \right)^{1-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j} \alpha^j y_{t+j} \prod_{i=1}^j \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta}$$

$$\text{since } D_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}}$$

⇕

$$x_{t+1}^2 \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \tilde{p}_{t+1}^\eta \left( \frac{1}{\pi_{t+1}} \right)^{1-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j} \alpha^j y_{t+j} \prod_{i=1}^j \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta}$$

↓

$$E_t \left[ x_{t+1}^2 \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \tilde{p}_{t+1}^\eta \left( \frac{1}{\pi_{t+1}} \right)^{1-\eta} \right] = E_t \sum_{j=1}^{\infty} D_{t,t+j} \alpha^j y_{t+j} \prod_{i=1}^j \left( \frac{1}{\pi_{t+i}^{\eta/(\eta-1)}} \right)^{1-\eta}$$

due to the law of iterated expectations,  $E_t E_{t+1}[\cdot] = E_t[\cdot]$ .

### SD.3 The central bank

We assume a standard Taylor rule of the form

$$\log\left(\frac{R_t}{R_{ss}}\right) = \rho_r \log\left(\frac{R_{t-1}}{R_{ss}}\right) + (1 - \rho_r) \left( \kappa_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \kappa_y \log\left(\frac{y_t}{y_{ss}}\right) \right).$$

### SD.4 Aggregation: Final good producer

From the final good producer, we have

$$y_{i,t} = \left( \frac{P_{i,t}}{P_t} \right)^{-\eta} y_t$$

⇕

$$a_t k_{i,t}^\theta h_{i,t}^{1-\theta} = \left( \frac{P_{i,t}}{P_t} \right)^{-\eta} y_t.$$

We next notice that:

1.  $\int_0^1 h_{i,t} di \equiv h_t$ ,
2.  $\int_0^1 k_{i,t} di \equiv k_t$
3.  $\text{cons}_t = k_t / h_t$

Doing the summation with respect to  $i$ , we get

$$\begin{aligned}
 \int_0^1 a_t k_{i,t}^\theta h_{i,t}^{1-\theta} di &= y_t \underbrace{\int_0^1 \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} di}_{s_{t+1}} \\
 \Downarrow \\
 \int_0^1 a_t h_{i,t} \left(\frac{k_{i,t}}{h_{i,t}}\right)^\theta di &= y_t s_{t+1} \\
 \Downarrow \\
 \int_0^1 a_t h_{i,t} (\text{constant})^\theta di &= y_t s_{t+1} \\
 \Downarrow \\
 a_t (\text{constant})^\theta \int_0^1 h_{i,t} di &= y_t s_{t+1} \\
 \Downarrow \\
 a_t \left(\frac{k_t}{h_t}\right)^\theta h_t &= y_t s_{t+1} \\
 \Downarrow \\
 a_t (k_t)^\theta h_t^{1-\theta} &= y_t s_{t+1} \\
 \Downarrow \\
 a_t (k_t)^\theta (h_t)^{1-\theta} &= y_t s_{t+1}
 \end{aligned}$$

and

$$\begin{aligned}
 s_{t+1} &\equiv \int_0^1 \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} di \\
 &= \underbrace{(1-\alpha) \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta}}_{\text{opt. in period } t} + \underbrace{(1-\alpha)\alpha \left(\frac{\tilde{P}_{t-1}}{P_t}\right)^{-\eta}}_{\text{opt. in period } t-1} + \underbrace{(1-\alpha)\alpha^2 \left(\frac{\tilde{P}_{t-2}}{P_t}\right)^{-\eta}}_{\text{opt. in period } t-2} + \dots \\
 &= (1-\alpha) \sum_{j=0}^{\infty} \alpha^j \left[\frac{\tilde{P}_{t-j}}{P_t}\right]^{-\eta} \\
 &= (1-\alpha) \tilde{p}_t^{-\eta} + P_t^\eta \sum_{j=1}^{\infty} \alpha^j [\tilde{P}_{t-j}]^{-\eta} \quad \text{where } \tilde{p}_{t-j} \equiv \frac{\tilde{P}_{t-j}}{P_t} \\
 &= (1-\alpha) \tilde{p}_t^{-\eta} + P_t^\eta (s_t \alpha P_{t-1}^{-\eta}); \quad \text{see below:} \\
 &= (1-\alpha) \tilde{p}_t^{-\eta} + \alpha (P_t/P_{t-1})^\eta s_t
 \end{aligned}$$

⇕

$$s_{t+1} = (1 - \alpha)\tilde{p}_t^{-\eta} + \alpha(\pi_t)^\eta s_t.$$

We used that

$$s_t = (1 - \alpha) \sum_{j=0}^{\infty} \alpha^j \left[ \frac{\tilde{P}_{t-j-1}}{P_{t-1}} \right]^{-\eta}$$

⇕ change of index:  $j + 1 = l$

$$s_t P_{t-1}^{-\eta} = (1 - \alpha) \sum_{l=1}^{\infty} \alpha^{l-1} [\tilde{P}_{t-l}]^{-\eta}$$

⇕

$$s_t P_{t-1}^{-\eta} \alpha = (1 - \alpha) \sum_{l=1}^{\infty} \alpha^l [\tilde{P}_{t-l}]^{-\eta}.$$

So the resource constraints in the goods market are:

$$(1) \quad a_t k_t^\theta (h_t)^{1-\theta} = y_t s_{t+1},$$

$$(2) \quad s_{t+1} = (1 - \alpha)\tilde{p}_t^{-\eta} + \alpha\pi_t^\eta s_t.$$

#### SD.5 *The aggregate resource constraint*

Recall that firm dividends in equilibrium are given by

$$\begin{aligned} \text{div}_{i,t} &\equiv \left[ \left( \frac{P_{i,t}}{P_t} \right) y_{i,t} - r_t^k k_{i,t} - w_t h_{i,t} \right] \\ &= y_t - r_t^k k_t - w_t h_t. \end{aligned}$$

From the household budget constraint, we have

$$b_t + c_t + i_t = \frac{R_{t-1} b_{t-1}}{\pi_t} + h_t w_t + r_t^k k_t + \text{div}_t$$

⇕

$$c_t + i_t = h_t w_t + r_t^k k_t + \text{div}_t$$

as bonds are in zero net supply in equilibrium. Inserting for firm dividends implies

$$\begin{aligned} c_t + i_t &= h_t w_t + r_t^k k_t + (y_t - r_t^k k_t - w_t h_t) \\ &= y_t. \end{aligned}$$

SD.6 *The goods market: The relation between the optimal price and the price index*

We start by noticing: the number of firms is by construction very large, so there is a fraction of  $1 - \alpha$  firms that reoptimize their prices and the remaining fraction has the same price as in the previous period. This implies

$$\begin{aligned}
 P_t &\equiv \left[ \int_0^1 P_{i,t}^{1-\eta} di \right]^{\frac{1}{1-\eta}} \\
 &\Downarrow \\
 P_t^{1-\eta} &= \int_0^1 \{ (1-\alpha)\tilde{P}_t^{1-\eta} + \alpha P_{i,t-1}^{1-\eta} \} di \\
 &= (1-\alpha)\tilde{P}_t^{1-\eta} + \alpha \int_0^1 P_{i,t-1}^{1-\eta} di \\
 &= (1-\alpha)\tilde{P}_t^{1-\eta} + \alpha \underbrace{\int_0^1 P_{i,t-1}^{1-\eta} di}_{P_{t-1}^{1-\eta}} \\
 &\Downarrow \\
 P_t^{1-\eta} &= (1-\alpha)\tilde{P}_t^{1-\eta} + \alpha P_{t-1}^{1-\eta} \\
 &\Downarrow \\
 1 &= (1-\alpha) \left( \frac{\tilde{P}_t}{P_t} \right)^{1-\eta} + \alpha \left( \frac{P_{t-1}}{P_t} \right)^{1-\eta} \\
 &\Downarrow \\
 1 &= (1-\alpha)\tilde{p}_t^{1-\eta} + \alpha \left( \frac{1}{\pi_t} \right)^{1-\eta}, \quad \text{since } \tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t}.
 \end{aligned}$$

SD.7 *The functional forms*

The households' utility functions

$$\begin{aligned}
 u(c_t, 1 - h_t) &= \frac{1}{1-\phi_2} (c_t)^{1-\phi_2} + \phi_0 \frac{(1 - h_t)^{1-\phi_1}}{1 - \phi_1}, \\
 u_c(c_t, 1 - h_t) &= c_t^{-\phi_2}, \\
 u_{1-h}(c_t, 1 - h_t) &= \phi_0 (1 - h_t)^{-\phi_1}.
 \end{aligned}$$



## SD.8 Summarizing

---

	<i>The households</i>
1	$\lambda_t = c_t^{-\phi_2}$
2	$q_t \lambda_t = E_t \beta \lambda_{t+1} [r_{t+1}^k + q_{t+1}(1 - \delta) - q_{t+1} \frac{\kappa}{2} (\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}})^2 + q_{t+1} \kappa (\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}}) \frac{i_{t+1}}{k_{t+1}}]$
3	$\phi_0 (1 - h_t)^{-\phi_1} = \lambda_t w_t$
4	$1 = q_t (1 - \kappa_2 (\frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}}))$
5	$\lambda_t = \beta R_t E_t [\frac{\lambda_{t+1}}{\pi_{t+1}}]$
	<i>The firms</i>
6	$m c_t a_t (1 - \theta) (\frac{h_t}{k_t})^{-\theta} = w_t$
7	$a_t m c_t \theta (\frac{h_t}{k_t})^{1-\theta} = r_t^k$
8	$\frac{(\eta-1)x_t^2}{\eta} = y_t m c_t \tilde{p}_t^{-\eta-1} + E_t [\alpha \beta \frac{\lambda_{t+1}}{\lambda_t} (\frac{\tilde{p}_t}{\tilde{p}_{t+1}})^{-\eta-1} (\frac{1}{\pi_{t+1}})^{-\eta} \frac{(\eta-1)x_{t+1}^2}{\eta}]$
9	$x_t^2 = y_t \tilde{p}_t^{-\eta} + E_t [\alpha \beta \frac{\lambda_{t+1}}{\lambda_t} (\frac{\tilde{p}_t}{\tilde{p}_{t+1}})^{-\eta} (\frac{1}{\pi_{t+1}})^{1-\eta} x_{t+1}^2]$
10	$1 = (1 - \alpha) \tilde{p}_t^{1-\eta} + \alpha (\frac{1}{\pi_t})^{1-\eta}$
	<i>The central bank</i>
11	$\log(\frac{R_t}{R_{ss}}) = \rho_r \log(\frac{R_{t-1}}{R_{ss}}) + (1 - \rho_r)(\kappa_\pi \log(\frac{\pi_t}{\pi_{ss}}) + \kappa_y \log(\frac{y_t}{y_{ss}}))$
	<i>Other relations</i>
12	$a_t k_t^\theta h_t^{1-\theta} = y_t s_{t+1}$
13	$s_{t+1} = (1 - \alpha) \tilde{p}_t^{-\eta} + \alpha \pi_t^\eta s_t$
14	$k_{t+1} = (1 - \delta) k_t + i_t - \frac{\kappa}{2} (\frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}})^2 k_t$
15	$y_t = c_t + i_t$
	<i>Exogenous processes</i>
16	$\log a_{t+1} = \rho_a \log a_t + \sigma_a \epsilon_{a,t+1}$

---

## SD.9 The intertemporal elasticity of substitution

The intertemporal elasticity of substitution (IES) is given by

$$\begin{aligned} \text{IES} &= -\frac{u_c(t)}{c_t u_{cc}(t)} \\ &= -\frac{(c_t)^{-\phi_2}}{c_t (-\phi_2 c_t^{\phi_2-1})} \\ &= \frac{1}{\phi_2}. \end{aligned}$$

## SD.10 The Frisch labor supply elasticity

Recall that this elasticity is given by

$$\text{elas}_F = \frac{u_h}{h \left( u_{hh} - \frac{u_{ch}}{u_{cc}} \right)}.$$

In our case

$$\begin{aligned} u_h &= -\phi_0 (1 - h_t)^{-\phi_1}, \\ u_{hh} &= -\phi_1 \phi_0 (1 - h_t)^{-\phi_1-1}, \\ u_{ch} &= 0. \end{aligned}$$

So

$$\begin{aligned}
 \text{elas}_F &= \frac{u_h}{h\left(u_{hh} - \frac{u_{ch}}{u_{cc}}\right)} = \frac{-\phi_0(1-h)^{-\phi_1}}{h(-\phi_1\phi_0(1-h)^{-\phi_1-1})} \\
 &= \frac{-(1-h)^{-\phi_1}}{h(-\phi_1(1-h)^{-\phi_1-1})} \\
 &= \frac{1}{h(\phi_1(1-h)^{-1})} \\
 &= \frac{1}{\phi_1} \frac{1-h}{h}.
 \end{aligned}$$

Hence, if  $h = \frac{1}{3}$ , then we get

$$\text{elas}_F = \frac{1}{\phi_1} \frac{1 - \frac{1}{3}}{\frac{1}{3}} = \frac{1}{\phi_1} \frac{3-1}{1} = \frac{2}{\phi_1}.$$

Thus, we get a Frisch labor elasticity of one by letting  $\phi_1 = 2$ .

#### SD.11 *The steady state*

This section solves for the steady state as a function of the structural parameters. We denote variables in steady state by subscript ss. The steady state value of labor, that is,  $h_{ss}$ , is assumed to be given and we then back out the value of  $\phi_0$ . We also consider  $\pi_{ss}$  as given.

*The optimal relative price,  $\tilde{p}_{ss}$ .*

From equation (10),

$$1 = (1 - \alpha)\tilde{p}_t^{1-\eta} + \alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}$$

↓

$$1 - \alpha\pi_{ss}^{-(1-\eta)} = (1 - \alpha)\tilde{p}_{ss}^{1-\eta}$$

⇕

$$\tilde{p}_{ss} = \left[ \frac{1 - \alpha\pi_{ss}^{(\eta-1)}}{1 - \alpha} \right]^{\frac{1}{1-\eta}}.$$

*The state variable for distortion due to price stickiness,  $s_{ss}$*

From equation (13),

$$s_{t+1} = (1 - \alpha)\tilde{p}_t^{-\eta} + \alpha\pi_t^\eta s_t$$

↓

$$\begin{aligned}
s_{ss} &= (1 - \alpha) \tilde{p}_{ss}^{-\eta} + \alpha \pi_{ss}^{\eta} s_{ss} \\
&\Downarrow \\
s_{ss} (1 - \alpha \pi_{ss}^{\eta}) &= (1 - \alpha) \tilde{p}_{ss}^{-\eta} \\
&\Downarrow \\
s_{ss} &= \frac{(1 - \alpha) \tilde{p}_{ss}^{-\eta}}{1 - \alpha \pi_{ss}^{\eta}}.
\end{aligned}$$

*The nominal one-period interest rate,  $R_{ss}$*

From equation (5),

$$\begin{aligned}
1 &= E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{R_t}{\pi_{t+1}} \right] \\
&\Downarrow \\
R_{ss} &= \frac{\pi_{ss}}{\beta}.
\end{aligned}$$

*The real price of capital,  $q_{ss}$*

From equation (4),

$$1 = q_t \left( 1 - \kappa \left( \frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}} \right) \right).$$

We immediately get that  $q_{ss} = 1$ .

*The real price of capital,  $r_{ss}^k$*

From equation (2),

$$q_t \lambda_t = E_t \beta \lambda_{t+1} \left[ r_{t+1}^k + q_{t+1} (1 - \delta) - q_{t+1} \frac{\kappa}{2} \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right)^2 + q_{t+1} \kappa \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right) \frac{i_{t+1}}{k_{t+1}} \right]$$

$\Downarrow$

$$q_{ss} = \beta [r_{ss}^k + q_{ss} (1 - \delta)]$$

$\Downarrow$

$$r_{ss}^k = q_{ss} \left( \frac{1}{\beta} - (1 - \delta) \right).$$

*The marginal costs in the firms,  $mc_{ss}$*

First, from equation (8),

$$\frac{(\eta - 1)x_t^2}{\eta} = y_t mc_t \tilde{p}_t^{-\eta-1} + E_t \left[ \alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \left( \frac{\tilde{p}_t}{\tilde{p}_{t+1}} \right)^{-\eta-1} \left( \frac{1}{\pi_{t+1}} \right)^{-\eta} \frac{(\eta - 1)x_{t+1}^2}{\eta} \right]$$

$\Downarrow$

$$\frac{(\eta - 1)x_{ss}^2}{\eta} = y_{ss} mc_{ss} \tilde{p}_{ss}^{-\eta-1} + \alpha \beta \left( \frac{1}{\pi_{ss}} \right)^{-\eta} \frac{(\eta - 1)x_{ss}^2}{\eta}$$

⇕

$$x_{ss}^2 \frac{(\eta - 1)}{\eta} \left( 1 - \alpha\beta \left( \frac{1}{\pi_{ss}} \right)^{-\eta} \right) = y_{ss} m c_{ss} \tilde{p}_{ss}^{-\eta-1}$$

⇕

$$x_{ss}^2 = \frac{y_{ss} m c_{ss} \tilde{p}_{ss}^{-\eta-1}}{\frac{(\eta - 1)}{\eta} (1 - \alpha\beta \pi_{ss}^\eta)}$$

and from equation (9),

$$x_t^2 = y_t \tilde{p}_t^{-\eta} + E_t \left[ \alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \left( \frac{\tilde{p}_t}{\tilde{p}_{t+1}} \right)^{-\eta} \left( \frac{1}{\pi_{t+1}} \right)^{1-\eta} x_{t+1}^2 \right]$$

⇓

$$x_{ss}^2 = y_{ss} \tilde{p}_{ss}^{-\eta} + \alpha\beta \left( \frac{1}{\pi_{ss}} \right)^{1-\eta} x_{ss}^2$$

⇕

$$x_{ss}^2 = \frac{y_{ss} \tilde{p}_{ss}^{-\eta}}{1 - \alpha\beta \left( \frac{1}{\pi_{ss}} \right)^{1-\eta}}.$$

So by setting the two equations equal to one another, we get

$$\frac{y_{ss} m c_{ss} \tilde{p}_{ss}^{-\eta-1}}{\frac{(\eta - 1)}{\eta} (1 - \alpha\beta \pi_{ss}^\eta)} = \frac{y_{ss} \tilde{p}_{ss}^{-\eta}}{1 - \alpha\beta \left( \frac{1}{\pi_{ss}} \right)^{1-\eta}}$$

⇕

$$m c_{ss} = \frac{\tilde{p}_{ss} \frac{(\eta - 1)}{\eta} (1 - \alpha\beta \pi_{ss}^\eta)}{1 - \alpha\beta \left( \frac{1}{\pi_{ss}} \right)^{1-\eta}}.$$

*The capital level,  $k_{ss}$*

From equation (7),

$$a_t m c_t \theta \left( \frac{h_t}{k_t} \right)^{1-\theta} = r_t^k$$

⇓

$$a_{ss} m c_{ss} \theta \left( \frac{h_{ss}}{k_{ss}} \right)^{1-\theta} = r_{ss}^k$$

⇕

$$\frac{1}{k_{ss}} = \left[ \frac{r_{ss}^k}{a_{ss} m c_{ss} \theta} \right]^{\frac{1}{1-\theta}} \frac{1}{h_{ss}}$$

$$\Downarrow$$

$$k_{ss} = \left[ \frac{r_{ss}^k}{a_{ss} m c_{ss} \theta} \right]^{\frac{1}{\theta-1}} h_{ss}.$$

*The wage level,  $W_{ss}$*

From equation (7),

$$m c_t a_t (1 - \theta) \left( \frac{h_t}{k_t} \right)^{-\theta} = w_t$$

$$\Downarrow$$

$$w_{ss} = m c_{ss} a_{ss} (1 - \theta) \left( \frac{h_{ss}}{k_{ss}} \right)^{-\theta}.$$

*The investment level,  $i_{ss}$*

From equation (14),

$$k_{t+1} = (1 - \delta) k_t + i_t - \frac{\kappa}{2} \left( \frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}} \right)^2 k_t$$

$$\Downarrow$$

$$k_{ss} = (1 - \delta) k_{ss} + i_{ss}$$

$$\Downarrow$$

$$i_{ss} = \delta k_{ss}.$$

*The output level,  $y_{ss}$*

From equation (12),

$$a_t k_t^\theta h_t^{1-\theta} = y_t s_{t+1}$$

$$\Downarrow$$

$$a_{ss} k_{ss}^\theta h_{ss}^{1-\theta} = y_{ss} s_{ss}$$

$$\Downarrow$$

$$y_{ss} = \frac{a_{ss} k_{ss}^\theta h_{ss}^{1-\theta}}{s_{ss}}.$$

*The consumption level:  $c_{ss}$*

From equation (15),

$$y_t = c_t + i_t$$

$$\Downarrow$$

$$c_{ss} = y_{ss} - i_{ss}.$$

The value of  $\lambda_t$

From equation (1),

$$\lambda_{ss} = c_{ss}^{-\phi_2}.$$

The value of  $\phi_0$

From equation (3),

$$\begin{aligned} \phi_0(1 - h_t)^{-\phi_1} &= \lambda_t w_t \\ \Downarrow \\ \phi_0 &= \lambda_{ss} w_{ss} (1 - h_{ss})^{\phi_1}. \end{aligned}$$

### SD.12 Model equation for accuracy study

To evaluate accuracy, we use the following list of equations, where the residuals are expressed in unit-free terms ( $\lambda_t = c_t^{-\phi_2}$ ):

---

	<i>The households</i>
1	$1 = E_t \frac{\beta \lambda_{t+1}}{q_t \lambda_t} [r_{t+1}^k + q_{t+1}(1 - \delta) - q_{t+1} \frac{\kappa}{2} (\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}})^2 + q_{t+1} \kappa (\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}}) \frac{i_{t+1}}{k_{t+1}}]$
2	$1 = \frac{\lambda_t w_t}{\phi_0(1 - h_t)^{-\phi_1}}$
3	$1 = q_t (1 - \kappa_2 (\frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}}))$
4	$1 = \beta R_t E_t [\frac{\lambda_{t+1}}{\lambda_t \pi_{t+1}}]$
	<i>The firms</i>
5	$1 = \frac{m c_t a_t (1 - \theta) (\frac{h_t}{k_t})^{-\theta}}{w_t}$
6	$1 = \frac{a_t m c_t \theta (\frac{h_t}{k_t})^{1-\theta}}{r_t^k}$
7	$1 = \frac{y_t m c_t \bar{p}_t^{-\eta-1} + E_t [\alpha \beta \frac{\lambda_{t+1}}{\lambda_t} (\frac{\bar{p}_t}{\bar{p}_{t+1}})^{-\eta-1} (\frac{1}{\pi_{t+1}})^{-\eta} \frac{(\eta-1)x_{t+1}^2}{\eta}]}{\frac{(\eta-1)x_t^2}{\eta}}$
8	$1 = \frac{y_t \bar{p}_t^{-\eta} + E_t [\alpha \beta \frac{\lambda_{t+1}}{\lambda_t} (\frac{\bar{p}_t}{\bar{p}_{t+1}})^{-\eta} (\frac{1}{\pi_{t+1}})^{1-\eta} x_{t+1}^2]}{\frac{x_t^2}{\eta}}$
9	$1 = (1 - \alpha) \tilde{p}_t^{1-\eta} + \alpha (\frac{1}{\pi_t})^{1-\eta}$
	<i>The central bank</i>
10	$\log(\frac{R_t}{R_{ss}}) = \rho_r \log(\frac{R_{t-1}}{R_{ss}}) + (1 - \rho_r) (\kappa_\pi \log(\frac{\pi_t}{\pi_{ss}}) + \kappa_y \log(\frac{y_t}{y_{ss}}))$
	<i>Other relations</i>
11	$1 = \frac{a_t k_t^\theta h_t^{1-\theta}}{y_t s_{t+1}}$
12	$1 = \frac{(1 - \alpha) \bar{p}_t^{-\eta} + \alpha \pi_t^\eta s_t}{s_{t+1}}$
13	$1 = \frac{(1 - \delta) k_t + i_t - \frac{\kappa}{2} (\frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}})^2 k_t}{k_{t+1}}$
14	$1 = \frac{c_t + i_t}{y_t}$
15	$1 = \frac{R_t}{(R_{t-1})_{t+1}}$ , link equation for $R_{t-1}$
	<i>Exogenous processes</i>
16	$\log a_{t+1} = \rho_a \log a_t + \sigma_a \epsilon_{a,t+1}$

---

### SD.13 The upper bound on inflation

The Calvo model for staggered pricing implies an upper bound on inflation. To see this, recall that

$$P_t^{1-\eta} \equiv \int_0^1 P_{i,t}^{1-\eta} di = (1 - \alpha) \tilde{P}_t^{1-\eta} + \alpha P_{t-1}^{1-\eta}$$

⇕

$$1 = (1 - \alpha) \left( \frac{\tilde{P}_t}{P_t} \right)^{1-\eta} + \alpha \left( \frac{1}{\pi_t} \right)^{1-\eta}.$$

Consider the limiting case, where marginal cost  $mc_t$ , and hence the new price of the optimizing firm  $\tilde{P}_t$  becomes arbitrary large. The constant elasticity of substitution index for prices with negative exponent, that is,  $P_t^{1-\eta} \equiv \int_0^1 P_{i,t}^{1-\eta} di$ , implies that the aggregate price index remains bounded, because the components with large prices  $(1 - \alpha) \left( \frac{\tilde{P}_t}{P_t} \right)^{1-\eta}$  tend to zero as  $\eta > 1$ . Isolating for inflation then gives the upper bound  $\pi^{\max}$

$$1 = \alpha \left( \frac{1}{\pi^{\max}} \right)^{1-\eta}$$

⇕

$$\frac{1}{\pi^{\max}} = \left( \frac{1}{\alpha} \right)^{\frac{1}{1-\eta}}$$

⇕

$$\pi^{\max} = \left( \frac{1}{\alpha} \right)^{\frac{1}{\eta-1}}.$$

Intuitively, consumers substitute away from the expensive goods with high prices to goods with old and lower prices. This substitution effect ensures that inflation across all goods in the economy remains bounded even when  $\tilde{P}_t$  becomes arbitrary large.

Consider the case with price indexation of firms that do not reoptimize their prices. Then we have

$$P_t^{1-\eta} \equiv \int_0^1 P_{i,t}^{1-\eta} di = (1 - \alpha) \tilde{P}_t^{1-\eta} + \alpha (\pi_{t-1}^\chi P_{t-1})^{1-\eta}$$

⇕

$$1 = (1 - \alpha) \left( \frac{\tilde{P}_t}{P_t} \right)^{1-\eta} + \alpha \left( \frac{\pi_{t-1}^\chi}{\pi_t} \right)^{1-\eta}.$$

Thus, with full indexation ( $\chi = 1$ ), we get an upper bound for the change in inflation as

$$1 = \alpha \left( \frac{\pi_{t-1}}{\pi_t} \right)^{1-\eta}.$$

⇕

$$\left( \frac{1}{\alpha} \right)^{1/(1-\eta)} = \frac{\pi_{t-1}}{\pi_t}$$

⇕

$$\log \left( \frac{\pi_t}{\pi_{t-1}} \right) = \log(\alpha^{1/(1-\eta)}).$$

## APPENDIX SE: A MULTICOUNTRY RBC MODEL

We consider the multicountry RBC model presented in [Juillard and Villemot \(2011\)](#). The problem for the social planner is (with the same notation as in [Juillard and Villemot \(2011\)](#))

$$\text{Max}_{\{c_t^j, i_t^j, k_{t+1}^j\}_{t=0,1,\dots,\infty, j=1,2,\dots,N}} \mathbb{E}_t \sum_{j=1}^N \tau^j \left( \sum_{i=1}^{\infty} \beta^i \frac{(c_{t+i}^j)^{1-1/\gamma}}{1-1/\gamma} \right)$$

subject to

$$k_{t+1}^j = (1 - \delta)k_t^j + i_t^j,$$

$$\sum_{j=1}^N (c_t^j + i_t^j - \delta k_t^j) = \sum_{j=1}^N \left( a_t^j A(k_t^j)^\alpha - \frac{\phi}{2} k_t^j \left( \frac{i_t^j}{k_t^j} - \delta \right)^2 \right).$$

The Lagrangian reads

$$\begin{aligned} \mathcal{L} = & \mathbb{E}_t \sum_{j=1}^N \tau^j \left( \sum_{i=1}^{\infty} \beta^i \frac{(c_{t+i}^j)^{1-1/\gamma}}{1-1/\gamma} \right) \\ & + \sum_{i=1}^{\infty} \beta^i \lambda_{t+i} q_{t+i} (k_{t+1+i}^j - (1 - \delta)k_{t+i}^j - i_{t+i}^j) \\ & + \sum_{i=1}^{\infty} \beta^i \lambda_{t+i} \left( \sum_{j=1}^N (c_{t+i}^j + i_{t+i}^j - \delta k_{t+i}^j) - \sum_{j=1}^N \left( a_{t+i}^j A(k_{t+i}^j)^\alpha - \frac{\phi}{2} k_{t+i}^j \left( \frac{i_{t+i}^j}{k_{t+i}^j} - \delta \right)^2 \right) \right). \end{aligned}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t^j} : \tau^j (c_t^j)^{-1/\gamma} = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial i_t^j} : -\lambda_t q_t + \lambda_t + \lambda_t \phi k_t^j \left( \frac{i_t^j}{k_t^j} - \delta \right) \frac{1}{k_t^j} = 0$$

⇔

$$\lambda_t \left[ 1 + \phi \left( \frac{i_t^j}{k_t^j} - \delta \right) \right] = \lambda_t q_t$$

⇔

$$q_t = 1 + \phi \left( \frac{i_t^j}{k_t^j} - \delta \right)$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}^j} = \lambda_t q_t + \mathbb{E}_t \left[ \beta \lambda_{t+1} \left( -q_{t+1} (1 - \delta) - \delta - \alpha a_{t+1}^j A(k_{t+1}^j)^{\alpha-1} \right) \right]$$



$$+ \frac{\phi}{2} \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right)^2 - \phi k_{t+1}^j \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \frac{i_{t+1}^j}{(k_{t+1}^j)^2} \Big] = 0$$

⇔

$$\begin{aligned} & \lambda_t \left[ 1 + \phi \left( \frac{i_t^j}{k_t^j} - \delta \right) \right] \\ & + \mathbb{E}_t \left[ \beta \lambda_{t+1} \left( - \left[ 1 + \phi \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \right] (1 - \delta) - \delta - \alpha a_{t+1}^j A(k_{t+1}^j)^{\alpha-1} \right. \right. \\ & \left. \left. + \frac{\phi}{2} \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right)^2 - \phi \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \frac{i_{t+1}^j}{k_{t+1}^j} \right) \right] = 0 \end{aligned}$$

⇔

$$\begin{aligned} & \lambda_t \left[ 1 + \phi \left( \frac{i_t^j}{k_t^j} - \delta \right) \right] \\ & + \mathbb{E}_t \left[ \beta \lambda_{t+1} \left( - (1 - \delta) - \phi \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) (1 - \delta) - \delta - \alpha a_{t+1}^j A(k_{t+1}^j)^{\alpha-1} \right. \right. \\ & \left. \left. + \frac{\phi}{2} \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right)^2 - \phi \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \frac{i_{t+1}^j}{k_{t+1}^j} \right) \right] = 0 \end{aligned}$$

⇔

$$\begin{aligned} & \lambda_t \left[ 1 + \phi \left( \frac{i_t^j}{k_t^j} - \delta \right) \right] \\ & + \mathbb{E}_t \left[ \beta \lambda_{t+1} \left( - 1 - \alpha a_{t+1}^j A(k_{t+1}^j)^{\alpha-1} - \phi \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) (1 - \delta) \right. \right. \\ & \left. \left. + \frac{\phi}{2} \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right)^2 - \phi \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \frac{i_{t+1}^j}{k_{t+1}^j} \right) \right] = 0 \end{aligned}$$

⇔

$$\begin{aligned} & \lambda_t \left[ 1 + \phi \left( \frac{i_t^j}{k_t^j} - \delta \right) \right] \\ & + \mathbb{E}_t \left[ \beta \lambda_{t+1} \left( - 1 - \alpha a_{t+1}^j A(k_{t+1}^j)^{\alpha-1} \right. \right. \\ & \left. \left. - \phi \left\{ (1 - \delta) + \frac{i_{t+1}^j}{k_{t+1}^j} - \frac{1}{2} \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \right\} \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \right) \right] = 0 \end{aligned}$$

⇕

$$1 + \phi \left( \frac{i_t^j}{k_t^j} - \delta \right) = \mathbb{E}_t \left[ \frac{\beta \lambda_{t+1}}{\lambda_t} \left( 1 + \alpha a_{t+1}^j A(k_{t+1}^j) \right)^{\alpha-1} \right. \\ \left. + \phi \left\{ (1 - \delta) + \frac{i_{t+1}^j}{k_{t+1}^j} - \frac{1}{2} \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \right\} \left( \frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \right].$$

Thus, the economy is given by

---

1	$\tau^j (c_t^j)^{-1/\gamma} = \lambda_t$
2	$\lambda_t [1 + \phi (\frac{i_t^j}{k_t^j} - \delta)] = \mathbb{E}_t [\beta \lambda_{t+1} (1 + \alpha a_{t+1}^j A(k_{t+1}^j) \right)^{\alpha-1} + \phi \{ (1 - \delta) + \frac{i_{t+1}^j}{k_{t+1}^j} - \frac{1}{2} (\frac{i_{t+1}^j}{k_{t+1}^j} - \delta) \} (\frac{i_{t+1}^j}{k_{t+1}^j} - \delta)]$
3	$k_{t+1}^j = (1 - \delta) k_t^j + i_t^j$
4	$\sum_{j=1}^N (c_t^j + i_t^j - \delta k_t^j) = \sum_{j=1}^N (a_t^j A(k_t^j)^\alpha - \frac{\phi}{2} k_t^j (\frac{i_t^j}{k_t^j} - \delta)^2)$
5	$\log a_{t+1}^j = \rho \log a_t^j + \sigma e_t^j$

---

For the steady state, we have

- $a_{ss}^j = 1$  from equation (5)
- $k_{ss}^j = 1$  from an assumption in [Juillard and Villemot \(2011\)](#)
- $c_{ss}^j = a_{ss}^j A(k_{ss}^j)^\alpha = A$

From equation (3),

$$k_{ss}^j = (1 - \delta) k_{ss}^j + i_{ss}^j$$

⇕

$$1 = 1 - \delta + i_{ss}^j$$

⇕

$$i_{ss}^j = \delta.$$

From equation (2),

$$1 + \phi \left( \frac{i_{ss}^j}{k_{ss}^j} - \delta \right) \\ = \frac{\beta \lambda_{t+ss}}{\lambda_{ss}} \left( 1 + \alpha a_{ss}^j A(k_{ss}^j) \right)^{\alpha-1} + \phi \left\{ (1 - \delta) + \frac{i_{ss}^j}{k_{ss}^j} - \frac{1}{2} \left( \frac{i_{ss}^j}{k_{ss}^j} - \delta \right) \right\} \left( \frac{i_{ss}^j}{k_{ss}^j} - \delta \right)$$

⇕

$$1 = \beta (1 + \alpha A)$$

⇕

$$1 = \beta + \alpha \beta A$$

⇕

$$A = \frac{1 - \beta}{\alpha\beta}.$$

From equation (4),

$$\sum_{j=1}^N (c_{ss}^j + i_{ss}^j - \delta k_{ss}^j) = \sum_{j=1}^N \left( a_{ss}^j A (k_{ss}^j)^\alpha - \frac{\phi}{2} k_{ss}^j \left( \frac{i_{ss}^j}{k_{ss}^j} - \delta \right)^2 \right)$$

⇕

$$\sum_{j=1}^N c_{ss}^j = \sum_{j=1}^N A$$

⇕

$$\sum_{j=1}^N A = \sum_{j=1}^N A$$

ok.

From equation (1),

$$\lambda_{ss} = \tau^j (c_{ss}^j)^{-1/\gamma}.$$

For  $\tau^j$ , we follow Juillard and Villemot (2011) and let

$$\tau^j = \frac{1}{(c_{ss}^j)^{-1/\gamma}} = A^{1/\gamma}.$$

The considered calibration is given by  $\gamma = 0.25$ ,  $\delta = 0.025$ ,  $\beta = 0.99$ ,  $\alpha = 0.36$ ,  $\rho = 0.99$ ,  $\sigma = 0.01$ , and  $\phi = 0.5$ .

## REFERENCES

- Adjemian, Stephane and Michel Juillard (2010), “Dealing with ZLB in DSGE models: An application to the Japanese economy.” ESRI Discussion Paper No. 258. [6]
- Andreasen, Martin M. (2012), “An estimated DSGE model: Explaining variation in nominal term premia, real term premia, and inflation risk premia.” *European Economic Review*, 56, 1656–1674. [6]
- Andreasen, Martin M. and Pawel Zabczyk (2015), “Efficient bond price approximations in non-linear equilibrium-based term structure models.” *Studies in Nonlinear Dynamics and Econometrics*, 19 (1), 1–34. [6]
- Binning, Andrew (2013), “Solving second and third-order approximations to DSGE models: A recursive Sylvester equation solution.” Norges Bank, Working Paper 18. [3]
- Boucekkine, Raouf (1995), “An alternative methodology for solving nonlinear forward looking models.” *Journal of Economic Dynamic and Control*, 19 (4), 711–734. [2, 5, 11, 19]

Fair, Ray C. and John B. Taylor (1983), "Solution and maximum likelihood estimation of dynamic nonlinear rational expectations models." *Econometrica*, 51 (4), 1169–1185. [2, 3]

Juillard, Michel and Sebastien Villemot (2011), "Multi-country real business cycle models: Accuracy tests and test bench." *Journal of Economic Dynamic and Control*, 35, 178–185. [48, 50, 51]

King, Rober G. and Sergio T. Rebelo (1999), "Resuscitating real business cycles." *Handbook of Macroeconomics*, 1, 927–1007. [2]

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