Supplement to "Concave-monotone treatment response and monotone treatment selection: With an application to the returns to schooling"

(*Quantitative Economics*, Vol. 5, No. 1, March 2014, 175–194)

Tsunao Okumura
Yokohama National University

Emiko Usui
Nagoya University and IZA

Appendix A: Proof of Proposition 1

Proof of part (a). (a.1) *Proof that the bounds on* $E[y(t)]$ *in Equations (4), (5), and (6) hold.*

For $u < s$, $E[y|z = s] = E[y(s)|z = s] \geq E[y(u)|z = s]$ by the MTR assumption; $E[y(u)|z = s] \geq E[y(u)|z = u] = E[y|z = u]$ by the MTS assumption. Hence,

$$E[y|z = s] \geq E\big[y(u)|z = s\big] \geq E[y|z = u]. \tag{S.1}$$

Because $y_j(\tau)$ is concave-MTR in $\tau \in T$ for all $j \in J$, $E[y(\tau)|z = s]$ is concave-MTR in τ.

Compare $E[y(t)|z = s]$ with the value of the function that describes the straight line joining the points $(s, E[y|z = s])$ and $(u, E[y|z = u])$, evaluated at t.

Because Equation (S.1) holds and $E[y(\tau)|z = s]$ is concave-MTR in τ, for $u \leq t < s$,

$$E\big[y(t)|z = s\big] \geq \frac{s-t}{s-u}E[y|z = u] + \frac{t-u}{s-u}E[y|z = s], \tag{S.2}$$

and for $u < s \leq t$,

$$E\big[y(t)|z = s\big] \leq \frac{s-t}{s-u}E[y|z = u] + \frac{t-u}{s-u}E[y|z = s]. \tag{S.3}$$

Because Equation (S.2) holds for any u that is not greater than t when t is smaller than s, then for $t < s$,

$$E\big[y(t)|z = s\big] \geq \max_{\{u|u\leq t\}} \frac{s-t}{s-u}E[y|z = u] + \frac{t-u}{s-u}E[y|z = s]. \tag{S.4}$$

Tsunao Okumura: okumura@ynu.ac.jp
Emiko Usui: usui@soec.nagoya-u.ac.jp

DOI: 10.3982/QE268

Similarly, because Equation (S.3) holds for any u that is smaller than s when s is not greater than t, then for $t \geq s$,

$$E\big[y(t)|z=s\big] \leq \min_{\{u|u<s\}} \frac{s-t}{s-u}E[y|z=u] + \frac{t-u}{s-u}E[y|z=s]. \tag{S.5}$$

The MTS assumption implies that for all $s' \leq s$, $E[y(t)|z=s'] \leq E[y(t)|z=s]$. Furthermore, for all $t < s' \leq s$, Equation (S.4) can be applied to the lower bound on $E[y(t)|z=s']$: for $t < s' \leq s$,

$$E\big[y(t)|z=s'\big] \geq \max_{\{u|u\leq t\}} \frac{s'-t}{s'-u}E[y|z=u] + \frac{t-u}{s'-u}E\big[y|z=s'\big]. \tag{S.6}$$

Therefore, for $t < s$,

$$\begin{aligned} E\big[y(t)|z=s\big] &\geq \max_{\{(u,s')|u\leq t<s'\leq s\}} \frac{s'-t}{s'-u}E[y|z=u] + \frac{t-u}{s'-u}E\big[y|z=s'\big] \\ &= \mathrm{LB}(s,t). \end{aligned} \tag{S.7}$$

Similarly, for $t > s$, by the MTS assumption and Equation (S.5),

$$\begin{aligned} E\big[y(t)|z=s\big] &\leq \min_{\{(u,s')|s\leq s'\leq t\wedge u<s'\}} \frac{s'-t}{s'-u}E[y|z=u] + \frac{t-u}{s'-u}E\big[y|z=s'\big] \\ &= \mathrm{UB}(s,t). \end{aligned} \tag{S.8}$$

Applying Equations (S.7) and (S.8) to the law of iterated expectations yields the second terms of the upper and lower bounds, respectively, on $E[y(t)]$ in Equation (4).

Manski (1997) and Manski and Pepper (2000) showed that under either the concave-MTR or the MTS–MTR assumptions, for $s \leq t$,

$$E\big[y(t)|z=s\big] \geq E[y|z=s], \tag{S.9}$$

and for $s \geq t$,

$$E\big[y(t)|z=s\big] \leq E[y|z=s]. \tag{S.10}$$

Applying Equations (S.9) and (S.10) to the law of iterated expectations yields the first terms of the lower and upper bounds, respectively, on $E[y(t)]$ in Equation (4).

These results thus yield the bounds on $E[y(t)]$ in Equation (4).

(a.2) *Proof that the bounds on $E[y(t)]$ in Equations (4), (5), and (6) are sharp.*

To show that the bounds on $E[y(t)]$ in Equation (4) are sharp, it suffices to demonstrate (i) that there exists a set of functions of $y_j(\tau)$ for $\tau \in T$ that satisfy the concave-MTR and MTS assumptions and that attain the lower bound, and (ii) that there also exists a set of functions of $y_j(\tau)$ for $\tau \in T$ that satisfy the concave-MTR and MTS assumptions and that attain the upper bound.

(a.2.1) *Proof of the existence of the functions $y_j(\tau)$ for $\tau \in T$ that satisfy the concave-MTR and MTS assumptions and that attain the lower bound.*

The proof is organized in the following seven steps: Step 1 specifies the functions $E[y(\tau)|z=s]$ for $\tau \in T$. Step 2 proves that these functions satisfy the concave-MTR assumption. Step 3 proves that these functions satisfy the MTS assumption. Steps 4 and 5 prove that these functions are equal to $E[y|z=s]$ when $\tau = s$. Step 6 proves that these functions attain the lower bound in Equation (4). Step 7 concludes.

Step 1. Define the following three functions: For $k = s, \widetilde{s}, t$ and $l = s, t$,

$$\begin{aligned}\big(u^*(k,t), s'^*(k,t)\big) = \arg \max_{\{(u,s')|u \le t < s' \le k\}} & \frac{s'-t}{s'-u} E[y|z=u] \\ & + \frac{t-u}{s'-u} E\big[y|z=s'\big],\end{aligned} \tag{S.11}$$

$$\begin{aligned}\mathrm{LB}(\tau,k,t) = & \frac{s'^*(k,t)-\tau}{s'^*(k,t)-u^*(k,t)} E\big[y|z=u^*(k,t)\big] \\ & + \frac{\tau-u^*(k,t)}{s'^*(k,t)-u^*(k,t)} E\big[y|z=s'^*(k,t)\big],\end{aligned} \tag{S.12}$$

$$\mathrm{LF}(\tau,l,t) = \min_{\{\widetilde{s}|\widetilde{s}\ge l\}} \min\big\{\mathrm{LB}(\tau,\widetilde{s},t), E[y|z=\widetilde{s}]\big\}, \tag{S.13}$$

For $s > t$, let the function $E[y(\tau)|z=s]$ be

$$\mathrm{LF}(\tau,s,t). \tag{S.14}$$

For $s \le t$, let the function $E[y(\tau)|z=s]$ be

$$\min\big\{E[y|z=s], \mathrm{LF}(\tau,t,t)\big\}. \tag{S.15}$$

Notice that for $k > t$, the function $\mathrm{LB}(t,k,t)$ in Equation (S.12) where $\tau = t$ weakly increases in k. This is because in Equation (S.11), the object is maximized over the set $\{(u,s')|u \le t < s' \le k\}$ such that the set $\{(u,s')|u \le t < s' \le k_1\}$ includes the set $\{(u,s')|u \le t < s' \le k_2\}$ for $k_1 \ge k_2$ and given u; the maximal value over the former set is therefore not smaller than that over the latter set. The function $\mathrm{LB}(t,k,t)$ is the maximal value in Equation (S.11). Notice also that $\mathrm{LF}(\tau,l,t)$ weakly increases in l. This is because in Equation (S.13), the object is minimized over the set $\{\widetilde{s}|\widetilde{s} \ge l\}$ such that the set $\{\widetilde{s}|\widetilde{s} \ge l_1\}$ includes the set $\{\widetilde{s}|\widetilde{s} \ge l_2\}$ for $l_1 \le l_2$; the minimal value over the former set is therefore not greater than that over the latter set.

Step 2. The functions (S.14) and (S.15) satisfy the concave-MTR assumption, because their graphs are the boundaries of the convex hulls (i.e., the intersection of the subgraph of the weakly increasing linear functions in τ) and because they weakly increase in τ.

Step 3. The functions (S.14) and (S.15) satisfy the MTS assumption, since $\mathrm{LF}(\tau,s,t)$ and $E[y|z=s]$ weakly increase in s.

Step 4. We now prove that when $s > t$, $\mathrm{LF}(s,s,t)$ in Equation (S.14) is equal to $E[y|z=s]$.

First, for $t < s \le \widetilde{s}$, by Equations (S.11) and (S.12) where $k = \widetilde{s}$ and $\tau = t$,

$$\mathrm{LB}(t,\widetilde{s},t) \ge \frac{s-t}{s-u^*(\widetilde{s},t)} E\big[y|z=u^*(\widetilde{s},t)\big] + \frac{t-u^*(\widetilde{s},t)}{s-u^*(\widetilde{s},t)} E[y|z=s]. \tag{S.16}$$

The left hand side of Equation (S.16) is the value of the line traversing $(u^*(\widetilde{s},t), E[y|z=u^*(\widetilde{s},t)])$ and $(s'^*(\widetilde{s},t), E[y|z=s'^*(\widetilde{s},t)])$, evaluated at t, whereas the right hand side of Equation (S.16) is the value of the line traversing $(u^*(\widetilde{s},t), E[y|z=u^*(\widetilde{s},t)])$ and $(s, E[y|z=s])$, evaluated at t. Notice that $u^*(\widetilde{s},t) \le t < s'^*(\widetilde{s},t) \le \widetilde{s}$ and $t < s \le \widetilde{s}$. Therefore, $E[y|z=s]$ is less than or equal to the value of the line traversing $(u^*(\widetilde{s},t), E[y|z=u^*(\widetilde{s},t)])$ and $(s'^*(\widetilde{s},t), E[y|z=s'^*(\widetilde{s},t)])$, evaluated at s, that is equal to the value $\mathrm{LB}(s,\widetilde{s},t)$, that is, for $t < s \le \widetilde{s}$,

$$\mathrm{LB}(s,\widetilde{s},t) \ge E[y|z=s]. \tag{S.17}$$

Second, for $\widetilde{s} \ge s$, because of the MTS–MTR assumption,

$$E[y|z=\widetilde{s}] \ge E[y|z=s]. \tag{S.18}$$

(Hereafter, we refer to this result as the monotonicity of $E[y|z]$ in z.)

Hence, for $s > t$, Equation (S.13) where $\tau = s$ and $l = s$, and Equations (S.17) and (S.18) imply that $\mathrm{LF}(s,s,t) = E[y|z=s]$. That is, when $s > t$ and $\tau = s$, the function (S.14) is equal to $E[y|z=s]$.

Step 5. We now prove that when $s \le t$ and $\tau = s$, the functions (S.15) is equal to $E[y|z=s]$.

First, for $s \le t \le \widetilde{s}$, by Equations (S.11) and (S.12) where $k = \widetilde{s}$ and $\tau = t$,

$$\mathrm{LB}(t,\widetilde{s},t) \ge \frac{s'^*(\widetilde{s},t) - t}{s'^*(\widetilde{s},t) - s} E[y|z=s] + \frac{t-s}{s'^*(\widetilde{s},t) - s} E\big[y|z=s'^*(\widetilde{s},t)\big]. \tag{S.19}$$

The left hand side of Equation (S.19) is the value of the line traversing $(u^*(\widetilde{s},t), E[y|z=u^*(\widetilde{s},t)])$ and $(s'^*(\widetilde{s},t), E[y|z=s'^*(\widetilde{s},t)])$, evaluated at t, whereas the right hand side of Equation (S.19) is the value of the line traversing $(s, E[y|z=s])$ and $(s'^*(\widetilde{s},t), E[y|z=s'^*(\widetilde{s},t)])$, evaluated at t. Notice that $u^*(\widetilde{s},t) \le t < s'^*(\widetilde{s},t) \le \widetilde{s}$ and $s \le t$. Therefore, $E[y|z=s]$ is less than or equal to the value of the line traversing $(u^*(\widetilde{s},t), E[y|z=u^*(\widetilde{s},t)])$ and $(s'^*(\widetilde{s},t), E[y|z=s'^*(\widetilde{s},t)])$, evaluated at s, that is equal to the value $\mathrm{LB}(s,\widetilde{s},t)$, that is, $\mathrm{LB}(s,\widetilde{s},t) \ge E[y|z=s]$.

Second, for $s \le t \le \widetilde{s}$, Equation (S.18) holds. Thus, for $s \le t$, $\mathrm{LF}(s,t,t) \ge E[y|z=s]$. Therefore, when $s \le t$ and $\tau = s$, the function (S.15) is equal to $E[y|z=s]$.

Step 6. We now prove that the functions (S.14) and (S.15) attain the lower bound in Equation (4). The proof is organized in the following five substeps: Substeps 6.1–6.3 prove that the function (S.14) is equal to $\mathrm{LB}(s,t)$ when $s > t$ and $\tau = t$. Substep 6.4 proves that the function (S.15) is equal to $E[y|z=s]$ when $s \le t$ and $\tau = t$. Substep 6.5 uses the previous substeps and the law of iterated expectations to prove that the functions (S.14) and (S.15) attain the lower bound in Equation (4).

Substep 6.1. Since the function $\mathrm{LB}(t,k,t)$ weakly increases in k, for $\widetilde{s} \ge s$,

$$\mathrm{LB}(t,s,t) \le \mathrm{LB}(t,\widetilde{s},t). \tag{S.20}$$

Substep 6.2. When $k = \widetilde{s}$ in Equation (S.11), $u^*(\widetilde{s},t) \le t < s'^*(\widetilde{s},t) \le \widetilde{s}$. Therefore, the MTS–MTR assumption implies that $E[y|z=\widetilde{s}] \ge E[y|z=s'^*(\widetilde{s},t)]$ and $E[y|z=s'^*(\widetilde{s},t)] \ge E[y|z=u^*(\widetilde{s},t)]$. Hence, for $\widetilde{s} > t$,

$$\mathrm{LB}(t,\widetilde{s},t) \le E[y|z=\widetilde{s}]. \tag{S.21}$$

Substep 6.3. By Equation (S.13) where $\tau = t$ and $l = s$, and by Equations (S.20) and (S.21), it follows that for $s > t$,

$$\mathrm{LF}(t, s, t) = \mathrm{LB}(t, s, t) = \mathrm{LB}(s, t). \tag{S.22}$$

The last equality holds because of Equations (5), (S.11), and (S.12). Hence, when $s > t$ and $\tau = t$, the function (S.14) is equal to $\mathrm{LB}(s, t)$ in Equation (5).

Substep 6.4. We now prove that the function (S.15) is equal to $E[y|z = s]$ when $s \le t$ and $\tau = t$. The proof is constructed along lines that are similar to the proofs of Substeps 6.1–6.3. For $\widetilde{s} \ge t$, because (i) Equations (S.11) and (S.12) hold, (ii) $\mathrm{LB}(t, k, t)$ weakly increases in k for $k \ge t$, and (iii) $s'^{*}(t, t) = t$, it follows that

$$\mathrm{LB}(t, \widetilde{s}, t) \ge \mathrm{LB}(t, t, t) = E[y|z = t]. \tag{S.23}$$

Furthermore, for $\widetilde{s} \ge t$,

$$E[y|z = \widetilde{s}] \ge E[y|z = t]. \tag{S.24}$$

By Equation (S.13) where $\tau = t$ and $l = t$, and by Equations (S.23) and (S.24),

$$\mathrm{LF}(t, t, t) = E[y|z = t]. \tag{S.25}$$

For $s \le t$,

$$E[y|z = s] \le E[y|z = t]. \tag{S.26}$$

Hence, by Equations (S.25) and (S.26), when $s \le t$ and $\tau = t$, the function (S.15) is equal to $E[y|z = s]$.

Substep 6.5. When the function $E[y(\tau)|z = s]$ is (S.14) for $s > t$ and (S.15) for $s \le t$, by Substeps 6.3 and 6.4, together with the law of iterated expectations,

$$E\big[y(t)\big] = \sum_{s \le t} E[y|z = s]P(z = s) + \sum_{s > t} \mathrm{LB}(s, t)P(z = s). \tag{S.27}$$

The quantity (S.27) is the lower bound in Equation (4). Therefore, these functions attain the lower bound in Equation (4).

Step 7. By combining Steps 1–6, we conclude that the functions (S.14) and (S.15) satisfy the concave-MTR and MTS assumptions and attain the lower bound in Equation (4). Hence, there exists a set of functions of $y_j(\tau)$ for $\tau \in T$ that satisfy the concave-MTR and MTS assumptions and that attain the lower bound in Equation (4).

For $s > t$, $\mathrm{LB}(s, t)$ is the sharp lower bound on $E[y(t)|z = s]$, and for $s \le t$, $E[y|z = s]$ is the sharp lower bound on $E[y(t)|z = s]$. Hence, the sharp joint lower bound on $\{E[y(t)|z = s], s \in T\}$ is obtained by setting each of the quantities $E[y(t)|z = s]$, $s \in T$, at $\mathrm{LB}(s, t)$ for $s > t$ and at $E[y|z = s]$ for $s \le t$. Therefore, the lower bound in Equation (4) is the sharp lower bound on $E[y(t)]$.

(a.2.2) *Proof of the existence of the functions $y_j(\tau)$ for $\tau \in T$ that satisfy the concave-MTR and MTS assumptions and that attain the upper bound.*

The proof is organized in the following eight steps: Step 1 specifies functions $E[y(\tau)|z=s]$ for $\tau \in T$. Step 2 proves that these functions satisfy the concave-MTR assumption. Step 3 proves that these functions satisfy the MTS assumption. Steps 4 and 5 prove that these functions are equal to $E[y|z=s]$ when $\tau = s$. Steps 6, 7, and 8 prove that these functions attain the upper bound in Equation (4).

Step 1. For $s < t$, let the function $E[y(\tau)|z=s]$ for $\tau \in T$ be

$$\mathrm{UF}(\tau, s, t) = \min\left\{ \min_{\{\widetilde{s}|s \le \widetilde{s} < t\}} \frac{t-\tau}{t-\widetilde{s}} E[y|z=\widetilde{s}] + \frac{\tau - \widetilde{s}}{t-\widetilde{s}} \mathrm{UB}(\widetilde{s}, t), E[y|z=t] \right\}. \tag{S.28}$$

For $s \ge t$, let the function $E[y(\tau)|z=s]$ for $\tau \in T$ be

$$E[y|z=s]. \tag{S.29}$$

Notice that the functions (6), (S.28), and (S.29) weakly increase in s, and $\mathrm{UB}(s,t)$ weakly increases in s, because in Equation (6) the object is minimized over the set $\{(\eta_1, \eta_2)|s \le \eta_2 \le t \wedge \eta_1 < \eta_2\}$ such that the set $\{(\eta_1, \eta_2)|s_1 \le \eta_2 \le t \wedge \eta_1 < \eta_2\}$ includes the set $\{(\eta_1, \eta_2)|s_2 \le \eta_2 \le t \wedge \eta_1 < \eta_2\}$ for $s_1 \le s_2$ and given η_1; the minimal value over the former set is therefore not greater than that over the latter set. Similarly, $\mathrm{UF}(\tau, s, t)$ in Equation (S.28) weakly increases in s. The function (S.29) weakly increases in s because of the monotonicity of $E[y|z]$ in z.

Step 2. The function $\mathrm{UF}(\tau, s, t)$ in Equation (S.28) satisfies the concave-MTR assumption, since by definition its graph is the boundary of the convex hull. The function $E[y|z=s]$ in Equation (S.29) satisfies the concave-MTR assumption.

Step 3. The functions (S.28) and (S.29) satisfy the MTS assumption, since these functions weakly increase in s.

Step 4. We now prove that when $s < t$ and $\tau = s$, the function (S.28) (i.e., $\mathrm{UF}(s, s, t)$) is equal to $E[y|z=s]$.

First, by Equation (6), for $s < \widetilde{s} < t$,

$$\mathrm{UB}(\widetilde{s}, t) \le \frac{\widetilde{s}-t}{\widetilde{s}-s} E[y|z=s] + \frac{t-s}{\widetilde{s}-s} E[y|z=\widetilde{s}]. \tag{S.30}$$

The right hand side of Equation (S.30) is the value of the line traversing $(s, E[y|z=s])$ and $(\widetilde{s}, E[y|z=\widetilde{s}])$, evaluated at t. Therefore, $E[y|z=s]$ is less than or equal to the value of the line traversing $(\widetilde{s}, E[y|z=\widetilde{s}])$ and $(t, \mathrm{UB}(\widetilde{s}, t))$, evaluated at s, that is, for $s < \widetilde{s} < t$,

$$E[y|z=s] \le \frac{t-s}{t-\widetilde{s}} E[y|z=\widetilde{s}] + \frac{s-\widetilde{s}}{t-\widetilde{s}} \mathrm{UB}(\widetilde{s}, t). \tag{S.31}$$

Second, when $s = \widetilde{s}$, the right hand side of Equation (S.31) is equal to $E[y|z=s]$. Third, when $s < t$, $E[y|z=s] \le E[y|z=t]$ because of the monotonicity of $E[y|z]$ in z. Thus, when $s < t$, $\mathrm{UF}(s, s, t)$ is equal to $E[y|z=s]$.

Step 5. The function (S.29) is $E[y|z=s]$ when $s \ge t$ and $\tau = s$.

Step 6. We now prove that when $s < t$ and $\tau = t$, the function (S.28) (i.e., $\mathrm{UF}(t, s, t)$) is equal to $\mathrm{UB}(s, t)$, and that when $s \ge t$ and $\tau = t$, the function (S.29) is equal to $E[y|z=s]$.

When $s < t$, $\mathrm{UF}(t, s, t) = \min(\min_{\{\widetilde{s}|s\le\widetilde{s}<t\}} \mathrm{UB}(\widetilde{s}, t), E[y|z = t])$. Because $\mathrm{UB}(s, t)$ weakly increases in s, $\min_{\{\widetilde{s}|s\le\widetilde{s}<t\}} \mathrm{UB}(\widetilde{s}, t) = \mathrm{UB}(s, t)$, and $\mathrm{UB}(s, t) \le \mathrm{UB}(t, t) = E[y|z = t]$ for $s < t$. Thus, when $s < t$, $\mathrm{UF}(t, s, t) = \mathrm{UB}(s, t)$.

When $s \ge t$ and $\tau = t$, the function (S.29) is equal to $E[y|z = s]$ by its definition.

Step 7. In the case where the function $E[y(\tau)|z = s]$ is $\mathrm{UF}(\tau, s, t)$ in Equation (S.28) for $s < t$ and $E[y|z = s]$ in Equation (S.29) for $s \ge t$, it follows from Step 6 and the law of iterated expectations, and from the fact that $\mathrm{UB}(s, t) = E[y|z = s]$ for $s = t$, that

$$E\big[y(t)\big] = \sum_{s\ge t} E[y|z = s]P(z = s) + \sum_{s<t} \mathrm{UB}(s, t)P(z = s). \tag{S.32}$$

The quantity (S.32) is the upper bound in Equation (4). Therefore, these functions attain the upper bound in Equation (4).

Step 8. By combining Steps 1–7, we conclude that the function $E[y(\tau)|z = s]$ represented by Equations (S.28) and (S.29) satisfies the concave-MTR and MTS assumptions and attains the upper bound in Equation (4). Hence, there exists a set of functions of $y_j(\tau)$ for $\tau \in T$ that satisfy the concave-MTR and MTS assumptions and that attain the upper bound in Equation (4).

For $s < t$, $\mathrm{UB}(s, t)$ is the sharp upper bound on $E[y(t)|z = s]$, and for $s \ge t$, $E[y|z = s]$ is the sharp upper bound on $E[y(t)|z = s]$. Hence, the sharp joint upper bound on $\{E[y(t)|z = s], s \in T\}$ is obtained by setting each of the quantities $E[y(t)|z = s]$, $s \in T$, at $\mathrm{UB}(s, t)$ for $s < t$ and at $E[y|z = s]$ for $s \ge t$. Therefore, the upper bound in Equation (4) is the sharp upper bound on $E[y(t)]$.

Proof of part (b). A proof similar to that of part (a) can now apply to obtain the result of part (b). Note that when we add the assumptions that $T = [0, \delta]$ for some $\delta \in (0, \infty]$, $Y = [0, \infty]$, and $P(z = 0) = 0$ to the assumptions of part (a), then in addition to Equations (S.2) and (S.3), we obtain the following inequalities: For $t < s$, $E[y(t)|z = s] \ge E[yt/s|z = s]$, and for $s \le t$, $E[y(t)|z = s] \le E[yt/s|z = s]$. Therefore, Equations (S.2) and (S.3) hold when we define $E[y|z = 0] = 0$ whenever $P(z = 0) = 0$.

Proof of part (c). We now prove that our bounds in Equation (4) are narrower than or equal to both Manski's (1997) bounds, as represented in Equation (2), and Manski and Pepper's (2000) bounds, as represented in Equation (3).

The first terms of the lower bounds in Equations (2), (3), and (4) are the same, and the first terms of the upper bounds in these equations are also the same. Therefore, we now compare the second terms of the bounds in these equations.

(i) *Comparison with the bounds in Manski (1997) (Equation (2)).*

For part (a), in which T is an ordered set and Y is a closed subset of the extended real line, Manski (1997) showed that the sharp bounds on $E[y(t)]$ under the concave-MTR assumption are the same as those under the MTR assumption, which are shown in Equation (1).

Because $\mathrm{LB}(s, t) \ge y_0$ and $\mathrm{UB}(s, t) \le y_1$, the second term of the lower bound in Equation (4) is greater than or equal to that in Equation (1), and the second term of the upper bound in Equation (4) is less than or equal to that in Equation (1).

Therefore, our bounds in Equation (4) are narrower than or equal to Manski's (1997) bound as shown in Equation (1).

In part (b), T is an ordered set, $T = [0, \delta]$ for some $\delta \in (0, \infty]$, and $Y = [0, \infty]$. Because Equations (5) and (6) hold, and because $E[y|z = 0] \geq 0$, it follows that for $s > t$,

$$\mathrm{LB}(s, t) \geq \frac{s-t}{s} E[y|z=0] + \frac{t}{s} E[y|z=s] \geq E\left[\frac{y}{s} t \middle| z=s\right], \tag{S.33}$$

and for $s < t$,

$$\mathrm{UB}(s, t) \leq \frac{s-t}{s} E[y|z=0] + \frac{t}{s} E[y|z=s] \leq E\left[\frac{y}{s} t \middle| z=s\right]. \tag{S.34}$$

Taking Equations (S.33) and (S.34) together with the law of iterated expectations implies that the second term of the lower bound in Equation (4) is greater than or equal to that in Equation (2) and that the second term of the upper bound in Equation (4) is less than or equal to that in Equation (2). Therefore, our bounds in Equation (4) are narrower than or equal to Manski's (1997) bounds, as shown in Equation (2).

(ii) *Comparison with the bounds in Manski and Pepper (2000) (Equation (3)).*

In either part (a), in which T is an ordered set and Y is a closed subset of the extended real line, or part (b), in which T is an ordered set, $T = [0, \delta]$ for some $\delta \in (0, \infty]$ and $Y = [0, \infty]$, the sharp bounds on $E[y(t)]$ using only the MTR and MTS assumptions of Manski and Pepper (2000) are Equation (3).

By Equations (5) and (6), we obtain the following inequalities: for $s > t$,

$$\mathrm{LB}(s, t) \geq \frac{\eta_2 - t}{\eta_2 - t} E[y|z=t] + \frac{t-t}{\eta_2 - t} E[y|z=\eta_2] = E[y|z=t]; \tag{S.35}$$

for $s < t$,

$$\mathrm{UB}(s, t) \leq \frac{t-t}{t-\eta_1} E[y|z=\eta_1] + \frac{t-\eta_1}{t-\eta_1} E[y|z=t] = E[y|z=t]. \tag{S.36}$$

Taking Equations (S.35) and (S.36) together with the law of iterated expectations implies that the second term of the lower bound in Equation (4) is greater than or equal to that in Equation (3) and that the second term of the upper bound in Equation (4) is less than or equal to that in Equation (3). Therefore, our bounds in Equation (4) are narrower than or equal to Manski and Pepper's (2000) bounds, as shown in Equation (3). □

Appendix B: Proof of Proposition 2

Proof of part (a). The lower bound on $E[y(t_2)] - E[y(t_1)]$ in Equation (8) holds because $y_j(\tau)$ is monotone. It is sharp because the hypothesis $\{y_j(t_1) = y_j(t_2) = y_j, j \in J\}$ satisfies the concave-MTR and MTS assumptions (because $E[y|z = s]$ increases in s).

To obtain the sharp upper bound on $E[y(t_2)] - E[y(t_1)]$, let us first obtain the sharp upper bound on $E[y(t_2)|z = s] - E[y(t_1)|z = s]$.

For $(s, t_1, t_2) \in T^3$, to obtain the sharp upper bound on $E[y(t_2)|z = s] - E[y(t_1)| z = s]$, hold $E[y(t_2)|z = s]$ fixed and minimize $E[y(t_1)|z = s]$ subject to three conditions:

(a) that the function $E[y(\tau)|z=s]$ for $\tau \in T$ traverses the three points $(t_2, E[y(t_2)|z=s])$, $(t_1, E[y(t_1)|z=s])$, and $(s, E[y|z=s])$; (b) that this function satisfies the concave-MTR assumption; and (c) that this function satisfies the MTS assumption. This procedure yields the maximum of $E[y(t_2)|z=s] - E[y(t_1)|z=s]$ as a function of $E[y(t_2)|z=s]$. Then maximize this expression over $E[y(t_2)|z=s]$.

To implement this strategy, we use the following eight-step process: In Steps 1 and 2, we set $E[y(t_2)|z=s]$ at the sharp upper bound on $E[y(t_2)|z=s]$. In Steps 3, 4, and 5, given $E[y(t_2)|z=s]$, which is equal to the sharp upper bound, we minimize $E[y(t_1)|z=s]$ subject to the preceding conditions (a), (b), and (c). Thus, Steps 1–5 determine the value $E[y(t_2)|z=s] - E[y(t_1)|z=s]$. In Step 6, we show that this value is greater than or equal to other values $E[y(t_2)|z=s] - E[y(t_1)|z=s]$ such that $E[y(\tau)|z=s]$ satisfies conditions (a), (b), and (c). In Step 7, by combining Steps 1–6, we show the sharp upper bound on $E[y(t_2)|z=s] - E[y(t_1)|z=s]$. In Step 8, we conclude that the sharp upper bound on $E[y(t_2)] - E[y(t_1)]$ is the upper bound in Equation (8).

Step 1. Equations (S.8) and (S.10) in the proof of part (a) of Proposition 1 imply

$$E\big[y(t_2)|z=s\big] \leq \text{UB}(s, t_2) \quad \text{for } z=s<t_2 \tag{S.37}$$

and

$$E\big[y(t_2)|z=s\big] \leq E[y|z=s] \quad \text{for } t_2 \leq s = z. \tag{S.38}$$

This proof implies that these upper bounds are sharp.

Step 2. Set $E[y(t_2)|z=s]$ at $\text{UB}(s, t_2)$ in case (i), in which $z=s<t_2$, and at $E[y|z=s]$ in case (ii), in which $t_2 \leq s = z$. Then find the minimal value of $E[y(t_1)|z=s]$ subject to conditions (a), (b), and (c). Steps 3 and 4 obtain these minimal values.

Step 3. Define the following functions: for $s<t$,

$$\text{AT}_1(\tau, s, t) = \begin{cases} \dfrac{\tau - s}{t-s}\text{UB}(s,t) + \dfrac{t-\tau}{t-s}E[y|z=s], & \text{if } s \leq \tau < t, \\ \text{LB}(s,\tau), & \text{if } \tau < s < t, \\ \text{UB}(s,t), & \text{if } \tau \geq t, \end{cases} \tag{S.39}$$

and for $t \leq s$,

$$\text{AT}_2(\tau, s, t) = \begin{cases} \displaystyle\max_{\{(\eta_1,\eta_2)|\eta_1 \leq \tau < \eta_2 \leq t\}} \frac{\eta_2 - \tau}{\eta_2 - \eta_1} E[y|z=\eta_1] + \frac{\tau - \eta_1}{\eta_2 - \eta_1}\mu(\eta_2), \\ \qquad\qquad \text{if } \tau < t, \\ E[y|z=s], \quad \text{if } \tau \geq t, \end{cases} \tag{S.40}$$

where $\mu(\eta_2) = E[y|z=s]$ if $\eta_2 = t$ and $E[y|z=\eta_2]$ if $\eta_2 < t$.

Step 4. The claim of this step is as follows. In case (i), in which $z=s<t_2$, given $E[y(t_2)|z=s] = \text{UB}(s, t_2)$, then $\text{AT}_1(t_1, s, t_2)$ is the minimal value of $E[y(t_1)|z=s]$, subject to conditions (a), (b), and (c). We prove this claim using four substeps.

Substep 4.1. Proof that $\text{AT}_1(\tau, s, t_2)$ satisfies condition (a). Equation (S.39) implies that $\text{AT}_1(t_2, s, t_2) = \text{UB}(s, t_2)$ and $\text{AT}_1(s, s, t_2) = E[y|z=s]$. Therefore, $\text{AT}_1(\tau, s, t_2)$ traverses the three points $(t_2, \mathrm{UB}(s, t_2))$, $(t_1, \mathrm{AT}_1(t_1, s, t_2))$, and $(s, E[y|z=s])$, thereby satisfying condition (a).

Substep 4.2. Proof that $\mathrm{AT}_1(\tau, s, t_2)$ satisfies condition (b). We divide case (i), in which $z=s<t_2$, into three subcases: (i.1A), in which $s \le \tau < t_2$; (i.2A), in which $\tau < s < t_2$; (i.3A), in which $t_2 \le \tau$. We then prove this claim for each of these three subcases.

In subcase (i.1A), in which $s \le \tau < t_2$, Equation (6) implies that

$$\mathrm{UB}(s, t_2) \le \min_{\{\eta_1 | \eta_1 < s \le t_2\}} \frac{s-t_2}{s-\eta_1} E[y|z=\eta_1] + \frac{t_2-\eta_1}{s-\eta_1} E[y|z=s].$$

Therefore,

$$\begin{aligned} 0 &\le \frac{\mathrm{UB}(s, t_2) - E[y|z=s]}{t_2 - s} \le \min_{\{\eta_1 | \eta_1 < s \le t_2\}} \frac{E[y|z=s] - E[y|z=\eta_1]}{s-\eta_1} \\ &= -\max_{\{\eta_1 | \eta_1 \le \tau = s-1 < \eta_2 = s \le t_2\}} \frac{\eta_2-\tau}{\eta_2-\eta_1} E[y|z=\eta_1] + \frac{\tau-\eta_2}{\eta_2-\eta_1} E[y|z=\eta_2] \\ &= E[y|z=s] - \mathrm{AT}_1(s-1, s, t_2). \end{aligned}$$

Therefore, the slope of $\mathrm{AT}_1(\tau, s, t_2)$ for $s \le \tau < t_2$ is not greater than the slope of $\mathrm{AT}_1(\tau, s, t_2)$ for $s-1 \le \tau < s$. Thus, $\mathrm{AT}_1(\tau, s, t_2)$ is concave-MTR for $s-1 \le \tau < t_2$.

In subcase (i.2A), in which $\tau < s < t_2$, by Equation (S.39) and the monotonicity of $E[y|z]$ in z, $\mathrm{AT}_1(\tau, s, t_2)$ is a weakly increasing function in τ that describes the upper envelope of points $(u, E[y|z=u])$ for all $u \le s$ (i.e., a function in τ that describes the upper boundary of the convex hull for a set formed by these points). Therefore, $\mathrm{AT}_1(\tau, s, t_2)$ is a concave function.

In subcase (i.3A), in which $t_2 \le \tau$, because $0 \le \{\mathrm{UB}(s, t_2) - E[y|z=s]\}/(t_2-s)$, then $\mathrm{AT}_1(\tau, s, t_2)$ is concave-MTR.

Therefore, in case (i), where $z=s<t_2$, $\mathrm{AT}_1(\tau, s, t_2)$ is concave-MTR in $\tau \in T$.

Substep 4.3. The proof that $\mathrm{AT}_1(\tau, s, t_2)$ satisfies condition (c). We divide case (i), in which $z=s<t_2$ into three subcases: (i.1B), in which $\tau \le s' \le s < t_2$; (i.2B), in which $s' \le \tau < t_2$ and $s' \le s < t_2$; (i.3B), in which $s' \le s < t_2 \le \tau$. We then prove this claim for each of these three subcases.

In subcase (i.1B), in which $\tau \le s' \le s < t_2$, by the proof of part (a) in Appendix A, $\mathrm{LB}(s', t_2) \le \mathrm{LB}(s, t_2)$. Thus, by Equation (S.39) and the monotonicity of $E[y|z]$ in z,

$$\mathrm{AT}_1\left(\tau, s', t_2\right) \le \mathrm{AT}_1(\tau, s, t_2). \tag{S.41}$$

In subcase (i.2B), in which $s' \le \tau < t_2$ and $s' \le s < t_2$, by Equation (S.41), $\mathrm{AT}_1(s', s', t_2) \le \mathrm{AT}_1(s', s, t_2)$ and by the proof of part (a) in Appendix A, $\mathrm{UB}(s', t_2) \le \mathrm{UB}(s, t_2)$. Thus, between s' and t_2, the function that describes the segment linking point $(s', \mathrm{AT}_1(s', s, t_2))$ and point $(t_2, \mathrm{UB}(s, t_2))$ is not smaller than the function that describes the segment linking point $(s', \mathrm{AT}_1(s', s', t_2))$ $(= (s', E[y|z=s']))$ and point $(t_2, \mathrm{UB}(s', t_2))$. Thus, because Equation (S.39) holds and because $\mathrm{AT}_1(\tau, s, t_2)$ is concave-MTR in τ, it follows that Equation (S.41) holds for $s' \le \tau < t_2$ and $s' \le s < t_2$.

In subcase (i.3B), in which $s' \le s < t_2 \le \tau$, by Equation (S.39), Equation (S.41) holds.

Because Equation (S.41) holds for $\tau \in T$, then $\mathrm{AT}_1(\tau, s, t_2)$ is MTS.

Substep 4.4. The claim of this substep is that given $E[y(t_2)|z = s] = \mathrm{UB}(s, t_2)$, $\mathrm{AT}_1(t_1, s, t_2)$ is less than or equal to the value $E[y(t_1)|z = s]$ for any function $E[y(\tau)|z = s]$ that satisfies the conditions (a), (b), and (c) identified previously. We divide case (i), in which $z = s < t_2$, into two subcases: (i.1C), in which $z = s \leq t_1 < t_2$, and (i.2C), in which $t_1 < z = s < t_2$. We then prove this claim for each of these two subcases.

In subcase (i.1C), in which $s \leq t_1 < t_2$, if the concave-MTR function $E[y(\tau)|z = s]$ traverses points $(t_2, \mathrm{UB}(s, t_2))$ and $(s, E[y|z = s])$, then for $t_1 \in [s, t_2)$, the value $E[y(t_1)|z = s]$ is not less than the value $\mathrm{AT}_1(t_1, s, t_2)$ because the function $\mathrm{AT}_1(\tau, s, t_2)$ for $\tau \in [s, t_2)$ in Equation (S.39) describes the segment that links points $(t_2, \mathrm{UB}(s, t_2))$ and $(s, E[y|z = s])$.

In subcase (i.2C), in which $t_1 < s < t_2$, the function $E[y(\tau)|z = s]$ for $\tau \leq s$, which satisfies conditions (a), (b), and (c), describes the upper boundary of a convex set that contains the points $(u, E[y|z = u])$ for all $u \leq s$. In subcase (i.2A) of Substep 4.2, we prove that $\mathrm{AT}_1(t_1, s, t_2)$ is a weakly increasing function in t_1 that describes the upper boundary of the convex hull for a set formed by points $(u, E[y|z = u])$ for all $u \leq s$. The convex hull for a set formed by these points is the smallest convex set that contains these points. Therefore, the claim of this substep is true.

Combining Substeps 4.1–4.4, we conclude that in case (i), in which $z = s < t_2$, given $E[y(t_2)|z = s] = \mathrm{UB}(s, t_2)$, then $\mathrm{AT}_1(t_1, s, t_2)$ is the minimal value of $E[y(t_1)|z = s]$, subject to conditions (a), (b), and (c).

Step 5. The claim of this step is the following. In case (ii), in which $z = s \geq t_2$, given $E[y(t_2)|z = s] = E[y|z = s]$, then $\mathrm{AT}_2(t_1, s, t_2)$ is the minimal value of $E[y(t_1)|z = s]$, subject to conditions (a), (b), and (c), as specified earlier. Similar to our proof of case (i) in Step 4, we prove this claim using six substeps.

Substep 5.1. By Equation (S.40), $\mathrm{AT}_2(\tau, s, t_2)$ traverses points $(t_2, E[y|z = s])$, $(t_1, \mathrm{AT}_2(t_1, s, t_2))$, and $(s, E[y|z = s])$, and therefore satisfies condition (a).

Substep 5.2. The function $\mathrm{AT}_2(\tau, s, t_2)$ satisfies condition (b).

Substep 5.3. The function $\mathrm{AT}_2(\tau, s, t_2)$ satisfies condition (c).

Substep 5.4. For $s' \leq t_2 < s$, $\mathrm{AT}_1(\tau, s', t_2)$ and $\mathrm{AT}_2(\tau, s, t_2)$ satisfy condition (c).

The proofs of Substeps 5.2, 5.3, and 5.4 can be constructed along lines that are similar to the proof in Substeps 4.2 and 4.3 of Step 4 that shows that $\mathrm{AT}_1(\tau, s, t_2)$ satisfies conditions (b) and (c). (For the proof of Substep 5.4, we use the fact that for $s' \leq t_2 < s$, $\mathrm{UB}(s', t_2) \leq \mathrm{UB}(t_2, t_2) = E[y|z = t_2] \leq E[y|z = s]$ to show that $\mathrm{AT}_1(\tau, s', t_2) \leq \mathrm{AT}_2(\tau, s, t_2)$.)

Substep 5.5. The claim of this substep is that given $E[y(t_2)|z = s] = E[y|z = s]$, it follows that $\mathrm{AT}_2(t_1, s, t_2)$ is less than or equal to the value of $E[y(t_1)|z = s]$ for any function $E[y(\tau)|z = s]$ that satisfies conditions (a), (b), and (c). The function $E[y(\tau)|z = s]$ for $\tau \leq t_2$, which satisfies conditions (a), (b), and (c), describes the upper boundary of a convex set that contains points $(u, E[y|z = u])$ for all $u \leq t_2$ and $(t_2, E[y|z = s])$. Equation (S.40) implies that $\mathrm{AT}_2(t_1, s, t_2)$ is a function in t_1 that describes the upper boundary of the convex hull for a set formed by points $(u, E[y|z = u])$ for all $u \leq t_2$ and $(t_2, E[y|z = s])$. Therefore, the claim of this substep is true.

Substep 5.6. Combining Substeps 5.1–5.5, we conclude that given $E[y(t_2)|z = s] = E[y|z = s]$, then $\mathrm{AT}_2(t_1, s, t_2)$ is the minimal value of $E[y(t_1)|z = s]$, subject to conditions (a), (b), and (c).

Step 6. In Steps 1–5, we have shown that when $s < t_2$ and $E[y(t_2)|z = s] = \mathrm{UB}(s, t_2)$, the maximum of $E[y(t_2)|z = s] - E[y(t_1)|z = s]$ subject to conditions (a), (b), and (c) is $\mathrm{UB}(s, t_2) - \mathrm{AT}_1(t_1, s, t_2)$. Furthermore, when $s \geq t_2$ and $E[y(t_2)|z = s] = E[y|z = s]$, the maximum of $E[y(t_2)|z = s] - E[y(t_1)|z = s]$ subject to these conditions is $E[y|z = s] - \mathrm{AT}_2(t_1, s, t_2)$. In Step 6, we show that these maxima are greater than or equal to the maxima of $E[y(t_2)|z = s] - E[y(t_1)|z = s]$ such that $E[y(t_2)|z = s]$ is different from $\mathrm{UB}(s, t_2)$ for $s < t_2$ or different from $E[y|z = s]$ for $s \geq t_2$, and such that $E[y(\tau)|z = s]$ satisfies conditions (a), (b), and (c).

In case (i), in which $z = s < t_2$, suppose that we set $E[y(t_2)|z = s]$ at a value that is smaller than $\mathrm{UB}(s, t_2)$. Let this value be $\mathrm{VB}(s, t_2)$, where $\mathrm{VB}(s, t_2) < \mathrm{UB}(s, t_2)$. (Note that $\mathrm{UB}(s, t_2)$ is the sharp upper bound on $E[y(t_2)|z = s]$ in case (i) in Equation (S.37).)

Given $E[y(t_2)|z = s] = \mathrm{VB}(s, t_2)$, we now minimize $E[y(t_1)|z = s]$ such that $E[y(\tau)|z = s]$ satisfies conditions (a), (b), and (c). The process for obtaining the minimal value of $E[y(t_1)|z = s]$ is similar to that in Step 4.

In subcase (i.1C), in which $z = s \leq t_1 < t_2$, given $E[y(t_2)|z = s] = \mathrm{VB}(s, t_2)$, the minimal value of $E[y(t_1)|z = s]$ such that $E[y(\tau)|z = s]$ satisfies conditions (a), (b), and (c) is the value of the function that describes the segment that links points $(t_2, \mathrm{VB}(s, t_2))$ and $(s, E[y|z = s])$, evaluated at t_1. The function $\mathrm{AT}_1(\tau, s, t_2)$ for $\tau \in [s, t_2)$ describes the segment that links points $(t_2, \mathrm{UB}(s, t_2))$ and $(s, E[y|z = s])$. Therefore, the maximum of $\mathrm{VB}(s, t_2) - E[y(t_1)|z = s]$ subject to conditions (a), (b), and (c) is smaller than $\mathrm{UB}(s, t_2) - \mathrm{AT}_1(t_1, s, t_2)$.

In subcase (i.2C), in which $t_1 < z = s < t_2$, given $E[y(t_2)|z = s] = \mathrm{VB}(s, t_2)$, the minimal value of $E[y(t_1)|z = s]$ such that $E[y(\tau)|z = s]$ satisfies conditions (a), (b), and (c) is $\mathrm{AT}_1(t_1, s, t_2)$. Therefore, the value $\mathrm{VB}(s, t_2) - \mathrm{AT}_1(t_1, s, t_2)$ is smaller than the value $\mathrm{UB}(s, t_2) - \mathrm{AT}_1(t_1, s, t_2)$.

Thus, for case (i), in which $z = s < t_2$, $\mathrm{UB}(s, t_2) - \mathrm{AT}_1(t_1, s, t_2)$ is greater than or equal to a value $E[y(t_2)|z = s] - E[y(t_1)|z = s]$ for a function $E[y(\tau)|z = s]$ that satisfies conditions (a), (b), and (c).

In case (ii), in which $z = s \geq t_2$, suppose that we set $E[y(t_2)|z = s]$ at a value that is less than $E[y|z = s]$. Let this value be $\mathrm{WB}(s, t_2)$, where $\mathrm{WB}(s, t_2) < E[y|z = s]$. (Note that $E[y|z = s]$ is the sharp upper bound on $E[y(t_2)|z = s]$ in case (ii) in Equation (S.38).) A process similar to that in Step 5 can now be applied to obtain the minimal value of $E[y(t_1)|z = s]$ subject to conditions (a), (b), and (c), given $E[y(t_2)|z = s] = \mathrm{WB}(s, t_2)$.

As a result, given $E[y(t_2)|z = s] = \mathrm{WB}(s, t_2)$, the minimal value of $E[y(t_1)|z = s]$ such that $E[y(\tau)|z = s]$ satisfies conditions (a), (b), and (c) is the value of the function that describes the upper boundary of a convex hull for a set formed by points $(u, E[y|z = u])$, where $u \leq t_2$ and $E[y|z = u] \leq \mathrm{WB}(s, t_2)$, evaluated at t_1. Therefore, the maximum of $\mathrm{WB}(s, t_2) - E[y(t_1)|z = s]$ subject to conditions (a), (b), and (c) is smaller than $E[y|z = s] - \mathrm{AT}_2(t_1, s, t_2)$.

Thus, in case (ii), in which $z = s \geq t_2$, $E[y|z = s] - \mathrm{AT}_2(t_1, s, t_2)$ is greater than or equal to a value $E[y(t_2)|z = s] - E[y(t_1)|z = s]$ for a function $E[y(\tau)|z = s]$ that satisfies conditions (a), (b), and (c).

Step 7. By combining Steps 1–6, we draw the following conclusions. For case (i), in which $z = s < t_2$ and $t_1 < t_2$,

$$0 \leq E\big[y(t_2)|z = s\big] - E\big[y(t_1)|z = s\big] \leq \mathrm{UB}(s, t_2) - \mathrm{AT}_1(t_1, s, t_2). \qquad \text{(S.42)}$$

For case (ii), in which $t_1 < t_2 \le s = z$,

$$0 \le E\big[y(t_2)|z=s\big] - E\big[y(t_1)|z=s\big] \le E[y|z=s] - \mathrm{AT}_2(t_1, s, t_2). \tag{S.43}$$

These bounds are sharp.

Step 8. By Step 7, the sharp joint upper bound on $\{E[y(t_2)|z=s] - E[y(t_1)|z=s], s \in T\}$ is obtained by setting each of the quantities $E[y(t_2)|z=s] - E[y(t_1)|z=s]$, $s \in T$, at its upper bound in Equation (S.42) for $s < t_2$ and at its upper bound in Equation (S.43) for $s \ge t_2$. Therefore, by the law of iterated expectations, we conclude that the sharp upper bound on $E[y(t_2)] - E[y(t_1)]$ is the upper bound in Equation (8).

Proof of part (b). A proof similar to that of part (a) can now be applied to obtain the result of part (b). Specifically, when we add the assumptions that $T = [0, \delta]$ for some $\delta \in (0, \infty]$, $Y = [0, \infty]$, and $P(z=0) = 0$ to the assumptions of part (a), then in addition to Equations (S.42) and (S.43), we obtain the following inequalities: for $z = s < t_2$ and $t_1 < t_2$,

$$\begin{aligned} &E\big[y(t_2)|z=s\big] - E\big[y(t_1)|z=s\big] \\ &\le \begin{cases} E[y|z=s]\dfrac{t_2 - t_1}{s}, & \text{if } z = s \le t_1 < t_2, \\ \mathrm{UB}(s, t_2) - \max\limits_{\{\eta_2 | t_1 < \eta_2 \le s < t_2\}} \dfrac{t_1}{\eta_2} E[y|z=\eta_2], & \text{if } t_1 < z = s < t_2, \end{cases} \end{aligned} \tag{S.44}$$

and for $t_1 < t_2 \le s$,

$$E\big[y(t_2)|z=s\big] - E\big[y(t_1)|z=s\big] \le E[y|z=s] - \max_{\{\eta_2 | t_1 < \eta_2 \le t_2 \le s\}} \frac{t_1}{\eta_2}\mu(\eta_2). \tag{S.45}$$

Therefore, when we define $E[y|z=0] = 0$ whenever $P(z=0) = 0$ in Equations (9) and (10), Equations (S.42) and (S.43) hold.

Proof of part (c). We now prove that in either part (a), in which T is an ordered set and Y is a closed subset of the extended real line, or part (b), in which T is an ordered set, $T = [0, \delta]$ for some $\delta \in (0, \infty]$, and $Y = [0, \infty]$, our bounds in Equation (8) are narrower than or equal to the bounds in Manski (1997) and Manski and Pepper (2000).

(i) *Comparison with the bounds in Manski (1997).*

For part (a), in which T is an ordered set and Y is a closed subset of the extended real line, Manski (1997) showed that the sharp bounds on the average treatment effects obtained by using only the concave-MTR assumption are the same as the bounds obtained using only the MTR assumption. That is, they are

$$\begin{aligned} 0 &\le E\big[y(t_2)\big] - E\big[y(t_1)\big] \\ &\le \sum_{s \le t_1} \big\{y_1 - E[y|z=s]\big\} P(z=s) + \sum_{t_1 < s < t_2} (y_1 - y_0) P(z=s) \\ &\quad + \sum_{s \ge t_2} \big\{E[y|z=s] - y_0\big\} P(z=s), \end{aligned} \tag{S.46}$$

where $[y_0, y_1]$ is the range of Y.

Note that for $s \le t_1$, $\mathrm{UB}(s,t_2)-\mathrm{AT}_1(t_1,s,t_2) \le y_1 - E[y|z=s]$; for $t_1 < s < t_2$, $\mathrm{UB}(s,t_2)-\mathrm{AT}_1(t_1,s,t_2) \le y_1 - y_0$; and for $s \ge t_2$, $E[y|z=s]-\mathrm{AT}_2(t_1,s,t_2) \le E[y|z=s]-y_0$. By using these inequalities and the law of iterated expectations, the upper bound in Equation (8) is less than or equal to the upper bound in Equation (S.46).

In part (b), in which T is an ordered set, $T=[0,\delta]$ for some $\delta \in (0,\infty]$, and $Y=[0,\infty]$, the sharp bounds on the average treatment effects obtained by using only the concave-MTR assumption of Manski (1997) are

$$\begin{aligned} 0 &\le E\big[y(t_2)\big]-E\big[y(t_1)\big] \\ &\le \sum_{s<t_2} E\left[\frac{y}{z}\middle| z=s\right](t_2-t_1)P(z=s) \\ &\quad + \sum_{s\ge t_2}\left\{E[y|z=s]-E\left[\frac{y}{t_2}t_1\middle| z=s\right]\right\}P(z=s). \end{aligned} \tag{S.47}$$

The upper bound in Equation (S.47) minus the upper bound in Equation (8) (in which $E[y|z=0]=0$ whenever $P(z=0)=0$) is equal to the quantity

$$\begin{aligned} &\sum_{s<t_2}\left\{E\left[\frac{y}{s}t_2\middle| z=s\right]-\mathrm{UB}(s,t_2)+\mathrm{AT}_1(t_1,s,t_2)-E\left[\frac{y}{s}t_1\middle| z=s\right]\right\}P(z=s) \\ &\quad + \sum_{s\ge t_2}\left\{\mathrm{AT}_2(t_1,s,t_2)-E\left[\frac{y}{t_2}t_1\middle| z=s\right]\right\}P(z=s). \end{aligned} \tag{S.48}$$

For $s \le t_1$, $E[yt_2/s|z=s]-\mathrm{UB}(s,t_2)+\mathrm{AT}_1(t_1,s,t_2)-E[yt_1/s|z=s]=\{E[yt_2/s|z=s]-\mathrm{UB}(s,t_2)\}(t_2-t_1)/(t_2-s)\ge 0$ because of Equation (S.34). For $t_1<s<t_2$, because Equation (9) holds and $E[y|z=0]\ge 0$, $\mathrm{AT}_1(t_1,s,t_2)\ge (s-t_1)/sE[y|z=0]+t_1/sE[y|z=s]\ge E[yt_1/s|z=s]$. Therefore, $E[yt_2/s|z=s]-\mathrm{UB}(s,t_2)+\mathrm{AT}_1(t_1,s,t_2)-E[yt_1/s|z=s]\ge 0$. For $s\ge t_2$, because Equation (10) holds and $E[y|z=0]\ge 0$, $\mathrm{AT}_2(t_1,s,t_2)\ge (t_2-t_1)/t_2E[y|z=0]+t_1/t_2E[y|z=s]\ge E[yt_1/t_2|z=s]$.

Therefore, by the law of iterated expectations, Equation (S.48) is nonnegative. Thus, the upper bound in Equation (8) is less than or equal to the upper bound in Equation (S.47).

(ii) *Comparison with the bounds in Manski and Pepper (2000).*

In either part (a), where T is an ordered set and Y is a closed subset of the extended real line, or part (b), where T is an ordered set, $T=[0,\delta]$ for some $\delta\in(0,\infty]$, and $Y=[0,\infty]$, the sharp bounds on the average treatment effects using only the MTR and MTS assumptions of Manski and Pepper (2000) are

$$\begin{aligned} 0 &\le E\big[y(t_2)\big]-E\big[y(t_1)\big] \\ &\le \sum_{s\le t_1}\big\{E[y|z=t_2]-E[y|z=s]\big\}P(z=s) \\ &\quad + \sum_{t_1<s<t_2}\big\{E[y|z=t_2]-E[y|z=t_1]\big\}P(z=s) \\ &\quad + \sum_{s\ge t_2}\big\{E[y|z=s]-E[y|z=t_1]\big\}P(z=s). \end{aligned} \tag{S.49}$$

We now compare the upper bounds in Equations (8) and (S.49). The upper bound in Equation (S.49) minus the upper bound in Equation (8) is equal to Equation (11). We obtain four results: (a) Equation (6) implies that the first term of Equation (11) is nonnegative (because of $E[y|z=0] \geq 0$). (b) For $s \leq \eta_2 \leq t_2$ and $\eta_1 < \eta_2$, $\{(\eta_2 - t_2)E[y|z=\eta_1] + (t_2 - \eta_1)E[y|z=\eta_2]\}/(\eta_2 - \eta_1) \geq E[y|z=\eta_2] \geq E[y|z=s]$ (because of the monotonicity of $E[y|z]$ in z). Thus, by Equation (6), $\mathrm{UB}(s, t_2) \geq E[y|z=s]$. Therefore, by Equation (9), the second term of Equation (11) is nonnegative. (c) Equation (9) implies that the third term of Equation (11) is nonnegative. (d) Equation (10) implies that the fourth term of Equation (11) is nonnegative. By combining results (a)–(d), we conclude that Equation (11) is nonnegative.

Therefore, the upper bound in Equation (8) is less than or equal to the upper bound in Equation (S.49). □

Appendix C: The bounds obtained using a combination of the IV or MIV assumptions with the concave-MTR and MTS assumptions

Let T be ordered and let Y be a closed subset of the extended real line. Assume that $y_j(\cdot)$, $j \in J$, satisfies the concave-MTR and MTS assumptions. Furthermore, let the variable $\kappa \in K$ be the instrumental variable.

We then define the functions

$$\mathrm{LB}(s,t,k) = \max_{\{(\eta_1,\eta_2)|\eta_1 \leq t < \eta_2 \leq s\}} \frac{\eta_2 - t}{\eta_2 - \eta_1} E[y|z=\eta_1, \kappa=k] + \frac{t-\eta_1}{\eta_2-\eta_1} E[y|z=\eta_2, \kappa=k],$$

$$\mathrm{UB}(s,t,k) = \min_{\{(\eta_1,\eta_2)|s \leq \eta_2 \leq t \wedge \eta_1 < \eta_2\}} \frac{\eta_2 - t}{\eta_2 - \eta_1} E[y|z=\eta_1, \kappa=k] + \frac{t-\eta_1}{\eta_2-\eta_1} E[y|z=\eta_2, \kappa=k],$$

$$\mathrm{LBIV}(t,k) = \sum_{s \leq t} E[y|z=s, \kappa=k] P(z=s|\kappa=k) + \sum_{s>t} \mathrm{LB}(s,t,k) P(z=s|\kappa=k),$$

$$\mathrm{UBIV}(t,k) = \sum_{s \geq t} E\big[y|z=s, \kappa=k\big] P(z=s|\kappa=k) + \sum_{s<t} \mathrm{UB}(s,t,k) P(z=s|\kappa=k).$$

For $s < t_2$,

$$\mathrm{AT}_1(t_1,s,t_2,k) = \begin{cases} \dfrac{t_1-s}{t_2-s}\mathrm{UB}(s,t_2,k) + \dfrac{t_2-t_1}{t_2-s} E[y|z=s,\kappa=k], & \text{if } s \leq t_1 < t_2, \\ \mathrm{LB}(s,t_1,k), & \text{if } t_1 < s < t_2. \end{cases}$$

For $t_2 \leq s$,

$$\mathrm{AT}_2(t_1,s,t_2,k) = \max_{\{(\eta_1,\eta_2)|\eta_1 \leq t_1 < \eta_2 \leq t_2\}} \frac{\eta_2 - t_1}{\eta_2 - \eta_1} E[y|z=\eta_1, \kappa=k] + \frac{t_1-\eta_1}{\eta_2-\eta_1} \mu_{\mathrm{IV}}(\eta_2),$$

where $\mu_{\mathrm{IV}}(\eta_2) = E[y|z=s, \kappa=k]$ if $\eta_2 = t_2$ and $E[y|z=\eta_2, \kappa=k]$ if $\eta_2 < t_2$.

$$\begin{aligned}\mathrm{ATIV}(t_1, t_2, k) = &\sum_{s<t_2}\big[\mathrm{UB}(s, t_2, k) - \mathrm{AT}_1(t_1, s, t_2, k)\big]P(z=s|\kappa=k)\\ &+\sum_{s\geq t_2}\big\{E\big[y|z=s, \kappa=k\big] - \mathrm{AT}_2(t_1, s, t_2, k)\big\}P(z=s|\kappa=k).\end{aligned}$$

1. We now make the IV assumption as $E[y(t)|z=s, \kappa=k_1] = E[y(t)|z=s, \kappa=k_2]$ for each $t \in T$, each $s \in T$, and all $(k_1, k_2) \in K^2$.

Then, under the assumptions of concave-MTR, MTS, and IV, we obtain three results:

1.1. The sharp bounds on $E[y(t)]$ are

$$\max_{k\in K} \mathrm{LBIV}(t, k) \leq E\big[y(t)\big] \leq \min_{k\in K} \mathrm{UBIV}(t, k). \tag{S.50}$$

1.2. The sharp bounds on $E[y(t_2)] - E[y(t_1)]$ are

$$0 \leq E\big[y(t_2)\big] - E\big[y(t_1)\big] \leq \min_{k\in K} \mathrm{ATIV}(t_1, t_2, k). \tag{S.51}$$

1.3. Furthermore, (i) let $T = [0, \delta]$ for some $\delta \in (0, \infty]$, (ii) let $Y = [0, \infty]$, and (iii) let $E[y|z=0, \kappa=k] = 0$ whenever $P(z=0) = 0$. Then Equations (S.50) and (S.51) hold. These bounds are sharp.

2. We now make the MIV assumption as $E[y(t)|z=s, \kappa=k_1] \leq E[y(t)|z=s, \kappa=k_2]$ for each $t \in T$, each $s \in T$, and all $(k_1, k_2) \in K^2$ such that $k_1 \leq k_2$.

Then, under the assumptions of concave-MTR, MTS, and MIV, we obtain three results:

2.1. The sharp bounds on $E[y(t)]$ are

$$\sum_{k\in K} P(\kappa=k) \max_{k_1\leq k} \mathrm{LBIV}(t, k_1) \leq E\big[y(t)\big] \leq \sum_{k\in K} P(\kappa=k) \min_{k_2\geq k} \mathrm{UBIV}(t, k_2). \tag{S.52}$$

2.2. The bounds on $E[y(t_2)] - E[y(t_1)]$ are

$$\begin{aligned}&\max\Bigg[0, \sum_{k\in K} P(\kappa=k) \max_{k_1\leq k} \mathrm{LBIV}(t_2, k_1) - \sum_{k\in K} P(\kappa=k) \min_{k_2\geq k} \mathrm{UBIV}(t_1, k_2)\Bigg]\\ &\quad\leq E\big[y(t_2)\big] - E\big[y(t_1)\big]\\ &\quad\leq \min\Bigg[\sum_{k\in K} P(\kappa=k)\, \mathrm{ATIV}(t_1, t_2, k),\\ &\qquad \sum_{k\in K} P(\kappa=k) \min_{k_2\geq k} \mathrm{UBIV}(t_2, k_2) - \sum_{k\in K} P(\kappa=k) \max_{k_1\leq k} \mathrm{LBIV}(t_1, k_1)\Bigg].\end{aligned} \tag{S.53}$$

2.3. Furthermore, (i) let $T = [0, \delta]$ for some $\delta \in (0, \infty]$, (ii) let $Y = [0, \infty]$, and (iii) let $E[y|z=0, \kappa=k] = 0$ whenever $P(z=0) = 0$. Then Equations (S.52) and (S.53) hold. The bounds represented by Equation (S.52) are sharp.

Appendix D: Proof of Proposition 3

First, Appendix A shows that in Proposition 1, even in the case that the assumption that $y_j(t)$ is concave-MTR is replaced with the assumption that the conditional mean of $y_j(t)$ is concave-MTR (i.e., $E[y(t)|z]$ is concave-MTR in t), the sharp bounds on $E[y(t)]$ that are represented by Equations (4), (5), and (6) hold.

Second, Equation (13) is equivalent to the condition that the indicator function $1(y(t) > r)$ satisfies the MTS assumption. Equations (14) and (15), taken together, are equivalent to the condition that $E[1(y(t) > r)|z]$ satisfies the concave-MTR assumption.

Therefore, by Proposition 1, we obtain Equations (4), (5), and (6), where the function $y(t)$ is replaced with the function $1(y(t) > r)$. Thus, we obtain the inequality

$$\begin{aligned} &\sum_{s\leq t} P(y > r|z = s)P(z = s) + \sum_{s>t} \mathrm{LBT}(s, t)P(z = s) \\ &\quad \leq P\big(y(t) > r\big) \leq \sum_{s\geq t} P(y > r|z = s)P(z = s) + \sum_{s<t} \mathrm{UBT}(s, t)P(z = s), \end{aligned} \tag{S.54}$$

where

$$\begin{aligned} \mathrm{LBT}(s, t) = &\max_{\{(\eta_1,\eta_2)|\eta_1\leq t<\eta_2\leq s\}} \frac{\eta_2 - t}{\eta_2 - \eta_1} P(y > r|z = \eta_1) \\ &+ \frac{t - \eta_1}{\eta_2 - \eta_1} P(y > r|z = \eta_2), \end{aligned} \tag{S.55}$$

$$\begin{aligned} \mathrm{UBT}(s, t) = &\min_{\{(\eta_1,\eta_2)|s\leq\eta_2\leq t\wedge\eta_1<\eta_2\}} \frac{\eta_2 - t}{\eta_2 - \eta_1} P(y > r|z = \eta_1) \\ &+ \frac{t - \eta_1}{\eta_2 - \eta_1} P(y > r|z = \eta_2). \end{aligned} \tag{S.56}$$

Appendix A implies that $\mathrm{LBT}(s, t)$ is the lower bound on $P(y(t) > r|z = s)$ for $s > t$, whereas $\mathrm{UBT}(s, t)$ is the upper bound on $P(y(t) > r|z = s)$ for $s < t$.

It follows from Equations (13) and (14) that for $\eta_1 \leq \eta_2$, $0 \leq P(y > r|z = \eta_1) = P(y(\eta_1) > r|z = \eta_1) \leq P(y(\eta_2) > r|z = \eta_1) \leq P(y(\eta_2) > r|z = \eta_2) = P(y > r|z = \eta_2) \leq 1$.

Therefore, in Equation (S.55), because $(\eta_2 - t)/(\eta_2 - \eta_1) > 0$, $(t - \eta_1)/(\eta_2 - \eta_1) \geq 0$, and $(\eta_2 - t)/(\eta_2 - \eta_1) + (t - \eta_1)/(\eta_2 - \eta_1) = 1$, then $0 \leq \mathrm{LBT}(s, t) \leq 1$. However, in Equation (S.56), because $(\eta_2 - t)/(\eta_2 - \eta_1) \leq 0$, $(t - \eta_1)/(\eta_2 - \eta_1) > 0$, and $(\eta_2 - t)/(\eta_2 - \eta_1) + (t - \eta_1)/(\eta_2 - \eta_1) = 1$, then $0 \leq \mathrm{UBT}(s, t)$, but it is possible that $\mathrm{UBT}(s, t) > 1$. To obtain the sharp upper bound on $P(y(t) > r|z = s)$ for $s < t$, we impose the additional restriction

$$\mathrm{UBT}(s, t) \leq 1. \tag{S.57}$$

By Equations (S.54) and (S.57), we obtain the sharp bounds on $P(y(t) > r)$ as

$$\begin{aligned} &\sum_{s\leq t} P(y > r|z = s)P(z = s) + \sum_{s>t} \mathrm{LBT}(s, t)P(z = s) \\ &\quad \leq P\big(y(t) > r\big) \leq \sum_{s\geq t} P(y > r|z = s)P(z = s) + \sum_{s<t} \min\big\{1, \mathrm{UBT}(s, t)\big\}P(z = s). \end{aligned} \tag{S.58}$$

Furthermore, by Equations (17), (18), (S.55), and (S.56), we obtain

$$1 - \mathrm{LBT}(s,t) = \mathrm{LBP}(s,t), \tag{S.59}$$

$$1 - \min\{1, \mathrm{UBT}(s,t)\} = \max\{0, 1 - \mathrm{UBT}(s,t)\} = \mathrm{UBP}(s,t). \tag{S.60}$$

By Equations (S.58), (S.59), and (S.60), we obtain the sharp bounds on the distribution of outcomes $F_{y(t)}(r)$ as

$$\begin{aligned} &\sum_{s \geq t} F_y(r|z=s)P(z=s) + \sum_{s<t} \mathrm{UBP}(s,t)P(z=s) \\ &\quad \leq F_{y(t)}(r) \leq \sum_{s \leq t} F_y(r|z=s)P(z=s) + \sum_{s>t} \mathrm{LBP}(s,t)P(z=s). \end{aligned} \tag{S.61}$$

Therefore, Equations (16), (17), and (18) hold and these bounds are sharp. □

Appendix E: Finite-sample bias correction

E.1 *The KP method*

Kreider and Pepper (2007) and Manski and Pepper (2009) proposed a bias-corrected estimator by using a bootstrap distribution. We now use the subsampling distribution to adjust the bias.[1] Specifically, let T_n be the analog estimate of the bound and let $E^*(T_{n,b})$ be the mean of the estimates using the subsampling distribution; then the subsampling bias-corrected estimator is $(1 + \sqrt{b/n})T_n - \sqrt{b/n}E^*(T_{n,b})$, where n and b are the sizes of the sample and the subsample, respectively (see Politis, Romano, and Wolf (1999)). In the estimation, the size of the subsample is 20 percent of the sample. The estimation results are not sensitive to the choice of subsample size.

E.2 *The HT method*

Haile and Tamer (2003) proposed a bias adjustment that replaces the minimum (the maximum) with a smooth weighted average that is greater (less) than the minimum (the maximum) in a finite sample and that converges to the minimum (the maximum) as the sample size goes to infinity. We apply their method to Equations (5), (6), (9), and (10) to provide the bias-corrected estimates of the bounds on $E[y(t)]$ in Equation (4) and the bounds on $E[y(t_2)] - E[y(t_1)]$ in Equation (8).

Following Haile and Tamer (2003), we use the smooth weighted average

$$\lambda(x_1, \ldots, x_M; \rho_n) = \sum_{m=1}^{M} x_m \left[\frac{\exp(x_m \rho_n)}{\sum_{m=1}^{M} \exp(x_m \rho_n)} \right], \tag{S.62}$$

where $x_m \in \mathbb{R}$ for $m = 1, \ldots, M$, $\rho_n \in \mathbb{R}$, and n is the sample size. Then $\min(x_1, \ldots, x_M) < \lambda(x_1, \ldots, x_M; \rho_n) < \max(x_1, \ldots, x_M)$. Furthermore, $\lim_{\rho_n \to -\infty} \lambda(x_1, \ldots, x_M; \rho_n) =$

[1] Chernozhukov, Hong, and Tamer (2007), Andrews and Guggenberger (2009, 2010), and Romano and Shaikh (2010) indicated that subsampling procedures provide uniformly asymptotically valid inference for parameters on the boundary of the parameter space.

$\min(x_1, \ldots, x_M)$ and $\lim_{\rho_n \to +\infty} \lambda(x_1, \ldots, x_M; \rho_n) = \max(x_1, \ldots, x_M)$. Therefore, Equation (S.62) approximates $\min(x_1, \ldots, x_M)$ by letting the smoothing parameter ρ_n decrease to minus infinity as sample size n goes to infinity, whereas it approximates $\max(x_1, \ldots, x_M)$ by letting ρ_n increase to plus infinity as n goes to infinity. This approximation effectively adjusts both the downward bias of our upper bounds and the upward bias of our lower bounds. We approximate the minima in Equations (6) and (9) by Equation (S.62), where $\rho_n = -\sqrt{n}$. Similarly, we approximate the maxima in Equations (5), (9), and (10) by Equation (S.62), where $\rho_n = \sqrt{n}$.

We use the subsampling distribution of the bias-corrected estimates to provide confidence intervals. The size of the subsample is 20 percent of the sample. The estimation results are not sensitive to the choice of subsample size.

E.3 *The CLR method*

Chernozhukov, Lee, and Rosen (2013) proposed median unbiased estimators and confidence intervals of the bounds by adding to the estimated bounding functions appropriate critical values multiplied by their pointwise standard errors. They used asymptotic theory to provide formal justification of their estimators.

We use the method of Chernozhukov, Lee, and Rosen (referred to as the CLR method here) (2013) to estimate the bounds on $E[y(t)]$ in Equations (4), (5), and (6), and to estimate the bounds on $E[y(t_2)] - E[y(t_1)]$ in Equations (8), (9), and (10). In Section E.3.1 below we illustrate an algorithm required for median unbiased estimators and confidence intervals of the bounds on $E[y(t)]$, and in Section E.3.2, we illustrate an algorithm for the bounds on $E[y(t_2)] - E[y(t_1)]$.

E.3.1 *Implementation algorithm for the mean treatment response,* $E[y(t)]$ In Section E.3.1.1, we obtain the alternative representation of the bounds on $E[y(t)]$ that are obtained in Equations (4), (5), and (6), and to which the CLR method can be applied. In Section E.3.1.2, we describe a procedure to obtain a median unbiased estimator and confidence intervals for these bounds.

E.3.1.1 *Alternative representation of the bounds on* $E[y(t)]$ The quantities $(s, t) \in T^2$ are given. For $(\eta_1(s,t), \eta_2(s,t)) \in T^2$, we define the sets $H_{\mathrm{LB}}(s,t) := \{(\eta_1(s,t), \eta_2(s,t)) | \eta_1(s,t) \le t < \eta_2(s,t) \le s\}$ and $H_{\mathrm{UB}}(s,t) := \{(\eta_1(s,t), \eta_2(s,t)) | s \le \eta_2(s,t) \le t \wedge \eta_1(s,t) < \eta_2(s,t)\}$.

Then Equations (4), (5), and (6) are equivalent to

$$\begin{aligned}
&\max_{(\eta_1(s,t), \eta_2(s,t)) \in H_{\mathrm{LB}}(s,t)} \Bigg(\sum_{s \le t} E[y|z=s] P(z=s) \\
&\quad + \sum_{s>t} \Bigg\{ \frac{\eta_2(s,t) - t}{\eta_2(s,t) - \eta_1(s,t)} E\big[y|z=\eta_1(s,t)\big] \\
&\quad + \frac{t - \eta_1(s,t)}{\eta_2(s,t) - \eta_1(s,t)} E\big[y|z=\eta_2(s,t)\big] \Bigg\} P(z=s) \Bigg) \\
&\le E\big[y(t)\big] \qquad \text{(S.63)}
\end{aligned}$$

$$\leq \min_{(\eta_1(s,t),\eta_2(s,t))\in H_{\mathrm{UB}}(s,t)} \Bigg(\sum_{s\geq t} E[y|z=s]P(z=s)$$
$$+ \sum_{s<t} \Bigg\{ \frac{\eta_2(s,t)-t}{\eta_2(s,t)-\eta_1(s,t)} E\big[y|z=\eta_1(s,t)\big]$$
$$+ \frac{t-\eta_1(s,t)}{\eta_2(s,t)-\eta_1(s,t)} E\big[y|z=\eta_2(s,t)\big] \Bigg\} P(z=s) \Bigg).$$

We apply the CLR method to the bounds on $E[y(t)]$ obtained in Equation (S.63).

E.3.1.2 *Algorithm for the estimators of the bounds on $E[y(t)]$ obtained in Equation (S.63)*
Step 1. Set $\gamma_n \equiv 1 - 0.1/\log n$, where n is sample size. Simulate $R \times n$ times independent draws from $N(0,1)$, denoted by $\{\xi_{ir} : i = 1,\ldots,n, r = 1,\ldots,R\}$, where R is the number of simulation repetitions ($R = 10{,}000$).

Step 2. Compute the local constant/linear kernel estimator $\widehat{E}_n[y_i|z_i]$ using the quartic kernel and the rule-of-thumb bandwidth presented in Fan and Gijbels (1996). Define $\widehat{U}_i$ as the regression residual.

Step 3. For each $(\eta_1(s,t),\eta_2(s,t))_{s<t} \in H_{\mathrm{UB}} := \prod_{s<t} H_{\mathrm{UB}}(s,t)$, compute the estimators

$$\widehat{\mathrm{MU}}_n\big(\eta_1(s,t),\eta_2(s,t)\big) = \sum_{s\geq t} \widehat{E}_n[y|z=s]P(z=s)$$
$$+ \sum_{s<t} \Bigg\{ \frac{\eta_2(s,t)-t}{\eta_2(s,t)-\eta_1(s,t)} \widehat{E}_n\big[y|z=\eta_1(s,t)\big]$$
$$+ \frac{t-\eta_1(s,t)}{\eta_2(s,t)-\eta_1(s,t)} \widehat{E}_n\big[y|z=\eta_2(s,t)\big] \Bigg\} P(z=s),$$

$$\widehat{g}^{\mathrm{MU}}_{(\eta_1(s,t),\eta_2(s,t))}(U_i,Z_i) = \sum_{s\geq t} \widehat{U}_i \frac{K\left(\frac{s-Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(s)} P(z=s)$$
$$+ \sum_{s<t} \Bigg[\frac{\eta_2(s,t)-t}{\eta_2(s,t)-\eta_1(s,t)} \widehat{U}_i \frac{K\left(\frac{\eta_1(s,t)-Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_1(s,t))}$$
$$+ \frac{t-\eta_1(s,t)}{\eta_2(s,t)-\eta_1(s,t)} \widehat{U}_i \frac{K\left(\frac{\eta_2(s,t)-Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_2(s,t))} \Bigg] P(z=s),$$

where $\widehat{f}_n(\eta)$ is the kernel density estimator of the density of η and h_n is a bandwidth.

$$E_n\big[\widehat{g}^{\mathrm{MU}}_{(\eta_1(s,t),\eta_2(s,t))}\big]^2 = \frac{1}{n}\sum_{i=1}^{n}\big[\widehat{g}^{\mathrm{MU}}_{(\eta_1(s,t),\eta_2(s,t))}(U_i,Z_i)\big]^2, \tag{S.64}$$

$$s^{\mathrm{MU}}_n\big(\eta_1(s,t),\eta_2(s,t)\big) = \sqrt{\frac{E_n[\widehat{g}^{\mathrm{MU}}_{(\eta_1(s,t),\eta_2(s,t))}]^2}{nh_n}}. \tag{S.65}$$

For each $(\eta_1(s,t), \eta_2(s,t))_{s<t} \in H_{\text{UB}}$ and $r = 1, \dots, R$, compute the estimators

$$G_n\big(\widehat{g}^{\text{MU}}_{(\eta_1(s,t),\eta_2(s,t))}; r\big) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_{ir} \widehat{g}^{\text{MU}}_{(\eta_1(s,t),\eta_2(s,t))}(U_i, Z_i), \tag{S.66}$$

$$\psi_n^{\text{MU}}\big(\eta_1(s,t), \eta_2(s,t); r\big) = \frac{G_n(\widehat{g}^{\text{MU}}_{(\eta_1(s,t),\eta_2(s,t))}; r)}{\sqrt{E_n[\widehat{g}^{\text{MU}}_{(\eta_1(s,t),\eta_2(s,t))}]^2}}. \tag{S.67}$$

Step 4. Compute $k_{n,H_{\text{UB}}}(\gamma_n) = \gamma_n$-quantile of $\{\max_{(\eta_1(s,t),\eta_2(s,t))_{s<t} \in H_{\text{UB}}} \psi_n^{\text{MU}}(\eta_1(s,t), \eta_2(s,t); r), r = 1, \dots, R\}$ and

$$\begin{aligned}\widehat{H}_{n,\text{UB}} = \Big\{ &\big(\eta_1(s,t), \eta_2(s,t)\big)_{s<t} \in H_{\text{UB}} | \\ &\widehat{\text{MU}}_n\big(\eta_1(s,t), \eta_2(s,t)\big) \le \min_{(\eta_1(s,t),\eta_2(s,t))_{s<t} \in H_{\text{UB}}} \big[\widehat{\text{MU}}_n\big(\eta_1(s,t), \eta_2(s,t)\big) \\ &+ k_{n,H_{\text{UB}}}(\gamma_n) s_n^{\text{MU}}\big(\eta_1(s,t), \eta_2(s,t)\big)\big] \\ &+ 2k_{n,H_{\text{UB}}}(\gamma_n) s_n^{\text{MU}}\big(\eta_1(s,t), \eta_2(s,t)\big)\Big\}.\end{aligned}$$

Step 5. Compute $k_{n,\widehat{H}_{n,\text{UB}}}(p) = p$-quantile of $\{\max_{(\eta_1(s,t),\eta_2(s,t))_{s<t} \in \widehat{H}_{n,\text{UB}}} \psi_n^{\text{MU}}(\eta_1(s,t), \eta_2(s,t); r), r = 1, \dots, R\}$. Set

$$\begin{aligned}\widehat{\text{MU}}_n^0(p) = &\min_{(\eta_1(s,t),\eta_2(s,t))_{s<t} \in H_{\text{UB}}} \big[\widehat{\text{MU}}_n\big(\eta_1(s,t), \eta_2(s,t)\big) \\ &+ k_{n,\widehat{H}_{n,\text{UB}}}(p) s_n^{\text{MU}}\big(\eta_1(s,t), \eta_2(s,t)\big)\big].\end{aligned} \tag{S.68}$$

Step 6. For each $(\eta_1(s,t), \eta_2(s,t))_{s>t} \in H_{\text{LB}} := \prod_{s>t} H_{\text{LB}}(s,t)$, compute the estimators

$$\begin{aligned}\widehat{\text{ML}}_n\big(\eta_1(s,t), \eta_2(s,t)\big) = &\sum_{s\le t} \widehat{E}_n[y|z=s] P(z=s) \\ &+ \sum_{s>t} \Big\{ \frac{\eta_2(s,t) - t}{\eta_2(s,t) - \eta_1(s,t)} \widehat{E}_n\big[y|z=\eta_1(s,t)\big] \\ &+ \frac{t - \eta_1(s,t)}{\eta_2(s,t) - \eta_1(s,t)} \widehat{E}_n\big[y|z=\eta_2(s,t)\big] \Big\} P(z=s),\end{aligned}$$

$$\begin{aligned}\widehat{g}^{\text{ML}}_{(\eta_1(s,t),\eta_2(s,t))}(U_i, Z_i) = &\sum_{s\le t} \widehat{U}_i \frac{K\left(\dfrac{s - Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(s)} P(z=s) \\ &+ \sum_{s>t} \Bigg[\frac{\eta_2(s,t) - t}{\eta_2(s,t) - \eta_1(s,t)} \widehat{U}_i \frac{K\left(\dfrac{\eta_1(s,t) - Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_1(s,t))} \\ &+ \frac{t - \eta_1(s,t)}{\eta_2(s,t) - \eta_1(s,t)} \widehat{U}_i \frac{K\left(\dfrac{\eta_2(s,t) - Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_2(s,t))} \Bigg] P(z=s).\end{aligned}$$

Furthermore, by replacing $\widehat{g}^{\mathrm{MU}}_{(\eta_1(s,t),\eta_2(s,t))}(U_i, Z_i)$ in Equations (S.64) and (S.65) with $\widehat{g}^{\mathrm{ML}}_{(\eta_1(s,t),\eta_2(s,t))}(U_i, Z_i)$, compute the estimators $E_n[\widehat{g}^{\mathrm{ML}}_{(\eta_1(s,t),\eta_2(s,t))}]^2$ and $s_n^{\mathrm{ML}}(\eta_1(s,t), \eta_2(s,t))$.

For each $(\eta_1(s,t), \eta_2(s,t))_{s>t} \in H_{\mathrm{LB}}$ and $r = 1, \dots, R$, compute the estimators $G_n(\widehat{g}^{\mathrm{ML}}_{(\eta_1(s,t),\eta_2(s,t))}; r)$ and $\psi_n^{\mathrm{ML}}(\eta_1(s,t), \eta_2(s,t); r)$ by replacing $\widehat{g}^{\mathrm{MU}}_{(\eta_1(s,t),\eta_2(s,t))}(U_i, Z_i)$ in Equations (S.66) and (S.67) with $\widehat{g}^{\mathrm{ML}}_{(\eta_1(s,t),\eta_2(s,t))}(U_i, Z_i)$.

Step 7. Compute $k_{n,H_{\mathrm{LB}}}(\gamma_n) = \gamma_n$-quantile of $\{\max_{(\eta_1(s,t),\eta_2(s,t))_{s>t}\in H_{\mathrm{LB}}} \psi_n^{\mathrm{ML}}(\eta_1(s,t), \eta_2(s,t); r), r = 1, \dots, R\}$, and

$$\begin{aligned}\widehat{H}_{n,\mathrm{LB}} = \Big\{&\big(\eta_1(s,t), \eta_2(s,t)\big)_{s>t} \in H_{\mathrm{LB}} |\\ &\widehat{\mathrm{ML}}_n\big(\eta_1(s,t), \eta_2(s,t)\big) \geq \max_{(\eta_1(s,t),\eta_2(s,t))_{s>t}\in H_{\mathrm{LB}}} \big[\widehat{\mathrm{ML}}_n\big(\eta_1(s,t), \eta_2(s,t)\big)\\ &- k_{n,H_{\mathrm{LB}}}(\gamma_n) s_n^{\mathrm{ML}}\big(\eta_1(s,t), \eta_2(s,t)\big)\big]\\ &- 2k_{n,H_{\mathrm{LB}}}(\gamma_n) s_n^{\mathrm{ML}}\big(\eta_1(s,t), \eta_2(s,t)\big)\Big\}.\end{aligned}$$

Step 8. Compute $k_{n,\widehat{H}_{n,\mathrm{LB}}}(p) = p$-quantile of $\{\max_{(\eta_1(s,t),\eta_2(s,t))_{s>t}\in \widehat{H}_{n,\mathrm{LB}}} \psi_n^{\mathrm{ML}}(\eta_1(s,t), \eta_2(s,t); r), r = 1, \dots, R\}$. Set

$$\begin{aligned}\widehat{\mathrm{ML}}_n^0(p) = \max_{(\eta_1(s,t),\eta_2(s,t))_{s>t}\in H_{\mathrm{LB}}} &\big[\widehat{\mathrm{ML}}_n\big(\eta_1(s,t), \eta_2(s,t)\big)\\ &- k_{n,\widehat{H}_{n,\mathrm{LB}}}(p) s_n^{\mathrm{ML}}\big(\eta_1(s,t), \eta_2(s,t)\big)\big].\end{aligned} \tag{S.69}$$

Step 9. The median unbiased estimates of the lower and upper bounds on $E[y(t)]$ in Equation (S.63) are $\widehat{\mathrm{ML}}_n^0(0.5)$ and $\widehat{\mathrm{MU}}_n^0(0.5)$, respectively. The $(1-\alpha)$ confidence interval of $E[y(t)]$ is $[\widehat{\mathrm{ML}}_n^0(\alpha/2), \widehat{\mathrm{MU}}_n^0(1-\alpha/2)]$.

E.3.2 *Implementation algorithm for the average treatment effect,* $E[y(t_2)] - E[y(t_1)]$ In Section E.3.2.1, we obtain the alternative representation of the upper bound on $E[y(t_2)] - E[y(t_1)]$ that is obtained in Equations (8), (9), and (10), and to which the CLR method can be applied. In Section E.3.2.2, we describe a procedure to obtain a median unbiased estimator and confidence intervals of this bound.

E.3.2.1 *Alternative representation of the bounds on* $E[y(t_2)] - E[y(t_1)]$ We define the following sets: (i) The quantities (t_1, t_2) are given, where $(t_1, t_2) \in T^2$ and $t_1 < t_2$. For $s < t_2$, $H_{\mathrm{UB}}(s, t_2) := \{(\eta_{10}(s,t_2), \eta_{20}(s,t_2)) \in T^2 | s \leq \eta_{20}(s,t_2) \leq t_2 \wedge \eta_{10}(s,t_2) < \eta_{20}(s,t_2)\}$ and $H_{\mathrm{AT1}}(s, t_1) := \{(\eta_{11}(s,t_1), \eta_{21}(s,t_1)) \in T^2 | \eta_{11}(s,t_1) \leq t_1 < \eta_{21}(s,t_1) \leq s < t_2\}$. For $s \geq t_2$, $H_{\mathrm{AT2}}(t_1, t_2) := \{(\eta_{12}(t_1,t_2), \eta_{22}(t_1,t_2)) \in T^2 | \eta_{12}(t_1,t_2) \leq t_1 < \eta_{22}(t_1,t_2) \leq t_2 \leq s\}$. (ii) For $(s, t_1, t_2) \in T^3$, where $t_1 < t_2$, $H_{\mathrm{AU}} := \prod_{s<t_2} H_{\mathrm{UB}}(s,t_2) \times \prod_{t_1<s<t_2} H_{\mathrm{AT1}}(s,t_1) \times \prod_{t_2\leq s} H_{\mathrm{AT2}}(t_1,t_2)$ and $\boldsymbol{\eta}(s,t_1,t_2) := (\eta_{10}(s,t_2), \eta_{20}(s,t_2), \eta_{11}(s,t_1), \eta_{21}(s,t_1), \eta_{12}(t_1,t_2), \eta_{22}(t_1,t_2))$.

Then Equations (8), (9), and (10) are equivalent to

$$\begin{aligned}0 &\leq E\big[y(t_2)\big] - E\big[y(t_1)\big]\\ &\leq \min_{\boldsymbol{\eta}(s,t_1,t_2)\in H_{\mathrm{AU}}} \sum_{s\leq t_1} \frac{t_2 - t_1}{t_2 - s} \Bigg\{ \frac{\eta_{20}(s,t_2) - t_2}{\eta_{20}(s,t_2) - \eta_{10}(s,t_2)} E\big[y | z = \eta_{10}(s,t_2)\big]\end{aligned}$$

$$
\begin{aligned}
&+ \frac{t_2 - \eta_{10}(s,t_2)}{\eta_{20}(s,t_2) - \eta_{10}(s,t_2)} E\big[y|z = \eta_{20}(s,t_2)\big] - E[y|z=s]\Bigg\} P(z=s) \\
&+ \sum_{t_1 < s < t_2} \Bigg(\frac{\eta_{20}(s,t_2) - t_2}{\eta_{20}(s,t_2) - \eta_{10}(s,t_2)} E\big[y|z = \eta_{10}(s,t_2)\big] \\
&+ \frac{t_2 - \eta_{10}(s,t_2)}{\eta_{20}(s,t_2) - \eta_{10}(s,t_2)} E\big[y|z = \eta_{20}(s,t_2)\big] \\
&- \Bigg\{ \frac{\eta_{21}(s,t_1) - t_1}{\eta_{21}(s,t_1) - \eta_{11}(s,t_1)} E\big[y|z = \eta_{11}(s,t_1)\big] \\
&+ \frac{t_1 - \eta_{11}(s,t_1)}{\eta_{21}(s,t_1) - \eta_{11}(s,t_1)} E\big[y|z = \eta_{21}(s,t_1)\big]\Bigg\}\Bigg) P(z=s) \\
&+ \sum_{s \geq t_2} \Bigg(E[y|z=s] - \Bigg\{ \frac{\eta_{22}(t_1,t_2) - t_1}{\eta_{22}(t_1,t_2) - \eta_{12}(t_1,t_2)} E\big[y|z = \eta_{12}(t_1,t_2)\big] \\
&+ \frac{t_1 - \eta_{12}(t_1,t_2)}{\eta_{22}(t_1,t_2) - \eta_{12}(t_1,t_2)} \mu\big(\eta_{22}(t_1,t_2)\big)\Bigg\}\Bigg) P(z=s),
\end{aligned}
\tag{S.70}
$$

where $\mu(\eta_{22}(t_1,t_2)) = E[y|z=s]$ if $\eta_{22}(t_1,t_2) = t_2$ and $E[y|z=\eta_{22}(t_1,t_2)]$ if $\eta_{22}(t_1,t_2) < t_2$.

E.3.2.2 *Algorithm for the estimators of the bounds on $E[y(t_2)] - E[y(t_1)]$ obtained in Equation (S.70)* Steps 1 and 2. Perform Steps 1 and 2 as described in Section E.3.1.2.

Step 3. For each $(\eta_{10}(s,t_2), \eta_{20}(s,t_2)) \in H_{\text{UB}}(s,t_2)$, where $s \leq t_1$, compute the estimators

$$
\begin{aligned}
&\text{ATT}_1\big(\eta_{10}(s,t_2), \eta_{20}(s,t_2)\big) \\
&= \frac{t_2 - t_1}{t_2 - s} \Bigg\{ \frac{\eta_{20}(s,t_2) - t_2}{\eta_{20}(s,t_2) - \eta_{10}(s,t_2)} \widehat{E}_n\big[y|z = \eta_{10}(s,t_2)\big] \\
&+ \frac{t_2 - \eta_{10}(s,t_2)}{\eta_{20}(s,t_2) - \eta_{10}(s,t_2)} \widehat{E}_n\big[y|z = \eta_{20}(s,t_2)\big] \\
&- \widehat{E}_n[y|z=s]\Bigg\},
\end{aligned}
\tag{S.71}
$$

$$
\begin{aligned}
&\widehat{g}^{\text{ATT1}}_{(\eta_{10}(s,t_2),\eta_{20}(s,t_2))}(U_i, Z_i) \\
&= \frac{t_2 - t_1}{t_2 - s} \Bigg\{ \frac{\eta_{20}(s,t_2) - t_2}{\eta_{20}(s,t_2) - \eta_{10}(s,t_2)} \widehat{U}_i \frac{K\left(\frac{\eta_{10}(s,t_2) - Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_{10}(s,t_2))} \\
&+ \frac{t_2 - \eta_{10}(s,t_2)}{\eta_{20}(s,t_2) - \eta_{10}(s,t_2)} \widehat{U}_i \frac{K\left(\frac{\eta_{20}(s,t_2) - Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_{20}(s,t_2))} \\
&- \widehat{U}_i \frac{K\left(\frac{s - Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(s)}\Bigg\}.
\end{aligned}
\tag{S.72}
$$

For each $(\eta_{10}(s,t_2),\eta_{20}(s,t_2))\in H_{\mathrm{UB}}(s,t_2)$, where $t_1<s<t_2$, and each $(\eta_{11}(s,t_1),\eta_{21}(s,t_1))\in H_{\mathrm{AT1}}(s,t_1)$, where $t_1<s<t_2$, compute the estimators

$$\begin{aligned}
&\mathrm{ATT}_2\big(\eta_{10}(s,t_2),\eta_{20}(s,t_2),\eta_{11}(s,t_1),\eta_{21}(s,t_1)\big)\\
&\quad=\left\{\frac{\eta_{20}(s,t_2)-t_2}{\eta_{20}(s,t_2)-\eta_{10}(s,t_2)}\widehat{E}_n\big[y|z=\eta_{10}(s,t_2)\big]\right.\\
&\qquad\left.+\frac{t_2-\eta_{10}(s,t_2)}{\eta_{20}(s,t_2)-\eta_{10}(s,t_2)}\widehat{E}_n\big[y|z=\eta_{20}(s,t_2)\big]\right\}\\
&\qquad-\left\{\frac{\eta_{21}(s,t_1)-t_1}{\eta_{21}(s,t_1)-\eta_{11}(s,t_1)}\widehat{E}_n\big[y|z=\eta_{11}(s,t_1)\big]\right.\\
&\qquad\left.+\frac{t_1-\eta_{11}(s,t_1)}{\eta_{21}(s,t_1)-\eta_{11}(s,t_1)}\widehat{E}_n\big[y|z=\eta_{21}(s,t_1)\big]\right\},
\end{aligned}\tag{S.73}$$

$$\begin{aligned}
&\widehat{g}^{\mathrm{ATT2}}_{(\eta_{10}(s,t_2),\eta_{20}(s,t_2),\eta_{11}(s,t_1),\eta_{21}(s,t_1))}(U_i,Z_i)\\
&\quad=\left\{\frac{\eta_{20}(s,t_2)-t_2}{\eta_{20}(s,t_2)-\eta_{10}(s,t_2)}\widehat{U}_i\frac{K\left(\frac{\eta_{10}(s,t_2)-Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_{10}(s,t_2))}\right.\\
&\qquad\left.+\frac{t_2-\eta_{10}(s,t_2)}{\eta_{20}(s,t_2)-\eta_{10}(s,t_2)}\widehat{U}_i\frac{K\left(\frac{\eta_{20}(s,t_2)-Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_{20}(s,t_2))}\right\}\\
&\qquad-\left\{\frac{\eta_{21}(s,t_1)-t_1}{\eta_{21}(s,t_1)-\eta_{11}(s,t_1)}\widehat{U}_i\frac{K\left(\frac{\eta_{11}(s,t_1)-Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_{11}(s,t_1))}\right.\\
&\qquad\left.+\frac{t_1-\eta_{11}(s,t_1)}{\eta_{21}(s,t_1)-\eta_{11}(s,t_1)}\widehat{U}_i\frac{K\left(\frac{\eta_{21}(s,t_1)-Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_{21}(s,t_1))}\right\}.
\end{aligned}\tag{S.74}$$

For each $(\eta_{12}(t_1,t_2),\eta_{22}(t_1,t_2))\in H_{\mathrm{AT2}}(t_1,t_2)$, where $s\ge t_2$, compute the estimators

$$\begin{aligned}
&\mathrm{ATT}_3\big(\eta_{12}(t_1,t_2),\eta_{22}(t_1,t_2)\big)\\
&\quad=\widehat{E}_n[y|z=s]-\left\{\frac{\eta_{22}(t_1,t_2)-t_1}{\eta_{22}(t_1,t_2)-\eta_{12}(t_1,t_2)}\widehat{E}_n\big[y|z=\eta_{12}(t_1,t_2)\big]\right.\\
&\qquad\left.+\frac{t_1-\eta_{12}(t_1,t_2)}{\eta_{22}(t_1,t_2)-\eta_{12}(t_1,t_2)}\widehat{\mu}\big(\eta_{22}(t_1,t_2)\big)\right\},
\end{aligned}\tag{S.75}$$

$$\begin{aligned}
&\widehat{g}^{\mathrm{ATT3}}_{(\eta_{12}(t_1,t_2),\eta_{22}(t_1,t_2))}(U_i,Z_i)\\
&\quad=\widehat{U}_i\frac{K\left(\frac{s-Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(s)}-\left\{\frac{\eta_{22}(t_1,t_2)-t_1}{\eta_{22}(t_1,t_2)-\eta_{12}(t_1,t_2)}\widehat{U}_i\frac{K\left(\frac{\eta_{12}(t_1,t_2)-Z_i}{h_n}\right)}{\sqrt{h_n}\widehat{f}_n(\eta_{12}(t_1,t_2))}\right.\\
&\qquad\left.+\frac{t_1-\eta_{12}(t_1,t_2)}{\eta_{22}(t_1,t_2)-\eta_{12}(t_1,t_2)}\widehat{\mu}_U\big(\eta_{22}(t_1,t_2)\big)\right\},
\end{aligned}\tag{S.76}$$

where $\widehat{\mu}(\eta_{22}(t_1,t_2)) = \widehat{E}_n[y|z=s]$ if $\eta_{22}(t_1,t_2)=t_2$ and $\widehat{E}_n[y|z=\eta_{22}(t_1,t_2)]$ if $\eta_{22}(t_1,t_2) < t_2$, and $\widehat{\mu}_U(\eta_{22}(t_1,t_2)) = \widehat{U}_i K(\frac{s-Z_i}{h_n})/(\sqrt{h_n}\widehat{f}_n(s))$ if $\eta_{22}(t_1,t_2)=t_2$ and $\widehat{U}_i K(\frac{\eta_{22}(t_1,t_2)-Z_i}{h_n})/(\sqrt{h_n}\widehat{f}_n(\eta_{22}(t_1,t_2)))$ if $\eta_{22}(t_1,t_2) < t_2$.

For each $\boldsymbol{\eta}(s,t_1,t_2) \in H_{\text{AU}}$, compute the estimators

$$\begin{aligned}
&\widehat{\text{AU}}_n\big(\boldsymbol{\eta}(s,t_1,t_2)\big) \\
&\quad = \sum_{s\le t_1} \text{ATT}_1\big(\eta_{10}(s,t_2),\eta_{20}(s,t_2)\big)P(z=s) \\
&\qquad + \sum_{t_1<s<t_2} \text{ATT}_2\big(\eta_{10}(s,t_2),\eta_{20}(s,t_2),\eta_{11}(s,t_1),\eta_{21}(s,t_1)\big)P(z=s) \\
&\qquad + \sum_{s\ge t_2} \text{ATT}_3\big(\eta_{12}(t_1,t_2),\eta_{22}(t_1,t_2)\big)P(z=s),
\end{aligned} \tag{S.77}$$

$$\begin{aligned}
&\widehat{g}^{\text{AU}}_{(\boldsymbol{\eta}(s,t_1,t_2))}(U_i,Z_i) \\
&\quad = \sum_{s\le t_1} \widehat{g}^{\text{ATT1}}_{(\eta_{10}(s,t_2),\eta_{20}(s,t_2))}(U_i,Z_i)P(z=s) \\
&\qquad + \sum_{t_1<s<t_2} \widehat{g}^{\text{ATT2}}_{(\eta_{10}(s,t_2),\eta_{20}(s,t_2),\eta_{11}(s,t_1),\eta_{21}(s,t_1))}(U_i,Z_i)P(z=s) \\
&\qquad + \sum_{s\ge t_2} \widehat{g}^{\text{ATT3}}_{(\eta_{12}(t_1,t_2),\eta_{22}(t_1,t_2))}(U_i,Z_i)P(z=s).
\end{aligned} \tag{S.78}$$

Furthermore, by replacing $\widehat{g}^{\text{MU}}_{(\eta_1(s,t),\eta_2(s,t))}(U_i,Z_i)$ in Equations (S.64) and (S.65) with $\widehat{g}^{\text{AU}}_{(\boldsymbol{\eta}(s,t_1,t_2))}(U_i,Z_i)$ in Equation (S.78), compute the estimators $E_n[\widehat{g}^{\text{AU}}_{(\boldsymbol{\eta}(s,t_1,t_2))}]^2$ and $s^{\text{AU}}_n(\boldsymbol{\eta}(s,t_1,t_2))$.

For each $\boldsymbol{\eta}(s,t_1,t_2) \in H_{\text{AU}}$ and $r=1,\ldots,R$, compute the estimators $G_n(\widehat{g}^{\text{AU}}_{(\boldsymbol{\eta}(s,t_1,t_2))};r)$ and $\psi^{\text{AU}}_n(\boldsymbol{\eta}(s,t_1,t_2);r)$ by replacing $\widehat{g}^{\text{MU}}_{(\eta_1(s,t),\eta_2(s,t))}(U_i,Z_i)$ in Equations (S.66) and (S.67) with $\widehat{g}^{\text{AU}}_{(\boldsymbol{\eta}(s,t_1,t_2))}(U_i,Z_i)$ in Equation (S.78).

Step 4. Compute $k_{n,H_{\text{AU}}}(\gamma_n) = \gamma_n$-quantile of $\{\max_{\boldsymbol{\eta}(s,t_1,t_2)\in H_{\text{AU}}} \psi^{\text{AU}}_n(\boldsymbol{\eta}(s,t_1,t_2);r), r=1,\ldots,R\}$ and

$$\begin{aligned}
\widehat{H}_{n,\text{AU}} = \Big\{&\boldsymbol{\eta}(s,t_1,t_2)\in H_{\text{AU}} | \\
&\widehat{\text{AU}}_n\big(\boldsymbol{\eta}(s,t_1,t_2)\big) \le \min_{\boldsymbol{\eta}(s,t_1,t_2)\in H_{\text{AU}}} \big[\widehat{\text{AU}}_n\big(\boldsymbol{\eta}(s,t_1,t_2)\big) \\
&+ k_{n,H_{\text{AU}}}(\gamma_n) s^{\text{AU}}_n\big(\boldsymbol{\eta}(s,t_1,t_2)\big)\big] \\
&+ 2k_{n,H_{\text{AU}}}(\gamma_n) s^{\text{AU}}_n\big(\boldsymbol{\eta}(s,t_1,t_2)\big)\Big\}.
\end{aligned}$$

Step 5. Compute $k_{n,\widehat{H}_{n,\text{AU}}}(p) = p$-quantile of $\{\max_{\boldsymbol{\eta}(s,t_1,t_2)\in \widehat{H}_{n,\text{AU}}} \psi^{\text{AU}}_n(\boldsymbol{\eta}(s,t_1,t_2);r), r=1,\ldots,R\}$. Set

$$\widehat{\text{AU}}^0_n(p) = \min_{\boldsymbol{\eta}(s,t_1,t_2)\in H_{\text{AU}}} \big[\widehat{\text{AU}}_n\big(\boldsymbol{\eta}(s,t_1,t_2)\big) + k_{n,\widehat{H}_{n,\text{AU}}}(p) s^{\text{AU}}_n\big(\boldsymbol{\eta}(s,t_1,t_2)\big)\big].$$

Step 6. The median unbiased estimate of the upper bound on $E[y(t_2)] - E[y(t_1)]$ in Equation (S.70) is $\widehat{\mathrm{AU}}_n^0(0.5)$. The one-sided $(1-\alpha/2)$ confidence interval of the upper bound on $E[y(t_2)] - E[y(t_1)]$ is $\widehat{\mathrm{AU}}_n^0(1-\alpha/2)$.

References

Andrews, D. W. K. and P. Guggenberger (2009), "Validity of subsampling and 'plug-in asymptotic' inference for parameters defined by moment inequalities." *Econometric Theory*, 25 (3), 669–709. [18]

Andrews, D. W. K. and P. Guggenberger (2010), "Asymptotic size and a problem with subsampling and with the *m* out of *n* bootstrap." *Econometric Theory*, 26 (2), 426–468. [18]

Chernozhukov, V., H. Hong, and E. Tamer (2007), "Estimation and confidence regions for parameter sets in econometric models." *Econometrica*, 75 (5), 1243–1284. [18]

Chernozhukov, V., S. Lee, and A. M. Rosen (2013), "Intersection bounds: Estimation and inference." *Econometrica*, 81 (2), 667–737. [19]

Fan, J. and I. Gijbels (1996), *Local Polynomial Modelling and Its Applications*. Chapman & Hall, London. [20]

Haile, P. A. and E. Tamer (2003), "Inference with an incomplete model of English auctions." *Journal of Political Economy*, 111 (1), 1–51. [18]

Kreider, B. and J. V. Pepper (2007), "Disability and employment: Reevaluating the evidence in light of reporting errors." *Journal of the American Statistical Association*, 102 (478), 432–441. [18]

Manski, C. F. (1997), "Monotone treatment response." *Econometrica*, 65 (6), 1311–1334. [2, 7, 8, 13, 14]

Manski, C. F. and J. V. Pepper (2000), "Monotone instrumental variables: With an application to the returns to schooling." *Econometrica*, 68 (4), 997–1010. [2, 7, 8, 13, 14]

Manski, C. F. and J. V. Pepper (2009), "More on monotone instrumental variables." *Econometrics Journal*, 12 (S1), S200–S216. [18]

Politis, D. N., J. P. Romano, and M. Wolf (1999), *Subsampling*. Springer-Verlag, New York. [18]

Romano, J. P. and A. M. Shaikh (2010), "Inference for the identified set in partially identified econometric models." *Econometrica*, 78 (1), 169–211. [18]

Submitted April, 2012. Final version accepted June, 2013.