

Inference on Heterogeneous Treatment Effects in High-Dimensional Dynamic Panels under Weak Dependence: Online Supplement

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Appendix A presents and results on independence couplings. Appendix B develops concentration results for weakly-dependent panel data. Appendix C presents the results for high-dimensional CLT for weakly dependent data. Appendix D contains proofs for Section 4, and Appendix E for Section 5. Appendix F contains tail bounds for empirical rectangular matrices in operator norm. Appendix G contains the analysis of OLS used in stage 3 of our inference procedure.

KEYWORDS. Orthogonal Learning, Residual Learning, CATE, Dynamic Panel Data, Time Series, Mixing, Cross-fitting, Neighbours-Left-Out.

JEL CLASSIFICATION. C14, C23, C33.

Notation. We use the standard notation for numeric and stochastic dominance. For two numeric sequences $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$, $a_n \lesssim b_n$ stands for $a_n = O(b_n)$. For two sequences of random variables $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$, $a_n \lesssim_P b_n$ stands for $a_n \lesssim_P(b_n)$. For a random vector V , let $V^0 := V - E[V]$ be the demeaned vector. Let $[N] := \{1, 2, \dots, N\}$, $[T] := \{1, 2, \dots, T\}$ and $[j] := \{1, 2, \dots, d\}$. Let $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Matrix and Vector Norms. For a vector $v \in \mathbb{R}^d$, denote the ℓ_2 -norm of v as $\|v\|_2 := \sqrt{\sum_{j=1}^d v_j^2}$, the ℓ_1 -norm of v as $\|v\|_1 := \sum_{j=1}^d |v_j|$, the ℓ_∞ -norm of v as $\|v\|_\infty := \max_{1 \leq j \leq d} |v_j|$, and the ℓ_0 -norm of v as $\|v\|_0 := \sum_{j=1}^d 1_{\{v_j \neq 0\}}$. Denote a unit sphere

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in \mathbb{R}^d as $\mathcal{S}^{d-1} = \{\alpha \in \mathbb{R}^d : \|\alpha\| = 1\}$. For a matrix $A = (a_{ij}) \in \mathbb{R}^{d \times d}$, let its operator norm be $\|A\|_2 = \sup_{\alpha \in \mathcal{S}^{d-1}} \|A\alpha\|_2$, the elementwise norm be $\|A\|_\infty = \max_{1 \leq i, j \leq d} |a_{ij}|$, and the maximal ℓ_1 -row-norm:

$$\|A\|_{1,\infty} = \max_{1 \leq j \leq d} \sum_{i=1}^d |a_{ij}|.$$

Empirical Process Notation In what follows, we use the standard empirical process notation. For a generic measurable function $f : \mathcal{W} \rightarrow \mathbb{R}$ and a generic sample $\{\{W_{it}\}_{t=1}^T\}_{i=1}^N$, where W_{it} 's take values in \mathcal{W} , define the empirical expectation

$$\mathbb{E}_{NT} f(W_{it}) = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T f(W_{it})$$

and the empirical process:

$$\mathbb{G}_{NT} f(W_{it}) = \sqrt{NT} \mathbb{E}_{NT} [f(W_{it}) - \mathbb{E}_{W_{it}} f(W_{it})].$$

APPENDIX A: TOOLS: STRASSEN AND BERBEE COUPLINGS. IMPLICATIONS FOR CROSS-FITTING

A.1 Strassen's Coupling: Weak and Strong Form via Dudley-Philipp

Let S be a Polish space and $P_{Z,W}$ be a law on $S \times S$, with marginal laws P_Z on S and P_W on S . Let (Ω, \mathcal{B}, P) be a probability space and Z be a random variable on Ω with values in S and law $\mathcal{L}(Z) = P_Z$. Assume that (Ω, \mathcal{B}, P) has been extended to carry a random variable U on Ω , independent of Z , with values in $[0, 1]$ and law $U(0, 1)$. The total variation norm of a signed measure ν on the Polish space T is defined as

$$\|\nu\|_{TV} = \sup_{F \text{ closed}} \nu(F).$$

The total variation distance between laws P and Q defined on the Polish space T is defined by taking $\nu = P - Q$ in the definition above.

We also make use of the following Strassen's weak coupling result (e.g, [Villani \(2007\)](#), p.7):

$$\min_{Z^*, W^*} \{P(Z^* \neq W^*) : \mathcal{L}(Z^*) = P_Z, \mathcal{L}(W^*) = P_W\} = \frac{1}{2} \|P_Z - P_W\|_{TV}, \quad (\text{A.1})$$

where minimization is done over space of random variables Z^* and W^* defined on the probability space (Ω, \mathcal{B}, P) . Note that the problem above is the optimal transportation problem for 0-1 cost; see [Villani \(2007\)](#) for discussion. The above is a special case of Strassen's original result; [Schwarz \(1980\)](#) (Theorem 1) provides another proof of (A.1).

We now recall the following result.

LEMMA A.1 (Strong Coupling; Lemma 2.11, [Dudley and Philipp \(1983\)](#)). *Let S and T be Polish spaces and Q a law on $S \times T$, with marginal P_Z on S . Let (Ω, \mathcal{B}, P) be a probability space and Z be a random variable on Ω with values in S and law $\mathcal{L}(Z) = P_Z$. Assume that*

there is a random variable U on Ω , independent of Z , with values in a separable metric space R and law $\mathcal{L}(U)$ on R having no atoms. Then there exists a random variable W on Ω with values in T and law $\mathcal{L}((Z, W)) = Q$.

This result is quoted with minor adaptation of notation. This lemma implies the strong form of Strassen's weak coupling (A.1) as stated in the following lemma.

LEMMA A.2 (Strong Form of Strassen's Coupling). *Given the setup above with a given random variable Z , there exists a random variable W taking values in S , defined on the same probability space, and having law $\mathcal{L}(W) = P_W$ such that:*

$$\mathbb{P}(Z \neq W) = \frac{1}{2} \|P_Z - P_W\|_{TV}. \quad (\text{A.2})$$

Proof. Strassen's weak coupling implies that there is a pair of random variables (Z^*, W^*) with law Q and marginals P_Z and P_W such that:

$$\mathbb{P}(Z^* \neq W^*) = \frac{1}{2} \|P_Z - P_W\|_{TV}.$$

Application of the Dudley-Philipp lemma with $S = T$ and U taken to be uniform random variable implies that for the given Z there is a pairing random variable W , such that law of (Z, W) is Q . Therefore,

$$\mathbb{P}(Z \neq W) = \mathbb{P}(Z^* \neq W^*) = \frac{1}{2} \|P_Z - P_W\|_{TV}.$$

■.

A.2 Independence Coupling

Consider a special case of the setup above with $S = S_1 \times S_2$ and $T = S$, where S_1 and S_2 are Polish spaces, and where $Z = (X, Y)$ is a pair of random variables such that $\mathcal{L}(X) = P_X$ on S_1 and $\mathcal{L}(Y) = P_Y$ on S_2 , and $\mathcal{L}(X, Y) = P_{X,Y}$.

LEMMA A.3 (Strong Coupling with Independence via Strassen-Dudley-Philip). *Consider the setup above. We can construct \tilde{Y} and \tilde{X} that are independent of each other with laws $\mathcal{L}(X) = P_X$ and $\mathcal{L}(Y) = P_Y$ such that*

$$\mathbb{P} \left\{ (X, Y) \neq (\tilde{X}, \tilde{Y}) \right\} = \frac{1}{2} \|P_{X,Y} - P_X \times P_Y\|_{TV}.$$

Proof. In the previous lemma take $Z = (X, Y)$ and $W = (\tilde{X}, \tilde{Y})$, and note that $P_W = P_X \times P_Y$. ■

A.3 Berbee Coupling Extended

Let (X, Y) be a pair of random variables taking values in the Polish space $S_1 \times S_2$ as in the setup above. Define their coefficient of dependence as:

$$\gamma(X, Y) = \frac{1}{2} \|P_{X,Y} - P_X \times P_Y\|_{TV}.$$

This coefficient vanishes if and only if X and Y are independent.

The following lemma is a minor extension of Lemma 2.1 of Berbee from real-valued to Polish-space valued random variables.

LEMMA A.4 (Berbee Coupling on Polish Spaces). *Let $X = (X_i)_{i=1}^n$ be a collection of random variables taking values on the Polish space $S = (S_1 \times \dots \times S_n)$, and defined on the same probability space $(\Omega, \mathcal{B}, \mathbb{P})$. Define for $1 \leq i < n$*

$$\gamma^{(i)} = \gamma(X_i, (X_{i+1}, \dots, X_n)).$$

The probability space can be extended so that there exist a collection of random variables $\tilde{X} = (\tilde{X}_i)_{i=1}^n$ that are mutually independent, such that each \tilde{X}_i has the same law as X_i and

$$\mathbb{P}(X \neq \tilde{X}) \leq \gamma^{(1)} + \dots + \gamma^{(n-1)}.$$

Proof. Assume that $(\Omega, \mathcal{B}, \mathbb{P})$ has been extended to carry a random variable U on Ω , independent of X , with values in $[0, 1]$ and law $U(0, 1)$.

1. Application of strong form of Strassen's coupling in Lemma A.2 implies that one can construct \tilde{X} as in the statement of the lemma such that

$$\mathbb{P}(X \neq \tilde{X}) = \frac{1}{2} \|P_X - P_{\tilde{X}}\|.$$

2. (Identical to Berbee). To prove the claim of the lemma, we have to estimate the right hand side. If X, Y and \tilde{Y} are random variables, with Y and \tilde{Y} having values in the same space, then

$$\begin{aligned} \|P_{X,Y} - P_{X,\tilde{Y}}\| &\leq \|P_{X,Y} - P_X \times P_Y\|_{TV} + \|P_X \times P_Y - P_X \times P_{\tilde{Y}}\|_{TV} \\ &= 2\gamma(X, Y) + \|P_Y - P_{\tilde{Y}}\|_{TV} \end{aligned}$$

Applying this rule successively one obtains

$$\begin{aligned} &\frac{1}{2} \|P_{X_1, \dots, X_n} - P_{X_1} \times \dots \times P_{X_n}\| \\ &\leq \gamma^{(1)} + \|P_{X_2, \dots, X_n} - P_{X_2} \times \dots \times P_{X_n}\| \\ &\leq \dots \leq \gamma^{(1)} + \dots + \gamma^{(n-1)} \end{aligned}$$

■

COROLLARY A.1 (Berbee's Coupling for Panel Data). *Let $\{X_{i1}, X_{i2}, \dots, X_{iL}\}_{i=1}^N$ be real random matrices of possibly distinct dimensions. Suppose the sequences $(X_{i1}, X_{i2}, \dots, X_{iL})$ are independent over i . For each i , $(X_{i1}, X_{i2}, \dots, X_{iL})$ is β -mixing whose coefficients are bounded as*

$$\sup_{1 \leq i \leq N} \sup_{1 \leq l \leq L-1} \gamma((W_{i,1}, \dots, W_{i,l-1}), (W_{i,l}, \dots, W_{i,L})) \leq \epsilon. \quad (\text{A.3})$$

The probability space can be extended with random variables X_{il}^* distributed as X_{il} such that X_{il}^* are independent over i, l and

$$P(X_{il} \neq X_{il}^* \text{ for some } i, l) \leq N(L-1)\epsilon. \quad (\text{A.4})$$

This follows immediately from the union bound and Lemma A.4. \blacksquare

A.4 Applications to Cross-Fitting

Here we recall the setup induced by the NLO construction given in the main text. Let \mathcal{M}_k and $\mathcal{M}_k^{\text{qc}}$ be two NLO subsets of time indices $\{1, 2, \dots, T\}$, for $k = 1, \dots, K$. Define the data blocks

$$\begin{aligned} B_k &= \cup_{i=1}^N B_{ik}, & B_{ik} &= \{W_{it}\}_{t \in \mathcal{M}_k}; \\ B_k^{\text{qc}} &= \cup_{i=1}^N B_{ik}^{\text{qc}}, & B_{ik}^{\text{qc}} &= \{W_{it}\}_{t \in \mathcal{M}_k^{\text{qc}}}. \end{aligned} \quad (\text{A.5})$$

By construction, the time periods in \mathcal{M}_k and $\mathcal{M}_k^{\text{qc}}$ are separated by at least $T_k \geq T_{\text{block}} := \lfloor T/(K-1) \rfloor$ time periods.

LEMMA A.5 (Approximate Independence of Separated Chunks). *Suppose Assumption 4.1 holds with*

$$\gamma(q) := \sup_{\bar{t} \leq T, i \leq N} \gamma\left(\{W_{it}\}_{t \leq \bar{t}}, \{W_{it}\}_{t \geq \bar{t}+q}\right) \leq C_\kappa \exp(-\kappa q) \quad (\text{A.6})$$

and $\log N/T_{\text{block}} = o(1)$. Then, there exist random elements B_k^* and $B_k^{\text{qc}*}$ such that (1) B_k^* and B_k are equal in law, $B_k^{\text{qc}*}$ and B_k^{qc} are equal in law, (2) B_k^* and $B_k^{\text{qc}*}$ are independent, and (3) the event

$$\mathcal{E}_{\text{berbee}} := \{(B_k, B_k^{\text{qc}}) = (B_k^*, B_k^{\text{qc}*}), \text{ for all } k = 1, \dots, K\} \quad (\text{A.7})$$

holds with probability $1 - o(1)$, $NT \rightarrow \infty$.

PROOF. Invoking Lemma A.3 shows that the required variables exist and obey

$$P\left((B_k, B_k^{\text{qc}}) \neq (B_k^*, B_k^{\text{qc}*})\right) \leq \gamma(B_k, B_k^{\text{qc}}) \leq NC_\kappa \exp(-\kappa T_{\text{block}}).$$

Invoking union bound over the partitions $k = 1, 2, \dots, K$ gives

$$P\left((B_k, B_k^{\text{qc}}) \neq (B_k^*, B_k^{\text{qc}*}), \text{ for some } k = 1, \dots, K\right) \leq KNC_\kappa \exp(-\kappa T_{\text{block}}).$$

Since K is finite and $\log N/T_{\text{block}} = o(1)$ gives $KNC_\kappa \exp(-\kappa T_{\text{block}}) = o(1)$. \blacksquare

COROLLARY A.2 (Convenient Rate Implications). *Consider the setup above. Suppose there exists a sequence V_{NT} such that for some $\psi(B_k^*, B_k^{\text{qc}*})$ is $O_P(V_{NT})$ for some measurable function ψ . Then, $\psi(B_k, B_k^{\text{qc}})$ is $O_P(V_{NT})$.*

PROOF OF COROLLARY A.2. Consider any sequence of constants such that $\ell_{NT} \rightarrow \infty$. Then

$$\begin{aligned} \mathbb{P}(\psi(B_k, B_k^{\text{qc}}) > \ell_{NT} V_{NT}) &\leq^i \mathbb{P}(\psi(B_k, B_k^{\text{qc}}) > \ell_{NT} V_{NT} \cap \mathcal{E}_{\text{berbee}}) + \mathbb{P}(\mathcal{E}_{\text{berbee}}^c) \\ &\leq^{ii} \mathbb{P}(\psi(B_k^*, B_k^{\text{qc}*}) > \ell_{NT} V_{NT}) + \mathbb{P}(\mathcal{E}_{\text{berbee}}^c) \\ &=^{iii} o(1), \end{aligned}$$

where (i) follows from union bound, (ii) holds since $\psi(B_k^*, B_k^{\text{qc}*}) = \psi(B_k, B_k^{\text{qc}})$ on $\mathcal{E}_{\text{berbee}}$, and (iii) is assumed in the statement of Lemma. \blacksquare

Consider the following setup. We assume all spaces to be separable and complete. Consider the parameter space \mathcal{T} with elements η , typically a space of functions. Consider also a measurable function (the estimation map) $b^{qc} \mapsto \bar{\eta}(b^{qc})$ that maps \mathcal{W}^{T-q+1} to \mathcal{T} . Here \mathcal{W} is the metric space containing realizations of W_{it} for all i and t . Let $\hat{\eta}_k = \bar{\eta}(B_k^{\text{qc}})$ denote an estimator constructed on the data B_k^{qc} . Let $b \mapsto \phi(b; \eta)$ be another measurable mapping, indexed by η , that maps \mathcal{W}^q to \mathbb{R}^{d_ϕ} . We assume that the composition map $(b, b^{qc}) \mapsto \phi(b; \bar{\eta}(b^{qc}))$ is measurable.¹

COROLLARY A.3. *Suppose there exists a sequence of sets $\{\bar{\mathcal{T}}_{N,T}\} \subseteq \bar{\mathcal{T}}$ obeying the conditions as $NT \rightarrow \infty$: (A) $\mathbb{P}(\hat{\eta}_k \in \bar{\mathcal{T}}_{N,T}) = 1 - o(1)$ and (B) for any sequence $\{\eta_{NT}\} \in \bar{\mathcal{T}}_{N,T}$, $\phi(B_k, \eta_{NT}) = O_P(V_{NT})$. Then, $\phi(B_k, \hat{\eta}_k) = O_P(V_{NT})$.*

PROOF OF COROLLARY A.3. Invoke Lemma A.2 with

$$\psi(b, b^{qc}) := \phi(b, \bar{\eta}(b^{qc})) \mathbb{1}_{\{\bar{\eta}(b^{qc}) \in \bar{\mathcal{T}}_{N,T}\}}.$$

Union bound implies

$$\begin{aligned} \mathbb{P}(\phi(B_k, \hat{\eta}_k) \geq \ell_{NT} V_{NT}) &\leq \mathbb{P}(\phi(B_k, \hat{\eta}_k) \geq \ell_{NT} V_{NT} \cap \hat{\eta}_k \in \bar{\mathcal{T}}_{N,T}) + \mathbb{P}(\hat{\eta}_k \notin \bar{\mathcal{T}}_{N,T}) \\ &= \mathbb{P}(\psi(B_k, B_k^{\text{qc}}) \geq \ell_{NT} V_{NT} \cap \hat{\eta}_k \in \bar{\mathcal{T}}_{N,T}) + \mathbb{P}(\hat{\eta}_k \notin \bar{\mathcal{T}}_{N,T}) \\ &\leq \mathbb{P}(\psi(B_k, B_k^{\text{qc}}) \geq \ell_{NT} V_{NT}) + o(1), \end{aligned}$$

where the last inequality holds by condition A. We have that

$$\mathbb{P}\left(\psi(B_k, B_k^{\text{qc}}) \geq \ell_{NT} V_{NT}\right) \leq \mathbb{P}\left(\psi(B_k^*, B_k^{\text{qc}*}) \geq \ell_{NT} V_{NT}\right) + o(1),$$

from the previous proof. By Condition B,

$$\mathbb{P}(\psi(B_k^*, B_k^{\text{qc}*}) > \ell_{NT} V_{NT} \mid B_k^{\text{qc}*}) = \mathbb{P}(\phi(B_k^*, \bar{\eta}(B_k^{\text{qc}*})) \mathbb{1}_{\{\bar{\eta}(B_k^{\text{qc}*}) \in \bar{\mathcal{T}}_{N,T}\}} > \ell_{NT} V_{NT} \mid B_k^{\text{qc}*}) = o_P(1).$$

Therefore, using LIE

$$\mathbb{P}\left(\psi(B_k^*, B_k^{\text{qc}*}) \geq \ell_{NT} V_{NT}\right) = \mathbb{E}\left[\mathbb{P}\left(\psi(B_k^*, B_k^{\text{qc}*}) \geq \ell_{NT} V_{NT} \mid B_k^{\text{qc}*}\right)\right] = o(1),$$

where the final conclusion holds by the boundedness (and therefore uniform integrability) of the integrand. \blacksquare

¹Otherwise, can use outer probability measures to work with the bounds below.

LEMMA A.6 (Bounds on Cross-Fit Sample Averages). *Let $w \mapsto A(w, \eta)$ be a generic (measurable) matrix-valued function defined on \mathcal{W} , indexed by the parameter $\eta \in \bar{\mathcal{T}}$. Define*

$$B_{Ak}(\eta) := (NT_k)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} \mathbb{E}_{W_{it}} A(W_{it}, \eta) \quad (\text{A.8})$$

$$V_{Ak}(\eta) := (NT_k)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} [A(W_{it}, \eta) - \mathbb{E}_{W_{it}} A(W_{it}, \eta)] \quad (\text{A.9})$$

Suppose there exist sequences of constants ζ_{NT}^B and ζ_{NT}^V so that as $NT \rightarrow \infty$ for each $k = 1, \dots, K$:

(1) $\mathbb{P}(\hat{\eta}_k \in \bar{\mathcal{T}}_{N,T}) = 1 - o(1)$.

(2) for any sequence $\{\eta_{NT}\} \in \bar{\mathcal{T}}_{N,T}$ and any norm $\|\cdot\|$,

$$\|B_{Ak}(\eta_{NT})\| = O(\zeta_{NT}^B), \quad \|V_{Ak}(\eta_{NT})\| \lesssim_P (\zeta_{NT}^V).$$

Then,

$$\left\| (NT)^{-1} \sum_{i=1}^N \sum_{k=1}^K \sum_{t \in \mathcal{M}_k} A(W_{it}, \hat{\eta}_k) \right\| \lesssim_P (\zeta_{NT}^V + \zeta_{NT}^B).$$

In our case, we will either use $\|\cdot\| = \|\cdot\|_\infty$ (sup-norm) or $\|\cdot\| = \|\cdot\|_2$ (operator norm).

PROOF OF LEMMA A.6. We invoke Corollary A.3 with $\phi(B_k, \eta) := B_{Ak}(\eta) + V_{Ak}(\eta)$. The conditions (A) and (B) are directly assumed in Lemma A.6 as conditions (1) and (2), respectively. Therefore, for each $k \leq K$,

$$\|B_{Ak}(\hat{\eta}_k) + V_{Ak}(\hat{\eta}_k)\| \lesssim_P (\zeta_{NT}^V + \zeta_{NT}^B).$$

We next note that with probability converging to one,

$$(NT)^{-1} \sum_{i=1}^N \sum_{k=1}^K \sum_{t \in \mathcal{M}_k} A(W_{it}, \hat{\eta}) = \frac{T_k}{T} \sum_{k=1}^K [B_{Ak}(\hat{\eta}_k) + V_{Ak}(\hat{\eta}_k)].$$

Since $T_k \asymp T$, the claim holds by the triangle inequality and the union bound. \blacksquare

APPENDIX B: TOOLS: TAIL BOUNDS FOR MAXIMA OF SUMS FOR WEAKLY DEPENDENT PANELS

Here we collect and develop some useful lemmas, some of which can be of interest.

B.1 Properties of Products of Sub-Gaussians

A random variable ξ is (σ^2, α) -sub-Exponential if

$$\mathbb{E} e^{\lambda \xi} \leq e^{\lambda^2 \sigma^2 / 2} \text{ a.s. } \quad \forall \lambda : |\lambda| \leq \alpha^{-1}. \quad (\text{B.1})$$

A $(\sigma^2, 0)$ -sub-Exponential is σ^2 -sub-Gaussian. Lemma B.2 states concentration inequality for a sub-Exponential martingale difference sequence (m.d.s.).

LEMMA B.1 (Properties of sub-Gaussian random variables). (1) Let $\sigma_X, \sigma_Y > 0$. If X is σ_X^2 -sub-Gaussian and Y is σ_Y^2 -sub-Gaussian, then $X + Y$ is $(\sigma_X + \sigma_Y)^2$ -sub-Gaussian. (2) Let $\{X_m\}_{m=1}^M$ be a sequence of σ^2 -sub-Gaussian random variables. Then, (2a) $\sum_{m=1}^M X_m$ is $(M^2\sigma^2)$ -sub-Gaussian and (2b) $\max_{1 \leq m \leq M} X_m \lesssim_P (\sigma\sqrt{\log d})$. (3) Furthermore, $\sum_{m=1}^M X_m$ is $(M\sigma^2)$ -sub-Gaussian if $\{X_m\}_{m=1}^M$ are independent. (4) If $Y \in [-B, B]$ a.s., Y is B^2 -sub-Gaussian. (5) If X is σ_X^2 -sub-Gaussian conditional on Y , and $Y \in [-B, B]$ a.s., then $X \cdot Y$ is $\sigma_X^2 B^2$ -sub-Gaussian. (6) If X_{mn} are $\bar{\sigma}^2$ -sub-Gaussian for $n = 1, 2, \dots, \bar{N}$ (\bar{N} finite) and $m = 1, 2, \dots, M$, then $\max_{1 \leq m \leq M} \prod_{n=1}^{\bar{N}} |X_{mn}| \lesssim_P ((2\bar{\sigma})^{\bar{N}} \log^{\bar{N}/2}(M\bar{N}))$.

PROOF OF LEMMA B.1. We prove (1). By Holder inequality, for any p, q in $[1, \infty)$ such that $1/p + 1/q = 1$,

$$\mathbb{E}e^{\lambda(X+Y)} \leq (\mathbb{E}e^{\lambda p X})^{1/p} (\mathbb{E}e^{\lambda q Y})^{1/q} \leq e^{\lambda^2/2(p\sigma_X^2 + q\sigma_Y^2)}. \quad (\text{B.2})$$

Plugging $p = (\sigma_Y + \sigma_X)/\sigma_X$ and $q = (\sigma_Y + \sigma_X)/\sigma_Y$ into (B.2) gives (4.3) with $\sigma^2 = (\sigma_X + \sigma_Y)^2$. We prove (2a) by induction over M . The statement holds for $M = 1$. The inductive step follows from (1) with $\sigma_X = (M - 1)\sigma$ and $\sigma_Y = \sigma$. (2b) is Theorem 1.14 in Rigollet and Hutter (2017). The statements (3) and (4) are Theorem 1.6 and Lemma 1.8 in Rigollet and Hutter (2017), respectively. To see (5), observe that $\mathbb{E}[X | Y] = 0$ a.s. by assumption. LIE gives

$$\mathbb{E}_{X,Y}[X \cdot Y] = \mathbb{E}_Y \mathbb{E}[X | Y] Y = 0.$$

Furthermore,

$$\mathbb{E}_Y \mathbb{E}[e^{\lambda X Y} | Y] \leq \mathbb{E}_Y e^{\lambda^2 \sigma^2 Y^2 / 2} \leq e^{\lambda^2 \sigma^2 B^2 / 2},$$

which gives the result. (6) Invoking union bound for any $t > 0$

$$\begin{aligned} \mathbb{P} \left(\max_{1 \leq m \leq M} \prod_{n=1}^{\bar{N}} |X_{mn}| > t \right) &\leq \sum_{m=1}^M \mathbb{P} \left(\prod_{n=1}^{\bar{N}} |X_{mn}| > t \right) \\ &\leq \sum_{m=1}^M \sum_{n=1}^{\bar{N}} \mathbb{P} \left(|X_{mn}| > t^{1/\bar{N}} \right) \leq 2M\bar{N} e^{-t^{2/\bar{N}}/2\bar{\sigma}^2}. \end{aligned}$$

Taking $t := \bar{C}(2\bar{\sigma})^{\bar{N}} \log^{\bar{N}/2}(M\bar{N})$ and setting $\bar{C} \rightarrow \infty$ makes the R.H.S above $o(1)$. Conclude that

$$\max_{1 \leq m \leq M} \prod_{n=1}^{\bar{N}} |X_{mn}| \lesssim_P ((2\bar{\sigma})^{\bar{N}} \log^{\bar{N}/2}(M\bar{N})).$$

■

B.2 Tails Bounds for Maxima of Sums of Martingale Differences

LEMMA B.2 (Martingale Difference Sequences; Theorem 2.19 in [Wainwright \(2019\)](#)). (1) Let $\{(\xi_m, \Phi_m)\}_{m=1}^M$ be an m.d.s. obeying

$$\mathbb{E}[e^{\lambda \xi_m} \mid \Phi_{m-1}] \leq e^{\lambda^2 \sigma^2 / 2} \text{ a.s.}$$

for any λ such that $|\lambda| \leq \alpha^{-1}$. Then, the following bounds hold: (1) The sum $\sum_{m=1}^M \xi_m$ is $(\sigma^2 M, \alpha)$ -sub-Exponential and satisfies concentration inequality

$$\mathbb{P}\left(\left|\sum_{m=1}^M \xi_m\right| \geq t\right) \leq \begin{cases} 2e^{-t^2/(2M\sigma^2)}, & 0 \leq t \leq M\sigma^2/\alpha \\ 2e^{-t/(2\alpha)}, & t > M\sigma^2/\alpha \end{cases}$$

(2) For each $j : 1 \leq j \leq d$, let $\{(\xi_{mj}, \Phi_m)\}_{m=1}^M$ be an m.d.s obeying the conditions above with the same parameters (σ^2, α) . Then,

$$\mathbb{P}\left(\|M^{-1} \sum_{m=1}^M \xi_m\|_\infty > t\right) \leq \begin{cases} 2e^{\log d - t^2 M / (2\sigma^2)}, & 0 \leq t \leq \sigma^2/\alpha \\ 2e^{\log d - tM / (2\alpha)}, & t > \sigma^2/\alpha \end{cases} \quad (\text{B.3})$$

PROOF OF LEMMA B.2. Lemma B.2 is essentially Theorem 2.19 in [Wainwright \(2019\)](#). Replacing ξ by $c \cdot \xi$ in (B.1) shows that $c \cdot \xi$ is $(c^2 \sigma^2, c\alpha)$ -sub-Exponential. ■

LEMMA B.3. Let $1 \leq i \leq N$ and $1 \leq t \leq T$ be the unit and the time indices. Denote the index m as

$$m = m(i, t) = T(i - 1) + t. \quad (\text{B.4})$$

Consider a sequence

$$\xi_m = V_{it} U_{it}, \quad m = 1, 2, \dots, M = NT. \quad (\text{B.5})$$

Under Assumption 4.3,

A $\{\xi_m\}_{m=1}^M$ is a martingale difference sequence with respect to natural filtration:

$$\mathbb{E}[\xi_m \mid \Phi_{m-1}] := \mathbb{E}[\xi_m \mid \xi_1, \dots, \xi_{m-1}] = 0, \quad \forall m = 1, 2, \dots, M.$$

B Given a large enough constant C_{VU} large enough, there exists NT large enough such that the maximal norm of the empirical moment vector obeys:

$$\mathbb{P}\left(\|\mathbb{E}_{NT} V_{it} U_{it}\|_\infty > C_{VU} \sqrt{\log d / NT}\right) \leq 2/d = o(1). \quad (\text{B.6})$$

PROOF OF LEMMA B.3. By conditional sequential exogeneity (2.4) and independence over i

$$\mathbb{E}[U_{it} \mid \cup_{t < t'} V_{it'} U_{it'}, \cup_{j \neq i} \{(V_{jt}, U_{jt})_{t=1}^T\}] = 0 \quad \forall i, t.$$

Therefore, the martingale difference property A holds. Union bound and Assumption 4.3 imply

$$\mathbb{P}(|V_{it,j} U_{it}| > t) \leq \mathbb{P}(|V_{it,j}| > \sqrt{t}) + \mathbb{P}(|U_{it}| > \sqrt{t}) \leq 2e^{-t/2\bar{\sigma}^2}.$$

By Theorem 2.13 in [Wainwright \(2019\)](#), $V_{it,j}U_{it}$ is (σ^2, α) -sub-Exponential for some $\sigma, \alpha > 0$ that do not depend on j, N or T . Since the cutoff point σ^2/α in (B.3) does not depend on N, T , for C_{VU} large enough and sample size NT

$$t := C_{VU} \sqrt{\log d/NT} \leq \sigma^2/\alpha.$$

The bound follows

$$P\left(\|\mathbb{E}_{NT} V_{it} U_{it}\|_\infty > C_{VU} \sqrt{\log d/NT}\right) \leq 2/d = o(1).$$

■

B.3 Tail Bounds for Maxima of Sums of Sub-Gaussian products

LEMMA B.4 (Tail Bounds for Weakly Dependent Matrices, ℓ_∞ -norm). *Suppose Assumption 4.1(1) holds. For each $j = 1, 2, \dots, d$, let $\phi_j(W_{it})$ be centered σ^2 -sub-Gaussian random variable for all i, t where $\sigma = \sigma(N, T)$ and $\phi_j(W_{it})$ can depend on N, T . Then,*

$$\|S\|_\infty := \max_{1 \leq j \leq d} \left| (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \phi_j(W_{it}) \right| \lesssim_P \sigma \sqrt{\log(NT) \log d/NT}, \quad (\text{B.7})$$

Remark 2.1 (Triangular Arrays). Note that all variables and σ can be indexed by (N, T) , but we omit the indexing to keep the notation light. Thus, this lemma and all other lemmas stated below apply to triangular arrays.

PROOF OF LEMMA B.4. Let q be the block size such that $1 \leq q \leq T/2$ and let $L = \lceil T/2q \rceil$. Define the odd blocks

$$B_{i(2l-1)} := [W_{i,(2l-2)q+1}, W_{i,(2l-2)q+2}, \dots, W_{i,(2l-2)q+q}], \quad l = 1, 2, \dots, L \quad (\text{B.8})$$

the even blocks

$$B_{i(2l)} := [W_{i,(2l-1)q+1}, W_{i,(2l-1)q+2}, \dots, W_{i,(2l-1)q+q}], \quad l = 1, 2, \dots, L \quad (\text{B.9})$$

and the remainder block, which can be empty, as

$$B_{ir} := [W_{i,2Lq+1}, W_{i,2Lq+2}, \dots, W_{i,T}]. \quad (\text{B.10})$$

Note that $\{B_{i(2l-1)}\}_{l=1}^L$ obeys (A.3) and $\{B_{i(2l)}\}_{l=1}^L$ obeys (A.3) with $\epsilon = \gamma(q)$. Let $B_{i(2l-1)}^*$ be the Berbee copy of $B_{i(2l-1)}$. Define the Berbee event

$$\mathcal{I}_1 := \{B_{i(2l-1)}^* = B_{i(2l-1)} \text{ for all } i, l\}.$$

Likewise, let $B_{i(2l)}^*$ and \mathcal{I}_2 be the analogs of $B_{i(2l-1)}^*$ and \mathcal{I}_1 for even indices. Define the blockwise sum

$$\phi(B_{i(2l-1)}^*) := \sum_{t=(2l-2)q+1}^{t=(2l-2)q+q} \phi(W_{it}^*) \quad (\text{B.11})$$

$$S_{\text{odd}}^*(q) := (NT)^{-1} \sum_{i=1}^N \sum_{l=1}^L \phi(B_{i(2l-1)}^*). \quad (\text{B.12})$$

Let $S_{\text{even}}^*(q)$ be the analog of $S_{\text{odd}}^*(q)$ for even indices. If $T \neq 2Lq$, the remainder block is non-empty, in which case define

$$\phi(B_{ir}) := \sum_{t=2Lq+1}^T \phi(W_{it}) \quad (\text{B.13})$$

$$S_{\text{rem}}(q) := (NT)^{-1} \sum_{i=1}^N \phi(B_{ir}). \quad (\text{B.14})$$

On the event $\mathcal{I}_1 \cap \mathcal{I}_2$, the union bound gives

$$\|S\|_{\infty} \leq \|S_{\text{odd}}^*(q)\|_{\infty} + \|S_{\text{even}}^*(q)\|_{\infty} + \|S_{\text{rem}}(q)\|_{\infty}. \quad (\text{B.15})$$

Thus,

$$\begin{aligned} \mathbb{P}(\|S\|_{\infty} \geq 3t) &\leq \mathbb{P}\left(\left\|\sum_{i=1}^N \sum_{l=1}^L \phi(B_{i(2l-1)}^*)\right\|_{\infty} \geq t\right) \\ &+ \mathbb{P}\left(\left\|\sum_{i=1}^N \sum_{l=1}^L \phi(B_{i(2l)}^*)\right\|_{\infty} \geq t\right) + \mathbb{P}\left(\left\|\sum_{i=1}^N \phi(B_{ir})\right\|_{\infty} \geq t\right) + 2NL\gamma(q). \end{aligned}$$

For each j , $S_{\text{odd}j}^*(q)$ is $(NT)^{-2}(NL)q^2\sigma^2 \leq (q/NT)\sigma^2$ -sub-Gaussian by Lemma B.1; similarly, for each j , $S_{\text{even}j}^*(q)$ is $(q/NT)\sigma^2$ -sub-Gaussian. Note that here the dependency on L is linear and not square, because the Berbee blocks are independent. For the remainder block, for each j , $S_{\text{rem}j}(q)$ is $(NT)^{-2}(N)q^2\sigma^2 \leq (q/NT)\sigma^2$ -sub-Gaussian since $q \leq T$ by Lemma B.1, where we use only independence across i . Since $S_{\text{odd}j}^*(q)$ is $(q/NT)\sigma^2$ -sub-Gaussian for each j , $\|S_{\text{odd}}^*(q)\|_{\infty} \lesssim_P \sigma \sqrt{q \log d/NT}$ by Lemma B.1 (2b). Likewise, $\|S_{\text{rem}}(q)\|_{\infty} \lesssim_P \sigma \sqrt{q \log d/NT}$ by Lemma B.1 (2b). Given the parameter κ in mixing coefficient (4.1), we set block size q to be:

$$q = \lfloor (2/\kappa) \log(NT) \rfloor. \quad (\text{B.16})$$

Invoking the bound (4.1) in Assumption 4.1 gives

$$\begin{aligned} \mathbb{P}(\mathcal{I}_1^c) + \mathbb{P}(\mathcal{I}_2^c) &\leq 2N(L-1)\gamma(q) \leq (2NT/q)\gamma(q) \\ &\leq 2(NT/q)(NT)^{-2} = o((NT)^{-1}) = o(1), \quad NT \rightarrow \infty, \end{aligned}$$

which implies (B.7). ■

The following is an extension/clarification of a useful lemma due to [Kock and Tang \(2019\)](#).

LEMMA B.5 (Concentration of Products of sub-Gaussian Random Variables with Independent Blocks). *Suppose the random variables $Z_{n,m,v,j}$ are uniformly $\bar{\sigma}_n^2$ -sub-Gaussian as in (4.3) for $n = 1, 2, \dots, \bar{N}$, ($\bar{N} \geq 2$ is fixed and finite), $m = 1, 2, \dots, M$, $v = 1, 2, \dots, V$, and $j = 1, 2, \dots, d$. Suppose Z_{n_1, m_1, v_1, j_1} and Z_{n_2, m_2, v_2, j_2} are independent as long as $m_1 \neq m_2$ regardless of the values of other subscripts. Then,*

$$\max_{j,v,m} \mathbb{E} \left| \prod_{n=1}^{\bar{N}} Z_{n,m,v,j} \right| \leq C_A \prod_{n=1}^{\bar{N}} \bar{\sigma}_n,$$

for some positive constant C_A that depends on \bar{N} and with probability approaching 1,

$$\max_{1 \leq j \leq d} \left| (MV)^{-1} \sum_{m=1}^M \sum_{v=1}^V \left(\prod_{n=1}^{\bar{N}} Z_{n,m,v,j} - \mathbb{E} \prod_{n=1}^{\bar{N}} Z_{n,m,v,j} \right) \right| \leq C_V \sqrt{\log^{\bar{N}+1}(dV)/M} \prod_{n=1}^{\bar{N}} \bar{\sigma}_n,$$

for some positive constant C_V that depends only on \bar{N} .

LEMMA B.6 (Concentration of Products of sub-Gaussian Random Variables under Weak Dependence). *Suppose Assumption 4.1 (I) holds, and let $\varphi_{nj}(\cdot) : \mathcal{W} \rightarrow \mathbb{R}$ be a deterministic function. Suppose that $\varphi_{nj}(W_{it})$ are uniformly $\bar{\sigma}_n$ -sub-Gaussian as in (4.3) for $n = 1, 2, \dots, \bar{N}$ ($\bar{N} \geq 2$ is fixed and finite) and $j = 1, 2, \dots, d$ and any i, t . Then,*

$$\max_{j,i,t} \left| \mathbb{E} \left[\prod_{n=1}^{\bar{N}} \varphi_{nj}(W_{it}) \right] \right| \leq C_A \prod_{n=1}^{\bar{N}} \bar{\sigma}_n, \quad (\text{B.17})$$

for some positive constant C_A that depends on \bar{N} and with probability approaching 1,

$$\begin{aligned} \|S\|_\infty &:= \max_{1 \leq j \leq d} \left| (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \left[\prod_{n=1}^{\bar{N}} \varphi_{nj}(W_{it}) - \mathbb{E} \left[\prod_{n=1}^{\bar{N}} \varphi_{nj}(W_{it}) \right] \right] \right| \\ &\leq \bar{C}_V \sqrt{\log^{\bar{N}+1}(d \log(NT)) \log(NT)/NT} \prod_{n=1}^{\bar{N}} \bar{\sigma}_n. \end{aligned} \quad (\text{B.18})$$

for some positive constant \bar{C}_V that depends only on \bar{N} .

PROOF OF LEMMA B.6. Define

$$\phi(W_{it}) := \{\phi_j(W_{it})\}_{j=1}^d, \quad \phi_j(W_{it}) := \prod_{n=1}^{\bar{N}} \varphi_{nj}(W_{it}).$$

Let the block size q , the odd blocks, even blocks, and remainder blocks, and events \mathcal{I}_1 and \mathcal{I}_2 be as defined in the proof of Lemma B.4. Likewise, let $S_{\text{odd}}^*(q)$ be as in (B.11), that is,

$$\phi(B_{i(2l-1)}^*) := \sum_{t=(2l-2)q+1}^{t=(2l-2)q+q} \phi(W_{it}^*), \quad S_{\text{odd}}^*(q) := (NT)^{-1} \sum_{i=1}^N \sum_{l=1}^L \phi(B_{i(2l-1)}^*),$$

$S_{\text{even}}^*(q)$ be its analog for the even-numbered blocks, and $S_{\text{rem}}^*(q)$ be as in (B.13). The first claim (B.17) is immediate from the previous lemma. Lemma B.5 with $L \geq 2$ and $M = NL$ and $V = q$ implies that wp $1 - o(1)$

$$\begin{aligned} \|S_{\text{odd}}^*(q)\|_{\infty} &:= (Lq/T) \left\| (NLq)^{-1} \sum_{i=1}^N \sum_{l=1}^L \{\varphi(B_{i(2l-1)}^*) - \mathbb{E}\varphi(B_{i(2l-1)}^*)\} \right\|_{\infty} \\ &\leq (Lq/T) C_V \left(\sqrt{\log^{\bar{N}+1}(dq)/NL} \right) \leq i C_V \left(\sqrt{\log^{\bar{N}+1}(dq)q/NT} \right), \end{aligned}$$

where (i) follows from $L = \lfloor T/2q \rfloor \leq T/2q$ and $L \geq \lfloor T/2q \rfloor \geq T/2q - 1 \geq T/4q$. A similar bound holds for $S_{\text{even}}^*(q)$. If $T_{\text{rem}} \neq 0$, Lemma B.5 with $M = N$ and $V = T_{\text{rem}} \leq q$ implies that wp $1 - o(1)$:

$$\begin{aligned} \|S_{\text{rem}}(q)\|_{\infty} &:= T_{\text{rem}}/T \left\| (NT_{\text{rem}})^{-1} \sum_{i=1}^N (\varphi(B_{ir}) - \mathbb{E}\varphi(B_{ir})) \right\|_{\infty} \\ &\leq T_{\text{rem}}/T C_V \left(\sqrt{\log^{\bar{N}+1}(dT_{\text{rem}})/N} \right) \\ &\leq q/TC_V \left(\sqrt{\log^{\bar{N}+1}(dq)/N} \right). \end{aligned}$$

Plugging $q^2/NT^2 \leq q/NT$ into the R.H.S above gives the bound $C_V \left(\sqrt{\log^{\bar{N}+1}(dq)q/NT} \right)$.

Let NT be large enough so that $L = \lfloor T/2q \rfloor \geq 2$ and $(2/\kappa) \leq \log(NT)$ so that $q \leq \log^2(NT)$ and $dq \leq (d \log(NT))^2$. Collecting the bounds gives (B.18). Adding up the bounds and plugging choice of $q = (2/\kappa) \log(NT)$ as in (B.16) and noting that $L = \lfloor T/2q \rfloor \geq 2$ for T large enough gives (B.18). ■

COROLLARY B.1. *Suppose Assumption 4.1 (1) holds. Suppose $Z_{1,nit}$ and $Z_{2,nit}$ are d -vectors obtained as (measurable) transformations of W_{it} , whose entries are uniformly $\bar{\sigma}_1^2$ and $\bar{\sigma}_2^2$ -sub-Gaussian for $n = 1, 2, \dots, \bar{N}$. Let U_{it} be uniformly $\bar{\sigma}^2$ -sub-Gaussian and $g \geq 0$ be a finite power. Then,*

$$\max_{1 \leq k, j \leq d} \max_{i, t} \left| \mathbb{E} \left[\prod_{n=1}^{\bar{N}} Z_{1,nitk} Z_{2,nitj} U_{it}^{2g} \right] \right| \leq C_A (\bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}^{2g})^{\bar{N}} \quad (\text{B.19})$$

$$\begin{aligned} &\max_{1 \leq k, j \leq d} \left| (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \left[\prod_{n=1}^{\bar{N}} Z_{1,nitk} Z_{2,nitj} U_{it}^{2g} - \mathbb{E} \left[\prod_{n=1}^{\bar{N}} Z_{1,nitk} Z_{2,nitj} U_{it}^{2g} \right] \right] \right| \\ &\leq \bar{C}_V (\bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}^{2g})^{\bar{N}} \left(\sqrt{\log^{2\bar{N}+2g+1}(d^2 \log(NT)) \log(NT)/NT} \right) \quad (\text{B.20}) \end{aligned}$$

Remark 2.2. Suppose Assumptions 4.1 and 4.3 hold. Invoking (B.19) with $\bar{N} = 1$ and $Z_{1,it} = Z_{2,it} = V_{it}$ implies for some finite $\sigma_V < \infty$,

$$\max_{it} \|EV_{it}V_{it}'\|_{\infty} \leq \max_{itj} EV_{itj}^2 \leq \sigma_V^2.$$

Likewise, Assumption 4.3 implies for some finite $\sigma_{VU} < \infty$,

$$\sup_{it} \mathbb{E}[U_{it}^2 | V_{it}] \leq \sigma_{VU}^2 \text{ a.s. .}$$

B.4 Some Technical Lemmas

Here we provide technical extensions of the results in [Kock and Tang \(2019\)](#), keeping the notation as in the original [Kock and Tang \(2019\)](#) and references therein.

LEMMA B.7 (Theorem 2.1 in [Fan et al. \(2012\)](#), Proposition F1 in [Kock and Tang \(2019\)](#)). . . Let $\alpha \in (0, 1)$. Assume that $(X_i, \mathcal{F}_i)_{i=1}^n$ is a sequence of supermartingale differences satisfying $\sup_i \mathbb{E} \left[e^{|X_i| \frac{2\alpha}{1-\alpha}} \right] \leq C_1$ for some constant $C_1 \in (0, \infty)$. Define $S_k := \sum_{i=1}^k X_i$. Then, for all $\epsilon > 0$,

$$\mathbb{P} \left(\max_{1 \leq k \leq n} S_k \geq n\epsilon \right) \leq C(\alpha, n, \epsilon) e^{-(\epsilon/4)^{2\alpha} n^\alpha}$$

where

$$C(\alpha, n, \epsilon) := 2 + 35C_1 \left[\frac{1}{16^{1-\alpha} (n\epsilon^2)^\alpha} + \frac{1}{n\epsilon^2} \left(\frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right].$$

LEMMA B.8 (Proposition F2 in [Kock and Tang \(2019\)](#)). Let $\alpha \in (0, 1)$. Assume that $(X_i, \mathcal{F}_i)_{i=1}^n$ is a sequence of martingale differences satisfying $\sup_i \mathbb{E} \left[e^{D|X_i| \frac{2\alpha}{1-\alpha}} \right] \leq C_1$ for some positive constant D , where $C_1 \geq 1$ can change with the sample size n . Then, for all $\epsilon \geq 1/\sqrt{n}$,

$$\mathbb{P} \left(\left| \sum_{i=1}^n X_i \right| \geq n\epsilon \right) \leq C_1 A(\alpha) e^{-K(\epsilon^2 n)^\alpha}, \quad K = (D \frac{1-\alpha}{2\alpha} / 4)^{2\alpha}.$$

where

$$A = A(\alpha) = 2 + 35 \left[\frac{1}{16^{1-\alpha}} + \left(\frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right].$$

■

Lemma B.8 restates Proposition F2 in [Kock and Tang \(2019\)](#) with explicit constants in Tail Bounds.

Proof. Note that for some positive constant D ,

$$\mathbb{P} \left(\sum_{i=1}^n X_i \geq n\epsilon \right) = \mathbb{P} \left(\sum_{i=1}^n D \frac{1-\alpha}{2\alpha} X_i \geq nD \frac{1-\alpha}{2\alpha} \epsilon \right) = \mathbb{P} \left(\sum_{i=1}^n Y_i \geq n\delta \right),$$

where $Y_i := D^{\frac{1-\alpha}{2\alpha}} X_i$ and $\delta := D^{\frac{1-\alpha}{2\alpha}} \epsilon$. Now $(Y_i)_{i=1}^n$ is a sequence of martingale differences satisfying $\sup_i \mathbb{E} \left[e^{|Y_i| \frac{2\alpha}{1-\alpha}} \right] \leq C_1$. Invoking the preceding theorem, we have

$$\mathbb{P} \left(\sum_{i=1}^n Y_i \geq n\delta \right) \leq C(\alpha, n, \delta) e^{-(\delta/4)^{2\alpha} n^\alpha}.$$

$(-Y_i)_{i=1}^n$ is also a sequence of martingale differences satisfying the same exponential moment condition. Thus,

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{i=1}^n X_i \right| \geq n\epsilon \right) &= \mathbb{P} \left(\left| \sum_{i=1}^n Y_i \right| \geq n\delta \right) \leq 2C(\alpha, n, \delta) e^{-(\delta/4)^{2\alpha} n^\alpha} \\ &= 2C \left(\alpha, n, D^{\frac{1-\alpha}{2\alpha}} \epsilon \right) e^{-\left(D^{\frac{1-\alpha}{2\alpha}} \epsilon / 4 \right)^{2\alpha} n^\alpha} \leq C_1 A(\alpha) e^{-K \epsilon^{2\alpha} n^\alpha}, \end{aligned}$$

where we can select

$$A = A(\alpha) = 2 + 35 \left[\frac{1}{16^{1-\alpha}} + \left(\frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right]$$

and K as defined above. ■

The following Lemma is inspired by Prop E3 of [Kock and Tang \(2019\)](#). The difference is that the constants are made explicit to make the results applicable to arrays; and part of the proof was replaced by another argument (as we were not able to follow one step in their proof).²

LEMMA B.9. *Suppose we have random variables $Z_{l,i,t,j}$ uniformly $(K, \sigma_l^2) > 0$ subgaussian for $l = 1, \dots, L$ ($L \geq 2$ fixed), $i = 1, \dots, N$, $t = 1, \dots, T$ and $j = 1, \dots, p$ that is,*

$$\mathbb{P}(|\sigma_l^{-1} Z_{l,i,t,j}| \geq \varepsilon) \leq K \exp(-\varepsilon^2),$$

and Z_{l_2, i_2, t_2, j_2} are independent as long as $i_1 \neq i_2$ regardless of the values of other subscripts. Then, we have that (1)

$$\max_{j,t,i} \mathbb{E} \left| \prod_{l=1}^L Z_{l,i,t,j} \right| \leq \left(L! (\log 2)^{-1/2} (1+K)^{1/2} \right) \prod_{l=1}^L \sigma_l,$$

and (2) with probability $1 - A'(pT)^{-1/2}$,

$$\max_{1 \leq j \leq d} \left| \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \left(\prod_{l=1}^L Z_{l,i,t,j} - \mathbb{E} \left[\prod_{l=1}^L Z_{l,i,t,j} \right] \right) \right| \leq M \left(\sqrt{\frac{(\log(pT))^{L+1}}{N}} \right) \prod_{l=1}^L \sigma_l,$$

for $M > M'$, and some positive constants A' and M' that only depend on L and K .

²KT's proof uses the inequality $(x - (y \wedge x))^{2/L} \leq x^{2/L} - (y \wedge x)^{2/L}$, for $x > 0$ and $y > 0$. This inequality is not true (for example, with $x = 10$, $y = 1$, $L = 4$, the inequality implies $3 < 2.163$), so we changed the middle part of the proof; the end result is preserved; none of conclusions in KT are affected.

PROOF. Hölder's inequality gives

$$\max_{j,t,i} \mathbb{E} \left| \prod_{l=1}^L \sigma_l^{-1} Z_{l,i,t,j} \right| \leq \max_{j,t,i} \prod_{l=1}^L \left(\mathbb{E} \left| \sigma_l^{-1} Z_{l,i,t,j} \right|^L \right)^{\frac{1}{L}},$$

where

$$\begin{aligned} \left(\mathbb{E} \left| \sigma_l^{-1} Z_{l,i,t,j} \right|^L \right)^{\frac{1}{L}} &\leq L! \left\| \sigma_l^{-1} Z_{l,i,t,j} \right\|_{\psi_1} \\ &\leq L! (\log 2)^{-1/2} \left\| \sigma_l^{-1} Z_{l,i,t,j} \right\|_{\psi_2} \leq L! (\log 2)^{-1/2} (1+K)^{1/2} =: A, \end{aligned}$$

where the first two inequalities are from [van der Vaart and Wellner \(1996\)](#), p.95 and the third inequality from Lemma 2.2.1 in [van der Vaart and Wellner \(1996\)](#). Thus,

$$\max_{j,t,i} \mathbb{E} \left| \prod_{l=1}^L \sigma_l^{-1} Z_{l,i,t,j} \right| \leq \left(L! (\log 2)^{-1/2} (1+K)^{1/2} \right) =: A.$$

This implies the first claim, after multiplying both sides by $\prod_{l=1}^L \sigma_l$. Let

$$X_{i,t,j} = \prod_{l=1}^L \sigma_l^{-1} Z_{l,i,t,j} - \mathbb{E} \left[\prod_{l=1}^L \sigma_l^{-1} Z_{l,i,t,j} \right].$$

For every $\epsilon \geq 0$,

$$\begin{aligned} \mathbb{P} (|X_{i,t,j}| \geq 2\epsilon) &\leq \mathbb{P} \left(\left| \prod_{l=1}^L \sigma_l^{-1} Z_{l,i,t,j} \right| \geq \epsilon \right) + \mathbb{P} \left(\left| \mathbb{E} \prod_{l=1}^L \sigma_l^{-1} Z_{l,i,t,j} \right| \geq \epsilon \right) \\ &\leq \sum_{l=1}^L \mathbb{P} \left(\sigma_l^{-1} |Z_{l,i,t,j}| \geq \epsilon^{1/L} \right) + 1 (\epsilon \leq A) \\ &\leq LK e^{-\epsilon^{2/L}} + 1 (\epsilon^{2/L} \leq A^{2/L}) \\ &\leq LK e^{-\epsilon^{2/L}} + e^{A^{2/L}} e^{-\epsilon^{2/L}} = K' e^{-\epsilon^{2/L}}. \quad (K' := (LK + e^{A^{2/L}})). \end{aligned}$$

Let

$$X_{i,j} := \frac{1}{T} \sum_{t=1}^T \left(\prod_{l=1}^L \sigma_l^{-1} Z_{l,i,t,j} - \mathbb{E} \left[\prod_{l=1}^L \sigma_l^{-1} Z_{l,i,t,j} \right] \right).$$

For every $\epsilon \geq 0$,

$$\mathbb{P} (|X_{i,j}| \geq 2\epsilon) \leq \mathbb{P} \left(\max_{1 \leq t \leq T} |X_{i,t,j}| \geq 2\epsilon \right) \leq TK' e^{-\epsilon^{2/L}}.$$

Consider some positive constant $D < 1$, then as [van der Vaart and Wellner \(1996\)](#), p.96, using Fubini and change of order of integration:

$$\mathbb{E} \left[e^{D|X_{i,j}/2|^{2/L}} \right] = \int_{x \in \mathbb{R}} \int_0^{|x/2|^{2/L}} D e^{Ds} ds P(dx) + 1 = \int_0^\infty D e^{Ds} \mathbb{P} (|X_{i,j}| > 2s^{L/2}) ds + 1.$$

This is further bounded by

$$\int_0^\infty TK'De^{(D-1)s} ds + 1 = \frac{TK'D}{1-D} + 1 \leq BT; \quad (B := \frac{K'D}{1-D} + 1).$$

Then we can use independence across i to invoke the previous Lemma B.8 with $\alpha = \frac{1}{L+1}$ and $C_1 = BT$, for $\epsilon \geq \frac{1}{\sqrt{N}}$

$$P\left(\left|\sum_{i=1}^N \frac{1}{T} \sum_{t=1}^T X_{i,t,j}\right| \geq 2N\epsilon\right) \leq A'Te^{-K''(\epsilon^2 N)^{\frac{1}{L+1}}}$$

for positive constants A' and K'' that depend only on K , L , and D .

Setting

$$\epsilon = \sqrt{\frac{M(\log(pT))^{L+1}}{N}}$$

for some $M \geq 1$, we have

$$P\left(\max_{1 \leq j \leq p} \left|\sum_{i=1}^N \frac{1}{T} \sum_{t=1}^T X_{i,t,j}\right| \geq 2N\epsilon\right) \leq pA'Te^{-K''(\epsilon^2 N)^{\frac{1}{L+1}}} = A'(pT)^{1-K''M \frac{1}{L+1}}.$$

Therefore, with probability $1 - A'(pT)^{1-K''M \frac{1}{L+1}}$,

$$\max_j \left| \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T (X_{i,t,j}) \right| \leq 2M \left(\sqrt{\frac{(\log(pT))^{L+1}}{N}} \right),$$

for any $M \geq 1$. Setting M large enough such that

$$1 - K''M \frac{1}{L+1} < -\frac{1}{2},$$

guarantees that the bound holds with probability at most $A'(pT)^{-1/2}$. The bounds can be then be restated as in the statement of the theorem. ■

APPENDIX C: TOOLS: HIGH-DIMENSIONAL CENTRAL LIMIT THEOREMS FOR WEAKLY
DEPENDENT DATA

Let $\{X_m\}_{m=1}^M$ be a weakly dependent martingale difference sequence (m.d.s.) with respect to natural filtration. Define its β -mixing coefficient

$$\gamma_X(q) = \sup_{m \leq M} \gamma((X_1, \dots, X_{m-1}, X_m), (X_{m+q}, X_{m+q+1}, \dots)).$$

The scaled sum

$$S_X = M^{-1/2} \sum_{m=1}^M X_m$$

has the variance

$$\Sigma_G := M^{-1} \sum_{m=1}^M \mathbb{E} X_m X_m'. \quad (\text{C.1})$$

The distribution of the scaled sum over the cubes can be approximated by the Gaussian distribution $N(0, \Sigma_G)$ over the cubes, as shown in the lemma below.

We will introduce the following notation. For some numbers $\bar{r} = \bar{r}_{NT}, \bar{q} = \bar{q}_{NT}$ and $L = \lfloor M/(\bar{q} + \bar{r}) \rfloor$, define Bernstein's "large" and "small" blocks of size \bar{q} and \bar{r} :

$$P_l = \{(l-1)(\bar{q} + \bar{r}) + 1, \dots, (l-1)(\bar{q} + \bar{r}) + \bar{q}\}, \quad l = 1, 2, \dots, L$$

$$Q_l = \{(l-1)(\bar{q} + \bar{r}) + 1 + \bar{q}, \dots, l(\bar{q} + \bar{r})\}$$

and let

$$S_l := \sum_{m \in P_l} X_m, \quad U_l := \sum_{m \in Q_l} X_m, \quad U_{L+1} := \sum_{m=L(\bar{q}+\bar{r})+1}^M X_m.$$

Denote

$$\Sigma_P := (L\bar{q})^{-1} \sum_{l=1}^L \mathbb{E} S_l S_l' = (L\bar{q})^{-1} \sum_{l=1}^L \sum_{m \in P_l} \mathbb{E} X_m X_m' \quad (\text{C.2})$$

and observe that

$$\Sigma_G = (L\bar{q}/M) \Sigma_P + M^{-1} \sum_{l=1}^{L+1} \mathbb{E} U_l U_l'.$$

The following result is useful both in the proof below and also for performing Gaussian inference, where we replaced unknown variance-covariance matrix by an estimated one.

LEMMA C.1 (Comparison of distributions). *Let $X \sim N(0, \Sigma_X)$ and $Y \sim N(0, \Sigma_Y)$ be centered normal d -vectors, and let $\Delta_{XY} := \|\Sigma_X - \Sigma_Y\|_\infty$. Suppose $\min_{1 \leq j \leq d} (\Sigma_Y)_{jj} > 0$.*

Then,

$$\sup_{t \geq 0} |\mathbb{P}(\|X\|_\infty \leq t) - \mathbb{P}(\|Y\|_\infty \leq t)| \leq C' (\Delta_{XY} \log^2(2d))^{1/2}, \quad (\text{C.3})$$

where $C' > 0$ depends only on $\min_{1 \leq j \leq d} (\Sigma_Y)_{jj}$ and $\max_{1 \leq j \leq d} (\Sigma_Y)_{jj}$.

Lemma C.1 follows from Proposition 2.1 in Chernozhukov et al. (2019b) for vectors $\bar{X} = (X, -X)$ and $\bar{Y} = (Y, -Y)$ and

$$\Sigma_{\bar{X}} = \begin{pmatrix} \Sigma_X & -\Sigma_X \\ -\Sigma_X & \Sigma_X \end{pmatrix}, \quad \Sigma_{\bar{Y}} = \begin{pmatrix} \Sigma_Y & -\Sigma_Y \\ -\Sigma_Y & \Sigma_Y \end{pmatrix}, \quad \|\Sigma_{\bar{X}} - \Sigma_{\bar{Y}}\|_\infty = \Delta_{XY}.$$

Another result is the following anti-concentration property. This result is useful for showing that linearization errors do not impact the behavior of the key statistics. The statistics are approximate means, namely averages of some centered influence functions plus linearization errors.

LEMMA C.2 (Anti-concentration). *Let $X = (X_1, X_2, \dots, X_d)' \sim N(0, \Sigma_X)$ be a centered Gaussian random vector in \mathbb{R}^d . Assume $\min_{1 \leq j \leq d} (\Sigma_X)_{jj} > 0$. Then,*

$$\sup_{t \in \mathbb{R}} \mathbb{P}(\|X\|_\infty - t \leq \epsilon) \leq C \epsilon \sqrt{1 \vee \log(2d/\epsilon)}, \quad (\text{C.4})$$

where $C > 0$ depends on $\min_{1 \leq j \leq d} (\Sigma_X)_{jj}$ and $\max_{1 \leq j \leq d} (\Sigma_X)_{jj}$.

Lemma C.2 follows from Corollary 1 in Chernozhukov et al. (2015) with $\bar{X} = (X, -X)$.

The following result is a consequence of Theorem E.1 in Chernozhukov et al. (2019a) for martingale difference sequence.

LEMMA C.3 (High-dimensional CLT for martingale difference sequence under weak dependence). *Let $\{X_m\}_{m=1}^M$ be a weakly dependent m.d.s. of d -vectors obeying for $D_M \geq 1$:*

$$\sup_{m \leq M} \|X_m\|_\infty \leq D_M \text{ a.s.}$$

Suppose there exist constants $0 < a_1 \leq A_1$ and $0 < c_2 < 1/4$ such that

$$a_1 \leq \min_{1 \leq j \leq d} \min_{1 \leq m \leq M} \text{Var} X_{mj} \leq \max_{1 \leq j \leq d} \sup_{1 \leq m \leq M} \text{Var} X_{mj} \leq A_1, \quad (\text{C.5})$$

and let \bar{r} and \bar{q} be such that $\bar{r}/\bar{q} \leq A_1 M^{-c_2} \log^{-2} d$ and

$$\max\{\bar{r} D_M \log^{3/2} d, \bar{q} D_M \log^{1/2} d, \sqrt{\bar{q}} D_M \log^{7/2}(dM)\} \leq A_1 M^{1/2-c_2}. \quad (\text{C.6})$$

Then, there exist constants $c_X, C_X > 0$ depending only on a_1, A_1, c_2 such that

$$\sup_{t \geq 0} \left| \mathbb{P}(\|S_X\|_\infty \leq t) - \mathbb{P}(\|G_P\|_\infty < t) \right| \leq 2 \frac{M}{\bar{q} + \bar{r}} \gamma_X(\bar{r}) + C_X M^{-c_X}, \quad (\text{C.7})$$

where $G_P \sim N(0, \Sigma_P)$ is a centered normal d -vector.

Note that this result uses Σ_P as the variance in the Gaussian approximation. In our application, we will be using Σ_G in place of Σ_P (i.e., Lemma C.5) so as not to worry about omitting small blocks. Therefore below, we will provide a sequence of the results that allow this replacement.

PROOF OF LEMMA C.3. Let

$$\bar{X}_m := (X_m, -X_m), \quad m = 1, 2, \dots, M$$

be a sequence of $2d$ -vectors. Observe that $\{\bar{X}_m\}_{m=1}^M$ is an m.d.s. It obeys

$$\sup_{m \leq M} \|\bar{X}_m\|_\infty \leq D_M, \text{ a.s.}, \quad \gamma_{\bar{X}}(q) = \gamma_X(q) \quad \forall q.$$

By construction, for any integer r

$$\begin{aligned} \bar{\sigma}^2(r) &= \max_{1 \leq j \leq d} \max_I \text{Var}(r^{-1/2} \sum_{m \in I} X_{mj}) = \max_{1 \leq j \leq 2d} \max_I \text{Var}(r^{-1/2} \sum_{m \in I} \bar{X}_{mj}), \\ \underline{\sigma}^2(r) &= \min_{1 \leq j \leq d} \min_I \text{Var}(r^{-1/2} \sum_{m \in I} X_{mj}) = \min_{1 \leq j \leq 2d} \min_I \text{Var}(r^{-1/2} \sum_{m \in I} \bar{X}_{mj}), \end{aligned}$$

where \max_I and \min_I are taken over the sets $I = \{i+1, i+2, \dots, i+r\}$ of size r . Theorem E.1 in Chernozhukov et al. (2019a) requires

$$a_1 \leq \underline{\sigma}^2(\bar{q}) \leq \bar{\sigma}^2(\bar{q}) \vee \bar{\sigma}^2(\bar{r}) \leq A_1. \quad (\text{C.8})$$

Because $\{X_m\}_{m=1}^M$ is an m.d.s,

$$\text{Cov}(X_{m_1}, X_{m_2}) = 0 \in \mathbb{R}^{d \times d} \text{ for } m_1 \neq m_2.$$

Therefore, for any r and any $I = \{i+1, i+2, \dots, i+r\}$,

$$a_1 \leq \text{Var}(r^{-1/2} \sum_{m \in I} X_{mj}) = r^{-1} \sum_{m \in I} \text{Var}(X_{mj}) \leq A_1, \quad 1 \leq j \leq d,$$

which implies (C.8). All other conditions of Theorem E.1 in Chernozhukov et al. (2019a) are satisfied. Invoking Theorem E.1 in Chernozhukov et al. (2019a) with

$$T := \max_{1 \leq j \leq 2d} M^{-1/2} \sum_{m=1}^M \bar{X}_{mj} = \|S_X\|_\infty$$

and

$$\bar{G}_P \sim N(0, \Sigma_{G_P})$$

being a centered normal $(2d)$ -vector with

$$\Sigma_{G_P} = \begin{pmatrix} \Sigma_P & -\Sigma_P \\ -\Sigma_P & \Sigma_P \end{pmatrix}.$$

gives (C.7). ■

LEMMA C.4 (Comparison of distributions, cont.). *Consider the setup above with $\Sigma_X = \Sigma_G$ and $\Sigma_Y = \Sigma_P$, where Σ_G and Σ_P are as in (C.2) and (C.1) where*

$$\sup_{1 \leq m \leq M} \|EX_m X'_m\|_\infty \leq \sup_{1 \leq m \leq M} \sup_{1 \leq j \leq d} \text{Var}(X_{mj}) \leq A_1.$$

For some $c_2 \in (0, 1/4)$, assume that the growth condition holds:

$$D_M \log d \log M \log^{7/2}(dM) \lesssim M^{1/2-2c_2}.$$

and $\log^4 d \log^2 M = o(\sqrt{M})$. Then, the max distance $\Delta_{GP} := \|\Sigma_G - \Sigma_P\|_\infty$ obeys

$$(\Delta_{GP} \log^2 d)^{1/2} \lesssim M^{-c_2/2}.$$

PROOF OF LEMMA C.4. Observe that

$$\Sigma_G - \Sigma_P = (L\bar{q}/M - 1)\Sigma_P + M^{-1} \sum_{l=1}^{L+1} EU_l U'_l.$$

Since $L = \lfloor M/(\bar{q} + \bar{r}) \rfloor$, $L \geq M/(\bar{q} + \bar{r}) - 1$. Therefore,

$$1 - L\bar{q}/M \leq 1 - \bar{q}/(\bar{q} + \bar{r}) + \bar{q}/M = \bar{r}/(\bar{q} + \bar{r}) + \bar{q}/M \leq \bar{r}/\bar{q} + \bar{q}/M.$$

Furthermore, $(L+1)/M \leq 2L/M \leq 2/\bar{q}$. The following bound holds

$$\Delta_{GP} \leq ((1 - L\bar{q}/M) + (L+1)/M) \sup_{1 \leq m \leq M} \|EX_m X'_m\|_\infty = O(\bar{r}/\bar{q} \vee \bar{q}/M \vee 1/\bar{q}),$$

Taking $\bar{q} = M^{c_2} \log^2 d \log^2 M$ and $\bar{r} = (2/\kappa) \log M$ gives:

$$\begin{aligned} \bar{r}/\bar{q} &= (2/\kappa) M^{-c_2} \log^{-2} d \log^{-1} M = o(M^{-c_2} \log^{-2} d) \\ \bar{q}/M &= M^{c_2-1} \log^2 d \log^2 M = o(M^{-c_2} \log^{-2} d) \\ 1/\bar{q} &= M^{-c_2} \log^{-2} d \log^{-2} M = o(M^{-c_2} \log^{-2} d), \end{aligned}$$

where (i) follows from $c_2 < 1/4$ and

$$\log^4 d \log^2 M = o(M^{1-2 \cdot 1/4}) = o(M^{1-2c_2})$$

Plugging $\Delta_{GP} = o(M^{-c_2} \log^{-2} d)$ into $(\Delta_{GP} \log^2(2d))^{1/2}$ gives

$$(\Delta_{GP} \log^2(2d))^{1/2} = o(M^{-c_2/2}).$$

■

Remark 3.3 (Sufficient Growth Condition). If the growth condition holds

$$D_M \log d \log M \log^{7/2}(dM) \lesssim M^{1/2-2c_2} \tag{C.9}$$

holds, then

$$\bar{r} = \log M, \quad \bar{q} = M^{c_2} \log^2 d \log^2 M \quad (\text{C.10})$$

obeys (C.6) and $\bar{r}/\bar{q} \leq A_1 M^{-c_2} \log^{-2} d$ for M large enough.

PROOF OF REMARK 3.3. Let M be large enough such that $M^{-c_2/2} \leq A_1$ and $(2/\kappa) \log^{-1} M \leq A_1$. Then, the growth condition

$$D_M \log d \log M \log^{7/2}(dM) \lesssim M^{1/2-2c_2} \leq A_1 M^{1/2-3/2c_2}$$

implies the third inequality in (C.6)

$$\sqrt{\bar{q}} D_M \log^{7/2}(dM) \leq A_1 M^{1/2-c_2}.$$

Next, for $d \geq e$ such that $\log d \geq 1$, and

$$\begin{aligned} M^{-c_2} D_M \bar{q} \log^{1/2} d &= D_M \log^{5/2} d \log^2 M \\ &\leq D_M \log^{5/2}(dM) \log M \log(dM) \log d \leq A_1 M^{1/2-3/2c_2}. \end{aligned}$$

Multiplying both sides by M^{c_2} gives

$$D_M \bar{q} \log^{1/2} d \leq A_1 M^{1/2-c_2},$$

which coincides with the second inequality in (C.6). Finally,

$$D_M \bar{r} \log^{3/2} d = (2/\kappa) D_M \log M \log^{3/2} d \leq D_M \bar{q} \log^{1/2} d,$$

as long as $(2/\kappa) \leq \log M$, which verifies (C.6). For M large enough $\bar{r}/\bar{q} = 2/\kappa M^{-c_2} \log^{-2} d \log^{-1} M \leq A_1$. ■

LEMMA C.5 (Summary). Let $\{X_m\}_{m=1}^M$ be a weakly dependent m.d.s. of d -vectors obeying for $D_M \geq 1$:

$$\sup_{m \leq M} \|X_m\|_\infty \leq D_M \text{ a.s.}$$

Suppose there exist constants $0 < a_1 \leq A_1$ such that

$$a_1 \leq \min_{1 \leq j \leq d} \min_{1 \leq m \leq M} \text{Var} X_{mj} \leq \max_{1 \leq j \leq d} \sup_{1 \leq m \leq M} \text{Var} X_{mj} \leq A_1.$$

For some constant $c_2 \in (0, 1/4)$, the growth condition (C.5) holds, namely

$$D_M \log d \log M \log^{7/2}(dM) \lesssim M^{1/2-2c_2}.$$

and $\log^4 d \log^2 M = o(M^{1/2})$. Then, there exist constants $c_X, C_X > 0$ depending only on a_1, A_1, c_2 such that for $\bar{r} = (2/\kappa) \log M$ and $\bar{q} = M^{c_2} \log^2 d \log^2 M$

$$\sup_{t \geq 0} |\text{P}(\|S_X\|_\infty \leq t) - \text{P}(\|G_\Sigma\|_\infty < t)| \lesssim C_X M^{-c_X} + M^{-c_2/2}, \quad (\text{C.11})$$

where $G_\Sigma \sim N(0, \Sigma_G)$ is a centered normal d -vector.

Triangular inequality gives

$$\sup_{t \geq 0} |\mathbb{P}(\|S_X\|_\infty \leq t) - \mathbb{P}(\|G_\Sigma\|_\infty < t)| \quad (\text{C.12})$$

$$\begin{aligned} &\leq \sup_{t \geq 0} |\mathbb{P}(\|S_X\|_\infty \leq t) - \mathbb{P}(\|G_P\|_\infty < t)| + \sup_{t \geq 0} |\mathbb{P}(\|G_P\|_\infty < t) - \mathbb{P}(\|G_\Sigma\|_\infty < t)| \\ &\lesssim 2 \frac{M}{\bar{q} + \bar{r}} \gamma(\bar{r}) + C_X M^{-c_X} + M^{-c_2/2} = o(M^{-c_2/2} + M^{-c_X}). \end{aligned} \quad (\text{C.13})$$

APPENDIX D: PROOFS FOR SECTION 4

D.1 Bounds on Errors for Estimating Q and Gradient S .

Below, we define the following terms that appear in the analysis of \hat{Q} and the least squares gradient S . In what follows we use the notations defined in the main text heavily, without further warning.

Define the first-stage approximation error as a function of $d(\cdot)$ and $l(\cdot)$:

$$R_{it}(\mathbf{d}, \mathbf{l}) := l_{i0}(X_{it}) - l_i(X_{it}) - (d_{i0}(X_{it}) - d_i(X_{it}))' \beta_0. \quad (\text{D.1})$$

Define the first-order error terms

$$\bar{a} := \mathbb{E}_{NT} V_{it} (d_{i0}(X_{it}) - \hat{d}_i(X_{it})) = \mathbb{E}_{NT} V_{it} (\hat{V}_{it} - V_{it}) \quad (\text{D.2})$$

$$\bar{m} = \mathbb{E}_{NT} V_{it} (l_{i0}(X_{it}) - \hat{l}_i(X_{it})) = \mathbb{E}_{NT} V_{it} (\hat{Y}_{it} - \tilde{Y}_{it}) \quad (\text{D.3})$$

$$\bar{f} = \mathbb{E}_{NT} U_{it} (d_{i0}(X_{it}) - \hat{d}_i(X_{it})) = \mathbb{E}_{NT} U_{it} (\hat{V}_{it} - V_{it}), \quad (\text{D.4})$$

$$\bar{e} = \mathbb{E}_{NT} V_{it} R_{it}(\hat{\mathbf{d}}, \hat{\mathbf{l}}) = \bar{m} - \bar{a}' \beta_0. \quad (\text{D.5})$$

the second-order error terms

$$\bar{b} = \mathbb{E}_{NT} (d_{i0}(X_{it}) - \hat{d}_i(X_{it})) (d_{i0}(X_{it}) - \hat{d}_i(X_{it}))' \quad (\text{D.6})$$

$$\bar{z} = \mathbb{E}_{NT} (d_{i0}(X_{it}) - \hat{d}_i(X_{it})) (l_{i0}(X_{it}) - \hat{l}_i(X_{it})) \quad (\text{D.7})$$

$$\bar{g} = \mathbb{E}_{NT} (d_{i0}(X_{it}) - \hat{d}_i(X_{it})) R_{it}(\hat{\mathbf{d}}, \hat{\mathbf{l}}) = \bar{z} - \bar{b}' \beta_0. \quad (\text{D.8})$$

LEMMA D.1 (First-Order Terms). *Under Assumptions 4.1–4.5, we have that*

$$\|\bar{a}\|_\infty \lesssim_P \left(\mathbf{d}_{NT, \infty} \sqrt{\log(dNT)/NT} \right) \quad (\text{D.9})$$

$$\|\bar{m}\|_\infty \lesssim_P \left(\mathbf{l}_{NT, \infty} \sqrt{\log(dNT)/NT} \right) \quad (\text{D.10})$$

$$\|\bar{f}\|_\infty \lesssim_P \left(\mathbf{d}_{NT, \infty} \sqrt{\log(dNT)/NT} \right) \quad (\text{D.11})$$

$$\|\bar{e}\|_\infty \lesssim_P \left(\sqrt{\log(dNT)/NT} (\mathbf{d}_{NT, \infty} \|\beta_0\|_1 + \mathbf{l}_{NT, \infty}) \right). \quad (\text{D.12})$$

PROOF OF LEMMA D.1. Define

$$\zeta_{NT}^V := \mathbf{d}_{NT, \infty} \sqrt{\log(dNT)/NT}, \quad \zeta_{NT}^B = 0,$$

and the A -function as

$$A(W_{it}, \eta) = V_{it}(d_{i0}(X_{it}) - d_i(X_{it})).$$

Define $B_{Ak}(\eta)$ and $V_{Ak}(\eta)$ with $\eta = \mathbf{d}$ as in (A.8)–(A.9).

Consider any $\eta = \eta_{NT} \in D_{NT}$ in what follows. Since V_{it} obeys the martingale difference property by assumption, we have that

$$\mathbb{E}[V_{it} \mid \cup_{t' \leq t, t' \in \mathcal{M}_k} (V_{it'}, X_{it'})] = 0, \quad (\text{D.13})$$

and it follows that $\|B_{Ak}(\eta_{NT})\|_\infty = 0$. By Assumption 4.3 and Lemma B.1, each entry of $V_{it}(d_{i0}(X_{it}) - d_i(X_{it}))$ is $\bar{\sigma}^2 \mathbf{d}_{NT, \infty}^2$ -sub-Gaussian. Invoking Lemma B.2 gives

$$\|V_{Ak}(\eta_{NT})\|_\infty \lesssim_P (\bar{\sigma} \mathbf{d}_{NT, \infty} \sqrt{\log d / NT_k}) = o_P(\zeta_{NT}^V)$$

since $T_k \asymp T$ (as we keep number of blocks K fixed). By Assumption 4.5, we have that $P(\hat{\mathbf{d}}_k \in D_{NT}, \forall k = 1, \dots, K) \rightarrow 1$. We conclude by Lemma A.6 that (D.9) holds. Repeating the same argument for

$$A(W_{it}, \eta) = V_{it}(l_{i0}(X_{it}) - l_i(X_{it})) \text{ and } A(W_{it}, \eta) = U_{it}(d_{i0}(X_{it}) - d_i(X_{it}))$$

establishes claims (D.10) and (D.11). Finally, (D.12) holds by definition of $\bar{e} = \bar{m} - \bar{a}'\beta_0$ and Holder inequalities. \blacksquare

LEMMA D.2 (Second-Order Term). *Under Assumptions 4.1–4.5, we have that*

$$\|\bar{z}\|_\infty \lesssim_P \left(\mathbf{d}_{NT} \mathbf{l}_{NT} + \mathbf{d}_{NT, \infty} \mathbf{l}_{NT, \infty} \sqrt{(NT)^{-1} \log(NT) \log d} \right) \quad (\text{D.14})$$

$$\|\bar{b}\|_\infty \lesssim_P \left(\mathbf{d}_{NT}^2 + \mathbf{d}_{NT, \infty}^2 \sqrt{(NT)^{-1} \log(NT) \log d} \right) \quad (\text{D.15})$$

$$\|\bar{g}\|_\infty \lesssim_P \left(\|\beta_0\|_1 \mathbf{d}_{NT}^2 + \mathbf{d}_{NT} \mathbf{l}_{NT} + (\|\beta_0\|_1 \mathbf{d}_{NT, \infty}^2 + \mathbf{d}_{NT, \infty} \mathbf{l}_{NT, \infty}) \sqrt{(NT)^{-1} \log(NT) \log d} \right) \quad (\text{D.16})$$

PROOF OF LEMMA D.2. Define the A -function as

$$A(W_{it}, \eta) = (d_{i0}(X_{it}) - d_i(X_{it}))(l_{i0}(X_{it}) - l_i(X_{it})), \quad \eta = (\mathbf{d}, \mathbf{l}).$$

Let $B_{Ak}(\eta)$ and $V_{Ak}(\eta)$ be defined according to (A.8)–(A.9). Let

$$\zeta_{NT}^B = \mathbf{l}_{NT} \mathbf{d}_{NT}, \quad \zeta_{NT}^V = \sqrt{\mathbf{l}_{NT, \infty}^2 \mathbf{d}_{NT, \infty}^2 \log d \log NT / NT}.$$

For any i and t , m and j , the Cauchy-Schwarz inequality gives

$$\begin{aligned} & \mathbb{E}[|(d_{i0}(X_{it}) - d_i(X_{it}))_m (l_{i0}(X_{it}) - l_i(X_{it}))|] \\ & \leq \sqrt{\mathbb{E}(d_{i0}(X_{it}) - d_i(X_{it}))_m^2 \mathbb{E}(l_{i0}(X_{it}) - l_i(X_{it}))^2} =: \sqrt{a_{it}^2 b_{it}^2} = |a_{it}| |b_{it}| \end{aligned}$$

Another application of the Cauchy-Schwarz gives

$$\begin{aligned} (T_k N)^{-1} \sum_i \sum_{t \in \mathcal{M}_k} |a_{it}| |b_{it}| &\leq \sqrt{(T_k N)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} a_{it}^2} \sqrt{(T_k N)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} b_{it}^2} \\ &\leq \sqrt{(T_k N)^{-1} \sum_{i=1}^N \sum_{t=1}^T a_{it}^2} \sqrt{(T_k N)^{-1} \sum_{i=1}^N \sum_{t=1}^T b_{it}^2} \leq \mathbf{d}_{NT} \mathbf{l}_{NT} T / T_k. \end{aligned}$$

Therefore $\|B_{Ak}(\eta_{NT})\|_\infty = O(\zeta_{NT}^B)$. Furthermore, each entry of $A(W_{it}, \eta)$ is bounded by $\mathbf{d}_{NT, \infty} \mathbf{l}_{NT, \infty}$, and, therefore, is $\mathbf{d}_{NT, \infty}^2 \mathbf{l}_{NT, \infty}^2$ -sub-Gaussian. By Lemma B.4,

$$\|V_{Ak}(\eta_{NT})\|_\infty \lesssim_P (\zeta_{NT}^V),$$

since $T_k \asymp T$. Furthermore, by Assumption 4.5 $P((\widehat{\mathbf{d}}_k, \widehat{\mathbf{l}}_k) \in D_{NT} \times L_{NT}, \forall k = 1, \dots, K) \rightarrow 1$. We conclude by Lemma A.6 that (D.14) holds. The bound (D.15) follows from the same argument. Finally, the bound (4.17) follows from the definition $\bar{g} = \bar{z} - \bar{b}' \beta_0$ and Holder inequality and union bounds. We obtain

$$\begin{aligned} \|\bar{g}\|_\infty &\lesssim_P \left(\|\beta_0\|_1 \left(\mathbf{d}_{NT}^2 + \mathbf{d}_{NT, \infty}^2 \sqrt{(NT)^{-1} \log(NT) \log d} \right) \right. \\ &\quad \left. + \mathbf{d}_{NT} \mathbf{l}_{NT} + \mathbf{d}_{NT, \infty} \mathbf{l}_{NT, \infty} \sqrt{(NT)^{-1} \log(NT) \log d} \right) \end{aligned}$$

Then we rewrite the bound as in (4.17). ■

Define

$$\widehat{Q} = \mathbb{E}_{NT} \widehat{V}_{it} \widehat{V}'_{it}, \quad \widetilde{Q} = \mathbb{E}_{NT} V_{it} V'_{it}, \quad \widehat{S} := \mathbb{E}_{NT} \widehat{V}_{it} (\widehat{Y}_{it} - \widehat{V}'_{it} \beta_0), \quad S := \mathbb{E}_{NT} V_{it} U_{it}$$

and the following rates

$$\kappa_{NT} := \sqrt{\log^3(d^2 \log(NT)) \log(NT) / NT} \quad (\text{D.17})$$

$$q_{NT} := \mathbf{d}_{NT, \infty} \sqrt{\log(dNT) / NT} + \mathbf{d}_{NT}^2 + \mathbf{d}_{NT, \infty}^2 \sqrt{\log(NT) \log(d) / NT} \quad (\text{D.18})$$

We will also use the following rates defined in the Section 4 of main text

$$\rho_{NT} := \mathbf{d}_{NT, \infty} \sqrt{\log(dNT) / NT} + \sqrt{\log(dNT) / NT} (\mathbf{d}_{NT, \infty} \|\beta_0\|_1 + \mathbf{l}_{NT, \infty}) + r_{NT}$$

$$r_{NT} := \|\beta_0\|_1 \mathbf{d}_{NT}^2 + \mathbf{d}_{NT} \mathbf{l}_{NT} + (\|\beta_0\|_1 \mathbf{d}_{NT, \infty}^2 + \mathbf{l}_{NT, \infty}) \sqrt{(NT)^{-1} \log(NT) \log d}$$

LEMMA D.3 (Summary of Gram Matrix and Gradient Error Bounds). *Suppose Assumptions 4.1–4.5 hold. Then, the following bounds hold up to $1 - o(1)$*

$$\|\widetilde{Q} - Q\|_\infty \lesssim_P o(\kappa_{NT} \log(d^2 NT)). \quad (\text{D.19})$$

$$\|\widetilde{Q} - \widehat{Q}\|_\infty \lesssim_P (q_{NT}) = o_P((NT)^{-1/2}) \quad (\text{D.20})$$

$$\|\widehat{Q} - Q\|_\infty \lesssim_P o(\kappa_{NT} \log(d^2 NT)) \quad (\text{D.21})$$

$$\|\widehat{S} - S\|_\infty \lesssim_P (\rho_{NT}) = o_P((NT)^{-1/2}), \quad (\text{D.22})$$

PROOF OF LEMMA D.3. Decompose matrix first-stage estimation error gives

$$\begin{aligned} \widehat{Q} &= \mathbb{E}_{NT} \widehat{V}_{it} \widehat{V}'_{it} \\ &= \mathbb{E}_{NT} (V_{it} + (d_{i0}(X_{it}) - \widehat{d}_i(X_{it}))) (V_{it} + (d_{i0}(X_{it}) - \widehat{d}_i(X_{it})))' \\ &= \mathbb{E}_{NT} V_{it} V'_{it} \\ &\quad + \mathbb{E}_{NT} V_{it} (d_{i0}(X_{it}) - \widehat{d}_i(X_{it}))' + (\mathbb{E}_{NT} V_{it} (d_{i0}(X_{it}) - \widehat{d}_i(X_{it})))' \\ &\quad + \mathbb{E}_{NT} (d_{i0}(X_{it}) - \widehat{d}_i(X_{it})) (d_{i0}(X_{it}) - \widehat{d}_i(X_{it}))' = \widetilde{Q} + \bar{a} + \bar{a}' + \bar{b}. \end{aligned}$$

Then, an application of Lemma B.6 with $\bar{N} = 2$ gives wp $1 - o(1)$ $\|\widetilde{Q} - Q\|_\infty \leq \bar{C}_\kappa \kappa_{NT}$ for large enough \bar{C}_κ . The bounds on $\|\bar{a}\|_\infty$ and $\|\bar{b}\|_\infty$ are given in (D.9) and (D.15), respectively. Collecting terms gives the bound (D.20). The (D.21) follows from the triangle inequality and $q_{NT} = o_P(\kappa_{NT})$. We can decompose the gradient error $\widehat{S} - S$ as follows. Note that

$$\begin{aligned} \widehat{Y}_{it} - \widetilde{Y}_{it} &= Y_{it} - \widehat{l}_i(X_{it}) - (Y_{it} - l_{i0}(X_{it})) = l_{i0}(X_{it}) - \widehat{l}_i(X_{it}) \\ \widehat{V}_{it} - V_{it} &= D_{it} - \widehat{d}_i(X_{it}) - (D_{it} - d_{i0}(X_{it})) = d_{i0}(X_{it}) - \widehat{d}_i(X_{it}) \end{aligned}$$

The difference of the two equations is

$$\widehat{Y}_{it} - \widetilde{Y}_{it} - (\widehat{V}_{it} - V_{it})' \beta_0 = R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{l}}).$$

Therefore,

$$\widehat{Y}_{it} - \widehat{V}'_{it} \beta_0 = (\widetilde{Y}_{it} - V'_{it} \beta_0) + ((\widehat{Y}_{it} - \widetilde{Y}_{it}) - (\widehat{V}_{it} - V_{it})' \beta_0) = U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{l}}). \quad (\text{D.23})$$

Decompose the gradient:

$$\widehat{S} = \mathbb{E}_{NT} \widehat{V}_{it} (\widehat{Y}_{it} - \widehat{V}'_{it} \beta_0) = \mathbb{E}_{NT} V_{it} (\widetilde{Y}_{it} - \widehat{V}'_{it} \beta_0) + \mathbb{E}_{NT} (\widehat{V}_{it} - V_{it}) (\widehat{Y}_{it} - \widehat{V}'_{it} \beta_0) = \widehat{S}_1 + \widehat{S}_2,$$

where

$$\widehat{S}_1 = \mathbb{E}_{NT} V_{it} U_{it} + \mathbb{E}_{NT} V_{it} R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{l}}) = S + \bar{e},$$

$$\widehat{S}_2 = \mathbb{E}_{NT} (d_{i0}(X_{it} - \widehat{d}_i(X_{it})) U_{it} + \mathbb{E}_{NT} (d_{i0}(X_{it}) - \widehat{d}_i(X_{it})) R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{l}}) = \bar{f} + \bar{g}.$$

Invoking bounds on \bar{e} , \bar{f} , and \bar{g} in (D.12)–(D.16) gives the result. ■

D.2 Proof of Orthogonal Lasso Rate: Theorem 4.1

D.2.0.1 *Group Sparsity Notation.* We use the same notation as [Lounici et al. \(2011\)](#). Consider a generic covariate vector of size $g \cdot d$, where d is the number of groups and g is the group size. Partition the set of indices $\{1, 2, \dots, gd\}$ into d groups of size g :

$$J_j := \{j, d + j, \dots, (g-1)d + j\}, \quad j = 1, 2, \dots, d, \quad |J_j| = g.$$

For a group index j and a subset of group indices \mathcal{T} , and vector $\Delta \in \mathbb{R}^{gd}$, denote

$$\Delta^j = (\Delta_m)_{m \in J_j} \in \mathbb{R}^g, \quad \Delta^{\mathcal{T}} = (\Delta_m)_{\{m \in J_j, j \in \mathcal{T}\}} \in \mathbb{R}^{|\mathcal{T}| \cdot g}.$$

For any $\Delta \in \mathbb{R}^{gd}$, define the group-vector norms

$$\|\Delta\|_{2,\infty} = \max_{1 \leq j \leq d} \|\Delta^j\|_2, \quad \|\Delta\|_{2,1} = \sum_{j=1}^d \|\Delta^j\|_2.$$

For a symmetric matrix M , define

$$\|M\|_{2,\infty} = \|M'\|_{2,\infty} = \max_{1 \leq i \leq dg} \max_{1 \leq j \leq d} \left(\sum_{k \in J_j} M_{i,k}^2 \right)^{1/2}.$$

Define the group restricted cone as

$$\text{REG}(\bar{c}) := \left\{ \Delta \in \mathbb{R}^{gd} : \sum_{j \in \mathcal{T}^c} \|\Delta^j\|_2 \leq \bar{c} \sum_{j \in \mathcal{T}} \|\Delta^j\|_2, \quad \Delta \neq 0 \right\}.$$

Given a matrix $M \in \mathbb{R}^{gd} \times \mathbb{R}^{gd}$, define the restricted group-sparse eigenvalue

$$\kappa_g(M, \mathcal{T}, \bar{c}) = \min_{\Delta \in \text{REG}(\bar{c})} \frac{\sqrt{s}(\Delta' M \Delta)^{1/2}}{\|\Delta^{\mathcal{T}}\|_{2,1}}.$$

When the group size g is equal to 1, the objects above reduce to the following quantities:

$$\Delta^j = \Delta_j, \Delta^{\mathcal{T}} = \Delta_{\mathcal{T}} = (\Delta_m)_{\{m \in \mathcal{T}\}}, \quad \|M\|_{2,\infty} = \|M\|_{\infty},$$

the group restricted cone is regular restricted cone

$$\text{REG}(\bar{c}) = \text{RE}(\bar{c}) = \{\Delta \in \mathbb{R}^d : \|\Delta_{\mathcal{T}^c}\|_1 \leq \bar{c} \|\Delta_{\mathcal{T}}\|_1, \quad \Delta \neq 0\},$$

and the the restricted group-sparse eigenvalue reduces to restricted eigenvalue

$$\kappa_1(M, \mathcal{T}, \bar{c}) = \kappa(M, \mathcal{T}, \bar{c}) = \min_{\Delta \in \text{RE}(\bar{c})} \frac{\sqrt{s}(\Delta' M \Delta)^{1/2}}{\|\Delta_{\mathcal{T}}\|_1}.$$

Let $\bar{X}_{it} \in \mathbb{R}^{gd}$ be a generic covariate (dg)-vector and \bar{Y}_{it} be a generic outcome. Given a parameter $\bar{\beta}_0$, decompose

$$\bar{Y}_{it} = \bar{X}'_{it} \bar{\beta}_0 + U_{it}.$$

The least squares loss function is

$$\mathcal{Q}(\bar{\beta}) := 1/2(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T (\bar{Y}_{it} - \bar{X}'_{it}\bar{\beta})^2.$$

The group lasso estimator is

$$\hat{\bar{\beta}} := \arg \min_{\bar{\beta}} \mathcal{Q}(\bar{\beta}) + \lambda \|\bar{\beta}\|_{2,1}. \quad (\text{D.24})$$

The least squares gradient is

$$\mathcal{S}(\bar{\beta}_0) := \nabla_{\bar{\beta}_0} \mathcal{Q}(\bar{\beta}_0) = (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T (\bar{Y}_{it} - \bar{X}'_{it}\bar{\beta}_0) \bar{X}_{it}.$$

and the Hessian is

$$\mathcal{H}(\bar{\beta}_0) := (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \bar{X}_{it} \bar{X}'_{it}.$$

LEMMA D.4 (Grouped Norm Inequalities). *For any two vectors $a, b \in \mathbb{R}^{gd}$ and matrix $M \in \mathbb{R}^{gd} \times \mathbb{R}^{gd}$, the following inequalities hold*

$$|a'b| \leq \|a\|_{2,1} \|b\|_{2,\infty} \quad (\text{D.25})$$

$$|v'Mv| \leq \sqrt{g} \|v\|_{2,1}^2 \cdot \|M\|_{2,\infty} \quad (\text{D.26})$$

$$\|M\|_{2,\infty} \leq \|M\|_{\infty} \sqrt{g} \quad (\text{D.27})$$

PROOF. For each group $j = 1, 2, \dots, d$, Cauchy inequality gives

$$\left| \sum_{k \in J_j} a_k b_k \right| \leq \left(\sum_{k \in J_j} a_k^2 \right)^{1/2} \left(\sum_{k \in J_j} b_k^2 \right)^{1/2} \leq \max_{1 \leq j \leq d} \left(\sum_{k \in J_j} a_k^2 \right)^{1/2} \|b^j\|_2 = \left(\max_{1 \leq j \leq d} \|a^j\|_2 \right) \|b^j\|_2,$$

which implies

$$|a'b| \leq \sum_{j=1}^d \left| \sum_{k \in J_j} a_k b_k \right| \leq \left(\max_{1 \leq j \leq d} \|a^j\|_2 \right) \sum_{j=1}^d \|b^j\|_2 = \|a\|_{2,\infty} \|b\|_{2,1}$$

For each index $i, 1 \leq i \leq kg$, the following bound holds:

$$\begin{aligned} \left| \sum_{k=1}^{gd} M_{i,k} v_k \right| &\leq \sum_{j=1}^d \left| \sum_{k \in J_j} M_{i,k} v_k \right| \leq \sum_{j=1}^d \left(\sum_{k \in J_j} M_{i,k}^2 \right)^{1/2} \left(\sum_{k \in J_j} v_k^2 \right)^{1/2} \\ &\leq \max_{1 \leq j \leq d} \left(\sum_{k \in J_j} M_{i,k}^2 \right)^{1/2} \sum_{j=1}^d \left(\sum_{k \in J_j} v_k^2 \right)^{1/2} \leq \|M\|_{2,\infty} \|v\|_{2,1}. \end{aligned}$$

Then,

$$\begin{aligned} \|Mv\|_{2,\infty} &= \max_{1 \leq j \leq d} \|(Mv)^j\|_2 = \max_{1 \leq j \leq d} \left(\sum_{i \in J_j} |Mv|_i^2 \right)^{1/2} \\ &\leq \max_{1 \leq j \leq d} \left(\sum_{i \in J_j} \|v\|_{2,1}^2 \|M\|_{2,\infty}^2 \right)^{1/2} \leq \sqrt{g} \|v\|_{2,1} \|M\|_{2,\infty}. \end{aligned}$$

Therefore, we obtain (D.26) by combining inequalities above:

$$|v' M v| \leq \|v\|_{2,1} \|Mv\|_{2,\infty} \leq \sqrt{g} \|M\|_{2,\infty} \cdot \|v\|_{2,1}^2.$$

Finally, the bound (D.27) follows from

$$M_{2,\infty} = \max_{1 \leq j \leq d} \|M^j\|_2 \leq \max_{1 \leq j \leq d} \sqrt{g} \|M^j\|_\infty$$

using the fact that $\|v\|_2 \leq \sqrt{\dim(v)} \|v\|_\infty$. ■

LEMMA D.5 (First-Stage Effect on the Curvature). *Let $\mathbf{M}_1, \mathbf{M}_2 \in \mathbb{R}^{g^d \times g^d}$ be two matrices. Let $\lambda_{NT}^M := \|\mathbf{M}_1 - \mathbf{M}_2\|_\infty$. On the event $\kappa_g^2(\mathbf{M}_2, \mathcal{T}, \bar{c}) > 0$, for any $\Delta \in \text{REG}(\bar{c})$,*

$$|\kappa_g^2(\mathbf{M}_2, \mathcal{T}, \bar{c}) - \kappa_g^2(\mathbf{M}_1, \mathcal{T}, \bar{c})| \leq \lambda_{NT}^M (1 + \bar{c})^2 s g. \quad (\text{D.28})$$

PROOF OF LEMMA D.5. For any $\Delta \in \mathbb{R}^{g^d}$, the difference can be bounded as

$$|\Delta'(\mathbf{M}_1 - \mathbf{M}_2)\Delta| \leq^i \sqrt{g} \|\mathbf{M}_1 - \mathbf{M}_2\|_{2,\infty} \|\Delta\|_{2,1}^2 \leq^{ii} g \lambda_{NT}^M \|\Delta\|_{2,1}^2, \quad (\text{D.29})$$

where i follows from (D.26) and ii from (D.27). For any $\Delta \in \text{REG}(\bar{c})$,

$$\|\Delta\|_{2,1}^2 \leq (1 + \bar{c})^2 \|\Delta \mathcal{T}\|_{2,1}^2 \leq \frac{(1 + \bar{c})^2 s}{\kappa_g^2(\mathbf{M}_2, \mathcal{T}, \bar{c})} \Delta' \mathbf{M}_2 \Delta =: \gamma \cdot \Delta' \mathbf{M}_2 \Delta. \quad (\text{D.30})$$

Combining (D.29) and (D.30) gives

$$|\Delta'(\mathbf{M}_1 - \mathbf{M}_2)\Delta| \leq (g \lambda_{NT}^M \gamma) \cdot \Delta' \mathbf{M}_2 \Delta. \quad (\text{D.31})$$

Noting that $x \leq |x|$ gives

$$\Delta'(\mathbf{M}_1 - \mathbf{M}_2)\Delta \leq |\Delta'(\mathbf{M}_1 - \mathbf{M}_2)\Delta| \leq (g \lambda_{NT}^M \gamma) \cdot \Delta' \mathbf{M}_2 \Delta,$$

which implies

$$\Delta' \mathbf{M}_1 \Delta \leq \Delta' \mathbf{M}_2 \Delta (1 + g \lambda_{NT}^M \gamma). \quad (\text{D.32})$$

Noting that $-x \leq |x|$ gives

$$\Delta'(\mathbf{M}_2 - \mathbf{M}_1)\Delta \leq (g \lambda_{NT}^M \gamma) \cdot (\Delta' \mathbf{M}_2 \Delta) \quad (\text{D.33})$$

which implies

$$\Delta' \mathbf{M}_1 \Delta \geq \Delta' \mathbf{M}_2 \Delta \cdot (1 - g\lambda_{NT}^M \gamma). \quad (\text{D.34})$$

Rearranging (D.32) gives an upper bound on $\kappa_g(\mathbf{M}_1, \mathcal{T}, \bar{c})$:

$$\begin{aligned} \kappa_g(\mathbf{M}_1, \mathcal{T}, \bar{c}) &:= \min_{\Delta \in \text{REG}(\bar{c})} \frac{\sqrt{s}(\Delta' \mathbf{M}_1 \Delta)^{1/2}}{\|\Delta^{\mathcal{T}}\|_{2,1}} \\ &\leq \min_{\Delta \in \text{REG}(\bar{c})} \frac{\sqrt{s}(\Delta' \mathbf{M}_2 \Delta)^{1/2}}{\|\Delta^{\mathcal{T}}\|_{2,1}} \sqrt{1 + g\lambda_{NT}^M \gamma} \\ &= \kappa_g(\mathbf{M}_2, \mathcal{T}, \bar{c}) \sqrt{1 + g\lambda_{NT}^M \gamma}. \end{aligned}$$

A lower bound on $\kappa_g(\mathbf{M}_1, \mathcal{T}, \bar{c})$ follows analogously, that is,

$$\begin{aligned} \kappa_g(\mathbf{M}_1, \mathcal{T}, \bar{c}) &\geq \min_{\Delta \in \text{REG}(\bar{c})} \frac{\sqrt{s}(\Delta' \mathbf{M}_2 \Delta)^{1/2}}{\|\Delta^{\mathcal{T}}\|_{2,1}} \sqrt{1 - g\lambda_{NT}^M \gamma} \\ &= \kappa_g(\mathbf{M}_2, \mathcal{T}, \bar{c}) \sqrt{1 - g\lambda_{NT}^M \gamma}. \end{aligned}$$

Taking the squares of both sides of the inequality and rearranging gives (D.28). ■

LEMMA D.6 (Oracle Inequality for Group Lasso). *On the event $\mathcal{G}_1 := \{\lambda \geq c\sqrt{g}\|\mathcal{S}(\bar{\beta}_0)\|_\infty\}$, the error vector $\Delta = \widehat{\beta} - \bar{\beta}_0$ belongs to the restricted set:*

$$\Delta \in \text{REG}(\bar{c})$$

and obeys the bound

$$(\Delta' \mathcal{H}(\bar{\beta}_0) \Delta) \leq 2\lambda \bar{c} \|\Delta^{\mathcal{T}}\|_{2,1}, \quad (\text{D.35})$$

where $\bar{c} := (c+1)/(c-1)$.

PROOF OF LEMMA D.6. Assume the event \mathcal{G}_1 holds throughout, which implies $\lambda \geq c\|\mathcal{S}(\bar{\beta}_0)\|_{2,\infty}$. Negahban et al. (2012) establishes

$$\|\bar{\beta}_0\|_{2,1} - \|\widehat{\beta}\|_{2,1} \leq \|\Delta^{\mathcal{T}}\|_{2,1} - \|\Delta^{\mathcal{T}^c}\|_{2,1}, \quad (\text{D.36})$$

and shows that $\Delta \in \text{REG}(\bar{c})$, which implies

$$\|\Delta\|_{2,1} \leq (1 + \bar{c}) \|\Delta^{\mathcal{T}}\|_{2,1}. \quad (\text{D.37})$$

Note that $\widehat{\beta}$ solves group lasso minimization problem (D.24), so that

$$\mathcal{Q}(\widehat{\beta}) + \lambda \|\widehat{\beta}\|_{2,1} \leq \mathcal{Q}(\bar{\beta}_0) + \lambda \|\bar{\beta}_0\|_{2,1}.$$

Expanding the least squares criterion gives

$$\mathcal{Q}(\widehat{\beta}) - \mathcal{Q}(\bar{\beta}_0) = \mathcal{S}(\bar{\beta}_0)' \Delta + 1/2(\Delta' \mathcal{H}(\bar{\beta}_0) \Delta) \leq \lambda(\|\bar{\beta}_0\|_{2,1} - \|\widehat{\beta}\|_{2,1})$$

Invoking inequality (D.25) for $\mathcal{S}(\bar{\beta}_0)' \Delta$ gives

$$1/2(\Delta' \mathcal{H}(\bar{\beta}_0) \Delta) \leq \lambda(\|\bar{\beta}_0\|_{2,1} - \|\widehat{\beta}\|_{2,1}) + \|\mathcal{S}(\bar{\beta}_0)\|_{2,\infty} \|\Delta\|_{2,1}.$$

Then

$$\begin{aligned} 1/2(\Delta' \mathcal{H}(\bar{\beta}_0) \Delta) &\leq^i \lambda(\|\Delta^{\mathcal{T}}\|_{2,1} - \|\Delta^{\mathcal{T}^c}\|_{2,1}) + \lambda/c \|\Delta\|_{2,1} \\ &\leq \lambda \|\Delta^{\mathcal{T}}\|_{2,1} + 0 + \lambda/c \|\Delta\|_{2,1} \\ &\leq^{ii} \lambda \|\Delta^{\mathcal{T}}\|_{2,1} + (\lambda/c)(1 + \bar{c}) \|\Delta^{\mathcal{T}}\|_{2,1} \\ &=^{iii} \lambda \bar{c} \|\Delta^{\mathcal{T}}\|_{2,1}, \end{aligned} \tag{D.38}$$

where (i) follows from (D.36), (ii) from (D.37), and (iii) from

$$1 + c^{-1}(\bar{c} + 1) = (c + (c + 1)/(c - 1))/c = (c + 1)/(c - 1) = \bar{c}.$$

Since $\Delta \in \text{REG}(\bar{c})$, (D.35) follows. ■

PROOF OF THEOREM 4.1. We invoke Lemma D.6 with the group size $g = 1$, $\bar{\beta}_0 = \beta_0$ and $\bar{U}_{it} = U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{1}})$. The gradient $\mathcal{S}(\beta_0) = \widehat{S}$, the Hessian is $\mathcal{H}(\beta_0) = \widehat{Q}$ and the penalty $\lambda = \lambda_{\beta}$. Note that $\delta \in \text{RE}(\bar{c})$ has been established in the proof of Lemma D.6.

Step 1. Union bound implies

$$\begin{aligned} \mathbb{P}(\lambda_{\beta} \leq c\sqrt{g} \|\widehat{S}\|_{\infty}) &\leq \mathbb{P}(\lambda_{\beta}/2 \leq c\sqrt{g} \|S\|_{\infty}) + \mathbb{P}(\lambda_{\beta}/2 \leq c\sqrt{g} \|\widehat{S} - S\|_{\infty}) \\ &= P_S + P_{\widehat{S}-S} \leq o(1) + o(1), \end{aligned}$$

where $P_S \leq 2/d = o(1)$ is given in (B.6) and $P_{\widehat{S}-S} = o(1)$ since

$$\|\widehat{S} - S\|_{\infty} \lesssim_P (\rho_{NT}) = o_P(\sqrt{\log d/NT}).$$

Step 2. Let $\mathbf{M}_2 := Q = (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} V_{it} V_{it}'$ and $\mathbf{M}_1 := \widetilde{Q} = \mathbb{E}_{NT} V_{it} V_{it}'$. Observe that

$$\kappa^2(Q, \mathcal{T}, \bar{c}) = \min_{\delta \in \text{RE}(\bar{c})} \frac{s \delta' Q \delta}{\|\delta_{\mathcal{T}}\|_1^2} \geq \min_{\delta \in \text{RE}(\bar{c})} \frac{s \min \text{eig}(Q) \|\delta\|_2^2}{\|\delta_{\mathcal{T}}\|_1^2} \geq^i \min \text{eig}(Q), \tag{D.39}$$

where (i) follows from

$$s \|\delta\|_2^2 \geq s \|\delta_{\mathcal{T}}\|_2^2 \geq \|\delta_{\mathcal{T}}\|_1^2 \quad \forall \delta \in \mathbb{R}^d.$$

The bounds (D.28) and (D.19) imply

$$|\kappa^2(\widetilde{Q}, \mathcal{T}, \bar{c}) - \kappa^2(Q, \mathcal{T}, \bar{c})| \leq s \|\widetilde{Q} - Q\|_{\infty} (1 + \bar{c})^2 \lesssim_P (s \kappa_{NT}).$$

Therefore, the event $\mathcal{G}_2 := \{\kappa^2(\widetilde{Q}, \mathcal{T}, \bar{c}) > C_{\min}/2\}$ holds w.p. $1 - o(1)$.

Step 3. Invoke Lemma D.5 on the event \mathcal{G}_2 with $\mathbf{M}_2 := \tilde{Q}$ and $\mathbf{M}_1 := \hat{Q}$. (D.28) gives

$$|\kappa^2(\hat{Q}, \mathcal{T}, \bar{c}) - \kappa^2(\tilde{Q}, \mathcal{T}, \bar{c})| \leq s \|\hat{Q} - \tilde{Q}\|_\infty (1 + \bar{c})^2 \lesssim_P (sq_{NT}),$$

which implies

$$|\kappa^2(\hat{Q}, \mathcal{T}, \bar{c}) - \kappa^2(Q, \mathcal{T}, \bar{c})| \lesssim_P (s(q_{NT} + \kappa_{NT})).$$

Therefore, the event $\{\kappa^2(\hat{Q}, \mathcal{T}, \bar{c}) > C_{\min}/2\}$ holds w.p. $1 - o(1)$. Thus, the event

$$\mathcal{G}_3 := s \|\hat{Q} - \tilde{Q}\|_\infty (1 + \bar{c})^2 / \kappa^2(\hat{Q}, \mathcal{T}, \bar{c}) < 1/2$$

is well-defined and holds w.p. $1 - o(1)$.

Step 4. On the event $\mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3$, invoking (D.34) with $\mathbf{M}_2 = \tilde{Q}$ and $\mathbf{M}_1 = \hat{Q}$ gives

$$\delta' \hat{Q} \delta \geq (1/2) \cdot \delta' \tilde{Q} \delta$$

Combining inequality above with (D.35) gives

$$\delta' \tilde{Q} \delta \leq 2\delta' \hat{Q} \delta \leq 4\lambda_\beta \bar{c} \|\delta_{\mathcal{T}}\|_1 \leq \sqrt{s} \lambda_\beta \frac{4\bar{c}(\delta' \tilde{Q} \delta)^{1/2}}{\kappa(\tilde{Q}, \mathcal{T}, \bar{c})}.$$

Dividing LHS and RHS by $(\delta' \tilde{Q} \delta)^{1/2}$ gives

$$(\delta' \tilde{Q} \delta)^{1/2} \leq \sqrt{s} \lambda_\beta \frac{4\bar{c}}{\kappa(\tilde{Q}, \mathcal{T}, \bar{c})} \lesssim_P (\sqrt{s} \lambda_\beta)$$

and

$$\|\delta\|_1 \leq (1 + \bar{c}) \|\delta_{\mathcal{T}}\|_1 \leq (1 + \bar{c}) \frac{\sqrt{s}(\delta' \tilde{Q} \delta)^{1/2}}{\kappa(\tilde{Q}, \mathcal{T}, \bar{c})} \leq 4(1 + \bar{c}) \frac{s\lambda_\beta \bar{c}}{\kappa^2(\tilde{Q}, \mathcal{T}, \bar{c})}.$$

■

D.3 Proof of Theorem 4.2

In what follows, we use the notation $Q^{-1} = (\omega_{ij}^0)$ and $Q_{\cdot j}^{-1} := \omega_j^0$. Define the following quantities

$$s_j(\lambda) := \|1\{\|\omega_j^0\| \geq \lambda\}\|_1, \quad r_j(\lambda) := \|(\omega_j^0)1\{\|\omega_j^0\| \leq \lambda\}\|_1.$$

Remark 4.4. Assumption 4.6 implies the following bounds

$$\|Q^{-1}\|_{1,\infty} = \max_{1 \leq j \leq d} \|\omega_j^0\|_1 \leq A_Q \sum_{j=1}^p j^{-a_Q} \leq A_Q \int_1^\infty j^{-a_Q} dj \leq A_Q / (a_Q - 1). \quad (\text{D.40})$$

Furthermore, if $A_Q j^{-a_Q} \leq \lambda$, then $j \geq j_Q^* := (A_Q/\lambda)^{1/a_Q}$. This implies

$$s_j(\lambda) := \|1\{\|\omega_j^0\| \geq \lambda\}\|_1 \leq \|1\{A_Q j^{-a_Q} \geq \lambda\}\|_1 \leq \sum_{j=1}^{j_Q^*} 1 = j_Q^* = (A_Q/\lambda)^{1/a_Q}.$$

$$r_j(\lambda) \leq \int_{j_Q^*}^{\infty} A_Q j^{-a_Q} dj = A_Q \frac{(j_Q^*)^{1-a_Q}}{a_Q - 1} = A_Q \frac{(A_Q/\lambda)^{(1-a_Q)/a_Q}}{a_Q - 1} = \frac{A_Q^{1/a_Q}}{(a_Q - 1)} \lambda^{1-1/a_Q}.$$

PROOF OF LEMMA 4.2. **Step 0.** Suppose Assumptions 4.1–4.6 hold. We claim that the event

$$\mathcal{G}_Q := \left\{ \|\widehat{Q} - Q\|_{\infty} \|Q^{-1}\|_{1,\infty} \leq \lambda_Q \right\}, \quad (\text{D.41})$$

holds w.p. $1 - o(1)$. On this event \mathcal{G}_Q , by definition of $\widehat{\Omega}$, we have

$$\|\widehat{\Omega}\|_{1,\infty} \leq \|Q^{-1}\|_{1,\infty}, \quad (\text{D.42})$$

and, therefore,

$$\|\widehat{\Omega}^{\text{CLIME}}\|_{1,\infty} \leq \|\widehat{\Omega}\|_{1,\infty} \leq \|Q^{-1}\|_{1,\infty}. \quad (\text{D.43})$$

To show that $P(\mathcal{G}_Q) = 1 - o(1)$, decompose

$$\widehat{Q}Q^{-1} - I_d = \widehat{Q}Q^{-1} - QQ^{-1} = (\widehat{Q} - Q)Q^{-1}.$$

By Lemma D.3 for some $\bar{C}_\kappa > 0$, w.p. $1 - o(1)$,

$$\|\widehat{Q} - Q\|_{\infty} \leq \bar{C}_\kappa \kappa_{NT}.$$

Therefore, w.p. $1 - o(1)$,

$$\|\widehat{Q}Q^{-1} - I_d\|_{\infty} \leq \|\widehat{Q} - Q\|_{\infty} \|Q^{-1}\|_{1,\infty} \leq \bar{C}_\kappa 2\kappa_{NT} \|Q^{-1}\|_{1,\infty} \leq \lambda_Q \quad (\text{D.44})$$

as long as $C_Q \geq 2\bar{C}_\kappa \|Q^{-1}\|_{1,\infty}$. Since $\|Q^{-1}\|_{1,\infty} \leq A_Q/(a_Q - 1)$, $C_Q \geq 2\bar{C}_\kappa \|Q^{-1}\|_{1,\infty}$ holds by Assumption 4.6.

Step 1. We establish (4.10). Specifically, we show that, on the event \mathcal{G}_Q , we have

$$\left\| \widehat{\Omega}^{\text{CLIME}} - Q^{-1} \right\|_{\infty} \leq \left\| \widehat{\Omega} - Q^{-1} \right\|_{\infty} \leq \frac{4A_Q}{(a_Q - 1)} \lambda_Q.$$

The argument repeats the proof of equation (13) in Cai et al. (2011), Theorem 6. On the event \mathcal{G}_Q , the bound holds

$$\begin{aligned} \|Q\widehat{\Omega} - I_d\|_{\infty} &= \|Q(\widehat{\Omega} - Q^{-1})\|_{\infty} \leq \|(Q - \widehat{Q})(\widehat{\Omega} - Q^{-1})\|_{\infty} + \|\widehat{Q}(\widehat{\Omega} - Q^{-1})\|_{\infty} \\ &\leq \|Q - \widehat{Q}\|_{\infty} \|\widehat{\Omega} - Q^{-1}\|_{1,\infty} + \|\widehat{Q}\widehat{\Omega} - I_d\|_{\infty} + \|I_d - \widehat{Q}Q^{-1}\|_{\infty} \\ &\leq \|Q - \widehat{Q}\|_{\infty} (\|Q^{-1}\|_{1,\infty} + \|\widehat{\Omega}\|_{1,\infty}) + \lambda_Q + \|\widehat{Q}Q^{-1} - I_d\|_{\infty}. \end{aligned}$$

Invoking (D.42) and (D.44) gives

$$\|Q\widehat{\Omega} - I_d\|_{\infty} \leq 2\|Q - \widehat{Q}\|_{\infty} \|Q^{-1}\|_{1,\infty} + \lambda_Q + \lambda_Q \leq 2\lambda_Q + 2\lambda_Q = 4\lambda_Q. \quad (\text{D.45})$$

Pre-multiplying $Q\widehat{\Omega} - I_d$ by Q^{-1} and invoking (D.40) gives

$$\|\widehat{\Omega} - Q^{-1}\|_{\infty} = \|Q^{-1}(Q\widehat{\Omega} - I_d)\|_{\infty} \leq \|Q^{-1}\|_{\infty,1} \|Q\widehat{\Omega} - I_d\|_{\infty} \leq 4 \frac{A_Q}{a_Q - 1} \lambda_Q.$$

Since Q is a symmetric matrix, so is Q^{-1} , and

$$|\widehat{\Omega}_{mj}^{\text{CLIME}} - Q_{mj}^{-1}| \leq \max(|\widehat{\Omega}_{mj} - Q_{mj}^{-1}|, |\widehat{\Omega}_{jm} - Q_{jm}^{-1}|) \leq \|\widehat{\Omega} - Q^{-1}\|_{\infty},$$

which implies (4.10).

Step 2. We show that (4.11) holds. Specifically, we show that on the event \mathcal{G}_Q we have that

$$\|\widehat{\Omega}^{\text{CLIME}} - Q^{-1}\|_{1,\infty} \leq \bar{C}_Q \lambda_Q^{1-1/a_Q},$$

for some constant \bar{C}_Q that depends on Q . We closely follow the proof of (14), page 605 in Cai et al. (2011). Using their notation, let

$$\begin{aligned} t_n &:= \|\widehat{\Omega}^{\text{CLIME}} - Q^{-1}\|_{\infty}, \quad \omega_j^0 := Q_{\cdot,j}^{-1} \\ h_j &:= \widehat{\Omega}_{\cdot,j}^{\text{CLIME}} - \omega_j^0, \quad h_j^1 := (\widehat{\omega}_{ij} 1\{\widehat{\omega}_{ij} \geq 2t_n\})_{i=1}^p - \omega_j^0, \quad h_j^2 := h_j - h_j^1. \end{aligned}$$

By definition of CLIME, on the event \mathcal{G}_G ,

$$\begin{aligned} \|\omega_j^0\|_1 - \|h_j^1\|_1 + \|h_j^2\|_1 &\leq \|h_j^1 + \omega_j^0\|_1 + \|h_j^2\|_1 \\ &= \|h_j^2 + h_j^1 + \omega_j^0\|_1 = \|\widehat{\Omega}_{\cdot,j}^{\text{CLIME}}\|_1 \leq \|\widehat{\Omega}_{\cdot,j}\|_1 \leq \|\omega_j^0\|_1, \end{aligned}$$

where (i) follows from $h_j^1 + \omega_j^0$ and h_j^2 having non-overlapping support. This implies

$$\|h_j - h_j^1\|_1 := \|h_j^2\|_1 \leq \|h_j^1\|_1, \quad \|h_j\|_1 \leq 2\|h_j^1\|_1.$$

Then, the following bound holds:

$$\begin{aligned} \|h_j^1\|_1 &= \sum_{i=1}^d |\widehat{\omega}_{ij} 1\{\widehat{\omega}_{ij} \geq 2t_n\} - \omega_{ij}^0| \\ &\leq \sum_{i=1}^d |\omega_{ij}^0| 1\{\omega_{ij}^0 \leq 2t_n\} + \sum_{i=1}^d |\widehat{\omega}_{ij} 1\{\widehat{\omega}_{ij} \geq 2t_n\} - \omega_{ij}^0 1\{\omega_{ij}^0 \geq 2t_n\}| \\ &\leq r_j(2t_n) + t_n \sum_{i=1}^d 1\{\widehat{\omega}_{ij} \geq 2t_n\} + \sum_{i=1}^d |\omega_{ij}^0| |(1\{\widehat{\omega}_{ij} \geq 2t_n\} - 1\{\omega_{ij}^0 \geq 2t_n\})| \\ &\leq r_j(2t_n) + t_n \sum_{i=1}^d 1\{|\omega_{ij}^0| \geq t_n\} + \sum_{i=1}^d |\omega_{ij}^0| I\{|\omega_{ij}^0| - 2t_n \leq |\widehat{\omega}_{ij} - \omega_{ij}^0|\} \\ &\leq r_j(2t_n) + t_n s_j(t_n) + \sum_{i=1}^d |\omega_{ij}^0| 1\{\omega_{ij}^0 \leq 3t_n\} \\ &\leq r_j(2t_n) + t_n s_j(t_n) + r_j(3t_n) \end{aligned}$$

$$\leq C'_Q t_n^{1-1/a_Q}. \quad (C'_Q := \frac{A_Q^{1/a_Q}}{(a_Q - 1)}(2^{1-1/a_Q} + (a_Q - 1) + 3^{1-1/a_Q})).$$

Since $t_n \leq \|\widehat{\Omega} - Q^{-1}\|_\infty$ from Step 1, we have

$$\begin{aligned} \|(\widehat{\Omega}^{\text{CLIME}} - Q^{-1})'\|_{1,\infty} &= \|\widehat{\Omega}^{\text{CLIME}} - Q^{-1}\|_{1,\infty} := \max_{1 \leq j \leq d} \|h_j\|_1 \leq C'_Q (\|\widehat{\Omega} - Q^{-1}\|_\infty)^{1-1/a_Q} \\ &\leq \bar{C}_Q \lambda_Q^{1-1/a_Q} \end{aligned}$$

where $\bar{C}_Q = C'_Q(4A_Q/(a_Q - 1))^{1-1/a_Q}$ is a constant that depends on Q . Thus, (4.11) follows.

Step 3. We show (4.12). Specifically, we show that on the event \mathcal{G}_Q and $\|\widehat{Q} - Q\| \leq 1$ and once $\lambda_Q \leq 1$, we have that

$$\|I_d - \widehat{\Omega}^{\text{CLIME}} \widehat{Q}\|_\infty = \|I_d - \widehat{Q} \widehat{\Omega}^{\text{CLIME}}\|_\infty \leq C'_Q \lambda_Q^{1-1/a_Q},$$

for some constant C'_Q that depends only on Q . Indeed,

$$\begin{aligned} \|I_d - \widehat{Q} \widehat{\Omega}^{\text{CLIME}}\|_\infty &\leq \|I_d - \widehat{Q} Q^{-1}\|_\infty + \|\widehat{Q}(Q^{-1} - \widehat{\Omega}^{\text{CLIME}})\|_\infty \\ &\leq \|I_d - \widehat{Q} Q^{-1}\|_\infty + (\|Q\|_\infty + 1) \|\widehat{\Omega}^{\text{CLIME}} - Q^{-1}\|_{1,\infty} \quad (\text{D.46}) \\ &\leq \lambda_Q + (\|Q\|_\infty + 1) \bar{C}_Q \lambda_Q^{1-1/a_Q} \leq C'_Q \lambda_Q^{1-1/a_Q} \end{aligned}$$

for example, taking C'_Q to bound:

$$(\lambda_Q^{1-1/a_Q} + (\|Q\|_\infty + 1)C_Q) \leq (1 + (\|Q\|_\infty + 1)C_Q) =: C'_Q$$

■

LEMMA D.7 (Linearization in Sup-Norm). *Suppose Assumptions 4.1–4.6 hold. Then, the debiased estimator $\widehat{\beta}_{DL}$ is asymptotically linear*

$$\sqrt{NT}(\widehat{\beta}_{DL} - \beta_0) = Q^{-1} \mathbb{G}_{NT} V_{it} U_{it} + R_{NT}, \quad (\text{D.47})$$

$$\|R_{NT}\|_\infty \lesssim_P \lambda_Q^{1-1/a_Q} \sqrt{s^2 \log d} + \sqrt{NT} \rho_{NT} = o_P(1). \quad (\text{D.48})$$

PROOF OF LEMMA D.7. Step 1. Recall that

$$R_{it}(\mathbf{d}, \mathbf{1}) := l_{i0}(X_{it}) - l_i(X_{it}) - (d_{i0}(X_{it}) - d_i(X_{it}))' \beta_0.$$

and invoking (D.23), which states that

$$\widehat{Y}_{it} - \widehat{V}'_{it} \beta_0 = (\widetilde{Y}_{it} - V'_{it} \beta_0) + ((\widehat{Y}_{it} - \widetilde{Y}_{it}) - (\widehat{V}_{it} - V_{it})' \beta_0) = U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{1}}).$$

we can see that

$$\begin{aligned}\widehat{Y}_{it} - \widehat{V}'_{it}\widehat{\beta}_L &= \widehat{Y}_{it} - \widehat{V}'_{it}\beta_0 + \widehat{V}'_{it}(\beta_0 - \widehat{\beta}_L) \\ \mathbb{E}_{NT}\widehat{V}_{it}(\widehat{Y}_{it} - \widehat{V}'_{it}\widehat{\beta}_L) &= \mathbb{E}_{NT}\widehat{V}_{it}(U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{1}})) + \widehat{Q}(\beta_0 - \widehat{\beta}_L).\end{aligned}$$

Since

$$\widehat{\beta}_{DL} - \beta_0 = \widehat{\beta}_L - \beta_0 + \widehat{\Omega}^{\text{CLIME}}(\mathbb{E}_{NT}\widehat{V}_{it}(\widehat{Y}_{it} - \widehat{V}'_{it}\widehat{\beta}_L))$$

we have that

$$\begin{aligned}\widehat{\beta}_{DL} - \beta_0 &= \widehat{\Omega}^{\text{CLIME}}(\mathbb{E}_{NT}\widehat{V}_{it}(U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{1}})) + \widehat{\Omega}^{\text{CLIME}}\widehat{Q}(\beta_0 - \widehat{\beta}_L) + \widehat{\beta}_L - \beta_0) \\ &= \widehat{\Omega}^{\text{CLIME}}(\mathbb{E}_{NT}\widehat{V}_{it}(U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{1}})) + \underbrace{(I_d - \widehat{\Omega}^{\text{CLIME}}\widehat{Q})}_{L_3}(\widehat{\beta}_L - \beta_0)) \\ &= Q^{-1}\mathbb{E}_{NT}V_{it}U_{it} + (\widehat{\Omega}^{\text{CLIME}} - Q^{-1})\mathbb{E}_{NT}V_{it}U_{it} \\ &\quad + \widehat{\Omega}^{\text{CLIME}}(\mathbb{E}_{NT}[\widehat{V}_{it}(U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{1}})) - V_{it}U_{it}]) + L_3 \\ &= Q^{-1}\mathbb{E}_{NT}V_{it}U_{it} + L_1 + L_2 + L_3,\end{aligned}$$

where

$$\begin{aligned}L_1 &= (\widehat{\Omega}^{\text{CLIME}} - Q^{-1})\mathbb{E}_{NT}V_{it}U_{it} \\ L_2 &= \widehat{\Omega}^{\text{CLIME}}\mathbb{E}_{NT}[V_{it}R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{1}}) + (\widehat{V}_{it} - V_{it})(U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{1}}))] \\ L_3 &= (I_d - \widehat{\Omega}^{\text{CLIME}}\widehat{Q})(\widehat{\beta}_L - \beta_0).\end{aligned}$$

Term L_1 . The bounds (4.11) and (B.6) imply

$$\begin{aligned}\|L_1\|_\infty &\leq \|\widehat{\Omega}^{\text{CLIME}} - Q^{-1}\|_{1,\infty}\sqrt{NT}\|\mathbb{E}_{NT}V_{it}U_{it}\|_\infty \\ &\lesssim_P \lambda_Q^{1-1/a_Q}\sqrt{NT}\sqrt{\log d/NT} = o_P(1),\end{aligned}\tag{D.49}$$

because $\lambda_Q^{1-1/a_Q} = o(s^{-1}\log^{-1/2}d) = o(\log^{-1/2}d)$ as assumed in (4.11).

Term L_2 . The bounds (D.43) and the gradient error bound (D.22) imply

$$\begin{aligned}\|L_2\|_\infty &\leq \|\widehat{\Omega}^{\text{CLIME}}\|_{\infty,1}\|\sqrt{NT}\mathbb{E}_{NT}[V_{it}R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{1}}) + (\widehat{V}_{it} - V_{it})(U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{1}}))]\|_\infty \\ &\lesssim_P 1\sqrt{NT}\rho_{NT} = o(1).\end{aligned}$$

because $\sqrt{NT}\rho_{NT} = o(1)$ is implied by our assumption Assumptions 4.1–4.5.

Term L_3 . The conditions (4.12) and (4.8) imply

$$\begin{aligned}\sqrt{NT}\|L_3\|_\infty &= \sqrt{NT}\|(I_d - \widehat{\Omega}^{\text{CLIME}}\widehat{Q})(\widehat{\beta}_L - \beta_0)\|_\infty \\ &\leq \sqrt{NT}\|I_d - \widehat{\Omega}^{\text{CLIME}}\widehat{Q}\|_\infty\|\widehat{\beta}_L - \beta_0\|_1 \\ &\lesssim_P (\lambda_Q^{1-1/a_Q}\sqrt{NT}\sqrt{s^2\log d/NT}) = o(1),\end{aligned}$$

where $\|I_d - \widehat{\Omega}^{\text{CLIME}}\widehat{Q}\|_\infty \lesssim_P \lambda_Q^{1-1/a_Q} = o(s^{-1}\log^{-1/2}d)$ as assumed in (4.11). ■

PROOF OF THEOREM 4.2. **Step 1.** Let $\alpha \in \mathbb{R}^d$ be such that $\|\alpha\|_1 = K_\alpha = O(1)$ and $\|\alpha\|_2 = 1$. Lemma D.7 implies

$$\alpha'(\alpha'\Sigma\alpha)^{-1/2}(\sqrt{NT}(\widehat{\beta}_{DL} - \beta_0)) = \alpha'(\alpha'\Sigma\alpha)^{-1/2}Q^{-1}\sqrt{NT}\mathbb{E}_{NT}V_{it}U_{it} + o_P(1),$$

where $(\alpha'\Sigma\alpha)^{-1/2} = O(1)$ because

$$\alpha'\Sigma\alpha \geq \sigma^2\alpha'Q^{-1}\alpha \geq \sigma^2C_{\max}^{-1} > 0 \quad (\text{D.50})$$

by the assumptions of the Theorem, so that

$$|\alpha'(\alpha'\Sigma\alpha)^{-1/2}R_{NT}| \leq O(1)K_\alpha\|R_{NT}\|_\infty = o_P(1). \quad (\text{D.51})$$

Consider a sequence

$$\xi_m(\alpha) := \alpha'Q^{-1}(\alpha'\Sigma\alpha)^{-1/2}V_mU_m, \quad m = 1, 2, \dots, M$$

with

$$m = m(i, t) = T(i-1) + t, \quad 1 \leq t \leq T, 1 \leq i \leq N.$$

As shown in Corollary B.3, $\{\xi_m(\alpha)\}_{m=1}^M$ is a martingale difference sequence w.r.t. natural filtration with $M = NT$. By Law of Large Numbers in Hansen (2019) and the assumed Lindeberg condition

$$\frac{1}{NT} \sum_{m=1}^{NT} \xi_m^2(\alpha) \rightarrow_p \frac{\alpha'Q^{-1}\Gamma Q^{-1}\alpha}{\alpha'\Sigma\alpha} = 1.$$

As discussed in McLeish (1974), the Lindeberg condition assumed in the Theorem 4.2 implies conditions (i) and (ii) in Theorem 2.3 of McLeish (1974), which implies the first part of the Theorem:

$$P(\alpha'(\alpha'\Sigma\alpha)^{-1/2}\sqrt{NT}(\widehat{\beta}_{DL} - \beta_0) \leq t) \rightarrow \Phi(t).$$

By Polya's theorem, the convergence is uniform in $t \in \mathbb{R}$. Since the result holds for any sequence $\{\alpha\}$ (indexed by N, T obeying conditions above, the convergence is uniform over such sequences.)

Step 2. Let K_α be a finite constant in the statement of the theorem. Thus,

$$\sup_{\alpha: \|\alpha\|_2 \leq 1, \|\alpha\|_1 \leq K_\alpha} |\alpha'(\widehat{\Sigma} - \Sigma)\alpha| \leq K_\alpha^2 \|\widehat{\Sigma} - \Sigma\|_\infty = o_P(1)$$

by assumption. Since $\min_{\|\alpha\|_2=1} \alpha'\Sigma\alpha \geq \sigma^2C_{\max}^{-1}$, by assumption, we conclude that, for N and T large enough, the event

$$\mathcal{G}_K := \left\{ \inf_{\|\alpha\|_2=1, \|\alpha\|_1 \leq K_\alpha} \alpha'\widehat{\Sigma}\alpha > \sigma^2C_{\max}^{-1}/2 \right\}$$

occurs wp $1 - o(1)$. Hence wp $1 - o(1)$.

$$\alpha_{NT} - 1 := \frac{(\alpha' \Sigma \alpha)^{1/2}}{(\alpha' \widehat{\Sigma} \alpha)^{1/2}} - 1 \quad (\text{D.52})$$

obeys

$$|\alpha_{NT} - 1| \leq (\sigma^2 C_{\max}^{-1}/2)^{-1} K_{\alpha}^2 \|\widehat{\Sigma} - \Sigma\|_{\infty} = o_P(1)$$

which follows from the inequality

$$\left| 1 - \frac{\sqrt{x}}{\sqrt{y}} \right| = \frac{|\sqrt{x} - \sqrt{y}|}{\sqrt{y}} = \frac{|x - y|}{\sqrt{y}(\sqrt{x} + \sqrt{y})}; \quad x > 0, y > 0$$

Then

$$|(\alpha_{NT} - 1) \alpha' (\alpha' \Sigma \alpha)^{-1/2} \sqrt{NT} (\widehat{\beta}_L - \beta_0)| \leq |\alpha_{NT} - 1| |\alpha' (\alpha' \Sigma \alpha)^{-1/2} \sqrt{NT} (\widehat{\beta}_L - \beta_0)| = o_P(1) O_P(1).$$

Therefore,

$$\alpha' (\alpha' \widehat{\Sigma} \alpha)^{-1/2} \sqrt{NT} (\widehat{\beta}_{DL} - \beta_0) = \alpha' (\alpha' \Sigma \alpha)^{-1/2} \sqrt{NT} (\widehat{\beta}_L - \beta_0) + o_P(1).$$

Then convergence in distribution for the left side follows by Slutsky's lemma and Step 1. \blacksquare

D.4 Estimation of Σ : Proof of Lemma 4.3

Define the following terms

$$\bar{b}_1 = \mathbb{E}_{NT} V_{it} V'_{it} U_{it}^2 - \Gamma \quad (\text{D.53})$$

$$\bar{b}_2 = \mathbb{E}_{NT} (d_{i0}(X_{it}) - \widehat{d}_i(X_{it})) V_{it} U_{it}^2 \quad (\text{D.54})$$

$$\bar{b}_3 = \mathbb{E}_{NT} (d_{i0}(X_{it}) - \widehat{d}_i(X_{it})) (d_{i0}(X_{it}) - \widehat{d}_i(X_{it}))' U_{it}^2 \quad (\text{D.55})$$

$$\bar{b}_4 = \mathbb{E}_{NT} V_{it} V'_{it} (\widehat{U}_{it}^2 - U_{it}^2) \quad (\text{D.56})$$

$$\bar{b}_5 = \mathbb{E}_{NT} (d_{i0}(X_{it}) - \widehat{d}_i(X_{it})) V_{it} (\widehat{U}_{it}^2 - U_{it}^2) \quad (\text{D.57})$$

$$\bar{b}_6 = \mathbb{E}_{NT} (d_{i0}(X_{it}) - \widehat{d}_i(X_{it})) (d_{i0}(X_{it}) - \widehat{d}_i(X_{it}))' (\widehat{U}_{it}^2 - U_{it}^2) \quad (\text{D.58})$$

The following Lemma establishes tail bound on \bar{b}_1 . Recall that κ_{NT} from Lemma D.3 is

$$\kappa_{NT} := \sqrt{\log^3(d^2 \log(NT)) \log NT / NT}.$$

LEMMA D.8 (Higher-Order Term \bar{b}_1). *Under Assumptions 4.1–4.5 and 4.7,*

$$\|\Gamma\|_{\infty} = \max_{1 \leq m, j \leq d} |\mathbb{E} V_{itj} V'_{itm} U_{it}^2| = O(1) \quad (\text{D.59})$$

$$\|\bar{b}_1\|_{\infty} \lesssim_P \sqrt{\log^5(d^2 \log(NT)) \log NT / NT} \leq \kappa_{NT} \log(d^2 NT) = o(1). \quad (\text{D.60})$$

Proof. The bound (D.59) and (D.60) follow from (B.19) and (B.20) with $Z_{1,nit} = Z_{2,nit} = V_{it}$, $\bar{N} = 1$ and $g = 1$. ■

LEMMA D.9 (Higher-Order Term \bar{b}_2). *Under Assumptions 4.1–4.5 and 4.7,*

$$\|\bar{b}_2\|_\infty \lesssim_P (NT)^{-1/4}. \quad (\text{D.61})$$

PROOF OF LEMMA D.9. **Step 1.** For \bar{b} as in (D.6) and q_{NT} as in (D.18),

$$P_1^2 := \max_{1 \leq j \leq d} \mathbb{E}_{NT} (d_{i0}(X_{it}) - \hat{d}_i(X_{it}))_j^2 \leq \|\bar{b}\|_\infty \lesssim_P q_{NT}.$$

Invoking the convergence requirement (4.14) gives

$$\left(1 + \sqrt{\log^7(d^2 \log(NT)) \log(NT)/NT}\right) \lesssim 1 + \kappa_{NT} \log^2(d^2 NT) \lesssim 1.$$

Invoking the bounds (B.19)–(B.20) with $Z_{1,nit} = 1$ and $Z_{2,nit} = V_{it}$ and $\bar{N} = 2$ and $g = 2$ gives

$$P_2^2 := \max_{1 \leq m \leq d} \mathbb{E}_{NT} V_{itm}^2 U_{it}^4 \lesssim_P \left(1 + \sqrt{\log^7(d^2 \log(NT)) \log(NT)/NT}\right) \lesssim_P 1.$$

Cauchy inequality implies

$$\begin{aligned} & \max_{1 \leq m, j \leq d} |\mathbb{E}_{NT} (d_{i0}(X_{it}) - \hat{d}_i(X_{it}))_j| |V_{itm} U_{it}^2| \\ & \leq \max_{1 \leq j \leq d} (\mathbb{E}_{NT} (d_{i0}(X_{it}) - \hat{d}_i(X_{it}))_j^2)^{1/2} \max_{1 \leq m \leq d} (\mathbb{E}_{NT} V_{itm}^2 U_{it}^4)^{1/2} \\ & \lesssim_P \sqrt{q_{NT} \cdot 1} = o_P((NT)^{-1/4}), \end{aligned}$$

where the last bound is established in (D.20). ■

LEMMA D.10 (Higher-Order Term \bar{b}_3). *Under Assumptions 4.1–4.5 and 4.7,*

$$\|\bar{b}_3\|_\infty \lesssim_P o((NT)^{-1/4}). \quad (\text{D.62})$$

PROOF OF LEMMA D.10. On the event $\sup_{it} |d_{i0}(X_{it}) - \hat{d}_i(X_{it})| \leq \mathbf{d}_{NT, \infty} \leq 1$, which happens with probability $1 - o(1)$,

$$\begin{aligned} & \max_{1 \leq m, j \leq d} \mathbb{E}_{NT} |(d_{i0}(X_{it}) - \hat{d}_i(X_{it}))_j| |(d_{i0}(X_{it}) - \hat{d}_i(X_{it}))_m| U_{it}^2 \\ & \leq \max_{1 \leq j \leq d} \mathbb{E}_{NT} |(d_{i0}(X_{it}) - \hat{d}_i(X_{it}))_j| U_{it}^2 \\ & \leq \max_{1 \leq j \leq d} (\mathbb{E}_{NT} (d_{i0}(X_{it}) - \hat{d}_i(X_{it}))_j^2)^{1/2} (\mathbb{E}_{NT} U_{it}^4)^{1/2} \\ & \leq \sqrt{P_1^2} (\mathbb{E}_{NT} U_{it}^4)^{1/2} \lesssim_P o((NT)^{-1/4}). \end{aligned}$$

■

Recall that the first-order estimation error is

$$R_{it}(\mathbf{d}, \mathbf{l}) := l_{i0}(X_{it}) - l_i(X_{it}) - (d_{i0}(X_{it}) - d_i(X_{it}))' \beta_0.$$

LEMMA D.11 (Squared Error). *Under Assumptions 4.1–4.5, we have that*

$$\mathbb{E}_{NT} R_{it}^2(\widehat{\mathbf{d}}, \widehat{\mathbf{l}}) \lesssim_P \mathbf{1}_{NT}^2 + o((NT)^{-1/2}). \quad (\text{D.63})$$

PROOF OF LEMMA D.11. **Step 1.** Consider a term \bar{z} in (D.7) in a special case when

$$d_{i0}(X_{it}) := l_{i0}(X_{it}) \cdot (1, 1), \quad d = 2.$$

Then, \bar{z} reduces to a 2-vector

$$\bar{z} := \mathbb{E}_{NT}(l_{i0}(X_{it}) - \widehat{l}_i(X_{it}))^2 \cdot (1, 1),$$

and

$$\|\bar{z}\|_\infty = \mathbb{E}_{NT}(l_{i0}(X_{it}) - \widehat{l}_i(X_{it}))^2.$$

Invoking (F47) with \mathbf{d}_{NT} and $\mathbf{d}_{NT, \infty}$ replaced by \mathbf{l}_{NT} and $\mathbf{l}_{NT, \infty}$ gives the bound.

Step 2. The following bound holds

$$\begin{aligned} \mathbb{E}_{NT} R_{it}^2(\widehat{\mathbf{d}}, \widehat{\mathbf{l}}) &\leq 2\mathbb{E}_{NT}((d_{i0}(X_{it}) - \widehat{d}_i(X_{it}))' \beta_0)^2 + 2\mathbb{E}_{NT}(l_{i0}(X_{it}) - \widehat{l}_i(X_{it}))^2 \\ &= 2\beta_0' \bar{b} \beta_0 + 2\mathbb{E}_{NT}(l_{i0}(X_{it}) - \widehat{l}_i(X_{it}))^2 \\ &\leq 2\|\bar{b}\|_\infty \|\beta_0\|_1^2 + 2\mathbb{E}_{NT}(l_{i0}(X_{it}) - \widehat{l}_i(X_{it}))^2 \\ &\lesssim_P \|\beta_0\|_1^2 \left(\mathbf{d}_{NT}^2 + \mathbf{d}_{NT, \infty}^2 \sqrt{(NT)^{-1} \log(NT) \log d} \right) + \mathbf{1}_{NT}^2 + o((NT)^{-1/2}) \\ &\lesssim_P^i o((NT)^{-1/2}) + \mathbf{1}_{NT}^2 + o((NT)^{-1/2}), \end{aligned}$$

where (i) follows combining $\|\beta_0\|_1 \leq \bar{C}_\beta$ assumed in Assumption 4.5 (a) and $q_{NT} = o((NT)^{-1/2})$, established in (D.20). ■

LEMMA D.12 (Higher-Order Terms $\bar{b}_4, \bar{b}_5, \bar{b}_6$ with $\widehat{U}_{it}^2 - U_{it}^2$). *Under Assumptions 4.1–4.7,*

$$\sum_{k=4}^6 \|\bar{b}_k\|_\infty \lesssim_P ((NT)^{-1/4} + \mathbf{1}_{NT} + \sqrt{s \log d / NT} + \mathbf{1}_{NT}^2 \log(d^2 NT)) =: \gamma_{NT}. \quad (\text{D.64})$$

PROOF OF LEMMA D.12. **Step 1.** Decompose

$$\widehat{U}_{it}^2 - U_{it}^2 = (\widehat{U}_{it} - U_{it} + U_{it})^2 - U_{it}^2 = 2U_{it}(\widehat{U}_{it} - U_{it}) + (\widehat{U}_{it} - U_{it})^2.$$

Invoking (D.23) gives

$$\begin{aligned} \widehat{U}_{it} &= \widehat{Y}_{it} - \widehat{V}_{it}' \widehat{\beta}_L = (\widehat{Y}_{it} - \widehat{V}_{it}' \beta_0) + (\widehat{V}_{it}' \beta_0 - \widehat{V}_{it}' \widehat{\beta}_L) \\ &= U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{l}}) + \widehat{V}_{it}' (\beta_0 - \widehat{\beta}_L). \end{aligned}$$

Cauchy inequality implies

$$\begin{aligned}\mathbb{E}_{NT}(\widehat{U}_{it} - U_{it})^2 &\leq 2\mathbb{E}_{NT}R_{it}^2(\widehat{\mathbf{d}}, \widehat{\mathbf{1}}) + 2\mathbb{E}_{NT}(\widehat{V}'_{it}(\widehat{\beta}_L - \beta_0))^2 \\ &= 2\mathbb{E}_{NT}R_{it}^2(\widehat{\mathbf{d}}, \widehat{\mathbf{1}}) + 2(\widehat{\beta}_L - \beta_0)' \widehat{Q}(\widehat{\beta}_L - \beta_0) =: U_1 + U_2,\end{aligned}$$

where $U_1 \lesssim_P o(NT)^{-1/2} + \mathbf{1}_{NT}^2$ in established in Lemma D.11 and $U_2 \lesssim_P s \log d/NT$ is Theorem 4.1.

Step 2. Let $C(W_{it}, \eta) = (C_{mj}(W_{it}, \eta))$ be a $d \times d$ matrix. For any coordinates m and j , decompose

$$\begin{aligned}\mathbb{E}_{NT}C_{mj}(W_{it}, \widehat{\eta})(\widehat{U}_{it}^2 - U_{it}^2) &= 2\mathbb{E}_{NT}C_{mj}(W_{it}, \widehat{\eta})U_{it}(\widehat{U}_{it} - U_{it}) + \mathbb{E}_{NT}C_{mj}(W_{it}, \widehat{\eta})(\widehat{U}_{it} - U_{it})^2 \\ &=: 2D_{1mj}(\widehat{\eta}) + D_{2mj}(\widehat{\eta}).\end{aligned}$$

Cauchy inequality gives

$$\begin{aligned}|D_{1mj}(\widehat{\eta})| &\leq (\mathbb{E}_{NT}C_{mj}^2(W_{it}, \widehat{\eta})U_{it}^2)^{1/2}(\mathbb{E}_{NT}(\widehat{U}_{it} - U_{it})^2)^{1/2} \\ &\leq \max_{1 \leq m, j \leq d} (\mathbb{E}_{NT}C_{mj}^2(W_{it}, \widehat{\eta})U_{it}^2)^{1/2}(\mathbb{E}_{NT}(\widehat{U}_{it} - U_{it})^2)^{1/2}.\end{aligned}$$

Maximal inequality gives

$$|D_{2mj}(\widehat{\eta})| \leq \max_{it} \max_{mj} |C_{mj}(W_{it}, \widehat{\eta})| \mathbb{E}_{NT}(\widehat{U}_{it} - U_{it})^2. \quad (\text{D.65})$$

Step 3. If one can verify

$$\max_{1 \leq m, j \leq d} \mathbb{E}_{NT}C_{mj}^2(W_{it}, \widehat{\eta})U_{it}^2 \lesssim_P 1, \quad (\text{D.66})$$

we have that

$$\begin{aligned}\|D_1(\widehat{\eta})\|_\infty &= \max_{1 \leq m, j \leq d} |D_{1mj}(\widehat{\eta})| \\ &\leq O_P(1) \cdot O_P((NT)^{-1/4} + \mathbf{1}_{NT} + \sqrt{s \log d/NT}) \\ &\lesssim_P ((NT)^{-1/4} + \mathbf{1}_{NT} + \sqrt{s \log d/NT}).\end{aligned}$$

If one can verify another condition

$$\max_{it} \max_{mj} |C_{mj}(W_{it}, \widehat{\eta})| \lesssim_P (\log(d^2 NT)), \quad (\text{D.67})$$

we have that

$$\|D_2(\widehat{\eta})\|_\infty = O_P(\log(d^2 NT)) \cdot O_P(s \log d/NT + \mathbf{1}_{NT}^2 + (NT)^{-1/2}).$$

Step 4.1 Take $C(W_{it}, \widehat{\eta}) = V_{it}V'_{it}$, which corresponds to \bar{b}_4 in (D.56). Invoking (B.19) and (B.20) with $Z_{1,nit} = Z_{2,nit} = V_{it}$ and $\bar{N} = 2$ and $g = 1$ as well the assumed bound (4.14) gives

$$\max_{jk} |\mathbb{E}_{NT}V_{itk}^2 V_{itj}^2 U_{it}^2| \lesssim_P (1 + \kappa_{NT} \log^2(d^2 NT)) \lesssim_P 1,$$

which verifies (D.66). By Lemma B.1 (6),

$$\max_{it} \max_{mj} |V_{itk} V_{itj}| \lesssim_P (\log(d^2 NT)),$$

which verifies (D.67).

Step 4.2 Take $C(W_{it}, \hat{\eta}) = V_{it}(d_{i0}(X_{it}) - \hat{d}_i(X_{it}))$, which corresponds to \bar{b}_5 in (D.57). In what follows, we focus on the event $\sup_{it} |d_{i0}(X_{it}) - \hat{d}_i(X_{it})| \leq \mathbf{d}_{NT, \infty} \leq 1$. Invoking (B.18) with $\bar{N} = 2$ and $g = 1$ gives

$$\begin{aligned} & \max_{1 \leq m, j \leq d} |\mathbb{E}_{NT} V_{itm}^2 (d_{i0}(X_{it}) - \hat{d}_i(X_{it}))_j^2 U_{it}^2| \\ & \lesssim_P \mathbf{d}_{NT, \infty}^2 \max_{1 \leq m \leq d} |\mathbb{E}_{NT} V_{itm}^2 U_{it}^2| \\ & \lesssim_P d_{NT, \infty} \left(1 + \sqrt{\log^5(d \log NT) \log(NT)/NT} \right) \\ & = \mathbf{d}_{NT, \infty} (1 + \kappa_{NT} \log(d \log NT)) \lesssim \mathbf{d}_{NT, \infty}. \end{aligned}$$

Likewise,

$$\max_{it} \max_{mj} |C(W_{it}, \hat{\eta})| \leq \mathbf{d}_{NT, \infty} \max_{it} \max_{1 \leq j \leq d} |V_{itj}| \lesssim_P (\log(d^2 NT) \mathbf{d}_{NT, \infty})$$

verifies (D.67).

Step 4.3 Take $C(W_{it}, \hat{\eta}) = (d_{i0}(X_{it}) - \hat{d}_i(X_{it}))(d_{i0}(X_{it}) - \hat{d}_i(X_{it}))'$, which corresponds to \bar{b}_6 in (D.58). On the event $\sup_{it} |d_{i0}(X_{it}) - \hat{d}_i(X_{it})| \leq \mathbf{d}_{NT, \infty} \leq 1$, the condition (D.66) becomes

$$\max_{1 \leq m, j \leq d} \mathbb{E}_{NT} (d_{i0}(X_{it}) - \hat{d}_i(X_{it}))_m^2 (d_{i0}(X_{it}) - \hat{d}_i(X_{it}))_j^2 U_{it}^2 \leq \mathbb{E}_{NT} U_{it}^2 \lesssim_P 1.$$

Noting that $\max_{it} \max_{mj} |C(W_{it}, \hat{\eta})| \lesssim_P \mathbf{d}_{NT, \infty}^2 \lesssim_P 1$ verifies (D.67).

Step 5. (Conclusion). Collecting the bounds and invoking $(s \vee 1)\kappa_{NT} = o(1)$ gives

$$o(NT)^{-1/4} + \mathbf{1}_{NT} + \sqrt{s \log d/NT} + (\log(d^2 NT))(o(NT)^{-1/2} + \mathbf{1}_{NT}^2 + s \log d/NT) \lesssim \gamma_{NT}. \quad (\text{D.68})$$

For N and T large enough,

$$\log(d^2 NT)/(NT)^{-1/4} \leq 1,$$

which implies $\log(d^2 NT)(NT)^{-1/4} = o((NT)^{-1/4})$. Likewise,

$$\log(d^2 NT) \sqrt{s \log d/NT} \leq s \sqrt{\log^2(d^2 NT) \log d/NT} \leq (s \vee 1)\kappa_{NT} = o(1),$$

which gives (D.68). ■

LEMMA D.13 (Bound on $\|\widehat{\Gamma}(\widehat{\beta}_L) - \Gamma\|_\infty$). *Under Assumptions 4.1–4.5 and 4.7, we have that:*

$$\|\widehat{\Gamma}(\widehat{\beta}_L) - \Gamma\|_\infty \lesssim_P (\gamma_{NT} + \kappa_{NT} \log(d^2 NT)) = o_p(1). \quad (\text{D.69})$$

PROOF. Decompose the matrix first-stage error

$$\begin{aligned} \widehat{\Gamma}(\widehat{\beta}_L) - \Gamma &= \mathbb{E}_{NT} \widehat{V}_{it} \widehat{V}'_{it} \widehat{U}_{it}^2 - \Gamma \\ &= \mathbb{E}_{NT} \widehat{V}_{it} \widehat{V}'_{it} (\widehat{U}_{it}^2 - U_{it}^2) + \mathbb{E}_{NT} (\widehat{V}_{it} \widehat{V}'_{it} - V_{it} V'_{it}) U_{it}^2 + \mathbb{E}_{NT} V_{it} V'_{it} U_{it}^2 - \Gamma \\ &= \bar{b}_6 + \bar{b}_5 + \bar{b}'_5 + \bar{b}_4 + \bar{b}_3 + \bar{b}_2 + \bar{b}'_2 + \bar{b}_1. \end{aligned}$$

The bound on \bar{b}_1 is given in (D.60), Lemma D.8. The bound on \bar{b}_2 is given in (D.61), Lemma D.9. The bound on \bar{b}_3 is given in (D.62), Lemma D.10. The bounds on $\bar{b}_4 - \bar{b}_6$ are given in (D.64), Lemma D.12. Summing the bounds gives (D.69). \blacksquare

PROOF OF LEMMA 4.3. **Step 1.** Define the following bounds

$$\begin{aligned} \Sigma_1 &:= \|\widehat{\Omega}^{\text{CLIME}} - Q^{-1}\|_{\infty,1} \|\widehat{\Gamma}(\widehat{\beta}_L)\|_\infty \|\widehat{\Omega}^{\text{CLIME}}\|_{1,\infty} \\ \Sigma_2 &:= \|Q^{-1}\|_{\infty,1} \|\widehat{\Gamma}(\widehat{\beta}_L) - \Gamma\|_\infty \|\widehat{\Omega}^{\text{CLIME}}\|_{1,\infty} \\ \Sigma_3 &:= \|\widehat{\Omega}^{\text{CLIME}} - Q^{-1}\|_{\infty,1} \|\Gamma\|_\infty \|Q^{-1}\|_{1,\infty} \end{aligned}$$

and note that

$$\|\widehat{\Sigma}(\widehat{\beta}_L) - \Sigma\|_\infty = \|\widehat{\Omega}^{\text{CLIME}} \widehat{\Gamma}(\widehat{\beta}_L) \widehat{\Omega}^{\text{CLIME}} - Q^{-1} \Gamma Q^{-1}\|_\infty \leq \Sigma_1 + \Sigma_2 + \Sigma_3.$$

Step 2. Invoking (D.59) and (D.69) gives

$$\|\widehat{\Gamma}(\widehat{\beta}_L)\|_\infty \leq \|\Gamma\|_\infty + \|\widehat{\Gamma}(\widehat{\beta}_L) - \Gamma\|_\infty \lesssim_P 1 + \gamma_{NT} + \kappa_{NT} \log(d^2 NT) \lesssim_P 1.$$

Invoking (D.43) and (D.40) gives

$$\|\widehat{\Omega}^{\text{CLIME}}\|_{1,\infty} \leq \|\widehat{\Omega}\|_{1,\infty} \leq \|Q^{-1}\|_{1,\infty} \leq (A_Q / (a_Q - 1)).$$

As a result, invoking (4.12) gives

$$\Sigma_1 = O_P(\lambda_Q^{1-1/a_Q}) \cdot O_P(1) \cdot O_P(1).$$

Likewise,

$$\begin{aligned} \Sigma_2 &:= \|Q^{-1}\|_{\infty,1} \|\widehat{\Gamma}(\widehat{\beta}_L) - \Gamma\|_\infty \|\widehat{\Omega}^{\text{CLIME}}\|_{1,\infty} = O(1) \cdot O_P(\gamma_{NT} + \kappa_{NT} \log(d^2 NT)) \cdot O_P(1) \\ &\lesssim_P (\gamma_{NT} + \kappa_{NT} \log(d^2 NT)) \end{aligned}$$

$$\Sigma_3 := \|\widehat{\Omega}^{\text{CLIME}} - Q^{-1}\|_{\infty,1} \|\Gamma\|_\infty \|Q^{-1}\|_{1,\infty} = O(1) \cdot O_P(\gamma_{NT}) \cdot O_P(1) \lesssim_P (\lambda_Q^{1-1/a_Q}).$$

Collecting the terms gives

$$\begin{aligned} \|\widehat{\Sigma}(\widehat{\beta}_L) - \Sigma\|_\infty &= \|\widehat{\Omega}^{\text{CLIME}}\widehat{\Gamma}(\widehat{\beta}_L)\widehat{\Omega}^{\text{CLIME}} - Q^{-1}\Gamma Q^{-1}\|_\infty \\ &\leq \Sigma_1 + \Sigma_2 + \Sigma_3 \\ &\lesssim_P \lambda_Q^{1-1/a_Q} + \gamma_{NT} + \kappa_{NT} \log(d^2 NT) + \lambda_Q^{1-1/a_Q}. \end{aligned}$$

■

D.5 Proof of Theorem 4.3

The proof is divided in several steps. Step 1 outlines the proof. Step 2-5 establish (4.20). Steps 6-8 establish (4.22).

PROOF. Step 1. (Outline) Let $Z \sim N(0, \mathcal{C})$ and $\widehat{Z} | \widehat{\mathcal{C}} \sim N(0, \widehat{\mathcal{C}})$ be as defined in the Theorem. Define

$$T_{\Sigma, \beta} := \sqrt{NT} \Sigma_{jj}^{-1/2} (\widehat{\beta}_{DL, j} - \beta_0), \quad T_{\widehat{\Sigma}, \beta} := \sqrt{NT} \widehat{\Sigma}_{jj}^{-1/2} (\widehat{\beta}_{DL, j} - \beta_0)$$

and

$$T_\Sigma := \Sigma_{jj}^{-1/2} \mathbb{G}_{NT} V_{itj} U_{it}.$$

Define

$$\begin{aligned} O_1(t) &:= \mathbb{P}(\|T_{\Sigma, \beta}\|_\infty < t) - \mathbb{P}(\|T_\Sigma\|_\infty < t + \delta_1) \\ O_2(t) &:= \mathbb{P}(\|T_\Sigma\|_\infty \leq t + \delta_1) - \mathbb{P}(\|Z\|_\infty < t + \delta_1) \\ O_3(t) &:= \mathbb{P}(\|Z\|_\infty < t + \delta_1) - \mathbb{P}(\|Z\|_\infty < t) \end{aligned}$$

and note that for each t

$$\mathbb{P}(\|T_{\Sigma, \beta}\|_\infty < t) - \mathbb{P}(\|Z\|_\infty < t) = \sum_{k=1}^3 O_k(t).$$

Likewise, define

$$O_4(t) := \mathbb{P}(\|T_{\widehat{\Sigma}, \beta}\|_\infty < t) - \mathbb{P}(\|T_{\Sigma, \beta}\|_\infty < t)$$

and

$$O_5(t) := \mathbb{P}(\|Z\|_\infty < t) - \mathbb{P}(\|\widehat{Z}\|_\infty < t | \widehat{\mathcal{C}}).$$

Note that for each t ,

$$\mathbb{P}(\|T_{\widehat{\Sigma}, \beta}\|_\infty < t) - \mathbb{P}(\|\widehat{Z}\|_\infty < t | \widehat{\mathcal{C}}) = O_4(t) + \sum_{k=1}^3 O_k(t) + O_5(t).$$

Then, (4.20) is equivalent to

$$\sup_{t \geq 0} |\mathbb{P}(\|T_{\Sigma, \beta}\|_{\infty} < t) - \mathbb{P}(\|Z\|_{\infty} < t)| \rightarrow 0 \quad (\text{D.70})$$

and (4.22) is equivalent to

$$\sup_{t \geq 0} |\mathbb{P}(\|T_{\hat{\Sigma}, \beta}\|_{\infty} < t) - \mathbb{P}(\|\hat{Z}\|_{\infty} < t \mid \hat{\mathcal{C}})| \rightarrow_P 0. \quad (\text{D.71})$$

Step 2. We show that the elements of $\text{diag}\Sigma$ are bounded from above and below. By Assumption 4.3 (2), there exists a finite $\bar{\sigma}_{UV}$ such that $\max_{it} \mathbb{E}[U_{it}^2 \mid V_{it}] \leq \bar{\sigma}_{UV}^2$ a.s.. As a result, Assumption 4.3 gives

$$0 < \underline{\sigma}^2 \leq \min_{it} \mathbb{E}[U_{it}^2 \mid V_{it}] \leq \max_{it} \mathbb{E}[U_{it}^2 \mid V_{it}] \leq \bar{\sigma}_{UV}^2 < \infty \text{ a.s.},$$

which implies $\underline{\sigma}^2 Q \preceq \Gamma \preceq \bar{\sigma}_{UV}^2 Q$, and $\underline{\sigma}^2 Q^{-1} \preceq \Sigma \preceq \bar{\sigma}_{UV}^2 Q^{-1}$. As a result,

$$0 < \underline{\sigma}^2 C_{\max}^{-1} \leq c_{\Sigma} = \min_{1 \leq j \leq d} \Sigma_{jj} \leq C_{\Sigma} = \max_{1 \leq j \leq d} \Sigma_{jj} \leq \bar{\sigma}_{UV}^2 C_{\min}^{-1} < \infty.$$

Likewise, the elements of $(\text{diag}\Sigma)^{-1/2}$ are bounded from above by $c_{\Sigma}^{-1/2}$ and from below by $C_{\Sigma}^{-1/2}$.

Step 3. We bound $\sup_{t \geq 0} |O_1(t)|$ with $\delta_1 = \log^{-1/2} d \log^{-1/2} NT$. Decomposition (D.47) implies

$$\|T_{\Sigma}\|_{\infty} - \|R_{NT}\|_{\infty} \leq \|T_{\Sigma, \beta}\|_{\infty} \leq \|T_{\Sigma}\|_{\infty} + \|R_{NT}\|_{\infty},$$

and union bound gives

$$\begin{aligned} \mathbb{P}(\|T_{\Sigma, \beta}\|_{\infty} < t) &\leq \mathbb{P}(\|T_{\Sigma}\|_{\infty} \leq t + \delta_1) + \mathbb{P}(\|R_{NT}\|_{\infty} \geq \delta_1) \\ \mathbb{P}(\|T_{\Sigma}\|_{\infty} < t) &\leq \mathbb{P}(\|T_{\Sigma, \beta}\|_{\infty} \leq t + \delta_1) + \mathbb{P}(\|R_{NT}\|_{\infty} \geq \delta_1) \end{aligned}$$

which gives

$$\sup_{t \geq 0} |O_1(t)| \leq \mathbb{P}(\|R_{NT}\|_{\infty} \geq \delta_1) = o(1), \quad (\text{D.72})$$

where (i) follows from

$$\|R_{NT}\|_{\infty} \lesssim_P \lambda_Q^{1-1/a_Q} s \log^{1/2} d + \sqrt{NT} \rho_{NT} = o_P(\log^{-1/2} d \log^{-1/2} NT)$$

given in (D.48) and (4.19).

Step 4. We verify the conditions of Lemma C.5 for the m.d.s. with

$$m = m(i, t) = T(i-1) + t, \quad M = NT$$

and

$$X_m := (\text{diag}\Sigma)^{-1/2} V_m U_m, \quad m = 1, 2, \dots, M, \quad D_M = c_{\Sigma}^{-1} \pi_M^{VU}.$$

To verify the condition (C.5), we invoke Assumption 4.8 which gives

$$\text{Var}(X_{mj}) = \Sigma_{jj}^{-1/2} \text{EV}_{mj}^2 U_m^2 \Sigma_{jj}^{-1/2} \geq \underline{\sigma}^2 \min_{it} \|\text{EV}_{it} V'_{it}\|_{\infty} C_{\Sigma}^{-1} =: a_1 > 0$$

and Remark 2.2

$$\text{Var}(X_{mj}) = \Sigma_{jj}^{-1/2} \text{EV}_{mj}^2 U_m^2 \Sigma_{jj}^{-1/2} \leq \bar{\sigma}^2 \max_{it} \|\text{EV}_{it} V'_{it}\|_{\infty} c_{\Sigma}^{-1} =: A_1 < \infty.$$

By Assumption 4.8,

$$\bar{r} := (2/\kappa) \cdot \log(NT), \quad \bar{q} := (NT)^{c_2} \log^2 d \log^2(NT)$$

obey (4.18), which implies (C.9). By Lemma C.5, there exist constants $c_2 \in (0, 1/4)$ and c_X and C_X depending on $\underline{\sigma}, \bar{\sigma}, c_2, C_{\min}, C_{\max}$ such that

$$\sup_{t \geq 0} |O_2(t)| = \sup_{t \geq 0} |\text{P}(\|T_{\Sigma}\|_{\infty} \leq t) - \text{P}(\|Z\|_{\infty} \leq t)| \lesssim C_X (NT)^{-c_X} + (NT)^{-c_2/2}. \quad (\text{D.73})$$

Step 4. Bound on $\sup_{t \geq 0} |O_3(t)|$. Invoking Lemma C.2 gives

$$\begin{aligned} & \sup_{t \geq 0} |O_3(t)| \\ & \sup_{t \geq 0} |\text{P}(\|Z\|_{\infty} < t + \delta_1) - \text{P}(\|Z\|_{\infty} < t)| \\ & \leq \sup_{t \geq 0} |\text{P}(\|Z\|_{\infty} < t + \delta_1) - \text{P}(\|Z\|_{\infty} < t - \delta_1)| \\ & = \sup_{t \geq 0} \text{P}(\|\|Z\|_{\infty} - t\| \leq \delta_1) \leq C \delta_1 \sqrt{1 \vee \log(2d/\delta_1)}. \end{aligned}$$

Notice that the R.H.S is a non-decreasing function of δ_1 in some neighborhood of 0 and that $\sqrt{1 \vee (x+y)} \leq 1 + \sqrt{x} + \sqrt{y}$ for $x, y > 0$. Plugging in $\delta_1 = \log^{-1/2} d \log^{-1/2} NT$ gives

$$\begin{aligned} \sup_{t \geq 0} |O_3(t)| & \leq C \log^{-1/2} d \log^{-1/2} NT \sqrt{1 \vee (\log(2d) + \log(\log^{1/2} d \log^{1/2} NT))} \\ & \lesssim \log^{-1/2} NT + \log^{-1/2} d \log^{-1/2} NT \log^{1/2} \log(NT) = o(1). \end{aligned} \quad (\text{D.74})$$

Combining (D.72) and (D.73) and (D.74) gives (D.70). By a standard calculation we have $\text{E}\|Z\|_{\infty} \lesssim \sqrt{\log 2d}$. Invoking Gaussian concentration inequality (see, e.g., Ledoux (2001), Theorem 7.1 or Comment 4 in Chernozhukov et al. (2015), page 56) implies

$$\|Z\|_{\infty} \lesssim_P \log^{1/2}(2d) + \log^{1/2}(NT).$$

Since $\|Z\|_{\infty}$ and $\|T_{\Sigma, \beta}\|_{\infty}$ converge in distribution to the same limit,

$$\|T_{\Sigma, \beta}\|_{\infty} \lesssim_P \log^{1/2}(2d) + \log^{1/2}(NT). \quad (\text{D.75})$$

Step 5.1. We bound $\sup_{t \geq 0} |O_4(t)|$. Take $\rho_j = \Sigma_{jj}^{1/2} / \widehat{\Sigma}_{jj}^{1/2}$ and let $\rho := (\rho_1, \rho_2, \dots, \rho_d)'$ be a d -vector. Note that all Euclidean j -vectors e_j vectors obey $\|e_j\|_2 = \|e_j\|_1 = 1$ and

therefore belong to the set in Theorem 4.2 with $K_\alpha = 1$. Let $\alpha_{NT} = (\alpha \Sigma \alpha)^{1/2} / (\alpha \widehat{\Sigma} \alpha)^{1/2}$ be as in (D.52). Invoking (D.51) and the bound (4.15) in Lemma 4.3 gives

$$\max_j |\rho_j - 1| \leq \sup_{\alpha: \|\alpha\|_2 = \|\alpha\|_1 = 1} |\alpha_{NT} - 1| \lesssim_P \gamma_{NT}.$$

In particular, it implies that the even

$$\min_{1 \leq j \leq d} \rho_j > 1/2$$

occurs wp $1 - o(1)$. For any $\rho_j > 1/2$,

$$|\rho_j^{-1} - 1| = |1 - \rho_j| / |\rho_j| \leq 2|\rho_j - 1|.$$

Combining the bounds above on the event $\min_{1 \leq j \leq d} \rho_j > 1/2$ gives

$$\max_{1 \leq j \leq d} |\widehat{\Sigma}_{jj}^{-1/2} / \Sigma_{jj}^{-1/2} - 1| = \max_{1 \leq j \leq d} |\rho_j^{-1} - 1| \leq 2 \max_{1 \leq j \leq d} |\rho_j - 1| \lesssim_P \gamma_{NT}. \quad (\text{D.76})$$

Step 5.2. Let $v_1 \cdot v_2$ denote $(v_1 \cdot v_2)_j = v_{1j} \cdot v_{2j}$ for $j = 1, 2, \dots, d$. Note that

$$T_{\widehat{\Sigma}, \beta} = T_{\Sigma, \beta} \cdot \rho^{-1},$$

or, equivalently,

$$T_{\widehat{\Sigma}, \beta} - T_{\Sigma, \beta} = (\rho^{-1} - 1)T_{\Sigma, \beta}.$$

Invoking (D.76) and (D.75) gives

$$\|T_{\widehat{\Sigma}, \beta} - T_{\Sigma, \beta}\|_\infty \leq \max_{1 \leq j \leq d} |\rho_j^{-1} - 1| \|T_{\Sigma, \beta}\|_\infty = O_P(\zeta_{NT}) \cdot O_P(\log^{1/2} d + \log^{1/2} NT) =^i o_P(1),$$

where (i) follows from (4.21). Thus, $\|T_{\widehat{\Sigma}, \beta}\|_\infty$ and $\|T_{\Sigma, \beta}\|_\infty$ converge to the same limit in distribution.

Step 6. We bound $\sup_{t \geq 0} |O_5(t)|$. Invoking Lemma C.1 with $X \sim N(0, \mathcal{C})|\widehat{\mathcal{C}}$ and $Y \sim N(0, \mathcal{C})$ and $\widehat{\Delta} = \|\mathcal{C} - \widehat{\mathcal{C}}\|_\infty$

$$\sup_{t \geq 0} |O_5(t)| \leq C' (\widehat{\Delta} \log^2(2d))^{1/2},$$

where C depends only on the constants defined in Assumptions 4.2 and 4.3. In Step 7, we show that for ζ_{NT} in (4.15),

$$\widehat{\Delta} := \|\mathcal{C} - \widehat{\mathcal{C}}\|_\infty \lesssim_P^i \zeta_{NT} =^{ii} o_P(\log^{-2} d \log^{-1} NT), \quad (\text{D.77})$$

where (i) is verified in Steps 7-8 and (ii) is directly assumed in (4.21).

Step 7. Note that

$$\|\Sigma\|_\infty = \|Q^{-1} \Gamma Q^{-1}\|_\infty \leq \|Q^{-1}\|_{\infty, 1} \|\Gamma\|_\infty \|Q^{-1}\|_{1, \infty} \leq (A_Q / (a_Q - 1))^2 \|\Gamma\|_\infty = O(1).$$

As a result,

$$\|\widehat{\Sigma}\|_\infty \leq \|\widehat{\Sigma} - \Sigma\|_\infty + \|\Sigma\|_\infty \lesssim_P 1 + \gamma_{NT} \lesssim_P 1.$$

Likewise,

$$\|(\text{diag}\widehat{\Sigma})^{-1/2}\|_{\infty,1} = \|(\text{diag}\widehat{\Sigma})^{-1/2}\|_{1,\infty} = \max_{1 \leq j \leq d} \widehat{\Sigma}_{jj}^{-1/2} \lesssim_P \zeta_{NT} + c_{\Sigma}^{-1/2} \lesssim_P 1.$$

Step 8. Define

$$C_1 := \max_{1 \leq j \leq d} |\widehat{\Sigma}_{jj}^{-1/2} - \Sigma_{jj}^{-1/2}| \|\widehat{\Sigma}\|_{\infty} \|(\text{diag}\widehat{\Sigma})^{-1/2}\|_{1,\infty}$$

$$C_2 := \|(\text{diag}\Sigma)^{-1/2}\|_{\infty,1} \|\widehat{\Sigma} - \Sigma\|_{\infty} \|(\text{diag}\widehat{\Sigma})^{-1/2}\|_{1,\infty}$$

$$C_3 := \|(\text{diag}\Sigma)^{-1/2}\|_{\infty,1} \|\Sigma\|_{\infty} \max_{1 \leq j \leq d} |\widehat{\Sigma}_{jj}^{-1/2} - \Sigma_{jj}^{-1/2}|$$

and note that

$$\|\widehat{\mathcal{C}} - \mathcal{C}\|_{\infty} = \|(\text{diag}\widehat{\Sigma})^{-1/2} \widehat{\Sigma} (\text{diag}\widehat{\Sigma})^{-1/2} - (\text{diag}\Sigma)^{-1/2} \Sigma (\text{diag}\Sigma)^{-1/2}\|_{\infty} \leq C_1 + C_2 + C_3.$$

Invoking (D.76) and (4.15)

$$\max_{1 \leq j \leq d} |\widehat{\Sigma}_{jj}^{-1/2} - \Sigma_{jj}^{-1/2}| \lesssim_P \zeta_{NT}, \quad \|\widehat{\Sigma} - \Sigma\|_{\infty} \lesssim_P \zeta_{NT}$$

implies that each term C_j is a product of two $O_P(1)$ terms and a single $O_P(\zeta_{NT})$ term. Thus, $C_1 + C_2 + C_3 \lesssim_P \zeta_{NT}$ verifies (i) in (D.77). ■

PROOF OF LEMMA 4.4. We invoke Lemma D.6 with $\bar{V}_{it} = D_{it} - d_{i0}(Z_{it})$ and $\bar{Y}_{it} = Y_{it} - l_{i0}(Z_{it})$ and $\bar{\beta}_0 = (\beta_0, \rho_0)$ and $g = 2$. Steps 1, 2 and 3 are established similarly to the proof of Theorem 4.1. Thus, the bounds (4.27) hold for the Orthogonal Group Lasso. As a result, $\|\widehat{\beta}_L - \beta_0\|_1 \leq \sqrt{s^2 \log d / NT}$ wp $1 - o(1)$. As a result, the debiased Orthogonal Group Lasso obeys the uniform linearization result (D.47), and Theorems 4.2 and 4.3 hold. ■

APPENDIX E: PROOFS FOR SECTION 5

PROOF OF REMARK 5.1. To prove this, let $\|\cdot\|_{\psi_2}$ denote the Orlicz sub-Gaussian norm under the probability measure P (see van der Vaart and Wellner (1996)). Then

$$\| \|F_{it}\| \|_{\psi_2} \leq \| \| \Pi_{it} F_{i,t-1} \| \|_{\psi_2} + \| \| QT_{it} \| \|_{\psi_2} \leq (1 - \delta) \| \| F_{i,t-1} \| \|_{\psi_2} + A' \bar{\sigma}^2,$$

where A' is a numerical constant. Iterating on this inequality exactly t times we obtain

$$\| \| F_{it} \| \|_{\psi_2} \leq (1 - \delta)^t \| \| F_{i,0} \| \|_{\psi_2} + A' \sum_{\tilde{t}=1}^{t-1} (1 - \delta)^{\tilde{t}} \bar{\sigma}^2 \leq A' \frac{\bar{\sigma}^2}{1 - \delta}.$$

■

■

PROOF OF REMARK 5.3. Step 1 shows that $\mathbf{p}_{NT} \leq N^{-1/2}(2(B_{\max} + 1))^{1/2}\zeta_{NT,\infty}$. Step 2 shows that wp $1 - o(1)$,

$$\sup_{it} |p_i(X_{it}) - p_{i0}(X_{it})| \leq 2\zeta_{NT,\infty}.$$

Step 1. For any δ^P and $\xi \in \bar{P}_{NT}$,

$$\|\delta^P - \delta_0^P\|_2 \leq \|\delta^P - \delta_0^P\|_1 \leq N^{-1/2}\zeta_{NT,\infty} \quad (\text{E.1})$$

$$\|\xi - \xi_0\|_2 \leq \|\xi - \xi_0\|_1 \leq \zeta_{NT,\infty}. \quad (\text{E.2})$$

Cauchy inequality gives

$$(p_i(X_{it}) - p_{i0}(X_{it}))^2 = (X'_{it}(\delta^P - \delta_0^P) + \xi_i - \xi_{i0})^2 \leq 2(X'_{it}(\delta^P - \delta_0^P))^2 + 2(\xi_i - \xi_{i0})^2.$$

Summing over $i = 1, 2, \dots, N$ and $t = 1, 2, \dots, T$ gives

$$\begin{aligned} \mathbf{p}_{NT}^2 &\leq 2(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E}(X'_{it}(\delta^P - \delta_0^P))^2 + 2N^{-1}\|\xi - \xi_0\|^2 \\ &\leq 2B_{\max}N^{-1}\zeta_{NT,\infty}^2 + 2N^{-1}\zeta_{NT,\infty}^2. \end{aligned}$$

Wp $1 - o(1)$, $\max_{1 \leq i \leq N, 1 \leq t \leq T} \|X_{it}\|_\infty \leq C_X \sqrt{\log d_X NT}$ for some finite C_X by Lemma B.1.

Step 2. The following bound holds wp $1 - o(1)$,

$$\begin{aligned} \sup_{it} |p_i(X_{it}) - p_{i0}(X_{it})| &\leq \sup_{it} |X'_{it}(\delta^P - \delta_0^P)| + |\xi_i - \xi_{i0}| \\ &\leq \sup_{it} \|X_{it}\|_\infty \|\delta^P - \delta_0^P\|_1 + \|\xi - \xi_0\|_1 \\ &\leq C_X \sqrt{\log(d_X NT)} N^{-1/2} \zeta_{NT,\infty} + \zeta_{NT,\infty} \\ &\leq 2\zeta_{NT,\infty}, \end{aligned}$$

where the last step holds assuming N is large enough and $C_X \sqrt{\log(d_X NT)/N} \leq 1$. ■

PROOF OF REMARK 5.5. Step 1 shows that $\mathbf{l}_{NT} = O(N^{-1/2}(\zeta_{NT,\infty} + \zeta_{NT,\infty}^E))$. Step 2 shows that wp $1 - o(1)$,

$$\sup_{it} |l_i(X_{it}) - l_{i0}(X_{it})| \leq 2\bar{K}\|\beta_0\|_1 \zeta_{NT,\infty} + 2\zeta_{NT,\infty}^E.$$

Step 1. Decompose

$$l_i(X_{it}) - l_{i0}(X_{it}) = (d_i(X_{it}) - d_{i0}(X_{it}))' \beta_0 + X'_{it}(\delta^E - \delta_0^E) + \xi_i^E - \xi_{i0}^E + d_i(X_{it})'(\beta - \beta_0)$$

Cauchy inequality gives

$$(l_i(X_{it}) - l_{i0}(X_{it}))^2 \leq 4 \left(((d_i(X_{it}) - d_{i0}(X_{it}))' \beta_0)^2 + (X'_{it}(\delta^E - \delta_0^E))^2 + (\xi_i^E - \xi_{i0}^E)^2 + (d_i(X_{it})'(\beta - \beta_0))^2 \right).$$

Note that $d_i(X_{it}) = K(X_{it})p_i(X_{it}) = K(X_{it})(X'_{it}\delta^P + \xi_i)$. Summing over $i = 1, 2, \dots, N$ and $t = 1, 2, \dots, T$ gives

$$\begin{aligned} \mathbf{I}_{NT}^2 &\leq 4(NT)^{-1} \underbrace{\sum_{i=1}^N \sum_{t=1}^T (\delta^P - \delta_0^P)' E[(K'_{it}\beta_0)^2 X_{it} X'_{it}] (\delta^P - \delta_0^P)}_{(\delta^P - \delta_0^P)' \Psi_D (\delta^P - \delta_0^P)} \\ &\quad + 4(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T E(X'_{it}(\delta^E - \delta_0^E))^2 + 4N^{-1} \|\xi^E - \xi_0^E\|_2^2 + 4(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T E\|d_i(X_{it})\|_\infty^2 \|\beta - \beta_0\| \\ &\leq 4(B_{\max} N^{-1} \zeta_{NT,\infty}^2 + B_{\max} N^{-1} (\zeta_{NT,\infty}^E)^2 + N^{-1} (\zeta_{NT,\infty}^E)^2 + B_4 N^{-1} (\zeta_{NT,\infty}^E)^2) \\ &\leq 4(B_{\max} + 1 + B_4) N^{-1} (\zeta_{NT,\infty}^E)^2 + 4N^{-1} B_{\max} \zeta_{NT,\infty}^2. \end{aligned}$$

Note that for N, T large enough such that $\|\delta^P\|_2^2 \leq 2\|\delta_0^P\|_2^2 \leq 2\|\delta_0^P\|_1^2$, which is bounded,

$$\begin{aligned} B_4 &= (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T E\|d_i(X_{it})\|_\infty^2 \\ &\leq \bar{K}^2 (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T E(X'_{it}\delta^P + \xi_i)^2 \leq 2\bar{K}^2 (\delta^{P'} \Psi_X \delta^P + N^{-1} \|\xi\|_2^2) \\ &\leq 2\bar{K}^2 (B_{\max} \|\delta^P\|_2^2 + N^{-1} \|\xi\|_2^2) \\ &\leq 4\bar{K}^2 B_{\max} \|\delta_0^P\|_2^2 + 1. \end{aligned}$$

Step 2. For N, T large enough, wp $1 - o(1)$,

$$\begin{aligned} &\sup_{it} |l_i(X_{it}) - l_{i0}(X_{it})| \leq \sup_{it} |K'_{it}\beta_0| \sup_{it} |p_i(X_{it}) - p_{i0}(X_{it})| \\ &\quad + \sup_{it} \|X_{it}\|_\infty \|\delta^E - \delta_0^E\|_1 \\ &\quad + \sup_i |\xi_i - \xi_i^E| \\ &\quad + \sup_{it} \|K(X_{it})\|_\infty |X'_{it}\delta^E| \|\beta - \beta_0\|_1 \\ &\leq 2\bar{K} \|\beta_0\|_1 \zeta_{NT,\infty} + \|X_{it}\|_\infty (N^{-1/2} \zeta_{NT,\infty}^E + \bar{K} \|\delta^E\|_1 N^{-1/2} \zeta_{NT,\infty}^E) + \|\xi^E - \xi_0^E\|_1 \\ &\leq 2\bar{K} \|\beta_0\|_1 \zeta_{NT,\infty} + C_X \sqrt{\log(d_X NT)} N^{-1/2} (1 + \bar{K} \|\delta^E\|_1) \zeta_{NT,\infty}^E + \zeta_{NT,\infty}^E \\ &\leq 2\bar{K} \|\beta_0\|_1 \zeta_{NT,\infty} + 2\zeta_{NT,\infty}^E, \end{aligned}$$

where the last step holds assuming N is large enough and $\|\delta^E\|_1 \leq \|\delta_0^E\|_1$ and

$$C_X \sqrt{\log(d_X NT)/N} (1 + 2\bar{K} \|\delta_0^E\|_1) \leq 1. \quad \blacksquare$$

PROOF OF REMARK 5.6. Invoking Remark 5.3 and the bound (5.34) on $\zeta_{NT,\infty}^P$ in Lemma 5.1 gives

$$\begin{aligned} \sqrt{NT} \mathbf{p}_{NT}^2 &\lesssim \sqrt{NT} N^{-1} (\zeta_{NT,\infty}^P)^2 \lesssim (S^P)^2 N^{-1/2} T^{1/2} T^{\nu-1} \log^{3(1-\nu)}(d_X + N) \\ &\lesssim (S^P)^2 N^{-1/2} T^{\nu-1/2} \log^{3(1-\nu)}(d_X + N) = o(1). \end{aligned}$$

In addition, the bound (5.36) on $\zeta_{NT,\infty}^E$ in Lemma 5.2 gives

$$\begin{aligned} \sqrt{NT} \mathbf{p}_{NT} \mathbf{1}_{NT} &\lesssim \sqrt{NT} N^{-1} \zeta_{NT,\infty}^P \zeta_{NT,\infty}^E \\ &\lesssim S^P \cdot S^E N^{-1/2} T^{(\nu+\nu^E)/2-1} \log^{3(1-(\nu+\nu^E)/2)}(d_X + N) = o(1). \end{aligned} \quad \blacksquare$$

APPENDIX F: TOOLS: TAILS BOUNDS FOR EMPIRICAL RECTANGULAR MATRICES UNDER WEAK DEPENDENCE

LEMMA F1 (Rectangular Matrix Bernstein, Theorem 1.6 in Tropp (2012)). *Consider a finite sequence $\{\Xi_m\}_{m=1}^M$ of independent, random matrices with dimensions $d_1 \times d_2$. Assume that there exist constants R_Ξ and σ_Ξ such that*

$$\mathbb{E} \Xi_m = 0, \quad \|\Xi_m\| \leq R_\Xi \text{ a.s.} \quad (\text{F1})$$

Define

$$\sigma_\Xi^2 = \max \left(\left\| \mathbb{E} \sum_{m=1}^M \Xi'_m \Xi_m \right\|, \left\| \mathbb{E} \sum_{m=1}^M \Xi_m \Xi'_m \right\| \right). \quad (\text{F2})$$

Then, for all $t \geq 0$,

$$\mathbb{P} \left(\left\| \sum_{m=1}^M \Xi_m \right\| \geq t \right) \leq (d_1 + d_2) e^{-(t^2/2(\sigma_\Xi^2 + R_\Xi t/3))}. \quad (\text{F3})$$

LEMMA F2 (Tail Bounds for Weakly Dependent Sums, Operator Norm). *Consider the setup of Lemma B.4 with weakly dependent data $\{W_{it}\}$ and matrix-valued functions $\{\phi_i(\cdot)\}_{i=1}^N : \mathcal{W} \rightarrow \mathbb{R}^{d_1 \times d_2}$. Let $q = \lfloor (2/\kappa) \log(NT) \rfloor$ be as in (B.16) and $L = \lfloor T/2q \rfloor$. For $i = 1, 2, \dots, N$ and $l = 1, 2, \dots, L$, let the data blocks $B_{i(2l-1)}$, B_{i2l} and B_{ir} be as in (B.8)–(B.10). Let the full-sized odd-block sums $\phi_i(B_{i(2l-1)})$ be as in (B.11), that is,*

$$\phi_i(B_{i(2l-1)}) = \sum_{t=(2l-2)q+1}^{t=(2l-2)q+q} \phi_i(W_{it}), \quad \phi_i(B_{i2l}) = \sum_{t=(2l-1)q+1}^{t=2lq} \phi_i(W_{it})$$

and let $\phi_i(B_{i(2l-1)}^*)$ and $\phi_i(B_{i(2l)}^*)$ be their Berbee copies. In case $T \neq 2Lq$, the remainder block $\phi_i(B_{ir})$ as in (B.13), that is

$$\phi_i(B_{ir}) := \sum_{t=2Lq+1}^T \phi_i(W_{it})$$

Suppose that there exist constants R and σ such that the following conditions hold

$$\mathbb{E}\phi_i(W_{it}) = 0, \quad \sup_{it} \|\phi_i(W_{it})\| \leq R \text{ a.s.} \quad (\text{F.4})$$

and

$$\max \left(\left\| \sum_{i=1}^N \sum_{l=1}^L \mathbb{E}\phi_i(B_{i(2l)}^*)' \phi_i(B_{i(2l)}^*) \right\|, \left\| \sum_{i=1}^N \sum_{l=1}^L \mathbb{E}\phi_i(B_{i(2l)}^*) \phi_i(B_{i(2l)}^*)' \right\| \right) \leq qNT\sigma^2 \quad (\text{F.5})$$

$$\max \left(\left\| \sum_{i=1}^N \sum_{l=1}^L \mathbb{E}\phi_i(B_{i(2l-1)}^*)' \phi_i(B_{i(2l-1)}^*) \right\|, \left\| \sum_{i=1}^N \sum_{l=1}^L \mathbb{E}\phi_i(B_{i(2l-1)}^*) \phi_i(B_{i(2l-1)}^*)' \right\| \right) \leq qNT\sigma^2 \quad (\text{F.6})$$

$$\max \left(\left\| \sum_{i=1}^N \mathbb{E}\phi_i(B_{ir}) \phi_i(B_{ir})' \right\|, \left\| \sum_{i=1}^N \mathbb{E}\phi_i(B_{ir})' \phi_i(B_{ir}) \right\| \right) \leq qNT\sigma^2 \quad (\text{F.7})$$

Then, for any $t \geq 0$,

$$\mathbb{P}(\|(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \phi_i(W_{it})\| \geq 3t) \leq 3(d_1 + d_2)e^{-t^2NT/2(q\sigma^2 + qRt/3)} + 2NL\gamma(q) \quad (\text{F.8})$$

and under geometric beta-mixing condition (4.1),

$$\left\| \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \phi_i(W_{it}) \right\| \lesssim_P \frac{1}{\sqrt{NT}} \left(\sigma \sqrt{\log(NT) \log(d_1 + d_2)} + \frac{1}{\sqrt{NT}} \log(NT) R \log(d_1 + d_2) \right). \quad (\text{F.9})$$

Remark F.1. In what follows, we write $\phi(W_{it})$ in place of $\phi_i(W_{it})$, but subsume the dependence on i .

PROOF OF LEMMA F.2. Union bound gives

$$\begin{aligned} & \mathbb{P}(\| \sum_{i=1}^N \sum_{t=1}^T \phi(W_{it}) \| \geq 3t) \\ & \leq \mathbb{P}(\| \sum_{i=1}^N \sum_{l=1}^L \phi(B_{i(2l-1)}^*) \| \geq t) \\ & \quad + \mathbb{P}(\| \sum_{i=1}^N \sum_{l=1}^L \phi(B_{i(2l)}^*) \| \geq t) + \mathbb{P}(\| \sum_{i=1}^N \phi(B_{ir}) \| \geq t) + 2NL\gamma(q). \end{aligned} \quad (\text{F.10})$$

We first establish the bound for the odd-block sums. Define

$$m = m(i, l) = L \cdot (i - 1) + l, \quad M = NL, \quad \Xi_m := \phi(B_{i(2l-1)}^*).$$

Since $\phi(B_{i(2l-1)}^*)$ consists of q summands and W_{it}^* and W_{it} have the same marginal distributions, the bound (F4) gives

$$\|\phi(B_{i(2l-1)}^*)\| \leq qR \text{ a.s.},$$

which verifies (E1) with $R_{\Xi} = qR$. Likewise, (E5) directly verifies (E2) with the bound $\sigma_{\Xi}^2 = qNT\sigma^2$. Invoking Lemma F1 gives

$$\mathbb{P} \left(\left\| \sum_{i=1}^N \sum_{l=1}^L \phi(B_{i(2l-1)}^*) \right\| \geq t \right) \leq (d_1 + d_2) e^{-t^2/2(qNT\sigma^2 + qRt/3)}.$$

A similar bound holds for the even-numbered sums. For the remainder blocks, we take

$$m = i, \quad M = N, \quad \Xi_m = \phi(B_{ir}).$$

Since the remainder block has at most q elements,

$$\|\phi(B_{ir})\| \leq qR \text{ a.s.}$$

which implies (E1) with $R_{\Xi} = qR$. Likewise, (E7) directly verifies (E2) with the bound $\sigma_{\Xi}^2 = qNT\sigma^2$. Therefore,

$$\mathbb{P} \left(\left\| \sum_{i=1}^N \phi(B_{ir}) \right\| > t \right) \leq (d_1 + d_2) e^{-t^2/2(qNT\sigma^2 + qRt/3)}.$$

Invoking union bound (E10) gives

$$\mathbb{P} \left(\left\| \sum_{i=1}^N \sum_{t=1}^T \phi(W_{it}) \right\| \geq 3t \right) \leq 3(d_1 + d_2) e^{-t^2/2(qNT\sigma^2 + qRt/3)} + 2NL\gamma(q).$$

Plugging $t(NT)$ in place of t gives and dividing each side by NT gives

$$\begin{aligned} \mathbb{P} \left(\left\| \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \phi(W_{it}) \right\| \geq 3t \right) &\leq 3(d_1 + d_2) e^{-t^2(NT)^2/2(qNT\sigma^2 + qRNTt/3)} + 2NL\gamma(q) \\ &= 3(d_1 + d_2) e^{-t^2NT/2(q\sigma^2 + qRt/3)} + 2NL\gamma(q), \end{aligned}$$

which coincides with (F8). For geometric mixing, taking q as in (B.16) gives $NL\gamma(q) = o(1)$. Noting that

$$\begin{aligned} &3(d_1 + d_2) e^{-t^2NT/2(q\sigma^2 + qRt/3)} \\ &\leq \max \left(3(d_1 + d_2) e^{-(t^2NT/4q\sigma^2)}, 3(d_1 + d_2) e^{-(tNT/4qR/3)} \right). \end{aligned}$$

Plugging $t = C' \sigma \sqrt{q \log(d_1 + d_2) / NT}$ and taking C' large makes the first term in the max as small as desired. Plugging $t = C' \log(d_1 + d_2) q R / NT$ and taking C' large makes the second terms in the max as small as desired. Therefore,

$$\left\| \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \phi(W_{it}) \right\| \lesssim_P \frac{1}{\sqrt{NT}} \left(\sigma \sqrt{\log(NT) \log(d_1 + d_2)} + \frac{1}{\sqrt{NT}} \log(NT) R \log(d_1 + d_2) \right).$$

■

LEMMA F3. Let $\gamma(X) : X \rightarrow \mathbb{R}^{d_1 \times d_2}$ be a fixed matrix-valued function of a random vector X . Define the functional

$$\phi(X) = \gamma(X) - \mathbb{E}[\gamma(X)]. \quad (\text{E11})$$

Let γ_{NT}^∞ and γ_{NT} be numeric sequences obeying

$$\sup_{it} \|\gamma(X_{it})\| \leq \gamma_{NT}^\infty \text{ a.s.}, \quad (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \|\gamma(X_{it})\|^2 \leq \gamma_{2NT}^2 \quad (\text{E12})$$

Then, the conditions (F4) and (F5)–(F7) hold with

$$R = 2\gamma_{NT}^\infty, \quad \sigma^2 = 2\gamma_{2NT}^2. \quad (\text{E13})$$

As a result, the bound (F9) in Lemma F2 reduces to

$$\left\| \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \phi(X_{it}) \right\| \lesssim_P \frac{1}{\sqrt{NT}} \left(\gamma_{2NT} \sqrt{\log(NT) \log(d_1 + d_2)} + \frac{1}{\sqrt{NT}} \log(NT) \gamma_{NT}^\infty \log(d_1 + d_2) \right). \quad (\text{E14})$$

PROOF OF LEMMA F3 . **Step 1.** Let X and \bar{X} be two random vectors, and $\gamma(X)$ and $\gamma(\bar{X})$ be $d_1 \times d_2$ matrices. The following inequalities hold

$$\begin{aligned} \|\mathbb{E}\gamma(X)\gamma(\bar{X})'\| &\leq^i \mathbb{E}\|\gamma(X)\gamma(\bar{X})'\| \leq^{ii} \mathbb{E}\|\gamma(X)\| \|\gamma(\bar{X})'\| \\ &\leq^{iii} \sqrt{\mathbb{E}\|\gamma(X)\|^2 \mathbb{E}\|\gamma(\bar{X})'\|^2} \\ &\leq^{iv} 1/2(\mathbb{E}\|\gamma(X)\|^2 + \mathbb{E}\|\gamma(\bar{X})'\|^2) =^v 1/2(\mathbb{E}\|\gamma(X)\|^2 + \mathbb{E}\|\gamma(\bar{X})\|^2), \end{aligned} \quad (\text{E15})$$

where (i) follows from the convexity of the norm and Jensen's inequality, (ii) from sub-multiplicativity of operator norm $\|AB\| \leq \|A\| \|B\|$, (iii)–(iv) from Cauchy inequalities and (v) from $\|A'\| = \|A\|$. Likewise,

$$\begin{aligned} \|\mathbb{E}\gamma(X)\mathbb{E}\gamma(\bar{X})'\| &\leq^i \|\mathbb{E}\gamma(X)\| \|\mathbb{E}\gamma(\bar{X})'\| \\ &\leq^{ii} 1/2((\|\mathbb{E}\gamma(X)\|)^2 + (\|\mathbb{E}\gamma(\bar{X})'\|)^2) \\ &\leq^{iii} 1/2(\mathbb{E}\|\gamma(X)\|^2 + \mathbb{E}\|\gamma(\bar{X})'\|^2) =^{iv} 1/2(\mathbb{E}\|\gamma(X)\|^2 + \mathbb{E}\|\gamma(\bar{X})\|^2), \end{aligned} \quad (\text{E16})$$

where (i) follows from $\|AB\| \leq \|A\|\|B\|$, (ii) from Cauchy inequality, (iii) from the convexity of composition $t \rightarrow t^2$ and $\cdot \rightarrow \|\cdot\|$ and Jensen's inequality and (iv) from $\|A'\| = \|A\|$. Finally, since the RHS of (F15) and (F16) is invariant under transposition, the same bound holds for the transposed quantities:

$$\begin{aligned} \max(\|\mathbb{E}\gamma(X)'\gamma(\bar{X})\|, \|\mathbb{E}\gamma(X)\gamma(\bar{X})'\|) &\leq 1/2(\mathbb{E}\|\gamma(X)\|^2 + \mathbb{E}\|\gamma(\bar{X})\|^2) \\ \max(\|\mathbb{E}\gamma(X)'\mathbb{E}\gamma(\bar{X})\|, \|\mathbb{E}\gamma(X)\mathbb{E}\gamma(\bar{X})'\|) &\leq 1/2(\mathbb{E}\|\gamma(X)\|^2 + \mathbb{E}\|\gamma(\bar{X})\|^2). \end{aligned}$$

Step 2. For $\phi(X) = \gamma(X) - \mathbb{E}\gamma(X)$,

$$\begin{aligned} \mathbb{E}\phi(X)\phi(\bar{X})' &= \mathbb{E}\gamma(X)\gamma(\bar{X})' - \mathbb{E}\gamma(X)\mathbb{E}\gamma(\bar{X})' - \mathbb{E}\gamma(X)\mathbb{E}\gamma(\bar{X})' + \mathbb{E}\gamma(X)\mathbb{E}\gamma(\bar{X})' \quad (\text{F17}) \\ &= \mathbb{E}\gamma(X)\gamma(\bar{X})' - \mathbb{E}\gamma(X)\mathbb{E}\gamma(\bar{X})'. \end{aligned}$$

Let $\{X_{mz}\}_{m,z=1}^{M,Z}$ be a double-indexed sequence. For every value of m ,

$$\left(\sum_{z=1}^Z \gamma(X_{mz})\right) \left(\sum_{z'=1}^Z \gamma(X_{mz'})\right)' = \sum_{1 \leq z, z' \leq Z} \gamma(X_{mz})\gamma(X_{mz'})'.$$

Define

$$\begin{aligned} M_1 &:= \mathbb{E} \sum_{m=1}^M \left(\sum_{z=1}^Z \gamma(X_{mz})\right) \left(\sum_{z'=1}^Z \gamma(X_{mz'})\right)' = \sum_{m=1}^M \sum_{1 \leq z, z' \leq Z} \mathbb{E}\gamma(X_{mz})\gamma(X_{mz'})' \\ M_2 &:= \sum_{m=1}^M \left(\mathbb{E} \sum_{z=1}^Z \gamma(X_{mz})\right) \left(\mathbb{E} \sum_{z'=1}^Z \gamma(X_{mz'})\right)' = \sum_{m=1}^M \sum_{1 \leq z, z' \leq Z} \mathbb{E}\gamma(X_{mz})\mathbb{E}\gamma(X_{mz'})' \end{aligned}$$

Invoking (F17) gives

$$\sum_{m=1}^M \mathbb{E} \left[\sum_{z=1}^Z \phi(X_{mz}) \right] \left[\sum_{z'=1}^Z \phi(X_{mz'}) \right]' = M_1 - M_2.$$

Step 3. The bound on $\|M_1\|$ is

$$\begin{aligned} \|M_1\| &\leq \sum_{m=1}^M \sum_{1 \leq z, z' \leq Z} \|\mathbb{E}\gamma(X_{mz})\gamma(X_{mz'})'\| \leq 1/2 \sum_{m=1}^M \sum_{z=1}^Z \sum_{z'=1}^Z (\mathbb{E}\|\gamma(X_{mz})\|^2 + \mathbb{E}\|\gamma(X_{mz'})\|^2) \\ &= Z/2 \left(\sum_{m=1}^M \sum_{z=1}^Z \mathbb{E}\|\gamma(X_{mz})\|^2 + \sum_{m=1}^M \sum_{z'=1}^Z \mathbb{E}\|\gamma(X_{mz'})\|^2 \right) = Z \sum_{m=1}^M \sum_{z=1}^Z \mathbb{E}\|\gamma(X_{mz})\|^2. \end{aligned} \quad (\text{F18})$$

Likewise,

$$\|M_2\| \leq \sum_{m=1}^M \sum_{1 \leq z, z' \leq Z} \|\mathbb{E}\gamma(X_{mz})\mathbb{E}\gamma(X_{mz'})'\|$$

$$\begin{aligned}
&\leq 1/2 \sum_{m=1}^M \sum_{z=1}^Z \sum_{z'=1}^Z (\mathbb{E}\|\gamma(X_{mz})\|^2 + \mathbb{E}\|\gamma(X_{mz'})\|^2) \\
&= Z \sum_{m=1}^M \sum_{z=1}^Z \mathbb{E}\|\gamma(X_{mz})\|^2.
\end{aligned} \tag{F.19}$$

As a result,

$$\|M_1 - M_2\| \leq \|M_1\| + \|M_2\| \leq 2Z \sum_{m=1}^M \sum_{z=1}^Z \mathbb{E}\|\gamma(X_{mz})\|^2.$$

Because the bounds (F.15) and (F.16) are invariant to transpositions of $\gamma(X)$ and/or $\gamma(\bar{X})$,

$$\left\| \sum_{m=1}^M \mathbb{E} \left[\sum_{z=1}^Z \phi(X_{mz})' \right] \left[\sum_{z=1}^Z \phi(X_{mz'}) \right] \right\| \leq 2Z \sum_{m=1}^M \sum_{z=1}^Z \mathbb{E}\|\gamma(X_{mz})\|^2. \tag{F.20}$$

Step 4. We first verify the condition (F.5) for the odd-numbered full-sized blocks. We note that the L.H.S of (F.5) is a special case of the L.H.S of (F.20) with

$$\begin{aligned}
m &= m(i, l) = L \cdot (i - 1) + l, \quad M = NL, \quad Z = q \\
X_{mz} &:= X_{i, (2l-2)q+z} \\
\phi(B_{i(2l-1)}) &= \sum_{t=(2l-2)q+1}^{t=(2l-2)q+q} \phi(X_{it}) = \sum_{z=1}^q \phi(X_{mz}).
\end{aligned}$$

As a result,

$$\left\| \sum_{i=1}^N \sum_{l=1}^L \mathbb{E} \phi(B_{i(2l-1)}^*) \phi(B_{i(2l-1)}^*)' \right\| \leq 2q \sum_{i=1}^N \sum_{l=1}^L \sum_{z=1}^q \mathbb{E}\|\gamma(X_{i(2l-1),z}^*)\|^2 \tag{F.21}$$

$$= 2q \sum_{i=1}^N \sum_{l=1}^L \sum_{z=1}^q \mathbb{E}\|\gamma(X_{i(2l-1),z})\|^2 \tag{F.22}$$

$$\leq 2q \sum_{i=1}^N \sum_{t=1}^T \mathbb{E}\|\gamma(X_{it})\|^2 = 2qNT\gamma_{2NT}^2. \tag{F.23}$$

A similar argument for even-numbered full-sized blocks and $\phi(B_{i(2l)}^*) = \sum_{t=(2l-1)q+1}^{t=(2l)q} \phi(X_{it}^*)$ verifies condition (F.6) of Lemma F.2. Finally, if the remainder block is non-empty, i.e., $T - 2Lq \neq 0$, invoking (F.20) with

$$m = i, \quad M = N, \quad Z := T - 2Lq$$

and noting that

$$\left\| \sum_{i=1}^N \mathbb{E} \phi(B_{ir}) \phi(B_{ir})' \right\| \leq 2q \sum_{i=1}^N \sum_{z=1}^{T-2Lq} \mathbb{E} \|\gamma(X_{iz})\|^2 \leq 2q \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \|\gamma(X_{it})\|^2 = 2qNT\gamma_{2NT}^2, \quad (\text{E24})$$

which verifies condition (F7) of Lemma F2. Finally, the condition (F4) follows from

$$\|\phi(B_{i(2l-1)}^*)\| \leq qR \text{ a.s.}, \|\phi(B_{i(2l)}^*)\| \leq qR \text{ a.s.}, \|\phi(B_{ir})\| \leq qR,$$

since each block has at most q summands. Plugging $R = 2\gamma_{NT}^\infty$ and $q = 2\gamma_{2NT}^2$ into (F9) gives (F14). ■

Corollaries F1 and F2 are special cases of Lemma F3 with various cases of γ -function.

COROLLARY F1 (Covariance Matrix Moments). *Let $\psi(X) : \mathcal{X} \rightarrow \mathbb{R}^{d \times 1}$ be a fixed vector function of a random vector X . Define*

$$\gamma(X) = \psi(X)\psi(X)' \quad (\text{E25})$$

and the ϕ -function

$$\phi(X) := \gamma(X) - \mathbb{E}[\gamma(X)] = \psi(X)\psi(X)' - \mathbb{E}[\psi(X)\psi(X)'].$$

Let the numeric sequences ψ_{NT}^∞ and ψ_{4NT} obey

$$\sup_{it} \|\psi(X_{it})\| \leq \psi_{NT}^\infty \text{ a.s.} \quad (\text{E26})$$

$$(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \|\psi(X_{it})\|^4 \leq \psi_{4NT}^4 \quad (\text{E27})$$

Then, the bound (F12) hold with $\gamma_{NT}^\infty := (\psi_{NT}^\infty)^2$ and $\gamma_{2NT}^2 := \psi_{4NT}^4$. As a result, the rate (F14) reduces to

$$\|(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \phi(X_{it})\| \lesssim_P \sqrt{\psi_{4NT}^4 \log(NT) \log(2d)/NT} + \log(NT)(\psi_{NT}^\infty)^2 \log(2d)/NT. \quad (\text{E28})$$

PROOF OF COROLLARY F1. Noting that

$$\|\gamma(X_{it})\|_\infty \leq \|\psi(X_{it})\|_\infty^2 \leq (\psi_{NT}^\infty)^2$$

and

$$\|\gamma(X_{it})\|^2 = \|\psi(X_{it})\psi(X_{it})'\|^2 \leq \|\psi(X_{it})\|^2 \|\psi(X_{it})'\|^2 = \|\psi(X_{it})\|^4$$

Therefore,

$$(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \|\gamma(X_{it})\|^2 \leq (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \|\psi(X_{it})\|^4 \leq \psi_{4NT}^4.$$

Application of Lemma F3 yields the result. \blacksquare

COROLLARY F2 (Product Moments). *Let $\psi(X) : \mathcal{X} \rightarrow \mathbb{R}^{d \times 1}$ be a fixed vector function of a random vector X and $\xi(X)$ be a random variable. Define*

$$\gamma(X) := \psi(X) \cdot \xi(X)$$

Let the numeric sequences $\psi_{NT}^\infty, \xi_{NT}^\infty$ and ψ_{4NT}, ξ_{4NT} obey

$$\sup_{it} \|\psi(X_{it})\| \leq \psi_{NT}^\infty \text{ a.s.}, \quad \sup_{it} |\xi(X_{it})| \leq \xi_{NT}^\infty \text{ a.s.} \quad (\text{F29})$$

$$(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \|\psi(X_{it})\|^4 \leq \psi_{4NT}^4, \quad (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \xi^4(X_{it}) \leq \xi_{4NT}^4. \quad (\text{F30})$$

Then, the bound (F12) holds with

$$\gamma_{NT}^\infty := (\psi_{NT}^\infty) \cdot \xi_{NT}^\infty, \quad \gamma_{2NT}^2 := 1/2(\psi_{4NT}^4 + \xi_{4NT}^4).$$

As a result, the rate (F14) reduces to

$$\begin{aligned} \|(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \phi(X_{it})\| &\lesssim_P \sqrt{(\psi_{4NT}^4 + \xi_{4NT}^4) \log(NT) \log(d+1)/NT} \\ &+ \log(NT) \psi_{NT}^\infty \xi_{NT}^\infty \log(d+1)/NT. \end{aligned} \quad (\text{F31})$$

PROOF OF COROLLARY F2 . Noting that

$$\|\gamma(X_{it})\|_\infty \leq \|\psi(X_{it})\|_\infty |\xi(X_{it})| \leq \psi_{NT}^\infty \xi_{NT}^\infty$$

and

$$\|\gamma(X_{it})\|^2 = \|\psi(X_{it})\|^2 \xi^2(X_{it}).$$

Cauchy inequality gives

$$\mathbb{E} \|\gamma(X_{it})\|^2 = \mathbb{E} \|\psi(X_{it})\|^2 \xi^2(X_{it}) \leq 1/2(\mathbb{E} \|\psi(X_{it})\|^4 + \mathbb{E} \xi^4(X_{it})).$$

Therefore,

$$(NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \|\gamma(X_{it})\|^2 \leq (NT)^{-1} \left(\sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \|\psi(X_{it})\|^4 + \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \xi^4(X_{it}) \right) \leq (\psi_{4NT}^4 + \xi_{4NT}^4)/2.$$

Application of Lemma F3 yields the result. \blacksquare

APPENDIX G: ADDITIONAL RESULTS ON ORTHOGONAL OLS

ASSUMPTION 7.1 (Tail Bound on Empirical Covariance Matrix in ℓ_2 norm). *For some sequence $v_{NT} = o(1)$, in the regime where $d \rightarrow \infty$, we have that*

$$\|\tilde{Q} - Q\| \lesssim_P v_{NT}. \quad (\text{E32})$$

Remark F.2. Suppose Assumptions 4.1–4.3 hold and $\sup_{it} \|V_{it}\|_\infty \leq R$ a.s. and

$$\max_{itj} \text{EV}_{itj}^4 \leq \sigma_{4V}^4$$

We invoke Corollary F.1 with $\psi(W_{it}) = V_{it}$ and $\psi_{NT}^\infty := \sqrt{d}R$ and (F.27) with $\psi_{4NT}^4 = d^2 \sigma_{4V}^4$. As a result, the rate bound (F.28) reduces to

$$v_{NT} = \sqrt{d^2 \log(2d) \log(NT)/NT + dR \log(2d) \log(NT)/NT}.$$

Further improvement of this rate may be possible under additional structure on V_{it} , see, e.g., Theorem 1 and Corollary 3 in [Banna et al. \(2016\)](#).

Let $D_{NT} \times L_{NT}$ be a sequence of realization sets such that the following conditions hold. Let $\mathbf{d}_{NT}, \mathbf{l}_{NT}, \mathbf{d}_{NT,4}, \mathbf{l}_{NT,4}$ be the numeric sequences obeying the following bounds

$$\begin{aligned} \sup_{\mathbf{d} \in D_{NT}} (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T (\mathbb{E} \|d_i(X_{it}) - d_{i0}(X_{it})\|^2)^{1/2} &\leq \mathbf{d}_{NT}, \\ \sup_{\mathbf{d} \in D_{NT}} (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T (\mathbb{E} \|d_i(X_{it}) - d_{i0}(X_{it})\|^4)^{1/4} &\leq \mathbf{d}_{NT,4}, \\ \sup_{\mathbf{l} \in L_{NT}} (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T (\mathbb{E} (l_i(X_{it}) - l_{i0}(X_{it}))^4)^{1/4} &\leq \mathbf{l}_{NT,4}. \end{aligned}$$

Define the following rates

$$r_{2NT} := \mathbf{d}_{NT} \mathbf{l}_{NT} + \sqrt{\frac{(\mathbf{d}_{NT,4}^4 + \mathbf{l}_{NT,4}^4) \log(NT) \log(d+1)}{NT}} + \sqrt{d} \log(d+1) \log(NT)/NT \quad (\text{E33})$$

$$\chi_{NT} := \mathbf{d}_{NT}^2 + \sqrt{d \mathbf{d}_{NT,4}^4 \log(2d) \log(NT)/NT + d \log(2d) \log(NT)/NT} \quad (\text{E34})$$

ASSUMPTION 7.2. *We suppose that the true parameter vector has bounded ℓ_2 -norm:*

$$\|\beta_0\|_2 \leq \bar{C}_\beta$$

for some finite constant \bar{C}_β ; We suppose that the reduced form estimators obey: $\hat{\mathbf{l}}(\cdot) \in L_{NT}$ and $\hat{\mathbf{d}}(\cdot) \in D_{NT}$ if such that $\mathbf{d}_{NT}, \mathbf{d}_{NT,4}, \mathbf{l}_{NT}, \mathbf{l}_{NT,4}$ decay sufficiently fast:

$$r_{2NT} + \chi_{NT} = o((NT)^{-1/2}) \quad (\text{E35})$$

Furthermore, the reduced form estimates are bounded as

$$\sup_{\mathbf{d} \in D_{NT}} \|d_i(X_{it})\| \leq \sqrt{d}D, \quad \sup_{\mathbf{l} \in L_{NT}} |l_i(X_{it})| \leq L \quad \forall i.$$

THEOREM F1 (Orthogonal Least Squares). *Suppose Assumptions 4.1–4.3, 7.1 and 7.2 hold. Then, the following statements hold.*

1. *The Orthogonal Least Squares estimator converges at the rate $\sqrt{d/NT}$:*

$$\|\widehat{\beta}_{OLS} - \beta_0\|_2 \lesssim_P \sqrt{d/NT}. \quad (\text{F36})$$

2. *For any deterministic sequence $\{\alpha\} = \{\alpha_{N,T}\}$ with $\|\alpha_{N,T}\| = 1$, the estimator $\alpha' \widehat{\beta}_{OLS}$ of $\alpha' \beta_0$ is asymptotically linear:*

$$\sqrt{NT} \alpha' (\widehat{\beta}_{OLS} - \beta_0) = \alpha' Q^{-1} \mathbf{G}_{NT} V_{it} U_{it} + o_P(1), \quad (\text{F37})$$

3. *If the Lindeberg condition holds for each $M > 0$:*

$$\limsup_{NT \rightarrow \infty} \sup_{\|\alpha\|_2=1} (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E}[(\alpha' V_{it} U_{it})^2 \mathbf{1}\{|\alpha' V_{it} U_{it}| > M\sqrt{NT}\}] = 0,$$

then the Orthogonal Least Squares estimator is asymptotically Gaussian:

$$\lim_{NT \rightarrow \infty} \sup_{\|\alpha\|_2=1} \sup_{t \in \mathbb{R}} \left| \mathbb{P} \left(\frac{\sqrt{NT} \alpha' (\widehat{\beta}_{OLS} - \beta_0)}{\|\alpha' \Sigma\|^{1/2}} < t \right) - \Phi(t) \right| = 0. \quad (\text{F38})$$

LEMMA F4 (First-Order Terms, ℓ_2 -norm). *Let $\bar{a}, \bar{m}, \bar{f}, \bar{e}$ be as in (D.2)–(D.5). Under Assumptions 4.1–4.3, the following bounds hold*

$$\|\bar{a}\| \lesssim_P \left(\sqrt{d/NT} \mathbf{d}_{NT} \right) \quad (\text{F39})$$

$$\|\bar{m}\| \lesssim_P \left(\sqrt{d/NT} \mathbf{l}_{NT} \right) \quad (\text{F40})$$

$$\|\bar{f}\| \lesssim_P \left(\mathbf{d}_{NT} (NT)^{-1/2} \right) \quad (\text{F41})$$

$$\|\bar{e}\| \lesssim_P \left(\sqrt{d/NT} (\mathbf{d}_{NT} + \mathbf{l}_{NT}) \right) \quad (\text{F42})$$

PROOF OF LEMMA F4. Define

$$\xi_{NT}^V := \sqrt{d/NT} \mathbf{d}_{NT}, \quad \xi_{NT}^B = 0,$$

and the A -function as

$$A(W_{it}, \eta) = V_{it} (d_{i0}(X_{it}) - d_i(X_{it})). \quad (\text{F43})$$

Define $B_{Ak}(\eta)$ and $V_{Ak}(\eta)$ with $\eta = \mathbf{d}$ as in (A.8)–(A.9). Consider any $\eta = \eta_{NT} \in D_{NT}$ in what follows. Since V_{it} obeys the martingale difference property (D.13), it follows that $\|B_{Ak}(\eta_{NT})\| = 0$. Furthermore, for any $1 \leq j, j' \leq d$,

$$\mathbb{E}[(\alpha' V_{it})(\alpha' V_{it'}) (d_{i0}(X_{it}) - d_i(X_{it}))_j (d_{i0}(X_{it'}) - d_i(X_{it'}))_{j'}] = 0. \quad (\text{F44})$$

Combining (F44) and Assumption 4.3

$$\begin{aligned}
\mathbb{E}[\|\alpha' V_{Ak}(\eta_{NT})\|^2] &= {}^i (NT_k)^{-2} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} \sum_{j=1}^d \mathbb{E}[(\alpha' V_{it})^2 (d_{i0}(X_{it}) - d_i(X_{it}))_j^2] \\
&\leq (NT_k)^{-2} \sup_{\mathbf{d} \in D_{NT}} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} \sum_{j=1}^d \mathbb{E} \left[\mathbb{E}[\|V_{it}\|^2 | \Phi_{it}, X_{it}] (d_{i0}(X_{it}) - d_i(X_{it}))_j^2 \right] \\
&\leq (NT_k)^{-2} \sup_{\mathbf{d} \in D_{NT}} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \|d_{i0}(X_{it}) - d_i(X_{it})\|^2 d\sigma_V^2 \\
&\leq {}^{ii} (d/NT_k) \sigma_V^2 (T/T_k) \mathbf{d}_{NT}^2,
\end{aligned}$$

where (i) follows from (F44) and (ii) follows from definition of \mathbf{d}_{NT} . By Assumption 4.5, we have that $\mathbb{P}(\widehat{\mathbf{d}}_k \in D_{NT}, \forall k = 1, \dots, K) \rightarrow 1$. Moreover, since the number of cross-fit folds is finite, the size T_k of each fold obeys

$$1 \lesssim T_k/T \leq 1.$$

We conclude by Lemma A.6 that (F39) holds. Repeating the same argument for

$$A(W_{it}, \eta) = V_{it}(l_{i0}(X_{it}) - l_i(X_{it})) \text{ and } A(W_{it}, \eta) = U_{it}(d_{i0}(X_{it}) - d_i(X_{it}))$$

establishes claims (F40) and (F41). Finally, (D.12) holds by definition of $\bar{e} = \bar{m} - \bar{a}'\beta_0$ and Holder inequalities. ■

In the Lemma below, abusing the notation, we treat l_i as some generic vector-valued function.

LEMMA F5 (Second-Order Bias). *Let $d_{i0}(X_{it})$ be a d_1 -vector and $l_{i0}(X_{it})$ be a d_2 -vector. Suppose that*

$$\begin{aligned}
\sup_{\mathbf{d} \in D_{NT}} (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T (\mathbb{E} \|d_i(X_{it}) - d_{i0}(X_{it})\|^2)^{1/2} &\leq \mathbf{d}_{NT}, \\
\sup_{\mathbf{l} \in D_{NT}} (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T (\mathbb{E} \|l_i(X_{it}) - l_{i0}(X_{it})\|^2)^{1/2} &\leq \mathbf{l}_{NT}
\end{aligned}$$

Consider the A -function as

$$A(W_{it}, \eta) = (d_{i0}(X_{it}) - d_i(X_{it}))(l_{i0}(X_{it}) - l_i(X_{it})), \quad \eta = (\mathbf{d}, \mathbf{l}) \quad (\text{F45})$$

and its bias $B_{Ak}(\eta)$ as in (A.8). Then, we have the bias bound:

$$\sup_{\eta \in (D_{NT}, L_{NT})} \|B_{Ak}(\eta)\|_2 \leq \mathbf{d}_{NT} \mathbf{l}_{NT} (T/T_k).$$

PROOF OF LEMMA F.5. Take $\alpha \in \mathcal{S}^{d-1}$. Let $X_{it}(\alpha) := \alpha'(d_{i0}(X_{it}) - d_i(X_{it}))$ and $Y_{itj} := (l_{i0}(X_{it}) - l_i(X_{it}))_j$ and

$$a_{it}^2 = \|d_{i0}(X_{it}) - d_i(X_{it})\|^2, \quad b_{it}^2 = \|l_{i0}(X_{it}) - l_i(X_{it})\|^2 = \sum_{j=1}^d b_{itj}^2$$

Recognize that

$$\begin{aligned} \alpha' B_{Akj}(\eta) &= (NT_k)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} E \alpha'(d_{i0}(X_{it}) - d_i(X_{it})) (l_{i0}(X_{it}) - l_i(X_{it}))_j \\ &= (NT_k)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} E X_{it}(\alpha) Y_{itj}. \end{aligned}$$

Cauchy inequality gives

$$|E X_{it}(\alpha) Y_{itj}| \leq \sqrt{E X_{it}^2(\alpha) E Y_{itj}^2} \leq \sqrt{E \|d_{i0}(X_{it}) - d_i(X_{it})\|^2 E Y_{itj}^2} =: \sqrt{a_{it}^2 b_{itj}^2}.$$

Summing over i and t and invoking Cauchy inequality gives

$$\begin{aligned} \alpha' B_{Akj}(\eta) &\leq (NT_k)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} \sqrt{a_{it}^2 b_{itj}^2} \leq (NT_k)^{-1} \sqrt{\sum_{i=1}^N \sum_{t \in \mathcal{M}_k} a_{it}^2} \sqrt{\sum_{i=1}^N \sum_{t \in \mathcal{M}_k} b_{itj}^2}. \\ \|\alpha' B_{Ak}(\eta)\|^2 &= \sum_{j=1}^d |\alpha' B_{Akj}(\eta)|^2 \leq (NT_k)^{-1} \left(\sum_{i=1}^N \sum_{t \in \mathcal{M}_k} a_{it}^2 \right) (NT_k)^{-1} \left(\sum_{j=1}^d \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} b_{itj}^2 \right) \\ &\leq (NT_k)^{-1} \left(\sum_{i=1}^N \sum_{t=1}^T a_{it}^2 \right) (NT_k)^{-1} \left(\sum_{i=1}^N \sum_{t=1}^T b_{it}^2 \right) \\ &\leq \mathbf{1}_{NT}^2 \mathbf{d}_{NT}^2 (T/T_k)^2. \end{aligned}$$

■

Next, we invoke Lemmas F.2 and F.3 and Corollary F.1-F.2 from Appendix F.

LEMMA F.6 (Second-Order Covariance Term). *Define*

$$\zeta_{NT}^B = \mathbf{d}_{NT}^2, \quad \zeta_{NT}^V = \sqrt{\mathbf{d}_{NT,4}^4 \log(2d) \log NT / NT} + dD \log(2d) \log(NT) / NT.$$

Under Assumptions 4.1–4.3, the following bounds hold for the term \bar{b} defined in (D.6)

$$\|\bar{b}\| \lesssim_P \zeta_{NT}^B + \zeta_{NT}^V = \chi_{NT}. \quad (\text{E.46})$$

PROOF OF LEMMA F.6. Define the A -function as

$$A(W_{it}, \eta) = (d_{i0}(X_{it}) - d_i(X_{it}))(d_{i0}(X_{it}) - d_i(X_{it}))', \quad \eta = \mathbf{d} = \mathbf{1}.$$

Let $B_{Ak}(\eta)$ and $V_{Ak}(\eta)$ be defined according to (A.8)–(A.9). Invoking Lemma F5 with $\mathbf{l} = \mathbf{d}$ gives $\|B_{Ak}(\eta_{NT})\|_\infty = O(\zeta_{NT}^B)$ for any partition k . Note that

$$\begin{aligned} V_{Ak}(\eta_{NT}) &= (NT_k)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} \left((d_{i0}(X_{it}) - d_i(X_{it}))(d_{i0}(X_{it}) - d_i(X_{it}))' \right. \\ &\quad \left. - \mathbb{E}[(d_{i0}(X_{it}) - d_i(X_{it}))(d_{i0}(X_{it}) - d_i(X_{it}))'] \right) \\ &=: (NT_k)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} \phi_i(X_{it}). \end{aligned}$$

Define

$$\begin{aligned} \psi_i(X_{it}) &= (d_{i0}(X_{it}) - d_i(X_{it})) \\ \gamma_i(X_{it}) &= \psi_i(X_{it})\psi_i(X_{it})' = (d_{i0}(X_{it}) - d_i(X_{it}))(d_{i0}(X_{it}) - d_i(X_{it}))' \\ \phi_i(X_{it}) &= \gamma_i(X_{it}) - \mathbb{E}[\gamma_i(X_{it})]. \end{aligned}$$

Note that $\psi_i(X_{it}) = (d_{i0}(X_{it}) - d_i(X_{it}))$ obeys the conditions (F26) and (F27) with

$$\psi_{NT}^\infty := \sqrt{d}D, \quad \psi_{NT,4} := \mathbf{d}_{NT,4}.$$

As a result, the bound (F28) reduces to ζ_{NT}^V for each partition k and $T = T_k$. Since $T_k/T \asymp 1$, the bound follows. ■

LEMMA F7 (Second-Order Covariance Term, cont.). *Suppose Assumptions 4.1–4.3 and 7.2 hold. Let \bar{z} and \bar{g} be as defined in (D.7) and (D.8). Then,*

$$\|\bar{z}\| \lesssim_P r_{2NT} \tag{F47}$$

$$\|\bar{g}\| \lesssim_P r_{2NT} + \chi_{NT}. \tag{F48}$$

PROOF OF LEMMA F7. Define the A -function as

$$A(W_{it}, \eta) = (d_{i0}(X_{it}) - d_i(X_{it}))(l_{i0}(X_{it}) - l_i(X_{it})), \quad \eta = (\mathbf{d}, \mathbf{l}).$$

Let $B_{Ak}(\eta)$ and $V_{Ak}(\eta)$ be defined according to (A.8)–(A.9). Let

$$\zeta_{NT}^B = \mathbf{d}_{NT}\mathbf{l}_{NT}, \quad \zeta_{NT}^V = \sqrt{\frac{(\mathbf{d}_{NT,4}^4 + \mathbf{l}_{NT,4}^4) \log(NT) \log(d+1)}{NT}} + \sqrt{d}D \log(2d) \log(NT)/NT.$$

Invoking Lemma F5 with $d_1 = d$ and $d_2 = 1$ gives $\|B_{Ak}(\eta_{NT})\|_\infty = O(\zeta_{NT}^B)$ for any partition k . Note that

$$V_{Ak}(\eta_{NT}) = (NT_k)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} \left((d_{i0}(X_{it}) - d_i(X_{it}))(l_{i0}(X_{it}) - l_i(X_{it})) \right.$$

$$\begin{aligned}
& - \mathbb{E}[(d_{i0}(X_{it}) - d_i(X_{it}))(l_{i0}(X_{it}) - l_i(X_{it}))'] \\
& =: (NT_k)^{-1} \sum_{i=1}^N \sum_{t \in \mathcal{M}_k} \phi_i(X_{it}).
\end{aligned}$$

Define

$$\begin{aligned}
\psi_i(X_{it}) &= (d_{i0}(X_{it}) - d_i(X_{it})), \quad \xi_i(X_{it}) = l_{i0}(X_{it}) - l_i(X_{it}) \\
\gamma_i(X_{it}) &= \psi_i(X_{it})\xi_i(X_{it}) = (d_{i0}(X_{it}) - d_i(X_{it}))(l_{i0}(X_{it}) - l_i(X_{it})) \\
\phi_i(X_{it}) &= \gamma_i(X_{it}) - \mathbb{E}[\gamma_i(X_{it})].
\end{aligned}$$

Note that $\psi_i(X_{it}) = (d_{i0}(X_{it}) - d_i(X_{it}))$ and $\xi_i(X_{it}) = l_{i0}(X_{it}) - l_i(X_{it})$ obey the conditions (F29) and (F30) with

$$\psi_{NT}^\infty := \sqrt{d}D, \quad \psi_{NT,4} := \mathbf{d}_{NT,4}, \quad \xi_{NT}^\infty := L, \quad \xi_{NT,4} := \mathbf{l}_{NT,4}.$$

As a result, the bound (F31) reduces to ζ_{NT}^V for each partition k and $T = T_k$. Since $T_k/T \asymp 1$, the bound (F47) follows. Recognizing that $\bar{g} = \bar{z} - \bar{b}'\beta_0$ and invoking $\|\beta_0\| \leq C_\beta$ as in Assumption 7.2 gives

$$\|\bar{g}\| \leq \|\bar{z}\| + \|\bar{b}\beta_0\| \leq \|\bar{z}\| + \|\bar{b}\|\|\beta_0\|,$$

(F48) follows. ■

PROOF OF THEOREM F.1 . Step 0. Let $R_{it}(\hat{d}, \hat{l})$ be as defined in (D.1). Let $\bar{a}, \bar{b}, \bar{e}, \bar{f}, \bar{g}$ be as defined in (D.2), (D.6), ..., (D.8). As shown in the proof of Lemma D.3, the Gram matrix estimation error

$$\widehat{Q} - \widetilde{Q} = \mathbb{E}_{NT} \widehat{V}_{it} \widehat{V}'_{it} - \mathbb{E}_{NT} V_{it} V'_{it} = \bar{a} + \bar{a}' + \bar{b}$$

and gradient estimation error

$$\widehat{S} - S = \mathbb{E}_{NT} \widehat{V}_{it} (U_{it} + R_{it}(\hat{\mathbf{d}}, \hat{\mathbf{l}})) - V_{it} U_{it} = \bar{e} + \bar{f} + \bar{g}.$$

We have that

$$\|\widehat{Q} - Q\| \leq \|\widehat{Q} - \widetilde{Q}\| + \|\widetilde{Q} - Q\| \lesssim_P^i (\chi_{NT} + \sqrt{d/NT} \mathbf{d}_{NT} + v_{NT}) = o(1),$$

where (i) follows from Lemmas F4-F7 and Assumption 7.1. Furthermore, by Lemmas

$$\|\widehat{S} - S\| = \|\bar{e} + \bar{f} + \bar{g}\| \lesssim_P (r_{2NT} + \chi_{NT}) = o(1/\sqrt{NT}),$$

where we used Assumptions 4.1–4.3 and 7.2 to conclude that $r_{2NT} + \chi_{NT} = o(1/\sqrt{NT})$.

Step 1. Since Q is invertible by assumption, \widehat{Q} is also invertible wp $1 - o(1)$ by Step 0. Therefore, we can decompose $\widehat{\beta}_{OLS} - \beta_0$ as

$$\widehat{\beta}_{OLS} - \beta_0 = \widehat{Q}^{-1} \mathbb{E}_{NT} [\widehat{V}_{it} \widehat{Y}_{it}] - \widehat{Q}^{-1} \widehat{Q}' \beta_0 = \widehat{Q}^{-1} \mathbb{E}_{NT} [\widehat{V}_{it} \widehat{Y}_{it}] - \widehat{Q}^{-1} (\mathbb{E}_{NT} \widehat{V}_{it} \widehat{V}'_{it}) \beta_0$$

$$\begin{aligned}
&= \widehat{Q}^{-1} \mathbb{E}_{NT} [\widehat{V}_{it} (\widehat{Y}_{it} - \widehat{V}_{it}' \beta_0)] \\
&= \widehat{Q}^{-1} \mathbb{E}_{NT} [\widehat{V}_{it} (U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{l}}))] \\
&= \widehat{Q}^{-1} \mathbb{E}_{NT} V_{it} U_{it} + \widehat{Q}^{-1} \mathbb{E}_{NT} [\widehat{V}_{it} (U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{l}})) - V_{it} U_{it}].
\end{aligned}$$

Therefore the following bound holds by triangle and Holder inequalities:

$$\|\widehat{\beta}_{OLS} - \beta_0\| \leq \|\widehat{Q}^{-1}\| \|\mathbb{E}_{NT} V_{it} U_{it}\| + \|\widehat{Q}^{-1}\| \|\widehat{S} - S\| =: \|\widehat{Q}^{-1}\| (L_1 + L_2).$$

The first term L_1 is bounded as

$$\begin{aligned}
\mathbb{E} \|\mathbb{E}_{NT} V_{it} U_{it}\|^2 &= \sum_{j=1}^d \mathbb{E} (\mathbb{E}_{NT} (V_{it})_j U_{it})^2 \\
&= (NT)^{-2} \sum_{j=1}^d \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} ((V_{it})_j U_{it})^2 \\
&\leq (NT)^{-2} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \|V_{it}\|^2 \sup_{it} \mathbb{E} [U_{it}^2 | V_{it}] \\
&\leq \bar{\sigma}^2 (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \text{trace}(\mathbb{E} V_{it} V_{it}') \\
&= \bar{\sigma}^2 \text{trace}(Q) \leq^{ii} (d/NT) C_{\max},
\end{aligned}$$

where (i) follows from the m.d.s property in Lemma B.3, and (ii) from $\max \text{eig}(Q) \leq C_{\max}$. Markov inequality gives $L_1 \lesssim_P (\sqrt{d/NT})$. The second term $L_2 := \|\widehat{S} - S\|$ is $o_P(1/\sqrt{NT})$ by Step 0. Step 0 implies $\max \text{eig}(\widehat{Q}^{-1}) < 2C_{\min}^{-1}$ w.p. $1 - o(1)$. Therefore, the rate bound (E36) follows.

Step 2. From Step 1,

$$\begin{aligned}
\alpha' (\widehat{\beta}_{OLS} - \beta_0) &= \alpha' \widehat{Q}^{-1} \mathbb{E}_{NT} [\widehat{V}_{it} (U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{l}}))] \\
&= \alpha' Q^{-1} \mathbb{E}_{NT} V_{it} U_{it} \\
&\quad + \alpha' (\widehat{Q}^{-1} - Q^{-1}) \mathbb{E}_{NT} V_{it} U_{it} + \alpha' \widehat{Q}^{-1} [\mathbb{E}_{NT} [\widehat{V}_{it} (U_{it} + R_{it}(\widehat{\mathbf{d}}, \widehat{\mathbf{l}})) - V_{it} U_{it}]] \\
&=: \alpha' Q^{-1} \mathbb{E}_{NT} V_{it} U_{it} + S_1(\alpha) + S_2(\alpha).
\end{aligned}$$

The bound on $S_1(\alpha)$ follows

$$\begin{aligned}
|S_1(\alpha)| &\leq \|\widehat{Q}^{-1} - Q^{-1}\| \|\mathbb{E}_{NT} V_{it} U_{it}\| \\
&\leq \|\widehat{Q}^{-1}\| \|\widehat{Q} - Q\| \|Q^{-1}\| \|\mathbb{E}_{NT} V_{it} U_{it}\| \\
&= O_P(1) \cdot o_P(1) \cdot O_P(1) \cdot O_P((NT)^{-1/2}) = o_P((NT)^{-1/2}),
\end{aligned}$$

where $O_P(\cdot)$ and $o_P(\cdot)$ bounds are established in Steps 0-2. The bound on $S_2(\alpha)$ follows from:

$$|S_2(\alpha)| \leq \|\alpha\| C_{\min}^{-1} \|\widehat{S} - S\| \lesssim_P (r_{2NT} + \chi_{NT}) = o_P((NT)^{-1/2}),$$

where we are using the results of Step 0. As a result,

$$\sqrt{NT} \alpha' (\widehat{\beta}_{OLS} - \beta_0) = \alpha' Q^{-1} \mathbb{G}_{NT} V_{it} U_{it} + o_P(1),$$

which gives (E37).

Step 3. The proof of pointwise normality follows similarly to Step 1 of the proof of Theorem 4.2, where the step (D.51) is replaced by

$$|\alpha' (\alpha' \Sigma \alpha)^{-1/2} R_{NT}| \leq \|\alpha\|_2 O(1) \|R_{NT}\|_2 \lesssim_P \sqrt{NT} (r_{2NT} + \chi_{NT}) = o_P(1).$$

■

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