

Fixed Effects Binary Choice Models with Three or More Periods*

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Abstract

We consider fixed effects binary choice models with a fixed number of periods T and regressors without a large support. If the time-varying unobserved terms are i.i.d. with known distribution F , Chamberlain (2010) shows that the common slope parameter is point identified if and only if F is logistic. However, he only considers in his proof $T = 2$. We show that the result does not generalize to $T \geq 3$: the common slope parameter can be identified when F belongs to a family including the logit distribution. Identification is based on a conditional moment restriction. Under restrictions on the covariates, these moment conditions lead to point identification of relative effects. If $T = 3$ and mild conditions hold, GMM estimators based on these conditional moment restrictions reach the semiparametric efficiency bound. Finally, we illustrate our method by revisiting Brender and Drazen (2008).

Keywords: Binary choice models, panel data, point identification, conditional moment restrictions.

JEL Codes: C14, C23, C25.

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1 Introduction

In this paper, we revisit the classical binary choice model with fixed effects. Specifically, let T denote the number of periods and let us suppose to observe, for individual i , $(Y_{it}, X_{it})_{t=1, \dots, T}$ with

$$Y_{it} = \mathbf{1}\{X'_{it}\beta_0 + \gamma_i - \varepsilon_{it} \geq 0\} \quad (1.1)$$

where $\beta_0 \in \mathbb{R}^K$ is unknown and $\varepsilon_{it} \in \mathbb{R}$ is an idiosyncratic shock. The nonlinear nature of the model and the absence of restriction on the distribution of γ_i conditional on $X_i := (X'_{i1}, \dots, X'_{iT})'$ renders the identification of β_0 difficult. [Rasch \(1960\)](#) shows that if the $(\varepsilon_{it})_{t=1, \dots, T}$ are i.i.d. with a logistic distribution, a conditional maximum likelihood can be used to identify and estimate β_0 . [Chamberlain \(2010\)](#) establishes a striking converse of Rasch's result: if the $(\varepsilon_{it})_{t=1, \dots, T}$ are i.i.d. with distribution F and the support of X_i is bounded, β_0 is point identified only if F is logistic. Other papers have circumvented such an impossibility result by either considering large support regressors (see in particular [Manski, 1987](#); [Honore and Lewbel, 2002](#)) or allowing for dependence between the shocks (see [Magnac, 2004](#)).

It turns out, however, that [Chamberlain \(2010\)](#) only proves his result for $T = 2$. And in fact, we show that his result does not generalize to $T \geq 3$. Specifically, we consider distributions F satisfying

$$\frac{F(u)}{1 - F(u)} = \sum_{k=1}^{\tau} w_k \exp(\lambda_k u) \quad \text{or} \quad \frac{1 - F(u)}{F(u)} = \sum_{k=1}^{\tau} w_k \exp(-\lambda_k u), \quad (1.2)$$

with $T \geq \tau + 1$, $(w_1, \dots, w_\tau) \in (0, \infty)^\tau$ and $1 = \lambda_1 < \dots < \lambda_\tau$. We study the identification of β_0 , assuming that $\lambda := (\lambda_1, \dots, \lambda_\tau)$ is known. The weights w_1, \dots, w_τ remain unknown, thus allowing for much more flexibility on the distribution of ε_{it} than in the logit case. In particular, it may either be left- or right-skewed, platykurtic or leptokurtic. Our main insight is that for any F satisfying (1.2), a conditional moment restriction holds. We also obtain some results on the corresponding identified set B . For instance, if, roughly speaking, X_i is continuous, we show that B includes at most $T! - 1$ points (2 if $T = 3$) and relative marginal effects are point identified. Note that [Johnson \(2004\)](#) considers the same family with $\tau = 2$ and $T = 3$. However, he does not study the general case and does not show any formal identification result based on the corresponding moment conditions.

Obviously, the conditional moment condition can be used to construct GMM estimators. This means, in particular, that \sqrt{n} -consistent estimation is possible beyond the logit case when $T > 2$, overturning again the impossibility results of [Chamberlain \(2010\)](#) and [Magnac \(2004\)](#). Further, we show that if $T = 3$ and mild additional restrictions hold, the optimal GMM estimator based on our conditional moment conditions reaches the semiparametric efficiency bound of the model. Hence, at least when $T = 3$, these moment conditions contain all the information of the model.

Finally, we showcase the empirical relevance of our approach by studying whether budget deficits and economic growth affect reelection, revisiting [Brender and Drazen \(2008\)](#). The authors investigate this issue using simple and fixed effects logit models. However, the assumption of logistic errors is not warranted, so we consider whether the results are robust to this assumption on the unobserved terms. Our results suggest that the relative effects of budget deficits and economic growth or other variables are fairly robust to the logistic assumption.

Our paper is related to the seminal work of [Bonhomme \(2012\)](#), who develops a unified approach for models where the conditional distribution of (Y_1, \dots, Y_T) given (X_i, γ_i) is parametrized by β_0 , but no restriction on the distribution of $\gamma_i|X_i$ is imposed. In such set-ups, he shows that the identification and estimation of β_0 depends on the existence of functions $m \neq 0$ satisfying

$$\mathbb{E}(m(Y, X, \beta_0)|X, \gamma) = 0.$$

This approach has been fruitfully applied to the dynamic logit model by [Kitazawa \(2022\)](#) and [Honoré and Weidner \(2020\)](#). Our paper may be seen as yet another application of this approach, focusing on static models but dropping the logistic assumption.

The remainder of the paper is organized as follows. Section 2 describes the moment condition we use for identification of β_0 and establishes some properties of the identified set based on these moments. Section 3 discusses GMM estimation of β_0 , links it with the semiparametric efficiency bound of the model and discusses the case of unbalanced panel data. Section 4 is devoted to the application. Section 5 concludes. All the proofs are collected in the appendix.

2 Identification

2.1 The model and moment conditions

We drop the subscript i in the absence of ambiguity and let $Y = (Y'_1, \dots, Y'_T)'$, $X = (X'_1, \dots, X'_T)'$, $X_t = (X_{1,t}, \dots, X_{K,t})'$, $X_{k,\cdot} = (X_{k,1}, \dots, X_{k,T})'$, $X_{-k} = (X_{k',t})_{k' \neq k, t=1, \dots, T}$, $X_{k,-t} = (X_{k,s})_{s \neq t}$, and $X_{-k,t} = (X_{k',t})_{k' \neq k}$. $\text{Supp}(X) \subset \mathbb{R}^{KT}$ denotes the support of the random variable X . For any set $A \subset \mathbb{R}^p$ (for any $p \geq 1$), we let $A^* := A \setminus \{0\}$ and denote by $|A|$ the cardinal of A . Hereafter, we maintain the following conditions.

Assumption 1 (Binary choice panel model) *Equation (1.1) holds and:*

1. (X, γ) and $(\varepsilon_t)_{1 \leq t \leq T}$ are independent and the $(\varepsilon_t)_{1 \leq t \leq T}$ are i.i.d. with a known cumulative distribution function (cdf) F .
2. For all (k, t) , $\mathbb{E}[X_{k,t}^2] < \infty$.
3. $\beta_0 \in \mathbb{R}^{K^*}$.

The first condition is also considered in [Chamberlain \(2010\)](#). The second condition is a standard moment restriction on the covariates. Finally, we exclude in the third condition the case $\beta_0 = 0$ here. This case can be treated separately, as the following proposition shows.

Proposition 2.1 *Suppose that Assumption 1 holds, F is strictly increasing on \mathbb{R} and there exist $(t, t') \in \{1, \dots, T\}^2$ such that $\mathbb{E}[(X_t - X_{t'})(X_t - X_{t'})']$ is nonsingular. Then $\beta_0 = 0$ if and only if*

$$\mathbb{P}(Y_t = 1, Y_{t'} = 0 | Y_t + Y_{t'} = 1, X_t, X_{t'}) = \frac{1}{2} \quad a.s. \quad (2.1)$$

Condition (2.1) can be tested by a specification test on the nonparametric regression of $D = Y_t(1 - Y_{t'})$ on $(X_t, X_{t'})$, conditional on the event $Y_t + Y_{t'} = 1$. See, e.g., [Bierens \(1990\)](#) or [Hong and White \(1995\)](#).

Turning to identification on \mathbb{R}^{K^*} , we first recall the impossibility result of [Chamberlain \(2010\)](#). We say below that F is logistic if $G(u) := F(u)/(1 - F(u)) = w \exp(\lambda u)$ for some $(w, \lambda) \in \mathbb{R}^{+*2}$.

Theorem 2.2 *Suppose that $T = 2$, X_t includes $\mathbb{1}\{t = 2\}$, Assumption 1.1 holds, F is strictly increasing on \mathbb{R} with bounded, continuous derivative and $\text{Supp}(X)$ is compact. If F is not logistic, there exists $\beta_0 \in \mathbb{R}^{K^*}$, a distribution of $\gamma|X$ and an open ball $B \subset \mathbb{R}^K$ such that β_0 is not identified compared to $\beta \in B$.*

This result implies in particular that when $T = 2$ and F is not logistic, relative effects β_{0j}/β_{0k} , for k such that $\beta_{0k} \neq 0$, may not be identified. Such relative effects are important as they are equal to relative marginal effects if both $X_{j,t}$ and $X_{k,t}$ are continuous. If only $X_{k,t}$ is continuous (say), $-\beta_{0j}/\beta_{0k}$ still corresponds to a compensating variation.¹

The key step in Chamberlain's proof is that if β_0 is identified for all data generating process satisfying the restrictions of the theorem, the conditional probabilities (conditional on X and γ) of the four possible trajectories for (Y_1, Y_2) are necessarily affinely dependent. Moreover, by letting $|\gamma|$ tend to infinity, the stable trajectories $(0, 0)$ and $(1, 1)$ disappear from this relationship. This leads to the following functional equation for G :

$$\psi_1(\alpha)G(u) + \psi_2(\alpha)G(u + \alpha) = 0, \quad (2.2)$$

for all $u \in \mathbb{R}$, α in an open subset of \mathbb{R} and some functions $\psi_1(\cdot), \psi_2(\cdot)$ such that for all α , $(\psi_1(\alpha), \psi_2(\alpha)) \neq (0, 0)$. The result follows by noting that the solutions necessarily have the form $u \mapsto w \exp(\lambda u)$.

Equation (2.2) relies on the time dummy variable $\mathbb{1}\{t = 2\}$. However, the proof of Theorem 2 of Chamberlain (2010) shows that even without such a dummy variable, (2.2) is necessary for the semiparametric efficiency bound not to be zero, or, equivalently, for the existence of regular, root- n consistent estimators of β_0 . In this case, α corresponds to $(x_2 - x_1)'\beta_0$, for (x_1, x_2) in a set of positive measure.

¹To see the first point, note that under Assumptions 1-2,

$$\mu_{k,t}(x) := \frac{\partial \mathbb{P}(Y_t = 1 | X_{k,t} = x_{k,t}, X_{k,-t} = x_{k,-t}, X_{-k} = x_{-k})}{\partial x_{k,t}} = \beta_{0k} \mathbb{E}[F'(x_t' \beta_0 + \gamma) | X = x]$$

and thus $\mu_{j,t}(x)/\mu_{k,t}(x) = \beta_{0j}/\beta_{0k}$. Also, $-\beta_{0j}/\beta_{0k}$ corresponds to the change in $X_{k,t}$ necessary to keep $\mathbb{P}(Y_t = 1 | X_t, \alpha)$ constant when $X_{j,t}$ increases by one unit.

In any case, the same reasoning with $T = 3$ leads to the following equation for G :

$$\begin{aligned} \psi_1(\boldsymbol{\alpha})G(u) + \psi_2(\boldsymbol{\alpha})G(u + \alpha_1) + \psi_3(\boldsymbol{\alpha})G(u + \alpha_2) + \psi_4(\boldsymbol{\alpha})G(u)G(u + \alpha_1) \\ + \psi_5(\boldsymbol{\alpha})G(u)G(u + \alpha_2) + \psi_6(\boldsymbol{\alpha})G(u + \alpha_1)G(u + \alpha_2) = 0, \end{aligned} \quad (2.3)$$

for all $u \in \mathbb{R}$, $\boldsymbol{\alpha} := (\alpha_1, \alpha_2)$ in an open subset of \mathbb{R}^2 and some functions $\psi_k(\cdot)$, $k = 1, \dots, 6$, such that for all $\boldsymbol{\alpha}$, $(\psi_1(\boldsymbol{\alpha}), \dots, \psi_6(\boldsymbol{\alpha})) \neq (0, \dots, 0)$. We now have $6 = 2^3 - 2$ terms instead of just $2 = 2^2 - 2$, and thus we can expect to have other solutions than just $u \mapsto w \exp(\lambda u)$. And indeed, one can check that if G has the form $u \mapsto w_1 \exp(\lambda_1 u) + w_2 \exp(\lambda_2 u)$, we can construct $(\psi_1(\boldsymbol{\alpha}), \psi_2(\boldsymbol{\alpha}), \psi_3(\boldsymbol{\alpha})) \neq (0, 0, 0)$ such that (2.3) holds, with $\psi_4(\boldsymbol{\alpha}) = \psi_5(\boldsymbol{\alpha}) = \psi_6(\boldsymbol{\alpha}) = 0$. Similarly, if $1/G$ has the form $u \mapsto w_1 \exp(\lambda_1 u) + w_2 \exp(\lambda_2 u)$, we can construct $(\psi_4(\boldsymbol{\alpha}), \psi_5(\boldsymbol{\alpha}), \psi_6(\boldsymbol{\alpha})) \neq (0, 0, 0)$ such that (2.3) holds, with $\psi_1(\boldsymbol{\alpha}) = \psi_2(\boldsymbol{\alpha}) = \psi_3(\boldsymbol{\alpha}) = 0$. Note that there may still be other solutions to (2.3) that are increasing and have a limit of ∞ (resp. 0) at ∞ (resp. at $-\infty$). The question of identifying all such solutions is left for future research.

Generalizing this reasoning to any $T > 2$, we see that combinations of at most $T - 1$ exponential functions satisfy the functional restrictions tantamount to (2.3) and which render identification of β_0 possible. This suggests that identification may be achieved for the corresponding family of distribution, which we now formally introduce. Hereafter, Λ_τ denotes a subset of $\{(\lambda_1, \dots, \lambda_\tau) \in \mathbb{R}^\tau : 1 = \lambda_1 < \dots < \lambda_\tau\}$.

Assumption 2 (“Generalized” logistic distributions)² *There exist known $\tau \in \{1, \dots, T - 1\}$ and $\lambda := (\lambda_1, \dots, \lambda_\tau)' \in \Lambda_\tau$ and unknown $w := (w_1, \dots, w_\tau)' \in (0, \infty)^\tau$ such that:*

$$\begin{aligned} \text{Either } F(u)/(1 - F(u)) &= \sum_{j=1}^{\tau} w_j \exp(\lambda_j u) \quad (\text{First type}), \\ \text{or } (1 - F(u))/F(u) &= \sum_{j=1}^{\tau} w_j \exp(-\lambda_j u) \quad (\text{Second type}). \end{aligned}$$

Fixing $\min\{\lambda_1, \dots, \lambda_\tau\}$ to 1 is without loss of generality, as we can always multiply β_0 , γ_i and ε_{it} by this factor. If F is of the second type, then one can show that the cdf of $-\varepsilon_{it}$ is of the first type. Thus, up to changing (Y_t, X_t) into $(1 - Y_t, -X_t)$, we can

²Though we use the same name, our family of distributions should not be confused with those introduced by Balakrishnan and Leung (1988) and Stukel (1988).

assume without loss of generality, as we do afterwards, that F is of the first type. We shall see that $\tau + 1$ periods are sufficient to achieve identification. Hence, we assume, again without loss of generality, that $T = \tau + 1$: if $T > \tau + 1$, we can always focus on $\tau + 1$ periods.

Before describing our identification strategy of β_0 when F is a generalized logistic distribution, two remarks are in order. First, we obtain our results below irrespective of the vector w .³ Hence, in contradistinction with the fixed effect logistic model, we do not fix the distribution of ε , but simply impose that it belongs to a family of distributions indexed by two parameters. Members of this family differ in particular by their skewness and kurtosis. In linear regressions, the residuals are often found to have a skewed distribution with either positive or negative excess kurtosis. Then, there is no reason why the latent variables corresponding to Y_{it} would not exhibit a similar pattern. On the other hand, we do fix λ . Identification of λ could also be of interest but is not addressed in this paper.

Now, the idea behind the identification of β_0 is to construct a function $m \neq 0$ such that $\mathbb{E}(m(Y, X, \beta_0)|X, \gamma) = 0$ almost surely. Thus, as mentioned in the introduction, we apply [Bonhomme \(2012\)](#)'s general idea of functional differencing. The function m is related to the functions ψ_k in (2.3) when $T = 3$, and the generalization of (2.3) when $T > 3$. For any $x = (x'_1, \dots, x'_T)' \in \mathbb{R}^{KT}$, let $x_s^{-t} = x_s$ if $s < t$, $x_s^{-t} = x_{s+1}$ else. We let

$$M_t(x; \beta) = (-1)^{t+1} \det \begin{pmatrix} \exp(\lambda_1 x_1^{-t'} \beta) & \dots & \exp(\lambda_1 x_{T-1}^{-t'} \beta) \\ \vdots & & \vdots \\ \exp(\lambda_{T-1} x_1^{-t'} \beta) & \dots & \exp(\lambda_{T-1} x_{T-1}^{-t'} \beta) \end{pmatrix}.$$

Then define, for any $(y, x, \beta) \in \{0, 1\}^T \times \text{Supp}(X) \times \mathbb{R}^{K*}$,

$$m(y, x; \beta) := \sum_{t=1}^T \mathbf{1}\{y_t = 1, y_{t'} = 0 \forall t' \neq t\} M_t(x; \beta).$$

Our first result shows that m , indeed, satisfies a conditional moment restriction:

³ We do impose however that all the components of w are non-zero, for normalization purposes. Otherwise, the model with $w = (w_1, 0)$ and β_0 , for instance, would be equivalent to the model with $w = (0, w_1)$ and β_0/λ_2 . A similar issue arises with, e.g., $w = (w_1, w_2, 0)$ if $\lambda_3/\lambda_2 = \lambda_2/\lambda_1$.

Theorem 2.3 *If Assumptions 1-2 hold, we have, almost surely,*

$$\mathbb{E}[m(Y, X; \beta_0)|X, \gamma] = \mathbb{E}[m(Y, X; \beta_0)|X] = 0. \quad (2.4)$$

Theorem 2.3 shows there exists a known moment condition which potentially identifies β_0 in a more general model than the logistic one. Also, as the number of periods T increases, the class of distributions F for which β_0 can be point identified increases. This is consistent with the idea that if $T = \infty$, β_0 is point identified for any F , by using variations in X_t of a single individual. Note however that the class of generalized logistic distribution is not dense for the set of all cdf's: any cdf F belonging to the closure of this class should be such that either $F/(1 - F)$ or $(1 - F)/F$ is convex. Theorem 2.3 also complements the results of Chernozhukov et al. (2013) showing that bounds on β_0 for general F shrink quickly as T increases.

Theorem 2.3 holds with $T = \tau + 1 = 2$. In such a case, the conditional moment condition can be written

$$\mathbb{E}[\mathbf{1}\{Y_1 > Y_2\} \exp(X_2' \beta_0) - \mathbf{1}\{Y_2 > Y_1\} \exp(X_1' \beta_0)|X] = 0.$$

This conditional moment generates the first-order conditions of the maximization of the theoretical conditional likelihood, since these the first-order conditions are equivalent to

$$\mathbb{E} \left[\frac{(X_1 - X_2)}{\exp(X_1' \beta_0) + \exp(X_2' \beta_0)} (\mathbf{1}\{Y_1 > Y_2\} \exp(X_2' \beta_0) - \mathbf{1}\{Y_2 > Y_1\} \exp(X_1' \beta_0)) \right] = 0.$$

2.2 Necessary and sufficient conditions for identification

The discussion above implies that with $T = \tau + 1 = 2$, β_0 is identified by (2.4) as soon as $\mathbb{E}[(X_1 - X_2)(X_1 - X_2)']$ is nonsingular. We now turn to the more difficult case where $T - 1 = \tau > 1$. Let B denote the identified set of β_0 obtained with our conditional moment conditions, namely

$$B := \left\{ b \in \mathbb{R}^{K*} : \mathbb{E}[m(Y, X; b)|X] = 0 \text{ a.s.} \right\}.$$

We also denote by $B_k := \{b_k : \exists b = (b_1, \dots, b_k, \dots, b_K) \in B\}$ ($k = 1, \dots, K$) the identified set of β_{0k} . Our first result shows that B is included in a set depending on

the distribution of X only. To define this set, let us introduce

$$D_j(x; b) := \det \begin{pmatrix} \exp(\lambda_j x'_1 \beta_0) & \dots & \exp(\lambda_j x'_T \beta_0) \\ \exp(\lambda_1 x'_1 b) & \dots & \exp(\lambda_1 x'_T b) \\ \vdots & & \\ \exp(\lambda_{T-1} x'_1 b) & \dots & \exp(\lambda_{T-1} x'_T b) \end{pmatrix}$$

and, for all $b \in \mathbb{R}^{K*}$, let

$$\mathcal{D}(b) = \left\{ x \in \text{Supp}(X) : \begin{aligned} & \max_{j=1, \dots, T-1} D_j(x; b) > \min_{j=1, \dots, T-1} D_j(x; b) \geq 0 \\ & \text{or } \min_{j=1, \dots, T-1} D_j(x; b) < \max_{j=1, \dots, T-1} D_j(x; b) \leq 0 \end{aligned} \right\}.$$

Because $D_j(x; \beta_0) = 0$ for all $x \in \text{Supp}(X)$, we have $\mathbb{P}(X \in \mathcal{D}(\beta_0)) = 0$. The following lemma shows that B is actually included in the set of b 's satisfying this property.

Lemma 2.4 *Suppose that Assumptions 1-2 hold. Then,*

$$B \subset \tilde{B} := \{b \in \mathbb{R}^{K*} : \mathbb{P}(X \in \mathcal{D}(b)) = 0\}.$$

This result follows because the moment condition can be written as a weighted sum of the $D_j(x; b)$'s, with positive weights. It shows that β_0 is identified if for all nonzero $b \neq \beta_0$, we can find some $x \in \text{Supp}(X)$ such that all nonzero $D_j(x; b)$ have the same sign, and the set of such nonzero determinants is not empty.

The set \tilde{B} is convenient in that it does not depend on the unknown distribution of $\gamma|X$; but it is hard to characterize in general. Nevertheless, we are able to obtain results under either of the conditions below.

Assumption 3 *For all $k \in \{1, \dots, K\}$, $\mathbb{P}(|\{X_{k,1}, \dots, X_{k,T}\}| = T, X_{-k} = 0) > 0$.⁴*

Assumption 4 *There exists $(s, t, x) \in \{1, \dots, T\}^2 \times \mathbb{R}^K$, $s < t$ and a neighborhood V of x such that $\text{Supp}(X) \cap [\mathbb{R}^{(s-1)K} \times V \times \mathbb{R}^{(t-s-1)K} \times V \times \mathbb{R}^{(T-t)K}]$ has a non-empty interior.*

⁴When $K = 1$, the condition $X_{-k} = 0$ should simply be omitted.

The first assumption corresponds to a case where all components of X are discrete. It imposes that for all k and t , the support of $X_{k,t}$ includes 0 and at least $T - 1$ additional elements. Because we can always replace $X_{k,\cdot}$ by $X_{k,\cdot} - c_k$ for any $c_k \in \mathbb{R}^T$, the condition $0 \in \text{Supp}(X_{k,t})$ for all k, t holds as long as $\cap_{t=1}^T \text{Supp}(X_{k,t})$ is not empty (for all k). The second condition imposes that all components of X_t are continuous. It also imposes that for at least two periods s and t , $\text{Supp}(X_s) \cap \text{Supp}(X_t)$ is not empty. This last condition holds for instance if $(X_t)_{t \geq 1}$ is strictly stationary.

Theorem 2.5 *Suppose that Assumptions 1-2 hold. Then:*

1. *If Assumption 3 also holds, $|B| < \infty$ and $B_k \subset \{c\beta_{0k} : c \in \{0\} \cup (1/\lambda_{T-1}, \lambda_{T-1})\}$.*
2. *If Assumption 4 also holds,*

$$B \subset \tilde{B} \subset R := \{c\beta_0 : c \in (1/\lambda_{T-1}, \lambda_{T-1})\}. \quad (2.5)$$

Moreover, $|B| \leq T! - 1$ and $|B| \leq 2$ when $T = 3$. All relative effects β_{0j}/β_{0k} , for k such that $\beta_{0k} \neq 0$, are point identified.⁵

Whether Assumption 3 or 4 holds, Theorem 2.5 shows that under-identification is at most finite, namely $|B| < \infty$. This implies that β_0 is locally identified in the sense that there exists a neighborhood of β_0 in which the unique solution to the equation $\mathbb{E}[m(Y, X; b)|X] = 0$ is $b = \beta_0$. Further, the first result of Theorem 2.5 shows that with discrete regressors satisfying Assumption 3, the “length” of the identified set on β_{0k} , defined as

$$\max_{(b_{1k}, b_{2k}) \in B_k^2} |b_{1k} - b_{2k}|,$$

cannot exceed $\beta_{0k}(\lambda_{T-1} - 1/\lambda_{T-1})$ if $0 \notin B_k$. Note that under Assumption 3, we can actually identify whether or not $\beta_{0k} = 0$ without relying on our conditional moments, since the sign of β_{0k} is equal to that of $\mathbb{E}[Y_t - Y_s | X_{-k,s} = X_{-k,t}, X_{k,t} > X_{k,s}]$. The second result on continuous regressors is stronger. It shows that if Assumption 4 holds, β_0 is identified up to a scale c , with c belonging at most to $(1/\lambda_{T-1}, \lambda_{T-1})$. This directly implies point identification of relative marginal effects. The second result also states that B includes at most $T! - 1$ points, and even only 2 points when

⁵The set of indices k such that $\beta_{0k} \neq 0$ is identified since by (2.5), $B_k = \{0\}$ when $\beta_{0k} = 0$, and $0 \notin B_k$ otherwise.

$T = 3$. Importantly, all these results hold for any possible distribution of $\gamma|X$. Thus, point identification may actually hold for many distributions of $\gamma|X$, a point we shall come back to below.

The proof of Theorem 2.5 relies on the following ideas. In the first case, when $b_k \notin \{c\beta_{0k} : c \in \{0\} \cup (1/\lambda_{T-1}, \lambda_{T-1})\}$, we construct a subset of $\text{Supp}(X)$ of positive probability such that all nonzero $D_j(x; b)$ have the same sign. The result then follows by Lemma 2.4. We use a similar reasoning to prove (2.5). To establish the upper bounds on $|B|$, we exploit the fact that the family of exponential functions $(v \mapsto \exp(\zeta_k v))_{k=1, \dots, K}$ with distinct coefficients ζ_k forms a Chebyshev system (see, e.g., Krein and Nudelman, 1977, Chapter II for the formal definition of such systems). This property implies that some key determinants do not vanish, and any non-zero “exponential polynomial” $v \mapsto \sum_{k=1}^K \alpha_k \exp(\zeta_k v)$ does not have more than $K - 1$ zeros.

We now turn to necessary conditions for identification. The following result is a partial converse of Lemma 2.4 and Theorem 2.5 above.

Theorem 2.6 *Suppose that Assumptions 1-2 hold and $T = \tau + 1 \geq 3$. Then:*

1. *If $\mathbb{P}(|\{X_1, \dots, X_T\}| = T) = 0$, then $B = \mathbb{R}^{K^*}$.*
2. *If $T = 3$, then, for any $b \in R$, there exists a distribution of $\gamma|X$ such that for the corresponding distribution of $Y|X$, $b \in B$.⁶*

The first result shows that for our conditional moments to have any identifying power, there must exist trajectories of $X = (X_1, \dots, X_T)$ with distinct values at all periods. Since we focus here on $T \geq 3$, this excludes in particular the case where X_t is binary. More generally, if all components of X_t are binary, one must have $K > \log(T)/\log(2)$ for our moment conditions to have some identifying power. The second result shows that when $T = 3$, one cannot improve (2.5), at least in a uniform sense over conditional distributions of γ . Specifically, for any $b \in R$, there exists a data generating process satisfying Assumptions 1-2 and for which $b \in B$. Note however that failure of point identification at b implies strong restrictions on the distribution of $\gamma|X$. If $b \in B$ with $b \neq \beta_0$, then, for almost all x ,

$$E [a_1(\gamma, x)D_1(x, b) + a_2(\gamma, x)D_2(x, b)|X = x] = 0, \quad (2.6)$$

⁶Note that B depends on the distribution of $\gamma|X$ but as before, we leave this dependence implicit.

where $a_i(\gamma, x)$ is defined in (A.20). Namely, the distribution of $\gamma|X$ should satisfy a conditional moment restriction (note that (2.6) trivially holds when b is replaced by β_0 , because $D_1(x, \beta_0) = D_2(x, \beta_0) = 0$). A violation of (2.6) on a set of x of positive measure is sufficient to discard b from B .⁷

3 GMM estimation

3.1 Efficiency bounds

We now suppose point identification based on (2.4) (namely, $B = \{\beta_0\}$) and discuss estimation of β_0 . Let $R(X) = \mathbb{E}[\nabla_{\beta} m(Y, X; \beta_0)|X]$, $\Omega(X) = \mathbb{V}[m(Y, X; \beta_0)|X]$ (so that $\Omega(X) \in \mathbb{R}$) and define, provided that it exists,

$$V_0 := \mathbb{E} \left[\Omega(X)^{-1} R(X) R(X)' \right]^{-1}.$$

As shown by Chamberlain (1987), asymptotically optimal estimators of β_0 based on (2.4) have an asymptotic variance equal to V_0 . The standard way to construct such estimators consists in two steps: first, one uses the unconditional moment $g(X)m(Y, X; \beta)$ for some $g(\cdot)$ and second, one estimates the optimal instruments $g^*(X) := R(X)/\Omega(X)$. Such estimators, however, are not consistent if

$$\mathbb{E}[g(X)m(Y, X; \beta)] = 0 \quad \text{or} \quad \mathbb{E}[g^*(X)m(Y, X; \beta)] = 0$$

for $\beta \neq \beta_0$; see Dominguez and Lobato (2004). Instead, we can use an efficient GMM estimator exploiting the continuum of moment conditions associated with (2.4). We refer in particular to Sections 4 in Hsu and Kuan (2011) and Section 2.5 in Lavergne and Patilea (2013) for the construction of such estimators.

These GMM estimators are optimal among those based on (2.4). However, it is not obvious that (2.4) actually exhausts all the possible restrictions induced by the model, and therefore that V_0 is the semiparametric efficiency bound of β_0 . Theorem 3.1 below shows that this is the case for $T = \tau + 1 = 3$ under the following conditions.

⁷Related to this, we establish point identification of β_0 under some restrictions on the conditional distribution of $\gamma|X$ in a [previous version](#) of the paper.

- Assumption 5**
1. *There exists $t \in \{1, \dots, T\}$ such that $\mathbb{E}[X_t X_t']$ is nonsingular.*
 2. *$\mathbb{E}[\Omega^{-1}(X)R(X)R(X)']$ exists and is nonsingular.*
 3. *$|\text{Supp}(\gamma|X)| \geq 10$ almost surely.*

The first condition is a mild restriction on X . The second condition is a local identifiability condition, which is neither weaker nor stronger than $B = \{\beta_0\}$. The third condition is weaker than that imposed by Chamberlain (2010), namely $\text{Supp}(\gamma|X) = \mathbb{R}$. Intuitively, if $\gamma|X$ has few points of support, moments of $\gamma|X$ are restricted, and we may exploit this to produce additional restrictions that would improve an estimation of β_0 based solely on (2.4).

Theorem 3.1 *Assume $T = \tau + 1 = 3$ with $\lambda_2 \neq 2$ and Assumptions 1, 2 and 5 hold. Then the semiparametric efficiency bound of β_0 , $V^*(\beta_0)$, is finite and satisfies $V^*(\beta_0) = V_0$.*

Intuitively, this result states that all the information content of the model is included in the conditional moment restriction $\mathbb{E}[m(Y, X; \beta_0)|X] = 0$. It complements, for $T = \tau + 1 = 3$, the result of Hahn (1997), which states that the conditional maximum likelihood estimator is the efficient estimator of β_0 if F is logistic. Note however that we cannot compare his bound with ours in the logistic case: for this distribution, $w_2 = 0$, and for identification reasons, this case is excluded from our family of generalized logistic distributions with $\tau = 2$. We refer to Footnote 3 above for more details about this.

3.2 Unbalanced panel

In many applications, as that considered below, panel data are unbalanced. To handle this case, we can simply consider, for each individual, all possible subsets of periods of size $\tau + 1$ and form the corresponding moment conditions. Specifically, suppose that the set of periods available for individual i is $\mathcal{T}_i \subset \{1, \dots, T\}$. Thus, we observe the sample $((Y_{it}, X_{it})_{t \in \mathcal{T}_i})_{i=1, \dots, n}$. Let us assume that the selection of periods is (conditionally) exogenous, namely

$$\mathcal{T}_i \perp\!\!\!\perp (Y_{it})_{t \geq 1} | (X_{it})_{t \geq 1}, \gamma_i. \tag{3.1}$$

Then, we basically get back to the case $T = \tau + 1$ by considering the moment vector

$$\psi(Y_i, X_i, \mathcal{T}_i, \beta) = \mathbb{1}\{|\mathcal{T}_i| \geq \tau + 1\} \sum_{\substack{t_1 < \dots < t_{\tau+1} \\ (t_1, \dots, t_{\tau+1}) \in \mathcal{T}_i^{\tau+1}}} g(X_{it_1}, \dots, X_{it_{\tau+1}}) m((Y_{it}, X_{it})_{t \in \{t_1, \dots, t_{\tau+1}\}}, \beta).$$

for some function $g(X_{it_1}, \dots, X_{it_{\tau+1}}) \in \mathbb{R}^L$, with $L \geq K$. Condition (3.1) ensures that $\mathbb{E}[\psi(Y_i, X_i, \mathcal{T}_i, \beta_0)] = 0$. Then, we can consider the GMM estimator

$$\hat{\beta} = \arg \min_{\beta} \left(\sum_{i=1}^n \psi(Y_i, X_i, \mathcal{T}_i, \beta) \right)' \widehat{W} \left(\sum_{i=1}^n \psi(Y_i, X_i, \mathcal{T}_i, \beta) \right), \quad (3.2)$$

for some symmetric positive definite \widehat{W} . This idea also applies to balanced panel data for which $T > \tau + 1$. In such a case, $\mathcal{T}_i = \{1, \dots, \tau + 1\}$ and (3.1) automatically holds.

4 Application to Brender and Drazen (2008)

Brender and Drazen (2008) study how budget deficits and economic growth affect reelection. To this end, they gather data from multiple sources on 74 countries, over the period 1960-2003. They use two definitions for their binary outcome variable REELECT, one where reelection is defined in a “narrow” sense and another where it is “expanded”, following here their terminology. This also leads to two different samples, as REELECT may be missing in the narrow sense but equal to 0 in the expanded sense. The covariates related to budget deficits are BALCH_term and BALCH_ey. BALCH_term corresponds to the change in ratio of the central government’s balance to GDP over the term in office. BALCH_ey is the change in the balance/GDP ratio between the year preceding the election and the election year. The variable GDPPC_gr is the average annual growth rate of real GDP per capita between two election years. The authors also include in their models two controls, namely a dummy for a new democracy and a dummy of having a majoritarian electoral system. We refer to Brender and Drazen (2008) for more details about the data.

In their main specification, Brender and Drazen (2008) consider a simple logit model, see Table 2 therein. Then, as a robustness check (see their Table 3), they estimate a fixed effect logit model. They show that their main results are robust to including fixed effects. However, the assumption of logistic errors is not warranted, so we

investigate whether the results are robust to this assumption, by considering instead the family of generalized logistic distribution, with $\tau = 2$. We focus on the sample of developed countries as the sample of less developed countries is very small, and thus leads to noisy estimates. Note that the data are not balanced at all: some countries are only observed for 4 periods in the narrow sample (resp. 5 in the expanded sample), while others are observed over 13 (resp. 14) periods. We thus apply the procedure mentioned in Section 3.2. The vector of instruments $g(X_{it_1}, X_{it_2}, X_{it_3})$ is simply the list of the corresponding 15 variables (as $X_{it} \in \mathbb{R}^5$), demeaned over these three periods. We consider $\lambda_2 = 1.2, 1.4, 1.6$ and 1.8 . We do not consider larger values of λ_2 as they seem to lead to numerical instabilities.⁸ Finally, as the GMM objective function may have local optima, we consider 200 random initial points and pick the vector of parameters minimizing the corresponding final objective function.

The results are presented in Table 1. Because the coefficients themselves are not comparable, we focus on the sign of BALCH_ey and on the relative effects with respect to BALCH_ey; note that we were able to recover the exact same estimates as Brender and Drazen (2008) in their Tables 2 and 3. We choose BALCH_ey as the reference variable for relative effects because its coefficient should not be 0, and it has the largest t-test on the logit and fixed effect logit model. For the three methods, the t -statistics of relative effects under the null hypothesis are obtained using the estimated asymptotic variance of $\hat{\beta}$.

Overall, at least two important results seem robust to the distributional assumption on the unobserved terms. First, the sign of BALCH_ey is always positive. Second, the relative effect of BALCH_term and BALCH_ey remain quite stable when considering our FE generalized logistic model, with fluctuations between 0.27 and 0.52 depending on the sample and value of λ_2 that we consider. At the 10% level, we cannot reject that the effect of BALCH_term is actually 0, except in the narrow sample with $\lambda_2 = 1.8$. But the test was already close to not being rejected with the FE logit model on the narrow sample (p-value=0.097), and not rejected with the simple and FE logit models based on the expanded sample (p-values=0.124 and 0.204 respectively). So the most important results seem overall robust to the change of specification we consider. Other results fluctuate slightly more: the fact of being a new democracy had a positive and

⁸This may be because $|M_t(x; \beta)|$ increases quickly with λ_2 , due to the exponential function.

borderline significant effect with the expanded sample (p-value=0.099). It is not significant anymore with our model, the coefficient being sometimes even negative.

Table 1: Estimates of relative effects of budget balances and growth on the probability of reelection in developed economies

λ_2	Logit	FE logit	FE generalized logit			
			1.2	1.4	1.6	1.8
Narrow sample						
Sign of BALCH_ey	>0	>0	>0	>0	>0	>0
BALCH_term/BALCH_ey	0.54 (2.34)	0.55 (1.82)	0.48 (0.03)	0.37 (0.06)	0.37 (0.03)	0.52 (2.49)
GDPPC_gr/BALCH_ey	-0.04 (0.17)	0.27 (0.70)	-0.35 (0.04)	-0.34 (0.07)	-0.33 (0.04)	-0.37 (1.04)
New democracies/BALCH_ey	0.03 (2.69)	0.07 (1.62)	0.05 (0.04)	0.05 (0.08)	0.05 (0.04)	-0.20 (0.03)
Majoritarian electoral system/BALCH_ey	0.02 (1.31)	0.07 (1.52)	-0.10 (0.03)	0.00 (0.02)	0.00 (0.01)	-0.03 (0.20)
Expanded sample						
Sign of BALCH_ey	>0	>0	>0	>0	>0	>0
BALCH_term/BALCH_ey	0.40 (1.44)	0.36 (1.35)	0.34 (0.85)	0.27 (1.64)	0.37 (0.72)	0.47 (0.42)
GDPPC_gr/BALCH_ey	0.09 (0.30)	0.46 (1.20)	-0.09 (0.22)	0.00 (0.00)	-0.14 (0.29)	-0.28 (0.61)
New democracies/BALCH_ey	0.04 (3.11)	0.09 (1.81)	0.02 (0.32)	0.01 (0.08)	-0.00 (0.00)	-0.15 (0.26)
Majoritarian electoral system/BALCH_ey	0.02 (1.74)	0.04 (1.12)	-0.15 (0.99)	-0.10 (1.54)	-0.21 (1.08)	-0.67 (1.14)

Notes: Analytical t-statistics of the coefficient ratios are under parentheses. The estimated asymptotic variance of the simple logit model is obtained through clustering at the country level. Both samples include 23 countries, with on average 7.1 (resp. 7.8) periods per country in the narrow (resp. expanded) sample.

5 Conclusion

This paper studies the identification and root-n estimation of the common slope parameter in a static panel binary model with exogenous and bounded regressors. We first show that when $T \geq 3$ and the unobserved terms belong to a family of generalized logistic distribution, a conditional moment restriction holds. Then, we study the identified set corresponding to these restrictions. In particular, under a restriction on the distribution of covariates only, relative effects are point identified, no matter the distribution of the individual effect. Our identification results lead to a GMM estimator that reaches the semiparametric efficiency bound when $T = 3$. Estimating this model may serve as a robustness check for the fixed effect logit model, something we illustrate in the application.

Our paper also leaves a few questions unanswered. A first one is whether the family of F considered here is the only one for which point identification can be achieved. Another one is whether the GMM estimator still reaches the semiparametric efficiency bound when $T > 3$. Both questions raise difficult issues and deserve future investigation.

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A Proofs of the results

For any real $a \in \mathbb{R}$, we let $\text{sgn}(a) := \mathbf{1}(a > 0) - \mathbf{1}(a < 0)$. For any subset A of a reference space E , we let A^c denote the complement of A in E . The following lemma on “exponential polynomials” is key in the proof of Theorems 2.5 and 3.1.

Lemma A.1 *Let $K \geq 1$, $(\zeta_1, \dots, \zeta_K)$ be K distinct real numbers, $(\alpha_1, \dots, \alpha_K)' \in \mathbb{R}^K$, $(\alpha_1, \dots, \alpha_K) \neq (0, \dots, 0)$ and $P(x) := \sum_{k=1}^K \alpha_k \exp(\zeta_k x)$. Then P has at most $K - 1$ distinct roots.*

The proof is by induction on K and Rolle’s theorem, see e.g. Chapter 2, section 2 of Krein and Nudelman (1977).

A.1 Proposition 2.1

The sufficient part is obvious. To prove necessity, suppose $\beta_0 \neq 0$. Since $\mathbb{E}[(X_t - X_{t'})(X_t - X_{t'})']$ is non singular, there exists a subset \mathcal{S} of the support of $(X_t, X_{t'})$ such that $\mathbb{P}(\mathcal{S}) > 0$ and for all $(x_t, x_{t'}) \in \mathcal{S}$, $(x_t - x_{t'})'\beta_0$ has constant, non-zero sign. Without loss of generality let us assume $(x_t - x_{t'})'\beta_0 > 0$. Let $G(u) := F(u)/(1 - F(u))$. Because G is strictly increasing, we have, for all $g \in \mathbb{R}$,

$$G(x_t'\beta_0 + g) > G(x_{t'}'\beta_0 + g).$$

Equivalently,

$$F(x_t'\beta_0 + g)(1 - F(x_{t'}'\beta_0 + g)) > F(x_{t'}'\beta_0 + g)(1 - F(x_t'\beta_0 + g)).$$

In other words,

$$\mathbb{P}(Y_1 = 1, Y_{t'} = 0 | X_t = x_t, X_{t'} = x_{t'}, \gamma = g) > \mathbb{P}(Y_1 = 0, Y_{t'} = 1 | X_t = x_t, X_{t'} = x_{t'}, \gamma = g),$$

and the result follows by integration over g .

A.2 Theorem 2.3

Let us define

$$A(x, \gamma; \beta) := \begin{pmatrix} \sum_{j=1}^{T-1} w_j \exp(\lambda_j(x'_1\beta + \gamma)) & \dots & \sum_{j=1}^{T-1} w_j \exp(\lambda_j(x'_T\beta + \gamma)) \\ \exp(\lambda_1 x'_1 \beta) & \dots & \exp(\lambda_1 x'_T \beta) \\ \vdots & & \vdots \\ \exp(\lambda_{T-1} x'_1 \beta) & \dots & \exp(\lambda_{T-1} x'_T \beta) \end{pmatrix}.$$

Let $A_i(x, \gamma; \beta)$ denote the i th row of $A(x, \gamma; \beta)$. Then

$$A_1(x, \gamma; \beta) = \sum_{j=1}^{T-1} w_j \exp(\lambda_j \gamma) A_{j+1}(x, \gamma; \beta).$$

It follows that for all $(x, \gamma) \in \text{Supp}(X) \times \mathbb{R}$,

$$\det A(x, \gamma; \beta_0) = 0.$$

By Assumption 2 and since we focus on the first type therein, we have $G(u) := F(u)/(1 - F(u)) = \sum_{j=1}^{T-1} w_j \exp(\lambda_j u)$. Now, developing $\det A(x, \gamma; \beta_0)$ with respect to the first row yields, by definition of the function m ,

$$\sum_{y \in \{0,1\}^T} m(y, x; \beta_0) \prod_{t: y_t=1} G(x'_t \beta_0 + \gamma) = 0.$$

Multiplying this equality by $\prod_t (1 - F(x'_t \beta_0 + \gamma))$ we obtain

$$\sum_{y \in \{0,1\}^T} \left[m(y, x; \beta_0) \prod_{t: y_t=1} F(x'_t \beta_0 + \gamma) \prod_{t: y_t=0} (1 - F(x'_t \beta_0 + \gamma)) \right] = 0.$$

This equation is equivalent to $\mathbb{E}[m(Y, X; \beta_0) | X, \gamma] = 0$ a.s. The result follows.

A.3 Lemma 2.4

Let $b \in \tilde{B}^c$ and let us prove that $b \notin B$. Fix $x \in \mathcal{D}(b)$ and let $\mathcal{J}_x := \{j \in \{1, \dots, T-1\} : D_j(x; b) \neq 0\}$ and

$$a_j(x) := w_j \mathbb{E} \left[\frac{\exp(\lambda_j \gamma)}{\prod_{t=1}^T (1 + \sum_{k=1}^{T-1} w_k \exp(\lambda_k(x'_t \beta_0 + \gamma)))} \middle| X = x \right]. \quad (\text{A.1})$$

Then $\mathcal{J}_x \neq \emptyset$ and

$$\mathbb{E}[m(Y, X; b)|X = x] = \sum_{j \in \mathcal{J}_x} a_j(x) D_j(x; b). \quad (\text{A.2})$$

Moreover, $a_j(x) > 0$ and all the $D_j(x; b)$ for $j \in \mathcal{J}_x$ have the same sign. Thus, $\mathbb{E}[m(Y, X; b)|X = x] \neq 0$. Because $b \in \tilde{B}^c$, we have, by definition of \tilde{B} , $\mathbb{P}(X \in \mathcal{D}(b)) > 0$. Thus, $\mathbb{E}[m(Y, X; b)|X = x] \neq 0$ with positive probability, implying $b \notin B$.

A.4 Theorem 2.5

Part 1

a. $B_k \subset R_k := \{c\beta_{0k} : c \in \{0\} \cup (1/\lambda_{T-1}, \lambda_{T-1})\}$.

Let us fix $k \in \{1, \dots, K\}$, $b = (b_1, \dots, b_K)$ and define

$$\mathcal{X}_{0k} := \{x \in \text{Supp}(X) : x_{j,1} = \dots = x_{j,T} = 0 \forall j \neq k, |\{x_{k,1}, \dots, x_{k,T}\}| = T\}.$$

First, suppose that $\beta_{0k} = 0$ and $b_k \neq 0$. Then, $D_j(x; b)$ does not depend on j . Moreover, because

$$|\{x'_1 b, \dots, x'_T b\}| = |\{x_{k,1} b_k, \dots, x_{k,T} b_k\}| = T,$$

we have $D_j(x; b) \neq 0$ by properties of Chebyshev systems. Thus, $x \in \mathcal{D}(b)$, implying that $\mathcal{X}_{0k} \subset \mathcal{D}(b)$. By Assumption 3, $\mathbb{P}(X \in \mathcal{X}_{0k}) > 0$. Hence, $\mathbb{P}(X \in \mathcal{D}(b)) > 0$. By Lemma 2.4, $b_k \notin B_k$ and $B_k \subset \{0\} = R_k$.

Now, suppose $\beta_{0k} \neq 0$. Then any $b_k \in \mathbb{R}$ can be written as $c\beta_{0k}$. We prove that if $c \notin \{0\} \cup (1/\lambda_{T-1}, \lambda_{T-1})$, then $\mathcal{X}_{0k} \subset \mathcal{D}(b)$. By Lemma 2.4 again, this shows that $B_k \subset R_k$. Let us first suppose that $c \notin \{1/\lambda_{T-1}, \lambda_{T-1}\}$ and fix $x \in \mathcal{X}_{0k}$. Let us show that for each $(j, j') \in \{1, \dots, T-1\}^2$,

$$\text{sign}(D_j(x; b)) = \text{sign}(D_{j'}(x; b)) \neq 0. \quad (\text{A.3})$$

If $c \in (-\infty, 0)$, we have

$$c\lambda_{T-1} < \dots < c\lambda_2 < c < 0 < 1 < \lambda_2 < \dots < \lambda_{T-1}. \quad (\text{A.4})$$

If $c \in (0, 1/\lambda_{T-1})$, we have

$$0 < c < c\lambda_2 < \dots < c\lambda_{T-1} < 1 < \lambda_2 < \dots < \lambda_{T-1}. \quad (\text{A.5})$$

Else, $c \in (\lambda_{T-1}, +\infty)$ and we have

$$1 < \lambda_1 < \dots < \lambda_{T-1} < c < c\lambda_2 < \dots < c\lambda_{T-1}. \quad (\text{A.6})$$

Let p_j denote the number of transpositions (ie permutations exchanging two elements, leaving the others fixed) needed to sort $\tilde{\lambda}^j := (\lambda_j, c, c\lambda_2, \dots, c\lambda_{T-1})'$ in ascending order. It is clear from Equations (A.4)-(A.6) that $p_j = p_{j'} = p$ for all $(j, j') \in \{1, \dots, T-1\}^2$. Let $\tilde{\lambda}^{sj}$ denote the sorted version of $\tilde{\lambda}^j$ and define

$$D_j(x; b, \lambda) =: \det \begin{pmatrix} \exp(\lambda_j x'_1 \beta_0) & \dots & \exp(\lambda_j x'_T \beta_0) \\ \exp(\lambda_1 x'_1 b) & \dots & \exp(\lambda_1 x'_T b) \\ \vdots & & \\ \exp(\lambda_{T-1} x'_1 b) & \dots & \exp(\lambda_{T-1} x'_T b) \end{pmatrix},$$

so that $D_j(x; b) = D_j(x; b, \lambda)$. Because $x \in \mathcal{X}_{0k}$, we have

$$D_j(x; b, \lambda) = \det \begin{pmatrix} \exp(\lambda_j x_{k,1} \beta_{0k}) & \dots & \exp(\lambda_j x_{k,T} \beta_{0k}) \\ \exp(c\lambda_1 x_{k,1} \beta_{0k}) & \dots & \exp(c\lambda_1 x_{k,T} \beta_{0k}) \\ \vdots & & \\ \exp(c\lambda_{T-1} x_{k,1} \beta_{0k}) & \dots & \exp(c\lambda_{T-1} x_{k,T} \beta_{0k}) \end{pmatrix} = D_j(x; \beta_0, \tilde{\lambda}^j).$$

Hence, for all $j \in \{1, \dots, T-1\}$,

$$\text{sgn}(D_j(x; b)) = \text{sgn}(D_j(x; \beta_0, \tilde{\lambda}^j)) = (-1)^p \text{sgn}(D_j(x; \beta_0, \tilde{\lambda}^{sj})).$$

Now, let \bar{p} be the number of pairwise coordinates permutations needed to sort the vector $(x'_1 \beta_0, \dots, x'_T \beta_0)'$ in ascending order, and let x^s denote a rearrangement of x such that $x_1^s \beta_0 < \dots < x_T^s \beta_0$. It follows that, for all $j \in \{1, \dots, T-1\}$,

$$\begin{aligned} \text{sgn}(D_j(x; b)) &= (-1)^p \text{sgn}(D_j(x; \beta_0, \tilde{\lambda}^{sj})) \\ &= (-1)^{p+\bar{p}} \text{sgn}(D_j(x^s; \beta_0, \tilde{\lambda}^{sj})) \\ &= (-1)^{p+\bar{p}}, \end{aligned}$$

where the last equality follows by properties of Chebyshev systems. The last equality implies that (A.3) holds. Hence $x \in \mathcal{D}(b)$, implying $\mathbb{P}(X \in \mathcal{D}(b)) > 0$.

Finally, consider the case where $b = c\beta_0$ with $c \in \{1/\lambda_{T-1}, \lambda_{T-1}\}$. By continuity of the determinant and (A.3), we either have $0 \leq \min_{j=1, \dots, T-1} D_j(x; b) \leq \max_{j=1, \dots, T-1} D_j(x; b)$ or $0 \geq \max_{j=1, \dots, T-1} D_j(x; b) \geq \min_{j=1, \dots, T-1} D_j(x; b)$. Moreover, $D_{T-1}(x; \beta_0/\lambda_{T-1}) \neq 0$ and $D_1(x; \lambda_{T-1}\beta_0) \neq 0$. Therefore, whatever the value of c ($1/\lambda_{T-1}$ or λ_{T-1}), we have $x \in \mathcal{D}(b)$. Then, again, $\mathbb{P}(X \in \mathcal{D}(b)) > 0$. The result follows.

b. $|B| < \infty$.

Because $|B| \leq \prod_{k=1}^K |B_k|$, it suffices to prove that for each k , $|B_k| < \infty$. Fix k . If $\beta_{0k} = 0$, then $B_k = \{0\}$ and we have nothing to prove. Otherwise, let $b = (b_1, \dots, b_K) \in B$ and fix $x = (x_1, \dots, x_T) \in \mathcal{X}_{0k}$. Let $c \in \{0\} \cup (1/\lambda_{T-1}, \lambda_{T-1})$ be such that $b_k = c\beta_{0k}$. By Equation (A.2), we have $\sum_{j=1}^{T-1} a_j(x)D_j(x; b) = 0$, where $a_j(x)$ is defined by (A.1). Moreover, by definition of \mathcal{X}_{0k} , we have $D_j(x; b) = D_j(x; c\beta_0)$. Then, c satisfies

$$\sum_{j=1}^{T-1} a_j(x)D_j(x; c\beta_0) = 0, \quad (\text{A.7})$$

Developing $D_j(x; c\beta_0)$ with respect to the first line, and using the definition of the determinant, we obtain

$$\sum_{t=1}^T (-1)^{t+1} \sum_{j=1}^{T-1} a_j(x) \exp(\lambda_j x'_t \beta_0) \sum_{\sigma \in \mathfrak{S}_t} \varepsilon(\sigma) \exp \left[\left(\sum_{s \neq t} \lambda_{\sigma(s)} x'_s \beta_0 \right) c \right] = 0, \quad (\text{A.8})$$

where \mathfrak{S}_t is the set of bijections from $\{1, \dots, T\} \setminus \{t\}$ to $\{1, \dots, T-1\}$ and $\varepsilon(\sigma)$ denotes the parity of σ (we can assimilate σ to a permutation by assimilating $\{1, \dots, T\} \setminus \{t\}$ with $\{1, \dots, T-1\}$, keeping the natural ordering of both sets). The left-hand side of (A.8) is a function of c of the form $\sum_{k=1}^K d_k \exp(b_k c)$, with $K \leq T!$ (the inequality arises because some coefficients in the exponential monomials may be equal). Let us show that $d_k \neq 0$ for at least one k . First, remark that $x'_t \beta_0 = x_{k,t} \beta_{0,k}$. Then, because $|\{x_{k,1}, \dots, x_{k,T}\}| = T$ and $\beta_{0k} \neq 0$, we can assume without loss of generality, up to a rearrangement of periods, that $x'_1 \beta_0 < \dots < x'_T \beta_0$. Let I_t be the element of \mathfrak{S}_t such that $I_t(s) = s - \mathbf{1}_{\{s \geq t+1\}}$. By the rearrangement inequality, for all $\sigma \in \mathfrak{S}_t \setminus \{I_t\}$,

$$\sum_{s \neq t} \lambda_{\sigma(s)} x'_s \beta_0 < \sum_{s \neq t} \lambda_{I_t(s)} x'_s \beta_0.$$

Moreover, for all $t \in \{1, \dots, T\}$, let

$$g(t) := \sum_{s \neq t} \lambda_{I_t(s)} x'_s \beta_0.$$

Because $I_t(s) = I_{t-1}(s)$ for all $t > 1$ and $s \leq t - 2$ or $s \geq t + 1$, we have

$$g(t) - g(t - 1) = \lambda_{t-1} x'_{t-1} \beta_0 - \lambda_{t-1} x'_t \beta_0 < 0.$$

Hence, the exponential monomial with highest coefficient in (A.8) is

$$\exp \left[\left(\sum_{s \neq 1} \lambda_{s-1} x'_s \beta_0 \right) c \right]$$

and we can obtain it only by letting $t = 1$ and $\sigma = I_1$. Because $a_j(x) > 0$ for all j , the coefficient of this monomial is $\sum_{j=1}^{T-1} a_j(x) \exp(\lambda_j x'_1 \beta_0) > 0$. Therefore, at least one d_k in the exponential polynomial $\sum_{k=1}^K d_k \exp(b_k c)$ satisfies $d_k \neq 0$. Then, by Lemma A.1, the equation $\sum_{k=1}^K d_k \exp(b_k c) = 0$ has at most $T! - 1$ solutions. Thus, $|B_k| \leq T! - 1$. The result follows.

Part 2

The point identification of relative marginal effects is obvious given the other results, which we prove in turn.

a. Equation (2.5) holds.

Let us define, for all $b \in \mathbb{R}^{K^*}$,

$$\begin{aligned} \mathcal{X}_1(b) = \{ & x = (x_1, \dots, x_T) \in \text{Supp}(X) : \exists (s, t) \in \{1, \dots, T\}^2 : x'_s b = x'_t b, x'_s \beta_0 \neq x'_t \beta_0, \\ & \text{and } \forall (s', t') \in \{1, \dots, T\}^2, s' \neq t', \{s', t'\} \neq \{s, t\} : x'_{s'} b \neq x'_{t'} b \}. \end{aligned}$$

The proof of is divided into three steps. First, we prove that $\mathcal{X}_1(b) \subset \mathcal{D}(b)$, for all $b \in \mathbb{R}^{K^*} \setminus \text{lin}(\beta_0)$. In a second step, we prove that $\tilde{B} \subset \text{lin}(\beta_0)$. Finally, the third step shows that $\tilde{B} \subset R$.

First step: $\mathcal{X}_1(b) \subset \mathcal{D}(b)$ for all $b \in \mathbb{R}^{K^*} \setminus \text{lin}(\beta_0)$.

Let $x \in \mathcal{X}_1(b)$ and (s, t) be as in the definition of $\mathcal{X}_1(b)$. Developing $D_j(x; b)$ according to the first row, we obtain, for all $j \in \{1, \dots, T - 1\}$,

$$D_j(x; b) = \sum_{\ell=1}^T (-1)^{\ell+1} \exp(\lambda_j x'_\ell \beta_0) D_j^{-\{1, \ell\}}(x; b),$$

where $D_j^{-\{1,\ell\}}(x; b)$ denotes the determinant of the matrix in $D_j(x; b)$ once its first row and ℓ th column have been removed. Remark that, for all $j \in \{1, \dots, T-1\}$, for all $\ell \in \{1, \dots, T\} \setminus \{s, t\}$, $D_j^{-\{1,\ell\}}(x; b) = 0$, and

$$D_j^{-\{1,t\}}(x; b) = (-1)^{|s-t|-1} D_j^{-\{1,s\}}(x; b).$$

As a result,

$$\begin{aligned} D_j(x; b) &= (-1)^{s+1} \exp(\lambda_j x'_s \beta_0) D_j^{-\{1,s\}}(x; b) + (-1)^{t+1} \exp(\lambda_j x'_t \beta_0) D_j^{-\{1,t\}}(x; b) \\ &= D_j^{-\{1,s\}}(x; b) \left[(-1)^{s+1} \exp(\lambda_j x'_s \beta_0) + (-1)^{t+1} \exp(\lambda_j x'_t \beta_0) (-1)^{|s-t|-1} \right] \\ &= D_j^{-\{1,s\}}(x; b) (-1)^{s+1} [\exp(\lambda_j x'_s \beta_0) - \exp(\lambda_j x'_t \beta_0)], \end{aligned}$$

where we have used $(-1)^{|s-t|+t} = (-1)^s$. Now, $D_j^{-\{1,s\}}(x; b)$ does not depend on j and by definition of Chebyshev systems, $D_j^{-\{1,s\}}(x; b) \neq 0$. Also, the sign of the term inside brackets is equal to the sign of $(x_s - x_t)' \beta_0$, and thus does not depend on j . Hence for all $(j, j') \in \{1, \dots, T-1\}^2$,

$$\text{sgn}(D_j(x; b)) = \text{sgn}(D_{j'}(x; b)) \neq 0,$$

which shows that $x \in \mathcal{D}(b)$.

Second step: $\tilde{B} \subset \text{lin}(\beta_0)$.

Fix $b \notin \text{lin}(\beta_0)$, $b \neq 0$ and let us prove that $\mathbb{P}(X \in \mathcal{D}(b)) > 0$. The result will then follow by Lemma 2.4.

Suppose without loss of generality that (s, t) in Assumption 4 is equal to $(1, 2)$. By that assumption, there exists $\tilde{x} := (x', x', x'_3, \dots, x'_T)' \in \text{Supp}(X)$ and a neighborhood \tilde{V} of \tilde{x} included in $\text{Supp}(X)$. Since b and β_0 are not collinear, there exists $(u'_1, u'_2)' \in \mathbb{R}^{2K}$ such that $(u_1 - u_2)' b = 0$ and $(u_1 - u_2)' \beta_0 \neq 0$. Moreover, up to replacing $(u'_1, u'_2)'$ by $c(u'_1, u'_2)'$ with $c \neq 0$, $(u'_1, u'_2)'$ can be chosen of arbitrarily small norm.

Now, let $x_1 = x_2 = x$ and

$$\begin{aligned} \mathcal{A}(u_1, u_2) &= \left\{ (u'_3, \dots, u'_T)' \in \mathbb{R}^{K(T-2)} : \forall (s, t) \in \{1, \dots, T\}^2, s \neq t, \{s, t\} \neq \{1, 2\} : \right. \\ &\quad \left. (u_s - u_t + x_s - x_t)' b \neq 0 \right\}, \end{aligned}$$

The set $\mathcal{A}(u_1, u_2)$ is dense as the intersection of open, dense subsets of $\mathbb{R}^{K(T-2)}$. Hence, there exists $(u'_3, \dots, u'_T)' \in \mathcal{A}(u_1, u_2)$ with arbitrarily small norm. Then, we can ensure

that $u := (u'_1, \dots, u'_T)'$ satisfies $x^* := \tilde{x} + u \in \tilde{V}$. Moreover, by construction, $x^* \in \mathcal{X}_1(b)$. Then, Step 1 implies $x^* \in \mathcal{D}(b)$ and $D_j(x; b) \neq 0$ for all j . By continuity of the map $x \mapsto D_j(x; b)$ and Assumption 4, there exists a neighborhood of x^* , $\mathcal{V} \subset \mathcal{D}(b)$ such that $\mathbb{P}(X \in \mathcal{V}) > 0$. Hence, $P(X \in \mathcal{D}(b)) > 0$.

Third step: $\tilde{B} \subset R$.

We just have to prove that if $b = c\beta_0$ with $c \in (-\infty, 1/\lambda_{T-1}] \cup [\lambda_{T-1}, +\infty)$ and $c \neq 0$ (since $\beta_0 \neq 0$), then $b \notin B$. The reasoning is exactly the same as in Part 1.a, with just one change: Instead of considering $x \in \mathcal{X}_{0k}$, we consider $x \in \mathcal{X}_0$, with

$$\mathcal{X}_0 := \{x \in \text{Supp}(X) : |\{x'_1\beta_0, \dots, x'_T\beta_0\}| = T\}.$$

b. $|B| \leq T! - 1$.

The reasoning is exactly the same as in Part 1.b, with just two changes. First, we reason directly on B , not on B_k . Second, instead of considering $x \in \mathcal{X}_{0k}$, we consider $x \in \mathcal{X}_0$.

c. $|B| \leq 2$ when $T = 3$.

For any $b = c\beta_0 \in B$, we have, as in Eq. (A.7),

$$a_1(x)D_1(x; c\beta_0) + a_2(x)D_2(x; c\beta_0) = 0 \tag{A.9}$$

for almost all $x \in \text{Supp}(X)$. Suppose there exist three distinct solutions $1, c_1, c_2$ to Equation (A.9), with $1/\lambda_2 < c_1 < c_2 < \lambda_2$. Multiply Eq. (A.9), evaluated at $c = c_1$, by $D_2(x; c_2\beta_0)$. Similarly, multiply Eq. (A.9), evaluated at $c = c_2$, by $D_2(x; c_1\beta_0)$. Subtracting the two expressions, we obtain, since $a_1(x) > 0$,

$$D_1(x; c_1\beta_0)D_2(x; c_2\beta_0) - D_1(x; c_2\beta_0)D_2(x; c_1\beta_0) = 0. \tag{A.10}$$

For any $x \in \mathcal{X}_0$, let $u_t := x'_t\beta_0$. Fixing u_2 and u_3 , (A.10) may be written as

$$P(u_1) := \sum_{k=1}^{13} \alpha_k \exp(\zeta_k u_1) = 0, \tag{A.11}$$

where the α_k and ζ_k are functions of (u_2, u_3) . Suppose first that $c_2 > 1$. Some tedious algebra shows that the smallest ζ_k is $1 + c_1$, and its associated coefficient is equal to

$$\begin{aligned} \alpha_k = & [\exp(c_2(u_2 + \lambda_2 u_3)) - \exp(c_2(u_3 + \lambda_2 u_2))] \\ & \times [\exp(\lambda_2(u_2 + c_1 u_3)) - \exp(\lambda_2(u_3 + c_1 u_2))]. \end{aligned}$$

Because $u_2 \neq u_3$ (as $x \in \mathcal{X}_0$), $\alpha_k \neq 0$. Hence P is nonzero and by Lemma A.1, it has at most 12 zeros. However, under Assumption 4.2 and the second part of Assumption 4.3,

$$|\text{Supp}(X'_1\beta_0|X'_2\beta_0 = u_2, X'_3\beta_0 = u_3) \setminus \{u_2, u_3\}| > 12.$$

Thus, in view of (A.11), P has strictly more than 12 zeros, a contradiction.

Second, suppose that $c_2 < 1$. Then, the largest ζ_k is $\lambda_2(1 + c_2)$, and its associated coefficient is equal to

$$\begin{aligned} \alpha_k = & - [\exp(u_2 + c_2u_3) - \exp(u_3 + c_2u_2)] \\ & \times [\exp(c_1(u_2 + \lambda_2u_3)) - \exp(c_1(u_3 + \lambda_2u_2))]. \end{aligned}$$

Again, $\alpha_k \neq 0$ and we reach a contradiction as before. The result follows.

A.5 Theorem 2.6

Part 1

Let us suppose that $\mathbb{P}(|\{X_1, \dots, X_T\}| = T) = 0$. Let T_1 and $T_2 > T_1$ denote the two random dates, functions of X only, such that $X_{T_1} = X_{T_2}$ almost surely. For all $t \in \{1, \dots, T\}$, let e_t denote the vector of $T - 1$ zeros and a 1 at coordinate t . Let $f(x; b) := \mathbb{E}[m(Y, X; b)|X = x]$. By definition,

$$f(X; b) = \sum_{y \in \{0,1\}^T} \mathbb{P}(Y = y|X) m(y, X; b). \quad (\text{A.12})$$

Moreover, almost surely,

$$\begin{aligned} \mathbb{P}(Y = e_{T_1}|X) &= \int F(X'_{T_1}\beta_0 + \gamma) \prod_{t \neq T_1} (1 - F(X'_t\beta_0 + \gamma)) dF_{\gamma|X}(\gamma) \\ &= \int F(X'_{T_2}\beta_0 + \gamma) \prod_{t \neq T_2} (1 - F(X'_t\beta_0 + \gamma)) dF_{\gamma|X}(\gamma) \\ &= \mathbb{P}(Y = e_{T_2}|X). \end{aligned} \quad (\text{A.13})$$

Next, remark that the matrices in $M_{T_1}(X; b)$ and $M_{T_2}(X; b)$ have the same columns but in different order, with $T_2 - T_1 - 1$ transpositions needed to obtain the same

ordering. Thus, by definition of the determinant, $M_{T_1}(X; b) = -M_{T_2}(X; b)$, which implies

$$m(e_{T_1}, X; b) = -m(e_{T_2}, X; b). \quad (\text{A.14})$$

Moreover, for all $s \notin \{T_1, T_2\}$, $m(e_s, X; b) = 0$ because M_s includes two identical columns (given that $X_{T_1} = X_{T_2}$). Finally, if $\sum_t y_t \neq 1$, we also have $m(y, X; b) = 0$. These last points, combined with (A.12)-(A.14), imply $f(b) = 0$. Thus, $b \in B$ and the result follows.

Part 2

The proof is in two steps. First, we show that for all $b \in R$,

$$\text{sgn}(D_1(X; b)) = -\text{sgn}(D_2(X; b)) \quad \text{a.s.} \quad (\text{A.15})$$

Second, we show that whenever (A.15) holds, we can construct a distribution of $\gamma|X$ such that (2.4) holds. The result then follows.

First step: (A.15) holds.

First, the result holds for $b = \beta_0$ since then $D_j(X; b) = 0$ for $j \in \{1, 2\}$. Otherwise, fix $b = c\beta_0 \in R$ and let $\tilde{\lambda} := (1, c, c\lambda_2)$ and $\check{\lambda} := (\lambda_2, c, c\lambda_2)$. Let p (resp. p') denote the minimal number of pairwise coordinate permutations needed to sort the vector $\tilde{\lambda}$ (resp. $\check{\lambda}$) and let $\tilde{\lambda}^s$ (resp. $\check{\lambda}^s$) be the corresponding vector, sorted in ascending order. If $c \in (1/\lambda_2, 1)$, we have $p = 1$ and $p' = 2$, whereas if $c \in (1, \lambda_2)$, $p = 0$ and $p' = 1$. Hence, in all cases, $p' = p + 1$.

Now, for any $x \in \text{Supp}(X)$, notice that

$$D_1(x; b, \lambda) = D_1(x; \beta_0, \tilde{\lambda}) = (-1)^p D_1(x; \beta_0, \tilde{\lambda}^s), \quad (\text{A.16})$$

$$D_2(x; b, \lambda) = D_2(x; \beta_0, \check{\lambda}) = (-1)^{p'} D_2(x; \beta_0, \check{\lambda}^s). \quad (\text{A.17})$$

Let p'' be the minimal number of pairwise coordinates permutations needed to sort the vector $(x'_1\beta_0, x'_2\beta_0, x'_3\beta_0)$ in ascending order, and let x^s denote the corresponding vector, i.e., such that $x'_{s1}\beta_0 \leq x'_{s2}\beta_0 \leq x'_{s3}\beta_0$. Then

$$D_1(x; \beta_0, \tilde{\lambda}^s) = (-1)^{p''} D_1(x^s; \beta_0, \tilde{\lambda}^s), \quad (\text{A.18})$$

$$D_2(x; \beta_0, \check{\lambda}^s) = (-1)^{p''} D_2(x^s; \beta_0, \check{\lambda}^s). \quad (\text{A.19})$$

Now, by properties of Chebyshev systems, $D_1(x^s; \beta_0, \tilde{\lambda}^s)$ and $D_2(x^s; \beta_0, \check{\lambda}^s)$ are both non-negative. Moreover, both are nonzero if and only if $|\{x'_1\beta_0, x'_2\beta_0, x'_3\beta_0\}| = 3$. The result follows by (A.16)-(A.19) and $(-1)^p = -(-1)^{p'}$.

Second step: if (A.15) holds, there exists a distribution of $\gamma|X$ such that (2.4) holds.

Let us define

$$a_i(\gamma, x) = \frac{w_i \exp(\lambda_i \gamma)}{\prod_{t=1}^T \left(1 + \sum_{j=1}^{T-1} w_j \exp(\lambda_j (x'_t \beta_0 + \gamma))\right)}. \quad (\text{A.20})$$

Then, we have

$$\mathbb{E}[m(Y, X, b)|X = x] = \mathbb{E}[a_1(\gamma, x)|X = x] D_1(x, b) + \mathbb{E}[a_2(\gamma, x)|X = x] D_2(x, b). \quad (\text{A.21})$$

Hence, if $D_1(x, b) = D_2(x, b) = 0$, any distribution of $\gamma|X = x$ satisfies $\mathbb{E}[m(Y, X, b)|X = x] = 0$. Now, suppose that $\text{sgn}(D_1(x, b)) = -\text{sgn}(D_2(x, b)) \neq 0$. Then $R(x) := -D_1(x, b)/D_2(x, b) > 0$. Let us define

$$\gamma_0 := \frac{\ln[w_1 R(x)/w_2]}{\lambda_2 - 1}.$$

Consider for $\gamma|X = x$ the Dirac distribution at γ_0 . Then, from (A.21), we obtain that $\mathbb{E}[m(Y, X, b)|X = x] = 0$. The result follows.

A.6 Theorem 3.1

Let us first summarize the proof. We link the current model with a “complete” model where γ is also observed. This model is fully parametric and thus can be analyzed easily. Specifically, we show in a first step that this complete model is differentiable in quadratic mean (see, e.g. [van der Vaart, 2000](#), pp.64-65 for a definition) and has a nonsingular information matrix. In a second step, we establish an abstract expression for the semiparametric efficiency bound. This expression involves in particular the kernel \mathcal{K} of the conditional expectation operator $g \mapsto \mathbb{E}[g(X, Y)|X, \gamma]$. In a third step, we show that

$$\mathcal{K} = \{(x, y) \mapsto q(x)m(x, y; \beta_0), \mathbb{E}[q^2(X)] < \infty\}. \quad (\text{A.22})$$

The fourth step of the proof concludes.

First step: the complete model is differentiable in quadratic mean and has a nonsingular information matrix. Let $p(y|x, g; \beta) := \mathbb{P}(Y = y|X = x, \gamma = g; \beta)$. We check that the conditions of Lemma 7.6 in [van der Vaart \(2000\)](#) hold. Under, Assumptions 1-2, we have

$$p(y|x, g; \beta) = \prod_{t:y_t=1} F(x'_t\beta + g) \prod_{t:y_t=0} (1 - F(x'_t\beta + g)),$$

where F is C^∞ on \mathbb{R} and takes values in $(0, 1)$. This implies that $\beta \mapsto \ln p(y|x, g; \beta)$ is differentiable. Let $S_\beta := \partial \ln p(Y|X, \gamma; \beta)/\partial \beta$ and let $S_{\beta k}$ denote its k -th component. We prove that $\mathbb{E}[S_{\beta k}^2] < \infty$. First, remark that

$$S_{\beta k} = \sum_{t=1}^T \frac{X_{k,t} F'(X'_t\beta + \gamma)}{[F(X'_t\beta + \gamma)][1 - F(X'_t\beta + \gamma)]} [Y_t - F(X'_t\beta + \gamma)].$$

Next, we have

$$\begin{aligned} |S_{\beta k}| &\leq \sum_{t=1}^T |X_{k,t}| \frac{F'(X'_t\beta + \gamma)}{F(X'_t\beta + \gamma)(1 - F(X'_t\beta + \gamma))} \\ &= \sum_{t=1}^T |X_{k,t}| \frac{\sum_{j=1}^{T-1} w_j \lambda_j e^{\lambda_j(X'_t\beta + \gamma)}}{\sum_{j=1}^{T-1} w_j e^{\lambda_j(X'_t\beta + \gamma)}} \\ &\leq \lambda_{T-1} \sum_{t=1}^T |X_{k,t}|, \end{aligned} \tag{A.23}$$

where we have used the triangle inequality and $|Y_t - F(X'_t\beta + \gamma)| \leq 1$ to obtain the first inequality. Equation (A.23) and Assumption 1.2 imply that $\mathbb{E}[S_{\beta k}^2] < \infty$. By the dominated convergence theorem and again (A.23), $\beta \mapsto \mathbb{E}[S_\beta S'_\beta]$ is continuous. Therefore, the conditions in Lemma 7.6 in [van der Vaart \(2000\)](#) hold, and the complete model is differentiable in quadratic mean. Moreover,

$$\mathbb{E}[S_\beta S'_\beta] = \mathbb{E}[\mathbb{V}(S_\beta|X, \gamma)] = \sum_{t=1}^T \mathbb{E} \left[\left(\frac{F'(X'_t\beta + \gamma)}{[F(X'_t\beta + \gamma)][1 - F(X'_t\beta + \gamma)]} \right)^2 X_t X'_t \right].$$

Then, if for some $v \in \mathbb{R}^K$, $v' \mathbb{E}[S_\beta S'_\beta] v = 0$, we would have $X'_t v = 0$ almost surely for all $t \in \{1, \dots, T\}$. By Assumption 5.1, this implies $v = 0$. Hence, the information matrix $\mathbb{E}[S_\beta S'_\beta]$ is nonsingular.

Second step: V^* depends on the orthogonal projection of $\mathbb{E}[S_{\beta_0}|X, Y]$ on \mathcal{K} . Let $\tilde{\psi} = (\tilde{\psi}_1, \dots, \tilde{\psi}_K)'$ denote the efficient influence function, as defined p.363

of van der Vaart (2000). Then $V^* = \mathbb{E}[\tilde{\psi}\tilde{\psi}']$ and $\mathbb{E}[\tilde{\psi}] = 0$. Let $\mathcal{S} := \text{span}(S_{\beta_0})$, $\mathcal{G} := \{q : \mathbb{E}[q^2(X, \gamma)] < \infty, \mathbb{E}[q(X, \gamma)] = 0\}$ and for any closed convex set A and any $h = (h_1, \dots, h_K)'$, let Π_A denote the orthogonal projection on A and $\Pi_A(h) = (\Pi_A(h_1), \dots, \Pi_A(h_K))'$. By Equation (25.29), Lemma 25.34 (since the complete model is differentiable in quadratic mean by the first step) and the same reasoning as in Example 25.36 of van der Vaart (2000), $\tilde{\psi}$ is the function of (X, Y) of minimal L^2 -norm satisfying

$$\tilde{\chi} = \Pi_{\mathcal{S}+\mathcal{G}}(\tilde{\psi}), \quad (\text{A.24})$$

where $\tilde{\chi}$ is the efficient influence function of the large model. Because this large model is parametric, we have

$$\tilde{\chi} = \mathbb{E}[S_{\beta_0} S'_{\beta_0}]^{-1} S_{\beta_0}. \quad (\text{A.25})$$

Equation (A.24) implies $\mathbb{E}[(\tilde{\psi} - \tilde{\chi})\tilde{\chi}'] = 0$. Thus, defining $\ell_{\beta_0} = \mathbb{E}[S_{\beta_0}|Y, X]$, we get

$$\mathbb{E}[\tilde{\psi}\ell'_{\beta_0}] = \mathbb{E}[\tilde{\psi}S'_{\beta_0}] = \text{Id}, \quad (\text{A.26})$$

Moreover, because $\mathbb{E}[S_{\beta_0}|X, \gamma] = 0$, \mathcal{S} and \mathcal{G} are orthogonal. Thus, (A.24) is equivalent to $\Pi_{\mathcal{S}}(\tilde{\chi}) = \Pi_{\mathcal{S}}(\tilde{\psi})$ and $\Pi_{\mathcal{G}}(\tilde{\chi}) = \Pi_{\mathcal{G}}(\tilde{\psi})$. Moreover, (A.25) implies that $\Pi_{\mathcal{G}}(\tilde{\chi}) = 0$. Hence, $\tilde{\psi} \in \mathcal{K}^K$. Now, because $\Pi_{\mathcal{K}}$ is an orthogonal projector, we have

$$\mathbb{E}[\tilde{\psi}\Pi_{\mathcal{K}}(\ell_{\beta_0})'] = \mathbb{E}[\Pi_{\mathcal{K}}(\tilde{\psi})\ell'_{\beta_0}] = \mathbb{E}[\tilde{\psi}\ell'_{\beta_0}] = \text{Id},$$

where the last equality follows by (A.26). Hence, if $\Pi_{\mathcal{K}}(\ell_{\beta_0})'\lambda = 0$ a.s., we would have $\lambda = 0$. In other words, $\mathbb{E}[\Pi_{\mathcal{K}}(\ell_{\beta_0})\Pi_{\mathcal{K}}(\ell_{\beta_0})']$ is nonsingular. Now, consider the set

$$\mathcal{F} := \left\{ \mathbb{E}[\Pi_{\mathcal{K}}(\ell_{\beta_0})\Pi_{\mathcal{K}}(\ell_{\beta_0})']^{-1}\Pi_{\mathcal{K}}(\ell_{\beta_0}) + v : \mathbb{E}[v\Pi_{\mathcal{K}}(\ell_{\beta_0})'] = 0 \right\}.$$

\mathcal{F} is thus the set of vector-valued functions ψ satisfying the equation $\mathbb{E}[\psi\Pi_{\mathcal{K}}(\ell_{\beta_0})] = \text{Id}$. Hence, $\tilde{\psi}$ being the element of \mathcal{F} with minimum L^2 -norm, we obtain

$$\tilde{\psi} = \mathbb{E}[\Pi_{\mathcal{K}}(\ell_{\beta_0})\Pi_{\mathcal{K}}(\ell_{\beta_0})']^{-1}\Pi_{\mathcal{K}}(\ell_{\beta_0}).$$

Finally, because $V^* = \mathbb{E}[\tilde{\psi}\tilde{\psi}']$,

$$V^* = \mathbb{E}[\Pi_{\mathcal{K}}(\ell_{\beta_0})\Pi_{\mathcal{K}}(\ell_{\beta_0})']^{-1}. \quad (\text{A.27})$$

Third step: (A.22) holds. Let $r \in \mathcal{K}$ and let us prove that $r(y, x) = q(x)m(y, x; \beta_0)$ for some q . First, by definition of \mathcal{K} , we have, for almost all $(g, x) \in \text{Supp}(\gamma, X)$,

$$\begin{aligned}
0 &= r((0, 0, 0), x_0) + r((1, 0, 0), x_0)G(x'_1\beta_0 + g) + r((0, 1, 0), x_0)G(x'_2\beta_0 + g) \\
&+ r((0, 0, 1), x_0)G(x'_3\beta_0 + g) + r((1, 1, 0), x_0)G(x'_1\beta_0 + g)G(x'_2\beta_0 + g) \\
&+ r((1, 0, 1), x_0)G(x'_1\beta_0 + g)G(x'_3\beta_0 + g) + r((0, 1, 1), x_0)G(x'_2\beta_0 + g)G(x'_3\beta_0 + g) \\
&+ r((1, 1, 1), x_0)G(x'_1\beta_0 + g)G(x'_2\beta_0 + g)G(x'_3\beta_0 + g). \tag{A.28}
\end{aligned}$$

Let $a_t := x'_t\beta_0$ for $t \in \{1, 2, 3\}$ and, for the sake of conciseness, let us remove the dependence of r on x . Then, using Assumption 2, we obtain, for almost all (g, x) ,

$$\begin{aligned}
0 &= A_1e^{0 \times g} + A_2e^g + A_3e^{\lambda_2 g} + A_4e^{2g} + A_5e^{2\lambda_2 g} + A_6e^{(1+\lambda_2)g} + A_7e^{3g} + A_8e^{(2+\lambda_2)g} \\
&+ A_9e^{(1+2\lambda_2)g} + A_{10}e^{3\lambda_2 g},
\end{aligned}$$

where

$$\begin{aligned}
A_1 &:= r(0, 0, 0), \\
A_2 &:= w_1 [r(1, 0, 0)e^{a_1} + r(0, 1, 0)e^{a_2} + r(0, 0, 1)e^{a_3}], \\
A_3 &:= w_2 [r(1, 0, 0)e^{\lambda_2 a_1} + r(0, 1, 0)e^{\lambda_2 a_2} + r(0, 0, 1)e^{\lambda_2 a_3}], \\
A_4 &:= w_1^2 [r(1, 1, 0)e^{(a_1+a_2)} + r(1, 0, 1)e^{(a_1+a_3)} + r(0, 1, 1)e^{(a_2+a_3)}], \\
A_5 &:= w_1 w_2 [r(1, 1, 0)(e^{a_1+\lambda_2 a_2} + e^{a_2+\lambda_2 a_1}) + r(1, 0, 1)(e^{a_1+\lambda_2 a_3} + e^{a_3+\lambda_2 a_1}) \\
&\quad + r(0, 1, 1)(e^{a_2+\lambda_2 a_3} + e^{a_3+\lambda_2 a_2})], \\
A_6 &:= w_2^2 [r(1, 1, 0)e^{\lambda_2(a_1+a_2)} + r(1, 0, 1)e^{\lambda_2(a_1+a_3)} + r(0, 1, 1)e^{\lambda_2(a_2+a_3)}], \\
A_7 &:= w_1^3 r(1, 1, 1)e^{a_1+a_2+a_3}, \\
A_8 &:= w_1^2 w_2 r(1, 1, 1) [e^{a_1+a_2+\lambda_2 a_3} + e^{a_1+\lambda_2 a_2+a_3} + e^{\lambda_2 a_1+a_2+a_3}], \\
A_9 &:= w_1 w_2^2 r(1, 1, 1) [e^{a_1+\lambda_2(a_2+a_3)} + e^{a_2+\lambda_2(a_1+a_3)} + e^{a_3+\lambda_2(a_1+a_2)}], \\
A_{10} &:= w_2^3 r(1, 1, 1)e^{\lambda_2(a_1+a_2+a_3)}.
\end{aligned}$$

Since $\lambda_2 = 2$ is excluded by assumption, there are three cases left depending on the number of different exponents in Equation (A.28).

First, we consider $\lambda_2 \notin \{3/2, 3\}$. By Lemma A.1 and because $|\text{Supp}(\gamma|X)| \geq 10$, we obtain $A_k = 0$ for all $k \in \{1, \dots, 10\}$. $A_1 = A_7 = 0$ imply that $r(0, 0, 0) = r(1, 1, 1) =$

0. Next, $A_4 = A_6 = 0$ implies that either $r(1, 0, 1) = r(1, 1, 0) = r(0, 1, 1) = 0$ or

$$\begin{cases} r(1, 1, 0) &= -r(1, 0, 1)e^{\lambda_2(a_3-a_2)} - r(0, 1, 1)e^{\lambda_2(a_3-a_1)}, \\ r(1, 1, 0) &= -r(1, 0, 1)e^{(a_3-a_2)} - r(0, 1, 1)e^{(a_3-a_1)}. \end{cases} \quad (\text{A.29})$$

Consider the second case. $A_5 = 0$ implies, since $(r(1, 0, 1), r(1, 1, 0), r(0, 1, 1)) \neq (0, 0, 0)$,

$$r(1, 1, 0) = -r(1, 0, 1) \frac{e^{a_1+\lambda_2 a_3} + e^{a_3+\lambda_2 a_1}}{e^{a_1+\lambda_2 a_2} + e^{a_2+\lambda_2 a_1}} - r(0, 1, 1) \frac{e^{a_2+\lambda_2 a_3} + e^{a_3+\lambda_2 a_2}}{e^{a_1+\lambda_2 a_2} + e^{a_2+\lambda_2 a_1}}.$$

By assumption, for almost every $x = (x_1, x_2, x_3)$, $a_3 \neq a_2$ and $a_3 \neq a_1$. Then, using the latter display with equation (A.29) yields, since $\lambda_2 \neq 1$,

$$\begin{aligned} r(1, 0, 1) &= r(0, 1, 1) \left[e^{\lambda_2(a_3-a_2)} - e^{a_3-a_2} \right]^{-1} \left[e^{a_3-a_1} - e^{\lambda_2(a_3-a_1)} \right], \\ r(1, 0, 1) &= r(0, 1, 1) \left[e^{\lambda_2(a_3-a_2)} - \frac{e^{a_1+\lambda_2 a_3} + e^{a_3+\lambda_2 a_1}}{e^{a_1+\lambda_2 a_2} + e^{a_2+\lambda_2 a_1}} \right]^{-1} \\ &\quad \times \left[\frac{e^{a_2+\lambda_2 a_3} + e^{a_3+\lambda_2 a_2}}{e^{a_1+\lambda_2 a_2} + e^{a_2+\lambda_2 a_1}} - e^{\lambda_2(a_3-a_1)} \right]. \end{aligned}$$

Since $(r(1, 1, 0), r(1, 0, 1), r(0, 1, 1)) \neq (0, 0, 0)$, these equalities and (A.29) imply that $r(1, 0, 1) \neq 0$ and $r(0, 1, 1) \neq 0$. Then

$$\frac{e^{(1-\lambda_2)a_2} e^{a_3+\lambda_2 a_2+(\lambda_2-1)a_1} - e^{\lambda_2(a_2+a_3)}}{e^{(1-\lambda_2)a_1} e^{\lambda_2(a_1+a_2)} - e^{(\lambda_2-1)a_2+\lambda_2 a_1+a_3}} = \frac{e^{a_3+\lambda_2 a_2+(\lambda_2-1)a_1} - e^{\lambda_2(a_2+a_3)}}{e^{\lambda_2(a_1+a_2)} - e^{(\lambda_2-1)a_2+\lambda_2 a_1+a_3}},$$

which is equivalent to $a_1 = a_2$. By assumption, the set of x for which this occurs is of probability zero. In other words, for almost every x ,

$$r((1, 1, 0), x) = r((1, 0, 1), x) = r((0, 1, 1), x) = 0.$$

$A_2 = A_3 = 0$ implies that either $r(1, 0, 0) = r(0, 1, 0) = r(0, 0, 1) = 0$ or

$$\begin{cases} r(0, 0, 1) &= -e^{(a_1-a_3)}r(1, 0, 0) - e^{(a_2-a_3)}r(0, 1, 0), \\ r(0, 0, 1) &= -e^{\lambda_2(a_1-a_3)}r(1, 0, 0) - e^{\lambda_2(a_2-a_3)}r(0, 1, 0). \end{cases}$$

In the first case, almost surely $r(Y, X) = 0 = 0 \times m(Y, X; \beta_0)$. In the second case, $r(Y, X) = q(X) \times m(Y, X; \beta_0)$ for some $q \in L_X^2$. The result follows.

Now, we turn to $\lambda_2 = 3/2$. Then, for almost all $(g, x) \in \text{Supp}(\gamma, X)$,

$$0 = A_1 e^{0 \times g} + A_2 e^g + A_3 e^{\frac{3}{2}g} + A_4 e^{2g} + (A_5 + A_7) e^{3g} + A_6 e^{\frac{5}{2}g} + A_8 e^{\frac{7}{2}g} + A_9 e^{4g} + A_{10} e^{\frac{9}{2}g}.$$

By Lemma A.1 and because $|\text{Supp}(\gamma|X)| \geq 9$, we obtain $A_5 + A_7 = 0$ and $A_k = 0$ for all $k \notin \{5, 7\}$. $A_1 = A_{10} = 0$ implies that $r(0, 0, 0) = r(1, 1, 1) = 0$ which in turn implies that $A_7 = 0$ and thus $A_5 = 0$. Hence, we have $A_k = 0$ for all $k \in \{1, \dots, 10\}$ and the same reasoning as when $\lambda_2 \notin \{3/2, 3\}$ allows us to obtain the result.

Finally, we consider $\lambda_2 = 3$. Then, for all (g, x) ,

$$0 = A_1 e^{0 \times g} + A_2 e^g + (A_3 + A_7) e^{3g} + A_4 e^{2g} + A_5 e^{6g} + A_6 e^{4g} + A_7 e^{5g} + A_8 e^{7g} + A_9 e^{9g},$$

By Lemma A.1 and because $|\text{Supp}(\gamma|X)| \geq 9$, we obtain $A_3 + A_7 = 0$ and $A_k = 0$ for all $k \notin \{3, 7\}$. $A_1 = A_{10} = 0$ implies that $r(0, 0, 0) = r(1, 1, 1) = 0$ which in turn implies that $A_7 = 0$ and thus $A_3 = 0$. Hence, $A_k = 0$ for all $k \in \{1, \dots, 10\}$ and the result follows again as when $\lambda_2 \notin \{3/2, 3\}$.

Fourth step: conclusion. By Steps 2 and 3, there exists $q_0(X)$ such that $\Pi_{\mathcal{K}}(\ell_{\beta_0}) = q_0(X)m(Y, X; \beta_0)$. Moreover, by definition of the orthogonal projection, $\Pi_{\mathcal{K}}(\ell_{\beta_0}) - \ell_{\beta_0} \in (\mathcal{K}^\perp)^K$. Hence, again by Step 3, we have, for all $q \in L_X^2$,

$$\mathbb{E}[q_0(X)q(X)m(Y, X; \beta_0)^2] = \mathbb{E}[\ell_{\beta_0}q(X)m(Y, X; \beta_0)].$$

This implies that

$$q_0(X)\Omega(X) = \mathbb{E}[\ell_{\beta_0}m(Y, X; \beta_0)|X].$$

As a result, because $\ell_{\beta_0} = \mathbb{E}[S_{\beta_0}|Y, X]$,

$$\begin{aligned} \Pi_{\mathcal{K}}(\ell_{\beta_0}) &= \Omega^{-1}(X)m(Y, X; \beta_0)\mathbb{E}[\ell_{\beta_0}m(Y, X; \beta_0)|X] \\ &= \Omega^{-1}(X)m(Y, X; \beta_0)\mathbb{E}[S_{\beta_0}m(Y, X; \beta_0)|X]. \end{aligned}$$

Then, using (A.27), we obtain

$$V^* = \mathbb{E} \left[\Omega^{-1}(X)\mathbb{E}[S_{\beta_0}m(Y, X; \beta_0)|X]\mathbb{E}[S_{\beta_0}m(Y, X; \beta_0)|X]' \right]^{-1}.$$

Now, by the end of the proof of Theorem 2.3, we have, for all β ,

$$0 = \mathbb{E}_\beta [m(Y, X; \beta)|X, \gamma].$$

As a result,

$$\begin{aligned} 0 &= \nabla_\beta \mathbb{E}_\beta [m(Y, X; \beta)|X, \gamma] \\ &= \mathbb{E}_\beta [\nabla_\beta m(Y, X; \beta)|X, \gamma] + \mathbb{E}_\beta [m(Y, X; \beta)S_\beta|X, \gamma]. \end{aligned}$$

Evaluating this equality at β_0 and integrating over γ yields:

$$\mathbb{E}[S_{\beta_0}m(Y, X; \beta_0)|X] = -\mathbb{E}[\nabla_{\beta}m(Y, X; \beta_0)|X] = -R(X).$$

We conclude that

$$V^* = \mathbb{E} \left[\Omega^{-1}(X)R(X)R(X)' \right]^{-1} = V_0,$$

which is a well-defined matrix by Assumption 5.1.