Supplement to “Difficulties in Testing for Capital Overaccumulation”

Narayana Kocherlakota*

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Abstract

Section A of this Online Appendix generalizes Definition 2 (of dynamic efficiency) to economics with aggregate risk. Section B describes a simple example economy that illustrates why interest rate-based tests of capital overaccumulation may be unreliable in the presence of financial market frictions.

Appendix A

This Appendix generalizes Definition 2 to the case in which $J > 1$. (It is stated for positive transition probability matrices, but could be relaxed to allow for non-negative Markov matrices.) The definition is somewhat complicated because it is necessary to allow for perturbed allocations that are fully history-dependent.

Consider a positive transition probability matrix P, so that $S(P) \equiv \{1, 2, ..., J\}$. An allocation $(c_t^y, c_t^o, K_{t+1})_{t=1}^{\infty}$ is a specification in each date and state of consumption for the

*University of Rochester and NBER.

two extant generations and of capital:

$$c_t^a : S(P)^t \to \mathbb{R}_+, a \in \{y, o\}, t \geq 1$$
$$K_t : S(P)^t \to \mathbb{R}_+, t \geq 1$$

An allocation is feasible if:

$$c_t^y(s^t) + c_t^o(s^t) + K_t(s^t) \leq e_t^y(s^t)(1+g)^t + e_{s_{t-1}}^o(1+g)^{t-1}$$
$$+ (1-\delta)K_{t-1}(s^{t-1}) + (1+g)^{(1-\alpha)(t-1)} A_{s_{t-1}} K_{t-1}(s^{t-1})^\alpha, \forall t, \forall s^t \in S(P)^t$$
$$c_1^y(s_1) + c_1^o(s_1) + K_1(s_1) \leq e^y(s_1)(1+g) + e_{init}^o, \forall s_1 \in S(P).$$

Suppose that, as described in the body of the paper, a stationary transfer plan τ induces a capital allocation K^*. Then:

$$c_t^y(s^t) = e_{s_t}^y(1+g)^t - \tau_{s_t}(1+g)^t - K_{s_t}^*(1+g)^t$$
$$c_t^o(s^t) = e_{s_{t-1}}^o(1+g)^{t-1} + \tau_{s_t}(1+g)^t + K_{s_{t-1}}^*(1-\delta)(1+g)^{t-1} + (1+g)^{t-1} A_{s_{t-1}} K_{s_{t-1}}^{*\alpha}$$
$$K_t(s^t) = K_{s_t}^*, t \geq 1$$

A feasible allocation is dynamically inefficient if there exists a feasible allocation $\{c_t^{y\prime}, c_t^{o\prime}, K_t^\prime\}_{t=1}^\infty$ such that:

$$c_t^{y\prime}(s^t) + c_t^{o\prime}(s^t) \geq c_t^y(s^t) + c_t^o(s^t) \forall t \geq 1, s^t \in S^t$$

with a strict inequality for some $s_1 \in S(P)$. A feasible allocation is dynamically efficient if it is not dynamically efficient.

Appendix B

I first describe an example overlapping generations economy in which all young agents face borrowing constraints, and only some are able to undertake physical investment. I then use the example to illustrate that the market interest rate r being low relative to g does not necessarily imply that capital is overaccumulated.

Description of an Overlapping Generations Model

Consider an infinite horizon overlapping generations economy in which a unit measure of two-period-lived agents is born at each date t. Each agent has the utility function:

$$ln(c^y) + ln(c^o)$$

where c^y is consumption when young and c^o is consumption when old. Each of the agents has an endowment of one unit of consumption when young. There is also a measure 1 of one-period-lived initial old agents who are each endowed with one unit of consumption in period 1.

Half of the young agents at each date are entrepreneurs who are endowed with an investment technology that translates x units, for any $x \geq 0$, of foregone consumption in period t into Ax units of consumption in period $(t + 1)$. These entrepreneurs have no goods endowment when old. The other half of the agents are non-entrepreneurs who are each endowed with y units of consumption when old, but have no investment technology. Note that there is no cross-generational growth in this example.

As in the body of the paper, a government can implement a transfer plan, in which it levies a lump-sum tax τ on each young agent and gives a lump-sum transfer τ to each old agent.

Equilibrium

We now consider the nature of equilibrium in this economy for the cases of a zero transfer plan and a slightly positive transfer plan. **No agents are allowed to borrow.** However, all young agents are allowed to buy one-period bonds. The equilibrium price of the bonds must be such that the demand for bonds equals the zero supply dictated by the borrowing constraint. Throughout, I assume that:

$$y < A$$

$$y < 1.$$

As we shall see, the latter restriction implies that the equilibrium real interest rate is negative, and the former restriction implies that the entrepreneurs never buy bonds (as they are dominated in return by physical investment).

Case 1: $\tau = 0$

Suppose the bond price at date t is q_t. The young entrepreneurs at date t solve the decision problem:

$$max_{k,b} ln(1 - k - q_t b) + ln(Ak + b)$$

$$s.t.\ k \geq 0, b \geq 0$$

The young non-entrepreneurs at date t solve the decision problem:

$$max_b ln(1 - q_t b) + ln(y + b)$$

$$\text{s.t. } b \geq 0.$$

The young entrepreneurs demand positive amounts of bonds if:

$$q_t < 1/y$$

But this positive demand is inconsistent with equilibrium (since nobody can issue bonds). Hence, the equilibrium price of bonds must be no smaller than $(1/y)$ at each date.

Indeed, it is readily seen that any sequence of bond prices $(q_t)_{t=1}^{\infty}$ such that $q_t \geq 1/y$ for all values of t is an equilibrium. In such an equilibrium, no agents demand bonds. The non-entrepreneurs thus consume their endowments. The return to physical investment exceeds that on bonds, as $A > y \geq 1/q_t$. Hence, entrepreneurs choose $k^* = 0.5$, and $b^* = 0$.

It follows that in this equilibrium, the real interest rate path satisfies:

$$r_t \leq y - 1 < 0, t = 1, 2, 3.. \tag{1}$$

and so the bond yield is less than the (zero) growth rate at each date.

Case 2: τ slightly positive

Suppose instead that the tax-transfer τ' is positive but near zero. Then, the equilibrium bond price satisfies:

$$q_t' \geq \frac{(1-\tau')}{(y+\tau')}$$

at all dates. The non-entrepreneurs chooses $b^* = 0$ and so their consumption profiles are given by $(1-\tau', y+\tau')$. The entrepreneurs choose $b^* = 0$ and set k^* so that:

$$\frac{1}{1-\tau'-k^*} = \frac{A}{\tau' + Ak^*}$$

or equivalently:

$$(0.5 - \frac{(A+1)\tau'}{2A}) = k^*. \tag{2}$$

Note that $k^* < 0.5$.

Pareto Dominance

The equilibrium induced by the zero transfer plan implies that the real interest rate is lower than the (zero) growth rate. But does that observation mean that the zero transfer plan is Pareto dominated by the positive transfer plan? We address that question for two distinct cases: $A > 1$ and $A < 1$.

Case 1: $A > 1$

The non-entrepreneurs' welfares under the positive transfer plan are given by:

$$ln(1 - \tau') + ln(y + \tau')$$

Since $y < 1$, this utility level exceeds their welfares:

$$ln(1) + ln(y)$$

under the zero transfer plan.

The entrepreneurs' welfares under the positive transfer plan are given by:

$$ln(1 - \tau' - k^*) + ln(1 + \tau' + Ak^*)$$

where k^* is defined as in (2). These welfares are lower than those under the zero transfer plan:

$$\begin{aligned} & ln(1 - \tau' - k^*) + ln(1 + \tau' + Ak^*) \\ < \; & ln(1 - \tau' - k^*) + ln(1 + A\tau' + Ak^*) \\ \leq \; & max_k ln(1 - k) + ln(1 + Ak). \end{aligned}$$

Hence, the positive transfer plan does not result in a Pareto superior allocation.

Case 2: $A < 1$

The non-entrepreneurs' welfares are unaffected by the size of A. Hence, as above, they are better off under the positive transfer plan.

The entrepreneurs' welfares under the zero transfer plan are given by:

$$ln(0.5) + ln(0.5A).$$

But they receive more utility under the positive transfer plan:

$$\begin{aligned} & ln(0.5) + ln(0.5A) \\ & < ln(0.5 - \tau') + ln(0.5A + \tau')(\text{since } A < 1) \\ & \leq max_{k \geq 0} ln(1 - k - \tau') + ln(Ak + \tau') \\ & = ln(1 - k^* - \tau') + ln(Ak^* + \tau') \end{aligned}$$

where k^* is defined as in (2).

Thus, in this case where $A < 1$, all agents (including the initial old) are better off under the positive transfer plan. The positive transfer plan also results in lower capital ($k^* < 0.5$), so that capital can be said to be overacccumulated in the zero-transfer equilibrium when $A < 1$.

Summary

In this deterministic economy, a zero transfer plan induces an equilibrium in which $r_t < g$ in all dates. Nonetheless, capital can only be said to be overaccumulated in the sense discussed in this paper if $A < 1$. The basic issue is that, with binding borrowing constraints, observed interest rates contain no information about the shadow interest rates of the entrepreneurs who have access to the investment technology.