Supplement to "A discrete choice model for partially ordered alternatives"

(*Quantitative Economics*, Vol. 13, No. 3, July 2022, 863–906)

Eleni Aristodemou
Department of Economics, University of Cyprus

Adam M. Rosen
Department of Economics, Duke University, CeMMAP, and Institute for Fiscal Studies

Section S1 of this supplement provides derivation of formulae used to compute price elasticities in the empirical application in Section 6 of the main text Aristodemou and Rosen (2022). Section S2 provides estimates of features of the distribution of elasticities using simulated data from DGP2 as described in Appendix D of Aristodemou and Rosen (2022).

S1. Derivation of elasticities

The elasticity of the quantity sold of brand b's product of quality y with respect to the price $p_{k\ell}$ of product (k, ℓ) is

$$\eta_{bykl} = \frac{\partial \wp_{by}}{\partial p_{k\ell}} \frac{p_{k\ell}}{\wp_{by}}.$$

We consider elasticities conditional on $Z = z$. Prices $p_{k\ell}$ can be plugged in directly and $\wp_{by} = \wp_{by}(z; \zeta)$ is given in (5.3) in the text. It is then additionally necessary to compute derivatives $\frac{\partial \wp_{by}(z;\zeta)}{\partial p_{k\ell}}$.

For this, we start with (5.3) with the function Δ the difference of bivariate normal CDF values defined in (5.4) and parameters m_1^-, m_2^-, m_1^+, m_2^+ defined in (5.6) and (5.5),

Eleni Aristodemou: aristodemou.d.eleni@ucy.ac.cy
Adam M. Rosen: adam.rosen@duke.edu

We are grateful to three anonymous referees for suggestions that led to substantial improvements in this paper. We thank Tim Christensen, Allan Collard-Wexler, Francesca Molinari, Lars Nesheim, and several conference and seminar audiences for helpful discussions and comments. Xinyue Bei, Khuong (Lucas) Do, and Muyang Ren provided excellent research assistance. We gratefully acknowledge financial support from the UK Economic and Social Research Council through a grant (RES-589-28-0001) to the ESRC Centre for Microdata Methods and Practice (CeMMAP) and through the funding of the "Programme Evaluation for Policy Analysis" node of the UK National Centre for Research Methods, as well as from the European Research Council (ERC) under grants ERC-2009-StG-240910-ROMETA and ERC-2009-AdG, grant agreement 249529. Eleni Aristodemou gratefully acknowledges financial support from ESRC and UCL. Data supplied by Kantar UK Ltd. The use of Kantar UK Ltd. data in this work does not imply the endorsement of Kantar UK Ltd. in relation to the interpretation or analysis of the data. All errors and omissions remain the responsibility of the authors.

 https://doi.org/10.3982/QE1497

respectively. Notation

$$g_{b,y} \equiv g_b(y;\, z,\, \theta), \qquad \tilde{g}_{b,y} \equiv \sigma_b^{-1} g_{b,y}, \qquad \tilde{z}^*_{by} \equiv \sigma_b^{-1} z^*_{by}$$

will also be used. There are three cases to consider, as follows, where Δ_j is used to denote the partial derivative of Δ with respect to its j_{th} argument, $j = 1, \ldots, 4$. The expressions below additionally make use of the fact that for any $(k,\, \ell)$: $\frac{\partial m_2^-}{\partial p_{k\ell}} = \frac{\partial m_2^+}{\partial p_{k\ell}} = 0$:

1. $g_{b,y} < z^*_{by} < g_{b,y+1}$.

$$\wp_{by}(z,\, \zeta) = \Delta\big(\tilde{g}_{b,y},\, \tilde{z}^*_{by},\, m_1^-,\, m_2^-\big) + \Delta\big(\tilde{z}^*_{by},\, \tilde{g}_{b,y+1},\, m_1^+,\, m_2^+\big).$$

Referring back to equations (5.1), (A.6), and (A.11) in the proof of Proposition 2, this can be equivalently written

$$\wp_{by}(z,\, \zeta) = \frac{1}{\sigma_b} \int_{g_b(y;z,\theta)}^{z^*_{by}} \Phi\left(\frac{h_b(y,\, z,\, v,\, \theta) - \rho \dfrac{\sigma_d}{\sigma_b} v}{\sigma_d \sqrt{1-\rho^2}} \right) \phi\left(\frac{v}{\sigma_b} \right) dv$$

$$+ \frac{1}{\sigma_b} \int_{z^*_{by}}^{g_b(y+1;z,\theta)} \Phi\left(\frac{h_b(y,\, z,\, v,\, \theta) - \rho \dfrac{\sigma_d}{\sigma_b} v}{\sigma_d \sqrt{1-\rho^2}} \right) \phi\left(\frac{v}{\sigma_b} \right) dv,$$

where it follows from steps in the proof of Proposition 2 that

$$\Delta\big(\tilde{g}_{b,y},\, \tilde{z}^*_{by},\, m_1^-,\, m_2^-\big) = \frac{1}{\sigma_b} \int_{g_b(y;z,\theta)}^{z^*_{by}} \Phi\left(\frac{h_b(y,\, z,\, v,\, \theta) - \rho \dfrac{\sigma_d}{\sigma_b} v}{\sigma_d \sqrt{1-\rho^2}} \right) \phi\left(\frac{v}{\sigma_b} \right) dv,$$

and

$$\Delta\big(\tilde{z}^*_{by},\, \tilde{g}_{b,y+1},\, m_1^+,\, m_2^+\big) = \frac{1}{\sigma_b} \int_{z^*_{by}}^{g_b(y+1;z,\theta)} \Phi\left(\frac{h_b(y,\, z,\, v,\, \theta) - \rho \dfrac{\sigma_d}{\sigma_b} v}{\sigma_d \sqrt{1-\rho^2}} \right) \phi\left(\frac{v}{\sigma_b} \right) dv.$$

Consequently, we see that

$$\frac{\partial \wp_{by}(z,\, \zeta)}{\partial z^*_{by}} = \Delta_2\big(\tilde{g}_{b,y},\, \tilde{z}^*_{by},\, m_1^-,\, m_2^-\big) + \Delta_1\big(\tilde{z}^*_{by},\, \tilde{g}_{b,y+1},\, m_1^+,\, m_2^+\big) = 0$$

and, therefore,

$$\frac{\partial \wp_{by}(z,\, \zeta)}{\partial p_{k\ell}} = \left\{ \begin{array}{c} \sigma_b^{-1} \left(\dfrac{\partial g_{b,y}}{\partial p_{k\ell}} \Delta_1\big(\tilde{g}_{b,y},\, \tilde{z}^*_{by},\, m_1^-,\, m_2^-\big) + \dfrac{\partial z^*_{by}}{\partial p_{k\ell}} \Delta_2\big(\tilde{g}_{b,y},\, \tilde{z}^*_{by},\, m_1^-,\, m_2^-\big) \right) \\ + \sigma_b^{-1} \left(\dfrac{\partial z^*_{by}}{\partial p_{k\ell}} \Delta_1\big(\tilde{z}^*_{by},\, \tilde{g}_{b,y+1},\, m_1^+,\, m_2^+\big) \right. \\ \left. + \dfrac{\partial g_{b,y+1}}{\partial p_{k\ell}} \Delta_2\big(\tilde{z}^*_{by},\, \tilde{g}_{b,y+1},\, m_1^+,\, m_2^+\big) \right) \\ + \dfrac{\partial m_1^-}{\partial p_{k\ell}} \Delta_3\big(\tilde{g}_{b,y},\, \tilde{z}^*_{by},\, m_1^-,\, m_2^-\big) + \dfrac{\partial m_1^+}{\partial p_{k\ell}} \Delta_3\big(\tilde{z}^*_{by},\, \tilde{g}_{b,y+1},\, m_1^+,\, m_2^+\big) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \sigma_b^{-1}\left(\dfrac{\partial g_{b,y}}{\partial p_{k\ell}} \Delta_1\big(\tilde{g}_{b,y}, \tilde{z}^*_{by}, m_1^-, m_2^-\big) + \dfrac{\partial g_{b,y+1}}{\partial p_{k\ell}} \Delta_2\big(\tilde{z}^*_{by}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) \right) \\ \quad + \dfrac{\partial m_1^-}{\partial p_{k\ell}} \Delta_3\big(\tilde{g}_{b,y}, \tilde{z}^*_{by}, m_1^-, m_2^-\big) + \dfrac{\partial m_1^+}{\partial p_{k\ell}} \Delta_3\big(\tilde{z}^*_{by}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) \end{array} \right\}$$

2. $g_{b,y} \le g_{b,y+1} \le z^*_{by}$. Using similar arguments as in case 1 above,

$$\wp_{by}(z, \zeta) = \Delta\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^-, m_2^-\big)$$

$$\Longrightarrow$$

$$\frac{\partial \wp_{by}(z, \zeta)}{\partial p_{k\ell}} = \left\{ \begin{array}{l} \sigma_b^{-1}\left(\dfrac{\partial g_{b,y}}{\partial p_{k\ell}} \Delta_1\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^-, m_2^-\big) \right. \\ \left. + \dfrac{\partial g_{b,y+1}}{\partial p_{k\ell}} \Delta_2\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^-, m_2^-\big) \right) \\ \quad + \dfrac{\partial m_1^-}{\partial p_{k\ell}} \Delta_3\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^-, m_2^-\big) \end{array} \right\}$$

3. $z^*_{by} \le g_{b,y} \le g_{b,y+1}$. Again, following similar steps as in case 1, it follows that:

$$\wp_{by}(z, \zeta) = \Delta\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big)$$

$$\Longrightarrow$$

$$\frac{\partial \wp_{by}(z, \zeta)}{\partial p_{k\ell}} = \left\{ \begin{array}{l} \sigma_b^{-1}\left(\dfrac{\partial g_{b,y}}{\partial p_{k\ell}} \Delta_1\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) \right. \\ \left. + \dfrac{\partial g_{b,y+1}}{\partial p_{k\ell}} \Delta_2\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) \right) \\ \quad + \dfrac{\partial m_1^+}{\partial p_{k\ell}} \Delta_3\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) \end{array} \right\}$$

Thus to compute these derivatives at a given value of parameters and a given value of observed variables, the following expressions (arguments suppressed) are needed:

$$\frac{\partial g_{b,y}}{\partial p_{k\ell}}, \qquad \frac{\partial g_{b,y+1}}{\partial p_{k\ell}}, \qquad \frac{\partial m_1^-}{\partial p_{k\ell}}, \qquad \frac{\partial m_1^+}{\partial p_{k\ell}}, \qquad \Delta_1, \Delta_2, \Delta_3.$$

Since $g_{b,y} = g_b(y; z, \theta) = \lambda_{b,y} - x_b \beta_b$ from (4.8), we have that

$$\frac{\partial g_{b,y}}{\partial p_{k\ell}} = \frac{\partial \lambda_{b,y}}{\partial p_{k\ell}}, \qquad \frac{\partial g_{b,y+1}}{\partial p_{k\ell}} = \frac{\partial \lambda_{b,y+1}}{\partial p_{k\ell}}.$$

In our application, with the linear specification $\alpha_{by} = \delta_b + \gamma_b p_{by}$, we have that at the estimated parameter vector $\alpha_{b2} > 2\alpha_{b1}$ for all observations. Under this inequality,

$$\lambda_{b1} = \alpha_{b1}, \qquad \lambda_{b2} = \alpha_{b2} - \alpha_{b1},$$

and it follows that

$$\frac{\partial g_{b,1}}{\partial p_{k\ell}} = \frac{\partial \lambda_{b1}}{\partial p_{k\ell}} = 1[b = k, \ell = 1] \cdot \gamma_b,$$

$$\frac{\partial g_{b,2}}{\partial p_{k\ell}} = \frac{\partial \lambda_{b2}}{\partial p_{k\ell}} = \big(1[b=k,\,\ell=2] - 1[b=k,\,\ell=1]\big)\cdot\gamma_b,$$

If there were observations for which instead $\alpha_{b2} \le 2\alpha_{b1}$, then for these observations we would have

$$\lambda_{b1} = \lambda_{b2} = \alpha_{b2}/2,$$

and given the linear specification $\alpha_{by} = \delta_b + \gamma_b p_{by}$, for each $y \in \{1,2\}$,

$$\frac{\partial g_{b,y}}{\partial p_{k\ell}} = \frac{\partial \lambda_{by}}{\partial p_{k\ell}} = 1[b=k,\,\ell=2]\cdot\frac{\gamma_b}{2}.$$

Now consider $\frac{\partial z^*_{by}}{\partial p_{k\ell}}$. Recall that $z^*_{by} \equiv \frac{\alpha_{d2}+\alpha_{by}-2\alpha_{d1}}{y} - x_b\beta_b$ and, therefore, for $y \in \{1,2\}$ and $\tilde{y} = 3-y$:

$$\frac{\partial z^*_{by}}{\partial p_{by}} = \frac{\gamma_b}{y}, \qquad \frac{\partial z^*_{by}}{\partial p_{d1}} = \frac{-2\gamma_d}{y}, \qquad \frac{\partial z^*_{by}}{\partial p_{d2}} = \frac{\gamma_d}{y}, \qquad \frac{\partial z^*_{by}}{\partial p_{b\tilde{y}}} = 0.$$

The variables m_1^- and m_1^+ are defined in (5.6) and (5.5). Applying the linear specification $\alpha_{by} = \delta_b + \gamma_b p_{by}$ and making explicit dependence on (b,y) yields

$$m_1^-(b,y) \equiv \frac{y x_b\beta_b + \delta_d + \gamma_d p_{d1} - \delta_b - \gamma_b p_{by} - x_d\beta_d}{\sqrt{\sigma_b^2 y^2 - 2\rho\sigma_b\sigma_d y + \sigma_d^2}}, \tag{S1}$$

$$m_1^+(b,y) \equiv \frac{y x_b\beta_b + \delta_d + \gamma_d p_{d2} - \delta_b - \gamma_b p_{by} - 2x_d\beta_d}{\sqrt{\sigma_b^2 y^2 - 4\rho\sigma_b\sigma_d y + 4\sigma_d^2}}. \tag{S2}$$

Thus, for $\tilde{y} \neq y$,

$$\frac{\partial m_1^-(b,y)}{\partial p_{by}} = \frac{-\gamma_b}{\sqrt{\sigma_b^2 y^2 - 2\rho\sigma_b\sigma_d y + \sigma_d^2}}, \quad \frac{\partial m_1^-(b,y)}{\partial p_{b\tilde{y}}} = 0,$$

$$\frac{\partial m_1^-(b,y)}{\partial p_{d1}} = \frac{\gamma_d}{\sqrt{\sigma_b^2 y^2 - 2\rho\sigma_b\sigma_d y + \sigma_d^2}}, \quad \frac{\partial m_1^-(b,y)}{\partial p_{d2}} = 0,$$

$$\frac{\partial m_1^+(b,y)}{\partial p_{by}} = \frac{-\gamma_b}{\sqrt{\sigma_b^2 y^2 - 4\rho\sigma_b\sigma_d y + 4\sigma_d^2}}, \quad \frac{\partial m_1^+(b,y)}{\partial p_{b\tilde{y}}} = 0,$$

$$\frac{\partial m_1^+(b,y)}{\partial p_{d2}} = \frac{\gamma_d}{\sqrt{\sigma_b^2 y^2 - 4\rho\sigma_b\sigma_d y + 4\sigma_d^2}}, \quad \frac{\partial m_1^+(b,y)}{\partial p_{d1}} = 0.$$

Finally, there is

$$\Delta(h,k,m_1,m_2) \equiv \Phi_2(k,m_1,m_2) - \Phi_2(h,m_1,m_2),$$

where $\Phi_2(x,y,\rho)$ denotes the CDF of a bivariate normal vector with standard normal marginals and correlation coefficient ρ evaluated at (x,y). Thus the partial derivatives

of $\Delta(h,k,m_1,m_2)$ are obtained as

$$\Delta_1(h,k,m_1,m_2) = -\frac{\partial\Phi_2}{\partial h}(h,m_1,m_2),$$
$$\Delta_2(h,k,m_1,m_2) = \frac{\partial\Phi_2}{\partial k}(k,m_1,m_2),$$
$$\Delta_3(h,k,m_1,m_2) = \frac{\partial\Phi_2}{\partial m_1}(k,m_1,m_2) - \frac{\partial\Phi_2}{\partial m_1}(h,m_1,m_2).$$

Expressions for the relevant partial derivatives of the bivariate normal CDF are

$$\frac{\partial\Phi_2}{\partial x}(x,y,\rho) = \phi(x)\Phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right),$$
$$\frac{\partial\Phi_2}{\partial y}(x,y,\rho) = \phi(y)\Phi\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right),$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal density and cumulative distribution function, respectively.

S2. Estimates of elasticities from simulated data

The data used in Section 6.3 of Aristodemou and Rosen (2022) is not publicly available, but code used to estimate features of the distribution of household elasticities is included in our replication code. In order to demonstrate usage, the file Compute_POR_Elasticities.R was used to compute quantities reported in Table 6 of the main text instead using data simulated from DGP2, which is described in Appendix D.[S1] The resulting estimates using 2000 observations generated from DGP2 are reported here in Table S1.

[S1] In DGP2 prices have bounded support on $\mathbb{R}_+$ and higher-quality products are priced higher than lower-quality products.

TABLE S1. Estimated means and 0.2, 0.5, and 0.8 quantiles of household elasticities using a sample of 2000 observations from DGP2.

	Price elasticities											
$\frac{\partial \log \wp_{by}}{\partial \log p_{k\ell}}$	p_{11}			p_{12}			p_{21}			p_{22}		
$\wp_{11}$												
Mean		−4.764			2.984			0.782			0.423	
Quantiles	−6.966	−3.768	−2.298	1.355	2.428	4.389	0.338	0.631	1.171	0.080	0.374	0.728
$\wp_{12}$												
Mean		1.888			−4.122			0.014			0.489	
Quantiles	0.991	1.780	2.523	−5.300	−3.790	−2.630	0.000	0.000	0.000	0.256	0.462	0.707
$\wp_{21}$												
Mean		1.263			0.024			−7.106			5.017	
Quantiles	0.865	1.252	1.632	0.000	0.000	0.000	−10.167	−5.761	−3.773	2.620	4.302	7.256
$\wp_{22}$												
Mean		0.236			0.402			1.464			−3.161	
Quantiles	0.051	0.216	0.395	0.204	0.379	0.589	0.674	1.331	2.098	−4.284	−2.845	−1.849
$\wp_0$												
Mean		1.086			0.000			1.329			0.000	
Quantiles	0.707	1.067	1.466	0.000	0.000	0.000	0.807	1.377	1.848	0.000	0.000	0.000

REFERENCES

Aristodemou, Eleni and Adam M. Rosen (2022), "A discrete choice model for partially ordered alternatives." *Quantitative Economics*, 13 (3), 863–906. [1, 5]

Co-editor Andres Santos handled this manuscript.

Manuscript received 29 November, 2019; final version accepted 19 January, 2022; available online 24 February, 2022.